On Some Generalized Rapoport–Zink Spaces

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Abstract. We enlarge the class of Rapoport–Zink spaces of Hodge type by modifying the centers of the associated p-adic reductive groups. Such obtained Rapoport–Zink spaces are said to be of abelian type. The class of Rapoport–Zink spaces of abelian type is strictly larger than the class of Rapoport–Zink spaces of Hodge type, but the two type spaces are closely related as having isomorphic connected components. The rigid analytic generic fibers of Rapoport–Zink spaces of abelian type can be viewed as moduli spaces of local G-shtukas in mixed characteristic in the sense of Scholze.

We prove that Shimura varieties of abelian type can be uniformized by the associated Rapoport–Zink spaces of abelian type. We construct and study the Ekedahl–Oort stratifications for the special fibers of Rapoport–Zink spaces of abelian type. As an application, we deduce a Rapoport–Zink type uniformization for the supersingular locus of the moduli space of polarized K3 surfaces in mixed characteristic. Moreover, we show that the Artin invariants of supersingular K3 surfaces are related to some purely local invariants.

1 Introduction

The theory of Rapoport–Zink spaces finds its origin in the work of Drinfeld in [12]. Let E be a finite extension of $\mathbb{Q}_p$, and let Ω_E^d be the complement of all E-rational hyperplanes in the p-adic projective space $\mathbb{P}^{d-1}$ over E. In [12], Drinfeld interpreted this rigid-analytic space Ω_E^d as the generic fibre of a formal scheme over $\mathcal{O}_E$ parametrizing certain p-divisible groups. He used this formal moduli scheme to p-adically uniformize certain Shimura curves and to construct étale coverings of Ω_E^d. In their foundational and seminal work [56], Rapoport and Zink greatly generalized the construction of Drinfeld by introducing general formal moduli spaces of p-divisible groups with EL/PEL structures, and proved these spaces $\breve{\mathcal{M}}$ can be used to uniformize certain pieces of general PEL type Shimura varieties. Moreover, Rapoport and Zink constructed étale coverings $\mathcal{M}_K$ of the generic fibers of these formal moduli spaces, and realized these rigid analytic spaces as étale coverings of more general non-archimedean period domains. Besides their importance in arithmetic geometry and p-adic Hodge theory, it was conjectured by Kottwitz that the ℓ-adic cohomology of these Rapoport–Zink spaces $\mathcal{M}_K$ realizes the local Langlands correspondence for the related local reductive group G; *cf.* [52, Section 5].

Recently, Kim [36] constructed more general formal moduli spaces of p-divisible groups with additional structures. (Here and throughout the rest of this introduction, we assume $p > 2$.) These formal schemes $\breve{\mathbb{M}}$ are called Rapoport–Zink spaces of Hodge type, associated with unramified local Shimura data of Hodge type $(G, [b], \{\mu\})$ (see below). The additional structures on p-divisible groups are given by the so-called crystalline Tate tensors (*cf.* [36, Definition 4.6]) generalizing the EL/PEL structures introduced by Rapoport–Zink (in the unramified case). Kim also constructed a tower $(\mathcal{M}_K)_K$ of rigid analytic spaces (as usual, $K \subset G(\mathbb{Q}_p)$ runs through open compact subgroups of $G(\mathbb{Q}_p)$), when passing to the generic fibers of these formal moduli schemes. These Rapoport–Zink spaces of Hodge type appear as local analogues of the recent work of Kisin [38] on integral canonical models of Shimura varieties of Hodge type. Kim [37] has proved that his Rapoport–Zink spaces of Hodge type can be used to uniformize certain pieces of Shimura varieties of Hodge type. If the unramified local Shimura datum of Hodge type comes from a Shimura datum of Hodge type, Howard and Pappas have given another (global) construction of the associated Hodge type Rapoport–Zink spaces. We refer the reader to [32] for more details.

In this note, we show that we can in fact go one step further: we will construct some (slightly) more general formal and rigid analytic Rapoport–Zink spaces, and we will show that these spaces can be used to uniformize (pieces of) Kisin's integral canonical models Shimura varieties of abelian type; *cf.* [38]. Moreover, we will give some interesting applications to the moduli spaces of K3 surfaces in mixed characteristic.

There are several motivations for our work here. In our previous work [67], we constructed perfectoid Shimura varieties of abelian type. One of the main motivations for this work is to study the local geometric structures of these perfectoid Shimura varieties, and to study the local geometric structures of Kisin's integral models of Shimura varieties of abelian type [38]. Another motivation is the recent developments in the theory of local Shimura varieties. In [57], Rapoport and Viehmann conjectured the existence of a rigid analytic tower $(\mathcal{M}_K)_K$ associated with a local Shimura datum $(G, [b], \{\mu\})$, where[1]

- G is a connected reductive group over $\mathbb{Q}_p$;
- $\{\mu\}$ is a conjugacy class of minuscule cocharacters $\mu \colon \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$;
- $[b]$ is a σ-conjugacy class in the Kottwitz set $B(G, \mu)$ (see [41, Section 6]).

These conjectural local Shimura varieties are intended to be generalizations of Rapoport–Zink spaces, and there should be a theory in the local situation as good as the classical theory of Shimura varieties [11]. Recently, using the theory of perfectoid spaces [60], and the developments in p-adic Hodge theory due to Fargues, Fargues–Fontaine, and Kedlaya–Liu [15, 20, 35], Scholze has almost given a solution for Rapoport and Viehmann's conjecture by constructing moduli of local G-shtukas in mixed characteristic (*cf.* [61]): $(\mathrm{Sht}_K)_K$ as some reasonable geometric objects. These geometric objects are called *diamonds* there, a generalization of perfectoid spaces and analytic adic spaces. Along the way of construction, we get an infinite level moduli

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[1]Here we have followed [57] to write a local Shimura datum as $(G, [b], \{\mu\})$.

space Sht$_\infty$, such that as diamonds we have

$$\text{Sht}_\infty = \varprojlim_K \text{Sht}_K.$$

In fact, Scholze proved more: one can allow the conjugacy class of cocharacters $\{\mu\}$ non-minuscule, contrary to the original requirement of Rapoport and Viehmann in [57], and, in fact, one can allow several $\{\mu\}$'s. Thus, this theory is the mixed characteristic analogue of the theory of moduli of shtukas in the function fields case [28, 42, 70].

Despite its great success, the method of [61] is purely generic:[2] a priori, one has no information on reduction mod p. In the case of EL/PEL Rapoport–Zink spaces $\mathcal{M}_K$, Scholze proved that the associated diamonds $\mathcal{M}_K^\diamond$ are isomorphic to his moduli spaces of local G-shtukas Sht$_K$. From the point of view of moduli, this means that one can switch p-divisible groups with additional structures to local G-shtukas. Thus, in these classical cases, one gets formal integral structures and can talk about reduction mod p. From now on, we assume that G is unramified over $\mathbb{Q}_p$ and fix a reductive integral model $G_{\mathbb{Z}_p}$ of G over $\mathbb{Z}_p$. Using Dieudonné theory, one can prove that the special fibers of formal Rapoport–Zink spaces (of EL/PEL/Hodge type) are closely related to the corresponding affine Deligne–Lusztig varieties

$$X_\mu^G(b) := \left\{ g \in G(L)/G(W) \mid g^{-1}b\sigma(g) \in G(W)\mu(p)G(W) \right\},$$

where $W = W(\overline{\mathbb{F}}_p), L = W_{\mathbb{Q}}, G(W) = G_{\mathbb{Z}_p}(W)$, and σ is the Frobenius. More precisely, in the above definition we have fixed a representative $b \in G(L)$ of the class $[b]$. On $X_\mu^G(b)$, we have an action of $J_b(\mathbb{Q}_p)$, where J_b is the σ-centralizer of b. These objects are defined purely group theoretically, and thus make sense for arbitrary $(G, [b], \{\mu\})$ (as in the case of Scholze's moduli of local G-shtukas). These affine Deligne–Lusztig varieties play a crucial role in understanding the reduction mod p of Shimura varieties; *cf.* [53].

In this paper, we introduce a class of local Shimura data, the so-called unramified local Shimura data of abelian type, and for each such datum $(G, [b], \{\mu\})$, we construct a formal scheme $\breve{\mathcal{M}}$ and a tower of rigid analytic spaces $(\mathcal{M}_K)_K$ such that

- the reduced special fiber is $\mathcal{M}_{\text{red}}(\overline{\mathbb{F}}_p) \simeq X_\mu^G(b)$;
- the rigid analytic (adic) generic fiber is $\breve{\mathcal{M}}_\eta^{\text{ad}} = \mathcal{M}_{G(\mathbb{Z}_p)}$;
- the associated diamonds are $\mathcal{M}_K^\diamond \simeq \text{Sht}_K$.

Moreover, we can prove that there exists a preperfectoid space $\mathcal{M}_\infty$ over L such that

$$\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K,$$

where the meaning of $\sim$ is as in [65, Definition 2.4.1]. This class of unramified local Shimura data of abelian type is strictly larger than the class of unramified local Shimura data of Hodge type. Thus, among all local Shimura data, we find a new and larger class such that

- there exists a formal model $\breve{\mathcal{M}}$, such that $\breve{\mathcal{M}}_\eta^{ad,\diamond} \simeq \text{Sht}_{G(\mathbb{Z}_p)}, \mathcal{M}_{\text{red}}(\overline{\mathbb{F}}_p) \simeq X_\mu^G(b)$;
- there exists a preperfectoid space $\mathcal{M}_\infty$, such that $\mathcal{M}_\infty^\diamond \simeq \text{Sht}_\infty$.

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[2]We have learnt very recently that Scholze's method also produces integral models of local Shimura varieties as v-sheaves; *cf.* [63, 64].

We remark that the analogues of the above two additional structures in the global situation of Shimura varieties of abelian type are known by [38, 67]. They are not known yet for general local Shimura data (or local shtuka data).

A local Shimura datum $(G, [b], \{\mu\})$ is called of unramified Hodge type if G is unramified and there exists an embedding $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}(V), [b'], \{\mu'\})$ of local Shimura data such that $\{\mu'\}$ corresponds to $(1^r, 0^{n-r})$ for some integral $1 \le r \le n = \dim V$. Roughly, the class of local Shimura data of Hodge type is the largest class for which the associated Rapoport–Zink spaces can be realized as moduli of p-divisible groups with additional structures. In this paper we introduce the following notion. A local Shimura datum $(G, [b], \{\mu\})$ is called of unramified abelian type if there exists an unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that we have an isomorphism of the associated adjoint local Shimura data $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\}) \simeq (G_1^{\mathrm{ad}}, [b_1^{\mathrm{ad}}], \{\mu_1^{\mathrm{ad}}\})$. This is the local analogue of a Shimura datum of abelian type.[3] We remark that although, by definition, we only change the centers of the groups, there no longer exists a local Hodge embedding for a general local Shimura datum of abelian type $(G, [b], \{\mu\})$ (as in the corresponding global situation of Shimura varieties). This means that the class of local Shimura data of (unramified) abelian type is strictly larger than the class of (unramified) Hodge type. By Serre's classification [66], the groups G in this larger class consist exactly of all classical groups; see Section 4.

Our first main theorem is as follows. See Theorem 4.6, Proposition 4.17, and Corollary 5.22.

Theorem 1.1 *Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type. Fix a representative $b \in G(L)$ of $[b]$. Then there exists a formal scheme $\breve{\mathcal{M}}(G, b, \mu)$, which is formally smooth, formally locally of finite type over W, such that*

$$\overline{\mathcal{M}}(G, b, \mu)^{\mathrm{perf}} \simeq X_\mu^G(b).$$

Here, $\overline{\mathcal{M}}(G, b, \mu)^{\mathrm{perf}}$ is the perfection of the special fiber $\overline{\mathcal{M}}(G, b, \mu)$, and $X_\mu^G(b)$ is the affine Deligne–Lusztig variety, considered as a perfect scheme by [1,78]. The formal scheme $\breve{\mathcal{M}} := \breve{\mathcal{M}}(G, b, \mu)$ is equipped with a transitive action of $J_b(\mathbb{Q}_p)$, compatible with the action of $J_b(\mathbb{Q}_p)$ on $X_\mu^G(b)$. Moreover, there exist a tower of rigid analytic spaces $(\mathcal{M}_K)_K$ and a preperfectoid space $\mathcal{M}_\infty$ over L such that

(i) $\breve{\mathcal{M}}_\eta^{\mathrm{ad}} = \mathcal{M}_{G(\mathbb{Z}_p)}$;
(ii) $\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K$;
(iii) $\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$;
(iv) *there exists a compatible system of étale morphism $\pi_{\mathrm{dR}} \colon \mathcal{M}_K \to \mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$;*
(v)   *there exists a Hodge–Tate period morphism $\pi_{HT} \colon \mathcal{M}_\infty \to \mathscr{F}\ell_{G,\mu^{-1}}$.*

Here $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$ is the admissible locus in the p-adic flag variety $\mathscr{F}\ell_{G,\mu}$ associated with $(G, \{\mu\})$ (*cf.* [54, Definition A.6] or [9, Definition 3.1]), and $\mathscr{F}\ell_{G,\mu^{-1}}$ is the p-adic flag variety associated with $(G, \{\mu^{-1}\})$. In fact, we will see in Corollary 5.22 that π_{HT} also factors through a locally closed subspace $\mathscr{F}\ell_{G,\mu^{-1}}^b \subset \mathscr{F}\ell_{G,\mu^{-1}}$.

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[3]More precisely, our local Shimura data of abelian type are the local analogues of Shimura data of preabelian type.

The construction of $\check{\mathcal{M}}(G, b, \mu)$ associated with $(G, [b], \{\mu\})$ as above is based on the following observations. Take any unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\}) \simeq (G_1^{\mathrm{ad}}, [b_1^{\mathrm{ad}}], \{\mu_1^{\mathrm{ad}}\})$. We have the associated formal Rapoport–Zink space $\check{\mathcal{M}}(G_1, b_1, \mu_1)$ constructed by Kim [36], by patching together Faltings's construction of deformation ring for p-divisible groups (with crystalline Tate tensors) with Artin's criterion for algebraic spaces. By [78, Proposition 3.11], $\overline{\mathcal{M}}(G_1, b_1, \mu_1)^{\mathrm{perf}} \simeq X_{\mu_1}^{G_1}(b_1)$. For any local Shimura datum $(G, [b], \{\mu\})$, we have a $J_b(\mathbb{Q}_p)$-equivariant surjective map

$$\omega_G \colon X_\mu^G(b) \longrightarrow c_{b,\mu} \pi_1(G)^\Gamma,$$

which factors through the set of connected components $\pi_0(X_\mu^G(b))$. Here $\pi_1(G)$ is the algebraic fundamental algebraic group of G and $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. See Subsection 2.2 for the construction of this map and the element $c_{b,\mu} \in \pi_1(G)$. Moreover, by [7, Theorem 1.2], $J_b(\mathbb{Q}_p)$ acts transitively on $\pi_0(X_\mu^G(b))$. For any local Shimura datum $(G, [b], \{\mu\})$, by [7, Corollary 2.4.2], we have a cartesian diagram

$$
\begin{array}{ccc}
X_\mu^G(b) & \longrightarrow & X_{\mu^{\mathrm{ad}}}^{G_1^{\mathrm{ad}}}(b^{\mathrm{ad}}) \\
\downarrow & & \downarrow \\
c_{b,\mu} \pi_1(G)^\Gamma & \longrightarrow & c_{b^{\mathrm{ad}}, \mu^{\mathrm{ad}}} \pi_1(G^{\mathrm{ad}})^\Gamma.
\end{array}
$$

In particular, we apply the above diagram to $(G, [b], \{\mu\})$ and $(G_1, [b_1], \{\mu_1\})$, as above. Let $X_{\mu_1}^{G_1}(b_1)^+ \subset X_{\mu_1}^{G_1}(b_1)$ be a fixed choice of fiber of the map $\omega_{G_1} \colon X_{\mu_1}^{G_1}(b_1) \to c_{b_1, \mu_1} \pi_1(G_1)^\Gamma$. This is isomorphic to the corresponding subset of $X_\mu^G(b)$. Let

$$\check{\mathcal{M}}(G_1, b_1, \mu_1)^+ \subset \check{\mathcal{M}}(G_1, b_1, \mu_1)$$

be the open and closed subspace corresponding to $X_{\mu_1}^{G_1}(b_1)^+$. As $X_\mu^G(b) = J_b(\mathbb{Q}_p) X_\mu^G(b)^+$, we get the formal scheme $\check{\mathcal{M}}(G, b, \mu)$ whose special fiber satisfies $\overline{\mathcal{M}}(G, b, \mu)^{\mathrm{perf}} \simeq X_\mu^G(b)$. By construction, this formal scheme does not depend on the choice of the Hodge type local Shimura datum $(G_1, [b_1], \{\mu_1\})$. The other properties can be proved similarly.

Let

$$\left(\mathcal{M}(G, b, \mu)_K\right)_{K \subset G(\mathbb{Q}_p)} \quad \text{and} \quad \left(\mathcal{M}(G_1, b_1, \mu_1)_{K_1}\right)_{K_1 \subset G_1(\mathbb{Q}_p)}$$

be the two towers associated with $(G, [b], \{\mu\})$ and $(G_1, [b_1], \{\mu_1\})$, as above. By construction, the two towers are locally isomorphic in the sense that there exist subtowers[4] $\left(\mathcal{M}(G, b, \mu)_K^+\right)_{K \subset G(\mathbb{Q}_p)}$ and $\left(\mathcal{M}(G_1, b_1, \mu_1)_{K_1}^+\right)_{K_1 \subset G_1(\mathbb{Q}_p)}$ such that

$$\mathcal{M}(G, b, \mu)_\infty^+ = \mathcal{M}(G_1, b_1, \mu_1)_\infty^+,$$

where $\mathcal{M}(G, b, \mu)_\infty^+$ is the preperfectoid space over L such that

$$\mathcal{M}(G, b, \mu)_\infty^+ \sim \varprojlim_K \mathcal{M}(G, b, \mu)_K^+,$$

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[4] Here a subtower $(Y_K)_K$ of a tower $(X_K)_K$ of inverse system of rigid analytic spaces is by definition given by an inverse system of subspaces $Y_K \subset X_K$ with compatible transition maps.

and similarly for $\mathcal{M}(G_1, b_1, \mu_1)_\infty^+$. This implies in particular that $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}} = \mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}}$. The tower $(\mathcal{M}(G, b, \mu)_K)_{K \subset G(\mathbb{Q}_p)}$ can be recovered from $(\mathcal{M}(G, b, \mu)_K^+)_{K \subset G(\mathbb{Q}_p)}$ and $\pi_1(G)^\Gamma$ by the action of either $G(\mathbb{Q}_p)$ or $J_b(\mathbb{Q}_p)$. We expect that such results hold true for any local shtuka data $(G, [b], \{\mu\})$ and $(G_1, [b_1], \{\mu_1\})$ with the same adjoint data.

We note that the above construction is simpler than the corresponding global situation (*cf.* [38,67]), where one has to make a quotient on each geometric connected component of Shimura varieties of Hodge type.

In Subsection 4.3 we will try to find a moduli interpretation for the formal scheme $\check{\mathcal{M}}(G, b, \mu)$ associated with $(G, [b], \{\mu\})$ as above (*cf.* Proposition 4.9), which is a priori non canonical, however. It is desirable to find a more canonical moduli interpretation for $\check{\mathcal{M}}(G, b, \mu)$. After the first version of this paper appeared on line, Bültel and Pappas have recently found an intrinsic moduli interpretation for $\check{\mathcal{M}}(G, b, \mu)$ with $(G, [b], \{\mu\})$ of Hodge type under a certain nilpotent condition; *cf.* [4]. They use a notion of (G, μ)-displays, which is purely group theoretical. We can naturally extend Bültel and Pappas's moduli interpretation to certain abelian type case $\check{\mathcal{M}}(G, b, \mu)$ studied in this paper; *cf.* Theorem 4.11. As mentioned above, the further recent progress of [64] will give a canonical moduli interpretation for the formal scheme $\check{\mathcal{M}}(G, b, \mu)$ in the general case, as moduli of local shtukas similar to that in Section 5; *cf.* Remark 5.24(iii).

If the unramified local Shimura datum of abelian type $(G, [b], \{\mu\})$ comes from a Shimura datum of abelian type (G, X), we can prove the following uniformization theorem. Let $K^p \subset G(\mathbb{A}_f^p)$ be a fixed sufficiently small open compact subgroup. Consider S_K, the Kisin integral canonical model over W of the Shimura variety Sh_K with $K = G(\mathbb{Z}_p)K^p$. Let $\phi \colon \mathfrak{Q} \to \mathfrak{G}_G$ be a Langlands–Rapoport parameter with the associated reductive group I_ϕ over $\mathbb{Q}$, such that $[b] = [b(\phi)]$ (see [39], 3.3.6) for the precise meaning of these objects, where a Langlands–Rapoport parameter is called an admissible morphism between the Galois gerbs $\mathfrak{Q}$ and $\mathfrak{G}_G$). Let $\check{\mathcal{M}} = \check{\mathcal{M}}(G, b, \mu)$. Fix a Langlands–Rapoport parameter $\phi_0 \colon \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ for the adjoint group such that $\phi^{\mathrm{ad}} = \phi_0$. In Section 6 we will construct a subspace $\mathcal{Z}_{\phi_0, K^p} \subset \overline{S}_K$, such that the formal completion of S_K along $\mathcal{Z}_{\phi_0, K^p}$ can be defined. The following theorem was proved by Rapoport and Zink in the PEL type case ([56]), and by Kim in the Hodge type case ([37]; see also [32]). It can be viewed as the geometric version of the Langlands–Rapoport description for the underlying $\overline{\mathbb{F}}_p$-points; *cf.* [39]. In fact, it was pointed out in the introduction of [57] that the works of Kisin [38,39] should yield new Rapoport–Zink spaces (comp. [32]). Here, we construct these spaces locally, and show that they admit global application (comp. [57, Remark 5.9]). See Theorems 6.7 and 6.13.

Theorem 1.2   *We have an isomorphism of formal schemes over W,*

$$\Theta \colon \coprod_{[\phi], \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \check{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \widehat{S_K}_{/\mathcal{Z}_{\phi_0, K^p}},$$

where $\phi_0 \colon \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ is a fixed Langlands–Rapoport parameter for the adjoint group G^{ad}, $[\phi]$ runs through the set of isomorphism classes of Langlands–Rapoport parameters ϕ for G such that $\phi^{\mathrm{ad}} = \phi_0$, and $[b] = [b(\phi)]$. When $[b]$ is basic, we have $\mathcal{Z}_{\phi_0, K^p} = \overline{S}_K^b$,

which is the basic locus, and the above isomorphism reduces to

$$\Theta \colon I_\phi(\mathbb{Q}) \backslash \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \widehat{S_K}_{/\overline{S}_K^b}.$$

Unsurprisingly, we apply the tricks of Kisin as in [39] to deduce the theorem from the Hodge type case. One can also deduce rigid analytic and perfectoid versions of the above uniformization theorem.

We consider the examples of basic GSpin and special orthogonal groups Rapoport–Zink spaces. Let $\breve{\mathcal{M}}_1 = \breve{\mathcal{M}}(\mathrm{GSpin}, b, \mu), \breve{\mathcal{M}} = \breve{\mathcal{M}}(\mathrm{SO}, b', \mu')$ be the associated basic Rapoport–Zink spaces, where $\mathrm{GSpin} = \mathrm{GSpin}(V, Q), \mathrm{SO} = \mathrm{SO}(V, Q)$ are unramified GSpin and special orthogonal groups associated with a quadratic space (V, Q) over $\mathbb{Q}_p$, with $\dim V = n + 2$ for some integer $n \geq 1$. By considering the G-zip associated with the universal p-divisible group with crystalline Tate tensors on the special fiber $\overline{\mathcal{M}_1}$ of $\breve{\mathcal{M}}_1$, we can define an Ekedahl–Oort stratification on $\overline{\mathcal{M}_1}$, and thus on $\mathcal{M}_{1\,\mathrm{red}}$ (the reduced special fiber), which is the local analogue of the Ekedahl–Oort stratification for Shimura varieties of Hodge type (*cf.* [77]). The index set of this stratification is a subset $^J\mathcal{W}^b$ of the absolute Weyl group of G_1, which is thus finite. In fact one can find by computation that, it is in bijection with some explicit finite set of integers. For each $w \in {}^J\mathcal{W}^b$, we have the associated Ekedahl–Oort stratum $\mathcal{M}_{1w}$ of $\mathcal{M}_{1\,\mathrm{red}}$. On the other hand, Howard and Pappas [32] introduced another stratification for the reduced special fiber $\mathcal{M}_{1\,\mathrm{red}}$:

$$\mathcal{M}_{1\,\mathrm{red}} = \coprod_\Lambda \mathcal{M}_{1\Lambda}^\circ,$$

where Λ runs through the set of vertex lattices; see loc. cit. Section 5. By Corollary 8.2 $\breve{\mathcal{M}} \simeq \breve{\mathcal{M}}_1/p^{\mathbb{Z}}$, we get the induced Ekedahl–Oort and Howard–Pappas stratifications for $\mathcal{M}_{\mathrm{red}}$. The following theorem is proved in Subsection 8.2: see Theorem 8.4 and Corollary 8.5 for more precise statements.

Theorem 1.3 *Each Ekedahl–Oort stratum $\mathcal{M}_{1w}$ of $\mathcal{M}_{1\,\mathrm{red}}$ is some (disjoint) union of Howard–Pappas strata. A similar result holds for $\mathcal{M}_{\mathrm{red}}$.*

For a similar result in the case of the basic unitary group $GU(1, n-1)$ Rapoport–Zink space, see [69, Theorem D].

In fact, in Subsection 7.1, we construct the Ekedahl–Oort stratification for the special fibers of arbitrary Rapoport–Zink spaces of abelian type, *cf.* Theorem 7.1. We can compare our geometric construction with the Ekedahl–Oort stratification for affine Deligne–Lusztig varieties (with hyperspecial levels) in [21]; *cf.* Proposition 7.2. In Subsection 7.2, we discuss a theorem of similar phenomena as Theorem 1.3 (*cf.* Theorem 7.5) for an unramified local Shimura datum of abelian type $(G, [b], \{\mu\})$, with $(G, \{\mu\})$ fully Hodge–Newton decomposable in the sense of [22, Definition 2.1]. Our discussion in this more general setting is indeed motivated by [22, Theorems 2.3 and 2.5], where a posteriori the classification there (for minuscule μ) lies in our class of local Shimura data of abelian type. The basic GSpin and special orthogonal groups Rapoport–Zink spaces are just special cases where one can make things more explicit (by the work of [32]).

Specializing further to the case of K3 surfaces, we have some interesting applications. Take an integer $d \geq 1$ such that $p \nmid 2d$. Let $M_{2d,K}$ be the moduli spaces of K3 surfaces $f \colon X \to S$ together with a primitive polarization ξ of degree $2d$ and a K-level structure over W. Recall that by the global integral Torelli theorem (*cf.* [47, Corollary 5.15]), the integral Kuga–Satake period map $\iota \colon M_{2d,K} \longrightarrow S_K$ is an open immersion, where S_K is the integral canonical model over W of the Shimura variety Sh_K for $G = \mathrm{SO}(2,19)$; see Subsection 8.3 for more details. Here, we assume that $K = K_p K^p$ with $K_p = G(\mathbb{Z}_p)$ is the fixed hyperspecial subgroup. Let X be a supersingular K3 surface over $\overline{\mathbb{F}}_p$, then the discriminant of its Néron–Severi lattice is equal to $-p^{2\sigma_0(X)}$ for some integer $1 \leq \sigma_0(X) \leq 10$. The integer $\sigma_0(X)$ is called the Artin invariant of X. The following corollary is a consequence of the above theorems. Note that the group G is adjoint and thus $\phi = \phi_0$.

Corollary 1.4 (Corollaries 8.12 and 8.14)   *(i) Let ϕ, $[b]$ and $\mathcal{Z}_{\phi,K^p}$ be as in Theorem 1.2, and let J_ϕ be the pullback of $\mathcal{Z}_{\phi,K^p}$ under the open immersion $\overline{M}_{2d,K} \hookrightarrow \overline{S}_K$ of special fibers. Then we have the identity*

$$\widehat{\overline{M}_{2d,K}}_{/J_\phi} = \coprod_{j \in I} \check{\mathcal{N}}/\Gamma_j,$$

where $\check{\mathcal{N}} \subset \check{\mathcal{M}}(G, b, \mu)$ is an open subspace, I is certain countable set, and for any $j \in I$, $\Gamma_j \subset J_b(\mathbb{Q}_p)$ is some discrete subgroup. If, moreover, $[b] = [b_0]$ is basic, then $J_\phi = \overline{M}_{2d,K}^{ss}$ which is the supersingular locus in $\overline{M}_{2d,K}$, and the above disjoint union is finite.

(ii) Let $x \in \overline{M}_{2d,K}^{ss}(\overline{\mathbb{F}}_p)$ be a point, and let X_x be the associated supersingular K3 surface over $\overline{\mathbb{F}}_p$. Then we have the identity between the Artin invariant $\sigma_0(X_x)$ and the type $t(\Lambda_x)$:

$$\sigma_0(X_x) = \frac{t(\Lambda_x)}{2},$$

where Λ_x is the vertex lattice attached to the special lattice associated with (X_x, ξ_x), cf. Subsection 8.5.

We briefly describe the structure of this article. In Section 2, we review some basics about affine Deligne–Lusztig varieties that will be used later. In Section 3, we first recall the Rapoport–Viehmann conjecture on the theory of local Shimura varieties; then we concentrate on the case of unramified local Shimura datum of Hodge type, and review the construction of Kim [36] on the associated Rapoport–Zink spaces of Hodge type. In Section 4, we introduce unramified local Shimura datum of abelian type and construct the associated formal and rigid analytic Rapoport–Zink spaces. Section 5 is devoted to a review of the general framework of moduli of local G-shtukas in mixed characteristic due to Scholze, to give a moduli interpretation of the generic fibers of our Rapoport–Zink spaces of abelian type. In Section 6, we turn to the global situation of Shimura varieties of abelian type, and prove a Rapoport–Zink type uniformization theorem in this setting. In Section 7, motivated by the study of Artin invariants of K3 surfaces, we construct the Ekedahl–Oort stratification for special fibers of Rapoport–Zink spaces. In Section 8, we discuss some applications of our theory. We work on the examples of basic GSpin and special orthogonal groups Rapoport–Zink spaces, and then more specially on the case of moduli spaces of K3 surfaces. These examples are

just (related to) special cases of the fully Hodge–Newton decomposable Shimura varieties introduced in [22] (see also [68]). Finally, we investigate p-adic period domains in the basic orthogonal case in the appendix following Fargues.

2 Affine Deligne–Lusztig Varieties in Mixed Characteristic

In this section, we recall some basic facts about affine Deligne–Lusztig varieties in mixed characteristic, which will be used later.

Fix a prime p. Let G be a connected reductive group over $\mathbb{Q}_p$, which we assume to be unramified. Fix $T \subset B$ a maximal torus inside a Borel subgroup of G. Let $W = W(\overline{\mathbb{F}}_p)$ be the ring of Witt vectors, and let $L = W_{\mathbb{Q}}$. Denote σ as the Frobenius on L and W. In the sequel we want to fix a hyperspecial subgroup $G(W) \subset G(L)$. To this end, we fix a reductive model $G_{\mathbb{Z}_p}$ of G over $\mathbb{Z}_p$ and set $G(W) = G_{\mathbb{Z}_p}(W)$. Sometimes by abuse of notation, we will also write G as the reductive group $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$.

2.1 Affine Deligne–Lusztig Varieties

For $b \in G(L)$ and a conjugacy class $\{\mu\}$ of cocharacters $\mu \colon \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$, we define the affine Deligne–Lusztig sets

$$X_\mu^G(b) = \left\{ g \in G(L)/G(W) \mid g^{-1}b\sigma(g) \in G(W)\mu(p)G(W) \right\},$$
$$X_{\leq\mu}^G(b) = \left\{ g \in G(L)/G(W) \mid g^{-1}b\sigma(g) \in \bigcup_{\mu' \leq \mu} G(W)\mu'(p)G(W) \right\}.$$

Here, we assume $\mu \in X_*(T)_+$ for the above choice of B, and for dominant elements $\mu, \mu' \in X_*(T)$, we say that $\mu' \leq \mu$ if $\mu - \mu'$ is a non-negative integral linear combination of positive coroots. Let J_b be the reductive group over $\mathbb{Q}_p$ such that for any $\mathbb{Q}_p$−algebra R,

$$J_b(R) = \left\{ g \in G(L \otimes_{\mathbb{Q}_p} R) \mid gb = b\sigma(g) \right\}.$$

Then $J_b(\mathbb{Q}_p)$ acts naturally on $X_\mu^G(b)$ and $X_{\leq\mu}^G(b)$. The isomorphism classes of $X_\mu^G(b)$, $X_{\leq\mu}^G(b)$ and J_b depend only on the σ-conjugacy class $[b]$ of b. By [73], $X_\mu^G(b)$ and $X_{\leq\mu}^G(b)$ are non-empty if and only if $[b] \in B(G,\mu)$. Here, $B(G,\mu)$ is the Kottwitz subset (*cf.* [41, Section 6]) inside $B(G)$, the set of all σ-conjugacy classes in $G(L)$. We assume $[b] \in B(G,\mu)$ from now on. The triple $(G,[b],\{\mu\})$ will be called a local shtuka datum in Section 5; *cf.* Definition 5.9. By construction, we have $X_\mu^G(b) \subset X_{\leq\mu}^G(b)$. When $\{\mu\}$ is minuscule, we have $X_{\leq\mu}^G(b) = X_\mu^G(b)$.

By the recent work of Zhu [78] and Bhatt–Schoze [1], there exist perfect scheme structures on the sets $X_\mu^G(b)$ and $X_{\leq\mu}^G(b)$. More precisely, $X_\mu^G(b)$ and $X_{\leq\mu}^G(b)$ are the sets of $\overline{\mathbb{F}}_p$-points of some perfect schemes over $\overline{\mathbb{F}}_p$, which are locally closed subschemes of the Witt vector affine Grassmannian Gr_G (*cf.* [1, 78]). It will be useful to briefly recall the related moduli interpretation. Denote $\mathcal{E}_0$ the trivial G-torsor on W. For any perfect $\overline{\mathbb{F}}_p$-algebra R, we have (*cf.* [78, 1.2 and 3.1]) $Gr_G(R) = \{(\mathcal{E},\beta)\}/\simeq$, where

- $\mathcal{E}$ is a G-torsor over $W(R)$,
- $\beta : \mathcal{E}[1/p] \simeq \mathcal{E}_0[1/p]$ is a trivialization over $W(R)[1/p]$,

and

$$X^G_{\leq\mu}(b)(R) = \left\{ (\mathcal{E}, \beta) \in Gr_G(R) \mid \mathrm{Inv}_x(\beta^{-1}b\sigma(\beta)) \leq \mu, \forall x \in \mathrm{Spec} R \right\},$$
$$X^G_{\mu}(b)(R) = \left\{ (\mathcal{E}, \beta) \in Gr_G(R) \mid \mathrm{Inv}_x(\beta^{-1}b\sigma(\beta)) = \mu, \forall x \in \mathrm{Spec} R \right\},$$

where Inv_x is the relative position at x, and μ is the dominant representative in the conjugacy class $\{\mu\}$. By abuse of notation, we denote also by $X^G_{\leq\mu}(b)$ and $X^G_{\mu}(b)$ the associated perfect schemes. By construction, $X^G_{\mu}(b) \subset X^G_{\leq\mu}(b)$ is an open subscheme.

Lemma 2.1 *Let $(G_1, [b_1], \{\mu_1\}) \rightarrow (G_2, [b_2], \{\mu_2\})$ be a morphism (cf. Definition 3.3). It induces a natural map*

$$X^{G_1}_{\leq\mu_1}(b_1) \longrightarrow X^{G_2}_{\leq\mu_2}(b_2).$$

If $G_1 \rightarrow G_2$ is a closed immersion, the above map is a closed immersion.

Proof The first statement is clear. For the second statement, see [36, Lemma 2.5.4 (1)] and [32, 2.4.4]. ∎

2.2 Connected Components

In [7, 8], Chen, Kisin, and Viehmann introduced a notion of connected components for the affine Deligne–Lusztig sets $X^G_{\leq\mu}(b)$ by some ad hoc methods, since the algebro-geometric structure on $X^G_{\leq\mu}(b)$ was not known at the time. We denote by $\pi_0(X^G_{\leq\mu}(b))$ the set of connected components defined by Chen, Kisin, and Viehmann in such a way. By resorting to the perfect scheme structure, we have a naturally defined notion of connected components for $X^G_{\leq\mu}(b)$. It is conjectured that the two definitions coincide; *cf.* [78, Remark 3.2] and [7, 2.3.5]. This was known in the case of unramified EL/PEL Rapoport–Zink spaces; *cf.* [7, Theorem 5.1.5]. This was recently proved by He and Zhou in the general case; *cf.* [31, Theorem A.4].

Let $\pi_1(G)$ be the quotient[5] of $X_*(T)$ by the coroot lattice of G. There is the Kottwitz homomorphism $\omega_G \colon G(L) \longrightarrow \pi_1(G)$ for which an element $g \in G(W)\mu(p)G(W) \subset G(L)$ is sent to the class of μ. Recall that for our pair (b, μ) we assume that $[b] \in B(G, \mu)$. Then there is an element $c_{b,\mu} \in \pi_1(G)$ such that $\omega_G(b) - \mu = (1 - \sigma)(c_{b,\mu})$. The $\pi_1(G)^\Gamma$-coset of $c_{b,\mu}$ is uniquely determined. Here and the sequel, $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the local Galois group. In particular, if $b \in G(W)\mu(p)G(W)$, then we can take $c_{b,\mu} = 1$. As ω_G is trivial on $G(W)$, when restricting to $X^G_{\leq\mu}(b) \subset G(L)/G(W)$, by [7, 2.3] (using the theory of Cartan decomposition in families of [7, 2.1]) we have a $J_b(\mathbb{Q}_p)$-equivariant morphism (of étale sheaves over $\overline{\mathbb{F}}_p$)

$$\omega_G \colon X^G_{\leq\mu}(b) \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma,$$

---

[5]We note that $\pi_1(G)$ is finite if G is semi-simple.

which factors through $\pi_0(X_{\leq\mu}^G(b))$. Thus, we get a commutative diagram

$$
\begin{array}{ccc}
X_{\leq\mu}^G(b) & & \\
\downarrow & \searrow^{\omega_G} & \\
\pi_0(X_{\leq\mu}^G(b)) & \longrightarrow & c_{b,\mu}\pi_1(G)^\Gamma.
\end{array}
$$

Therefore, the non-empty fibers of the map $\omega_G\colon X_{\leq\mu}^G(b) \to c_{b,\mu}\pi_1(G)^\Gamma$ are unions of connected components of $X_{\leq\mu}^G(b)$. Recall the following main theorem of [7].

Theorem 2.2 ([7, Theorems 1.2 and 1.1]) *Assume that μ is minuscule.*

(i) *$J_b(\mathbb{Q}_p)$ acts transitively on $\pi_0(X_\mu^G(b))$.*

(ii) *Assume that G^{ad} is simple and (μ,b) is Hodge–Newton indecomposable in G. Then ω_G induces a bijection*

$$
\pi_0(X_\mu^G(b)) \simeq c_{b,\mu}\pi_1(G)^\Gamma
$$

unless $[b] = [\mu(p)]$ with μ central, in which case $X_\mu^G(b) \simeq G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is discrete.

Recently, Nie has obtained similar results as above on $\pi_0(X_{\leq\mu}^G(b))$ for general μ (not necessary minuscule). We refer the reader to [49, Theorems 1.1 and 1.2] for the precise statements.

Assume that μ is minuscule. By (i) of the above theorem, all non-empty fibers of $\omega_G\colon X_{\leq\mu}^G(b) \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma$ are isomorphic to each other under the transition induced by the action of $J_b(\mathbb{Q}_p)$. Fix a point $x_0 \in \mathrm{Im}(\omega_G\colon X_\mu^G(b) \to c_{b,\mu}\pi_1(G)^\Gamma)$ (soon we will show that ω_G is surjective). Let $X_\mu^G(b)^+ \subset X_\mu^G(b)$ be the fiber of ω_G over x_0. By (i) of the above theorem, we have the equality $X_\mu^G(b) = J_b(\mathbb{Q}_p)X_\mu^G(b)^+$. In the sequel, we will not need to work on each connected component of $X_\mu^G(b)$. The subspace $X_\mu^G(b)^+$ and the equality above will be all that we need.

Now let μ be arbitrary. Since we assume $[b] \in B(G,\mu)$, the set $X_\mu^G(b) \neq \varnothing$. This means that there exists some $g \in G(L)$ such that $b' := g^{-1}b\sigma(g) \in G(W)\mu(p)G(W)$. Thus, after replacing b by b', we can assume that $c_{b,\mu} = 1$. (We note that the element $c_{b,\mu}$ can be defined for arbitrary μ.)

Lemma 2.3 (i) *The restriction of $\omega_G\colon G(L) \to \pi_1(G)$ to $G(\mathbb{Q}_p)$ induces a surjective map $\omega_G\colon G(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$.*

(ii) *The map $J_b(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$ is surjective.*

Proof For (i), this is contained in [39, Lemma 1.2.3].

For (ii), in the case where $(G, [b], \{\mu\})$ comes from a Hodge type Shimura datum $(\mathbb{G}, X)$ unramified at p (and $Z_\mathbb{G}$ is a torus), see [39, Lemma 4.6.4]. The arguments there also work in the general case. ∎

Proposition 2.4   *The map*

$$\omega_G \colon X^G_{\leq \mu}(b) \longrightarrow c_{b,\mu} \pi_1(G)^\Gamma$$

is surjective. In particular, we get a surjection

$$\pi_0\big(X^G_{\leq \mu}(b)\big) \twoheadrightarrow \pi_1(G)^\Gamma.$$

Proof   By [7, Lemma 2.3.6], the map ω_G is compatible with the $J_b(\mathbb{Q}_p)$-actions on both sides. By construction, $J_b(\mathbb{Q}_p)$ acts on $\pi_1(G)^\Gamma$ by left multiplication via the map $J_b(\mathbb{Q}_p) \to \pi_1(G)^\Gamma$, which is surjective by Lemma 2.3(ii). Thus, $\omega_G \colon X^G_{\leq \mu}(b) \to c_{b,\mu} \pi_1(G)^\Gamma$ is surjective.   ∎

We continue to assume that μ can be arbitrary. For a reductive group G over $\mathbb{Q}_p$, we write Z_G as its center.

Proposition 2.5   *Let $(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$ be a morphism. If $G_2 = G_1/Z$ for some central group $Z \subset Z_{G_1}$, we have the cartesian diagram*

$$
\begin{array}{ccc}
X^{G_1}_{\leq \mu_1}(b_1) & \longrightarrow & X^{G_2}_{\leq \mu_2}(b_2) \\
\Big\downarrow {\omega_{G_1}} & & \Big\downarrow {\omega_{G_2}} \\
c_{b_1,\mu_1} \pi_1(G_1)^\Gamma & \longrightarrow & c_{b_2,\mu_2} \pi_1(G_2)^\Gamma.
\end{array}
$$

Proof   This is contained in [7, Corollary 2.4.2].   ∎

Let the notations be as in the above proposition. Combined with Proposition 2.4, we get the following corollary.

Corollary 2.6   *Let $x_1 \in c_{b_1,\mu_1} \pi_1(G_1)^\Gamma$ be a point and let $x_2 \in c_{b_2,\mu_2} \pi_1(G_2)^\Gamma$ be its image under $c_{b_1,\mu_1} \pi_1(G_1)^\Gamma \to c_{b_2,\mu_2} \pi_1(G_2)^\Gamma$. Let $X^{G_1}_{\leq \mu_1}(b_1)^+$ and $X^{G_2}_{\leq \mu_2}(b_2)^+$ be the fibers of ω_{G_1} and ω_{G_2} at x_1 and x_2, respectively, which are non-empty by Proposition 2.4. Then the map $X^{G_1}_{\leq \mu_1}(b_1) \to X^{G_2}_{\leq \mu_2}(b_2)$ induces a bijection*

$$X^{G_1}_{\leq \mu_1}(b_1)^+ \xrightarrow{\sim} X^{G_2}_{\leq \mu_2}(b_2)^+.$$

We keep the above notation.

Lemma 2.7   *If $\pi_1(G_1)^\Gamma \to \pi_1(G_2)^\Gamma$ is surjective, then the map $X^{G_1}_{\leq \mu_1}(b_1) \to X^{G_2}_{\leq \mu_2}(b_2)$ induces an isomorphism*

$$X^{G_1}_{\leq \mu_1}(b_1)/Z(\mathbb{Q}_p) \simeq X^{G_2}_{\leq \mu_2}(b_2).$$

Proof   This is implied by the proof of [7, Corollaries 2.4.2 and 2.4.3]. Under the assumption that $\pi_1(G_1)^\Gamma \to \pi_1(G_2)^\Gamma$ is surjective, all fibers of $X^{G_1}_{\leq \mu_1}(b_1) \to X^{G_2}_{\leq \mu_2}(b_2)$ are torsors under $X_*(Z)^\Gamma$. The group $Z(\mathbb{Q}_p)$ acts on $X^{G_1}_{\leq \mu_1}(b_1)$ via the natural map $Z(\mathbb{Q}_p) \to X_*(Z)^\Gamma$.   ∎

3 Rapoport–Zink Spaces of Hodge Type

Following Rapoport and Viehmann, we first review the general conjecture on the theory of local Shimura varieties in [57]. Then we concentrate on the Hodge type case; *cf.* [4, 32, 36].

3.1 Local Shimura Data and Local Shimura Varieties

Recall the following definition of Rapoport and Viehmann.

Definition 3.1 ([57, Definition 5.1]) A local Shimura datum over $\mathbb{Q}_p$ is a triple $(G, [b], \{\mu\})$, where

- G is a connected reductive group over $\mathbb{Q}_p$;
- $[b] \in B(G)$ is a σ-conjugacy class;
- $\{\mu\}$ is a conjugacy class of cocharacters $\mu\colon \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$,

such that the following conditions are satisfied:

(i) $[b] \in B(G, \mu)$,
(ii) $\{\mu\}$ is minuscule.

Associated with a local Shimura datum, we have

- the reflex field $E = E(G, \{\mu\})$, which is the field of definition of $\{\mu\}$ inside the fixed algebraic closure $\overline{\mathbb{Q}}_p$;
- the flag variety $\mathscr{F}\ell_{G,\mu}$, considered as a rigid analytic space (or an adic space) over $\breve{E}$, the completion of the maximal unramified extension of E; here the associated parabolic subgroup P_μ is given by $P_\mu = \{g \in G \mid \lim_{t\to 0} \mu(t)g\mu(t)^{-1} \text{ exists}\}$;
- the reductive group J_b over $\mathbb{Q}_p$, for $b \in [b]$, which up to isomorphism only depends on $[b]$. The group $J_b(\mathbb{Q}_p)$ acts on $\mathscr{F}\ell_{G,\mu}$;
- the weakly admissible open subspace $\mathscr{F}\ell_{G,\mu}^{\mathrm{wa}} \subset \mathscr{F}\ell_{G,\mu}$ defined in [56, 1.35] and [10, Definition 9.5.4]. The action of $J_b(\mathbb{Q}_p)$ on $\mathscr{F}\ell_{G,\mu}$ stabilizes $\mathscr{F}\ell_{G,\mu}^{\mathrm{wa}}$.

In fact, if G is unramified, we also have (*cf.* the last section)

- the affine Deligne–Lusztig variety $X_\mu^G(b)$ over $\overline{\mathbb{F}}_p$ (which will be expected to be the special fiber of some formal model of the following local Shimura variety $\mathcal{M}_{G(\mathbb{Z}_p)}$; *cf.* Conjecture 3.2).

Let $(G, [b], \{\mu\})$ be a local Shimura datum, with local reflex field E. We have the following conjecture ([57] 5.1).

Conjecture 3.2 (Rapoport–Viehmann) *There is a tower of rigid analytic spaces over* $\mathrm{Sp}\breve{E}$, $(\mathcal{M}_K)_K$, *where K runs through all open compact subgroups of $G(\mathbb{Q}_p)$, with the following properties:*

(i) *the group $J_b(\mathbb{Q}_p)$ acts on each space $\mathcal{M}_K$;*
(ii) *the group $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_K$ as Hecke correspondences;*
(iii) *the tower is equipped with a Weil descent datum over E;*
(iv) *there exists a compatible system of étale and partially proper period maps*

$$\pi_K \colon \mathcal{M}_K \to \mathscr{F}\ell_{G,\mu}^{\mathrm{wa}},$$

which is equivariant for the action of $J_b(\mathbb{Q}_p)$.

In fact, in [57, 5.1] there is a more precise statement on the point (iv) of the conjecture. In particular, there should be an open subspace

$$\mathscr{F}\ell^{a}_{G,\mu} \subset \mathscr{F}\ell^{\mathrm{wa}}_{G,\mu},$$

which should be the image of the period maps π_K for all K. In fact, Rapoport and Zink conjecture that there exists a $\mathbb{Q}_p$-local system with G-structure over $\mathscr{F}\ell^{a}_{G,\mu}$ that interpolates the p-adic crystalline Galois representations attached to all classical points. Moreover, the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Q}_p)}$ should be the geometric realization (*i.e.*, spaces of lattices with level structures) of this universal $\mathbb{Q}_p$-local system with G-structure over $\mathscr{F}\ell^{a}_{G,\mu}$. We refer to [57, 5.1], [56, Section 1], [10, Conjecture 11.4.4], and [27, Conjecture 2.3] for more details. This conjecture is known for the local Shimura data that arise from local EL/PEL data ([56]), and the unramified local Shimura datum of Hodge type ([36]). In both cases, these spaces $\mathcal{M}_K$ are finite étale covers of the rigid analytic generic fibers of some formal schemes $\breve{\mathcal{M}}$ over $\mathrm{Spf}\,\mathcal{O}_{\breve{E}}$, which are formal moduli spaces of p-divisible groups with some additional structures. The special fibers of these formal schemes $\breve{\mathcal{M}}$ are the affine Deligne–Lusztig varieties that we introduced in the last section. In Section 5 we will discuss a partial solution of the above conjecture due to Scholze; *cf.* [61, 64].

It will be useful to define morphisms of local Shimura data.

Definition 3.3   Let $(G_1, [b_1], \{\mu_1\}), (G_2, [b_2], \{\mu_2\})$ be two local Shimura data. A morphism

$$\big(G_1, [b_1], \{\mu_1\}\big) \longrightarrow \big(G_2, [b_2], \{\mu_2\}\big)$$

is a homomorphism of algebraic groups $f\colon G_1 \to G_2$ sending $([b_1], \{\mu_1\})$ to $([b_2], \{\mu_2\})$.

If $(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$ is a morphism of local Shimura data, then it is conjectured ([57, Properties 5.3 (iv)]) that for any open compact subgroups $K_1 \subset G_1(\mathbb{Q}_p), K_2 \subset G_2(\mathbb{Q}_p)$ with $f(K_1) \subset K_2$, there exists a morphism of the associated local Shimura varieties

$$\mathcal{M}(G_1, b, \mu)_{K_1} \longrightarrow \mathcal{M}(G_2, b_2, \mu_2)_{K_2} \times \mathrm{Sp}\breve{E}_1,$$

and when $G_1 \to G_2$ is a closed immersion these are closed embeddings for $K_1 = K_2 \cap G_1(\mathbb{Q}_p)$.

3.2 Local Shimura Data of Hodge Type

Now we recall the definition of a special class of local Shimura data (*cf.* [57, Remark 5.4 (i)]).

Definition 3.4   A local Shimura datum $(G, [b], \{\mu\})$ is called of Hodge type if there exists an embedding $f\colon G \hookrightarrow \mathrm{GL}(V)$ and a local Shimura datum $(\mathrm{GL}(V), [b'], \{\mu'\})$ with $\{\mu'\}$ corresponding to $(1^r, 0^{n-r})$ for some integer $1 \le r \le n = \dim V$, such that $[b], \{\mu\}$ are mapped to $[b'], \{\mu'\}$ under f.

If G is, moreover, unramified, by [38, Lemma 2.3.1], we can find some $\mathbb{Z}_p$-lattice $V_{\mathbb{Z}_p} \subset V$ such that $G \hookrightarrow \mathrm{GL}(V)$ is induced by an embedding $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_p})$, where $G_{\mathbb{Z}_p}$ is a reductive model of G over $\mathbb{Z}_p$.

Definition 3.5 A local Shimura datum of Hodge type $(G, [b], \{\mu\})$ is called unramified if G is unramified.

We note that for an unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$, the local reflex field E is an unramified extension of $\mathbb{Q}_p$. Thus, $\breve{E} = L$, $\mathcal{O}_{\breve{E}} = \mathcal{O}_L = W$, where as before, $W = W(\overline{\mathbb{F}}_p)$, $L = W_{\mathbb{Q}}$. We will fix a reductive model $G_{\mathbb{Z}_p}$ of G over $\mathbb{Z}_p$.

Remark 3.6 The above definition of unramified local Shimura data of Hodge type is more general than that in [32, Definition 2.3.3]. Moreover, for an unramified local Shimura datum of Hodge type $(G, [b], \{\mu\})$ in the sense of [32], one always has $Z_G \supset \mathbb{G}_m$.

We want to classify local Shimura data of Hodge type. Let $(G, [b], \{\mu\})$ be a given local Shimura datum. Take any faithful representation V of G over $\mathbb{Q}_p$, so that we get an embedding $\rho \colon G \hookrightarrow \mathrm{GL}(V)$. Therefore we get a conjugacy class $\{\mu'\}$ of cocharacters, $\mu' = \rho_{\overline{\mathbb{Q}}_p} \circ \mu \colon \mathbb{G}_m \to \mathrm{GL}(V)_{\overline{\mathbb{Q}}_p}$. Let $N(G)$ be the set of Newton points of G; *cf.* [55, 1.7] (in [41, 4.1], it was denoted by $\overline{C}_{\mathbb{Q}}$). Recall that the maps

$$\nu_G \colon B(G) \longrightarrow N(G), \quad \kappa_G \colon B(G) \longrightarrow \pi_1(G)_\Gamma$$

are functorial in G, *cf.* [40, section 4], [41, 4.9 and 7.5], and [55, 1.9 and 1.15]. In particular, we get a map $B(G, \mu) \to B(\mathrm{GL}(V), \mu')$. Let $[b'] \in B(\mathrm{GL}(V), \mu')$ be the image of $[b]$ under this map. The triple $(\mathrm{GL}(V), [b'], \{\mu'\})$ is a local Shimura datum if and only if $\{\mu'\}$ is minuscule and corresponds to $(1^r, 0^{n-r})$ for some integer $1 \leq r \leq n = \dim V$, in which case $(G, [b], \{\mu\})$ is of Hodge type. As above, let G be a reductive group over $\mathbb{Q}_p$ and let $\{\mu\}$ be a conjugacy class of minuscule cocharacters $\mu \colon \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$. In [66] Serre classified the pair $(G, \{\mu\})$ for which there exists a faithful representation V of G such that the induced class of cocharacters $\{\mu'\}$ under the embedding $G \hookrightarrow \mathrm{GL}(V)$ corresponds to $(1^r, 0^{n-r})$ for some integer $1 \leq r \leq n = \dim V$. It turn out the simple factors of $G_{\overline{\mathbb{Q}}_p}^{\mathrm{ad}}$ are groups of type A, B, C, or D; *cf.* [66, section 3].

The following examples of local Shimura datum of Hodge type are standard.

Example 3.7 (i) Let $(G, [b], \{\mu\})$ be a local Shimura datum which comes from a local EL/PEL datum (*cf.* [56, 1.38]), then it is of Hodge type (*cf.* [36, 4.7]).

(ii) Let (G, X) be a Shimura datum of Hodge type, *i.e.*, there exists some embedding into the Siegel Shimura datum $(G, X) \hookrightarrow (\mathrm{GSp}, S^\pm)$. Let μ be the cocharacter associated with X. Take any $[b] \in B(G_{\mathbb{Q}_p}, \mu)$. Then the local Shimura datum $(G_{\mathbb{Q}_p}, [b], \{\mu\})$ is of Hodge type.

Here is an example of non-Hodge type local Shimura datum.

Example 3.8 (See [57, Example 5.5]) Let $G = \mathrm{PGL}_n$, μ be any nontrivial minuscule cocharacter, and let $[b] \in B(G, \mu)$ be arbitrary. Then the local Shimura datum $(G, [b], \{\mu\})$ is not of Hodge type.

3.3 Rapoport–Zink Spaces of Hodge Type

Throughout the rest of this section, we assume that $p > 2$. Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of Hodge type. Fix a representative $b \in G(L)$ of $[b]$. Kim ([36]) constructs a formal moduli scheme $\check{\mathcal{M}} = \check{\mathcal{M}}(G, b, \mu)$ over $\mathrm{Spf}\, W$ parametrizing p-divisible groups with crystalline Tate tensors. We briefly review the related constructions in this subsection. By abuse of notation, we write also G as the fixed associated reductive group scheme over $\mathbb{Z}_p$. Then there exists a faithful representation $\rho \colon G \hookrightarrow \mathrm{GL}(\Lambda)$, such that the induced cocharacter $\mu' = \rho_{\overline{\mathbb{Q}}_p} \circ \mu \colon \mathbb{G}_m \to \mathrm{GL}(\Lambda \otimes \overline{\mathbb{Q}}_p)$ is minuscule. Let $\Lambda^\vee$ be the dual lattice, and let $\Lambda^\otimes$ be the tensor algebra of $\Lambda \oplus \Lambda^\vee$. By [38, Proposition 1.3.2], there exists a finite collection of tensors $\{s_\alpha \in \Lambda^\otimes\}_{\alpha \in I}$ such that $\rho : G \subset \mathrm{GL}(\Lambda)$ is the schematic stabilizer of (s_α). We fix a representative μ. Let $\Lambda \otimes W = \Lambda^0 \oplus \Lambda^1$ be the decomposition of $\Lambda \otimes W$ according to the weights of μ, which in turn induces a filtration $\mathrm{Fil}^\bullet \Lambda \otimes W$ with $\mathrm{Fil}^0 \Lambda \otimes W = \Lambda \otimes W, \mathrm{Fil}^1 \Lambda \otimes W = \Lambda^1$. We assume that $\mathrm{rank}\, \Lambda = n, \mathrm{rank}\, \Lambda^1 = d$. We note that $P_\mu := \mathrm{Aut}(\Lambda, s_\alpha, \mathrm{Fil}^\bullet \Lambda \otimes W)$ is a parabolic subgroup of G_W.

By our assumption and the classical Dieudonné theory, there exists a p-divisible group X_0 of dimension d and height n over $\overline{\mathbb{F}}_p$, together with an isomorphism $\varepsilon : \mathbb{D}(X_0) \simeq (\Lambda \otimes W, b\sigma)$, where $\mathbb{D}(X_0)$ is the contravariant Dieudonné module of X_0. The pair (X_0, ε) is unique up to a unique isomorphism, and we fix it in the sequel. Then we can regard $s_\alpha \otimes 1$ as tensors in $\mathbb{D}(X_0)^\otimes$ via ε. We note that $b\sigma$ fixes $(s_\alpha \otimes 1)$ and $(s_\alpha \otimes 1)$ lie in $\mathrm{Fil}^1 \Lambda \otimes W$. Each $s_\alpha \otimes 1$ can be regarded as a map $\mathbb{1} := \mathbb{D}(\mathbb{Q}_p / \mathbb{Z}_p) \to \mathbb{D}(X_0)^\otimes$, compatible with the filtrations, and such that the induced map $\mathbb{1} \to \mathbb{D}(X_0)^\otimes[\frac{1}{p}]$ is Frobenius-invariant; *i.e.*, $s_\alpha \otimes 1$ is a crystalline Tate tensor of X_0; *cf.* [36, Definition 4.6].

Let Nilp_W be the category of W-algebras on which p is locally nilpotent. Let $R \in \mathrm{Nilp}_W$ and let X be a p-divisible group on $\mathrm{Spec}\, R$. Consider the contravariant Dieudonné crystal $\mathbb{D}(X)$ attached to X. Then as usual there is a decreasing (Hodge) filtration $\mathrm{Fil}^\bullet \mathbb{D}(X)_R$ on $\mathbb{D}(X)_R$ with locally free graded pieces over R. Here $\mathbb{D}(X)_R$ is the value of $\mathbb{D}(X)$ at the trivial PD-thickening $R \xrightarrow{\mathrm{id}} R$. Namely, $\mathrm{Fil}^0 \mathbb{D}(X)_R = \mathbb{D}(X)_R$, $\mathrm{Fil}^1 \mathbb{D}(X)_R = (\mathrm{Lie}\, X)^\vee$ and $\mathrm{Fil}^2 \mathbb{D}(X)_R = 0$. As above, a crystalline Tate tensor of X is a morphism $t_\alpha \colon \mathbb{1} \to \mathbb{D}(X)^\otimes$ of crystals, such that $t_{\alpha R} \colon \mathbb{1}_R \to \mathbb{D}(X)_R^\otimes$ is compatible with the filtrations, and the induced map $t_\alpha \colon \mathbb{1} \to \mathbb{D}(X)^\otimes[\frac{1}{p}]$ is Frobenius-invariant.

Denote by Nilp_W^{sm} the full subcategory of Nilp_W consisting of formally smooth formally finitely generated W/p^m-algebras for $m \geq 1$. We use the following version of Rapoport–Zink functor (*cf.* [78, Definition 3.8]), which is equivalent to [36, Definition 4.6].

Definition 3.9 The Rapoport–Zink space associated with the unramified local Shimura datum of Hodge type is the functor $\check{\mathcal{M}}$ on Nilp_W^{sm} defined by $\check{\mathcal{M}}(R) = \{(X, (t_\alpha)_{\alpha \in I}, \rho)\}/\simeq$ where

- X is a p-divisible group on $\operatorname{Spec} R$;
- $(t_\alpha)_{\alpha \in I}$ is a collection of crystalline Tate tensors of X;
- $\rho \colon X_0 \otimes R/J \to X \otimes R/J$ is a quasi-isogeny which sends $s_\alpha \otimes 1$ to t_α for $\alpha \in I$, where J is some ideal of definition of R,

such that the following condition holds: the R-scheme

$$\operatorname{Isom}\big(\big(\mathbb{D}(X)_R, (t_\alpha), \operatorname{Fil}^\bullet(\mathbb{D}(X)_R)\big), \big(\Lambda \otimes R, (s_\alpha \otimes 1), \operatorname{Fil}^\bullet \Lambda \otimes R\big)\big)$$

that classifies the isomorphisms between locally free sheaves $\mathbb{D}(X)_R$ and $\Lambda \otimes R$ on $\operatorname{Spec} R$ preserving the tensors and the filtrations is a $P_\mu \otimes R$-torsor.

Theorem 3.10 ([36, Theorem 4.9.1]) *The functor $\breve{\mathcal{M}}$ is represented by a separated formal scheme, formally smooth, and formally locally of finite type over W.*

In the classical EL/PEL case (and with ramification), see [56, Theorem 3.25]. In [36, 4.7], the unramified local EL/PEL data are explained as special examples of unramified Hodge type data. See also [32, Theorem 3.2.1] for the case where $(G, [b], \{\mu\})$ comes from a Shimura datum of Hodge type. When $\rho(b)$ has no slope 0, Bültel and Pappas have proved the above theorem by a different approach, see [4]. More precisely, they introduced notions of (G, μ)-displays and quasi-isogenies between such, and they proved that the similar moduli problem of (G, μ)-displays together with quasi-isogenies are representable. In the case $G = \operatorname{GL}_n$, the moduli problem of Bültel and Pappas is equivalent to the moduli problem of Rapoport and Zink, by the theorems of Zink [79] and Lau [43] that formal p-divisible groups over a p-adically complete and separated algebra R are classified by the associated nilpotent displays.

We denote also by $\breve{\mathcal{M}}$ the associated formal scheme and refer to it as the formal Rapoport–Zink space of Hodge type attached to $(G, [b], \{\mu\})$. Let $\mathcal{M}$ be the rigid analytic generic fiber over $L = W_{\mathbb{Q}}$ of the formal scheme $\breve{\mathcal{M}}$. In the rest of this paper, we will use the following convention: if G is an unramified reductive group over $\mathbb{Q}_p$, we will fix a reductive model over $\mathbb{Z}_p$ and write $G(\mathbb{Z}_p)$ for the associated hyperspecial group. In [36, 7.4], Kim explained how to construct a tower of rigid analytic spaces $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ that satisfies the list of properties in Conjecture 3.2. Moreover, $\mathcal{M}_{G(\mathbb{Z}_p)} = \mathcal{M}$, and $\mathcal{M}_K \to \mathcal{M}$ is finite étale for any open compact subgroup $K \subset G(\mathbb{Z}_p)$. In particular, for unramified local Shimura data of Hodge type, the Conjecture 3.2 is true.

Let $\overline{\mathcal{M}}$ be the special fiber over $\overline{\mathbb{F}}_p$ of $\breve{\mathcal{M}}$. Recall that in Section 2, attached to $(G, [b], \{\mu\})$, we introduced the affine Deligne–Lusztig variety $X_\mu^G(b)$ over $\overline{\mathbb{F}}_p$, viewed as a perfect scheme. The relation between $\breve{\mathcal{M}}$ and $X_\mu^G(b)$ is as follows.

Proposition 3.11 ([78, Proposition 3.11]) *$X_\mu^G(b)$ is the perfection $\overline{\mathcal{M}}^{\operatorname{perf}}$ of $\overline{\mathcal{M}}$.*

If $(G, [b], \{\mu\}) \hookrightarrow (\operatorname{GL}_n, [b'], \{\mu'\})$ is an embedding of unramified local Shimura data of Hodge type, by construction, we have the following embeddings

$$\breve{\mathcal{M}}(G, b, \mu) \hookrightarrow \breve{\mathcal{M}}(\operatorname{GL}_n, b', \mu'), \quad X_\mu^G(b) \hookrightarrow X_{\mu'}^{\operatorname{GL}_n}(b'),$$

which are compatible in the sense of the above proposition.

3.4 Connected Components

Let the notations be as above. Recall from Subsection 2.2 that we have the map

$$\omega_G \colon X_\mu^G(b) \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma.$$

By Proposition 3.11, we get an induced map of étale sheaves over W:

$$\omega_G \colon \check{\mathcal{M}} \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma.$$

Let $G^{\mathrm{der}} \subset G$ be the derived subgroup, and let G^{ab} be the abelian quotient G/G^{der}. Consider the exact sequence

$$1 \longrightarrow G^{\mathrm{der}} \longrightarrow G \longrightarrow G^{\mathrm{ab}} \longrightarrow 1,$$

which induces a map

$$c_{b,\mu}\pi_1(G)^\Gamma \longrightarrow c_{b,\mu}\pi_1(G^{\mathrm{ab}})^\Gamma = c_{b,\mu}X_*(G^{\mathrm{ab}})^\Gamma,$$

where $X_*(G^{\mathrm{ab}})$ is the cocharacter group of the torus G^{ab} over $\overline{\mathbb{Q}}_p$. Let $X^*_{\mathbb{Q}_p}(G)$ be the group of $\mathbb{Q}_p$-rational characters of G. Then we have

$$X^*_{\mathbb{Q}_p}(G) = X^*(G^{\mathrm{ab}})^\Gamma.$$

The Γ-equivariant pairing $X_*(G^{\mathrm{ab}}) \times X^*(G^{\mathrm{ab}}) \to \mathbb{Z}$ then induces a map

$$c_{b,\mu}X_*(G^{\mathrm{ab}})^\Gamma \longrightarrow \mathrm{Hom}(X^*(G^{\mathrm{ab}})^\Gamma, \mathbb{Z}) = \mathrm{Hom}(X^*_{\mathbb{Q}_p}(G), \mathbb{Z}).$$

In summary, we get a map by considering the composition

$$\varkappa_{\check{\mathcal{M}}} \colon \check{\mathcal{M}} \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma \longrightarrow c_{b,\mu}X_*(G^{\mathrm{ab}})^\Gamma \longrightarrow \mathrm{Hom}(X^*_{\mathbb{Q}_p}(G), \mathbb{Z}).$$

In the EL/PEL case, this is just the map constructed in [56, 3.52]. (See also [7, 5.1.3].)

If $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$ is an embedding of unramified local Shimura data of Hodge type, we get the commutative diagram

$$
\begin{array}{ccc}
X_\mu^G(b) & \longrightarrow & X_{\mu'}^{\mathrm{GL}_n}(b') \\
\downarrow & & \downarrow \\
c_{b,\mu}\pi_1(G)^\Gamma & \longrightarrow & c_{b',\mu'}\pi_1(\mathrm{GL}_n)^\Gamma.
\end{array}
$$

Moreover, we know $\pi_1(\mathrm{GL}_n)^\Gamma = \pi_1(\mathrm{GL}_n) \simeq \mathbb{Z}$.

Since by Proposition 3.11 $X_\mu^G(b)$ is the perfection $\overline{\mathcal{M}}^{\mathrm{perf}}$ of $\overline{\mathcal{M}}$, we have the isomorphism between the sets of connected components

$$\pi_0(\mathcal{M}_{\mathrm{red}}) \simeq \pi_0^{\mathrm{perf}}(X_\mu^G(b)).$$

Here, $\pi_0^{\mathrm{perf}}(X_\mu^G(b))$ denotes the set of connected components of the perfect scheme $X_\mu^G(b)$. On the other hand, we have also the set of connected components $\pi_0(X_\mu^G(b))$ defined in [7].

Proposition 3.12 *With the above notation, there is a bijection*

$$\pi_0(\mathcal{M}_{\mathrm{red}}) \simeq \pi_0\big(X_\mu^G(b)\big).$$

Proof See [78, Remark 3.2]. See also [31, Theorem A.4]. ∎

Let $\pi_0(\check{\mathcal{M}})$ be the set of connected components of the formal scheme $\check{\mathcal{M}}$, which is the same as $\pi_0(\mathcal{M}_{\mathrm{red}})$. On the other hand, we have also the set of connected components $\pi_0(\mathcal{M})$ of the generic fiber $\mathcal{M}$. As $\check{\mathcal{M}}$ is formally smooth and in particular normal, by [34, Theorem 7.4.1], we have a bijection

$$\pi_0(\mathcal{M}_{\mathrm{red}}) \simeq \pi_0(\mathcal{M}).$$

One can also consider the set of connected components $\pi_0(\mathcal{M}_K)$ for the finite étale cover $\mathcal{M}_K$ of $\mathcal{M}$. In [57], Rapoport and Viehmann made a conjecture on $\pi_0(\mathcal{M}_K \times \mathbb{C}_p)$ under the assumption that G^{der} is *simply connected*. We refer the reader to [57, Conjecture 4.26] for the precise statement on the existence of a determinant morphism for the tower $(\mathcal{M}_K)_K$. This conjecture is known in the unramified simple EL/PEL case; *cf.* [6, Theorem 6.3.1] (see also [7, Theorem 5.1.10 and Remark 5.1.11]). It will be interesting to consider the more general Hodge type case studied here.

Fix a point $x_0 \in c_{b,\mu}\pi_1(G)^\Gamma$. Let $\mathcal{M}_{\mathrm{red}}^+ \subset \mathcal{M}_{\mathrm{red}}$ be the fiber of ω_G over x_0. Then $\mathcal{M}_{\mathrm{red}}^+$ is some union of connected components of $\mathcal{M}_{\mathrm{red}}$. Let $\check{\mathcal{M}}^+ \subset \check{\mathcal{M}}$ be the associated subformal scheme, with generic fiber $\mathcal{M}^+$. For any open compact subgroup $K \subset G(\mathbb{Q}_p)$, let $\mathcal{M}_K^+ \subset \mathcal{M}_K$ be the pullback of $\mathcal{M}^+ \subset \mathcal{M}$. We get a tower $(\mathcal{M}_K^+)_{K \subset G(\mathbb{Z}_p)}$. We have the equalities

$$\check{\mathcal{M}} = J_b(\mathbb{Q}_p)\check{\mathcal{M}}^+, \quad \mathcal{M}_{\mathrm{red}} = J_b(\mathbb{Q}_p)\mathcal{M}_{\mathrm{red}}^+, \quad \mathcal{M} = J_b(\mathbb{Q}_p)\mathcal{M}^+$$

and $\mathcal{M}_K = J_b(\mathbb{Q}_p)\mathcal{M}_K^+$.

4 Rapoport–Zink Spaces of Abelian Type

We enlarge the class of Rapoport–Zink spaces of Hodge type in this section. They are constructed locally from Rapoport–Zink spaces of Hodge type. Throughout this section, we assume $p > 2$.

4.1 Local Shimura Data of Abelian Type

Let $(G, [b], \{\mu\})$ be a local Shimura datum. Consider the natural projection $G \to G^{\mathrm{ad}}$ from G to its associated adjoint group. We get induced $[b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\}$, so that $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\})$ is also a local Shimura datum and $(G, [b], \{\mu\}) \to (G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\})$ is a morphism of local Shimura data. We introduce the local analogue of a Shimura datum of abelian type (more precisely, of preabelian type) as follows.

Definition 4.1 A local Shimura datum $(G, [b], \{\mu\})$ is called of abelian type if there exists a local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that we have an isomorphism of the associated adjoint local Shimura data $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\}) \simeq (G_1^{\mathrm{ad}}, [b_1^{\mathrm{ad}}], \{\mu_1^{\mathrm{ad}}\})$.

Thus, any local Shimura datum of Hodge type is also of abelian type. The later class is strictly larger.

Example 4.2 Let $G = \mathrm{PGL}_n$. Consider a nontrivial minuscule cocharacter μ_1: $\mathbb{G}_m \to \mathrm{GL}_n$ and $[b_1] \in B(\mathrm{GL}_n, \mu_1)$. Take $\mu = \mu_1^{\mathrm{ad}}, [b] = [b_1^{\mathrm{ad}}]$. Then $(G, [b], \{\mu\})$ is of abelian type, but not of Hodge type; *cf.* Example 3.8.

Recall that for a local Shimura datum $(G, [b], \{\mu\})$, if G_i is a simple local factor of $G_{\overline{\mathbb{Q}}_p}^{\mathrm{ad}}$ such that the component μ_i^{ad} of μ^{ad} is not trivial, then G_i is a group of one of types A, B, C, D, E_6, E_7; *cf.* [66, Annexe]. By Serre's classification ([66, section 3]) and our definition, simple factors of G appearing in local Shimura data of abelian type consists exactly of local reductive groups of types A, B, C, D. This is compatible with Deligne's classification of Shimura data of abelian type in [11]; *cf.* Example 4.4.

4.2 The Associated Rapoport–Zink Spaces

To construct Rapoport–Zink spaces, we need the following unramified assumption.

Definition 4.3 A local Shimura datum of abelian type $(G, [b], \{\mu\})$ is called unramified, if G is unramified, and there exists an unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\}) \simeq (G_1^{\mathrm{ad}}, [b_1^{\mathrm{ad}}], \{\mu_1^{\mathrm{ad}}\})$.

For an unramified local Shimura datum of abelian type, the local reflex field E is an unramified extension of $\mathbb{Q}_p$. Thus, $\breve{E} = L, \mathcal{O}_{\breve{E}} = \mathcal{O}_L = W$ where, as before, $W = W(\overline{\mathbb{F}}_p), L = W_{\mathbb{Q}}$.

The following example is one of our main motivations.

Example 4.4 Let (G, X) be a Shimura datum of abelian type such that G is unramified at p (*cf.* [11, 38]). Take any $[b] \in B(G, \mu)$, the associated triple $(G_{\mathbb{Q}_p}, [b], \{\mu\})$ is an unramified local Shimura datum of abelian type.

Lemma 4.5 *Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type. Consider the associated adjoint local Shimura datum $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\})$. Fix a representative $b \in G(L)$ of $[b]$ with image $b^{\mathrm{ad}} \in G^{\mathrm{ad}}(L)$, and identify Z_G as a central subgroup of J_b. We have the following isomorphism of reductive groups over $\mathbb{Q}_p$:*

$$J_b/Z_G \simeq J_{b^{\mathrm{ad}}}.$$

Proof This follows from the definitions of J_b and $J_{b^{\mathrm{ad}}}$. ∎

Theorem 4.6 *Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type. Fix a representative $b \in G(L)$ of $[b]$. Then there exists a formal scheme $\breve{\mathcal{M}}(G, b, \mu)$, which is formally smooth, formally locally of finite type over W, such that*

$$\overline{\mathcal{M}}(G, b, \mu)^{\mathrm{perf}} \simeq X_\mu^G(b),$$

where $\overline{\mathcal{M}}(G, b, \mu)$ is the special fiber of $\breve{\mathcal{M}}(G, b, \mu)$. The formal scheme $\breve{\mathcal{M}}(G, b, \mu)$ is equipped with a transitive action of $J_b(\mathbb{Q}_p)$, compatible with the action of $J_b(\mathbb{Q}_p)$ on $X_\mu^G(b)$.

Proof   Take any unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ as in Definition 4.3 and fix a representative $b_1 \in [b_1]$. Consider the associated formal Rapoport–Zink space $\check{\mathcal{M}}(G_1, b_1, \mu_1)$ over $\operatorname{Spf} W$. Then its special fiber $\overline{\mathcal{M}}(G_1, b_1, \mu_1)$ satisfies

$$\overline{\mathcal{M}}(G_1, b_1, \mu_1)^{\mathrm{perf}} \simeq X_{\mu_1}^{G_1}(b_1).$$

Recall that we have the cartesian diagram (*cf.* Proposition 2.5)

$$
\begin{array}{ccc}
X_{\mu_1}^{G_1}(b_1) & \longrightarrow & X_{\mu_1^{\mathrm{ad}}}^{G_1^{\mathrm{ad}}}(b_1^{\mathrm{ad}}) \\
\downarrow{\scriptstyle \omega_{G_1}} & & \downarrow{\scriptstyle \omega_{G_1^{\mathrm{ad}}}} \\
c_{b_1, \mu_1} \pi_1(G_1)^{\Gamma} & \longrightarrow & c_{b_1^{\mathrm{ad}}, \mu_1^{\mathrm{ad}}} \pi_1(G_1^{\mathrm{ad}})^{\Gamma}.
\end{array}
$$

Let $X_{\mu_1}^{G_1}(b_1)^+ \subset X_{\mu_1}^{G_1}(b_1)$ be the fiber over c_{b_1, μ_1} under the map $\omega_{G_1} \colon X_{\mu_1}^{G_1}(b_1) \to c_{b_1, \mu_1} \pi_1(G_1)^{\Gamma}$. Let $\check{\mathcal{M}}(G_1, b_1, \mu_1)^+$ be the corresponding formal subscheme of $\check{\mathcal{M}}(G_1, b_1, \mu_1)$. On the other hand, we can also consider the fiber $X_\mu^G(b)^+ \subset X_\mu^G(b)$ over $c_{b, \mu}$ under $\omega_G \colon X_\mu^G(b) \to c_{b, \mu} \pi_1(G)^{\Gamma}$. Then by Corollary 2.6,

$$X_{\mu_1}^{G_1}(b_1)^+ \simeq X_\mu^G(b)^+.$$

We set

$$\check{\mathcal{M}}(G, b, \mu)^+ := \check{\mathcal{M}}(G_1, b_1, \mu_1)^+;$$

then $\overline{\mathcal{M}}(G, b, \mu)^{+, perf} \simeq X_\mu^G(b)^+$. By Theorem 2.2(i), we have

$$X_\mu^G(b) = J_b(\mathbb{Q}_p) X_\mu^G(b)^+.$$

Therefore, there exists a formal scheme $\check{\mathcal{M}}(G, b, \mu)$, equipped with an action of $J_b(\mathbb{Q}_p)$, such that

$$\check{\mathcal{M}}(G, b, \mu) = J_b(\mathbb{Q}_p) \check{\mathcal{M}}(G, b, \mu)^+,$$
$$\overline{\mathcal{M}}(G, b, \mu)^{\mathrm{perf}} \simeq X_\mu^G(b),$$

and the induced action of $J_b(\mathbb{Q}_p)$ on $\mathcal{M}(G, b, \mu)_{\mathrm{red}}$ is compatible with that on $X_\mu^G(b)$ under the above identification. In fact, we can take

$$\check{\mathcal{M}}(G, b, \mu) = \left[J_b(\mathbb{Q}_p) \times \check{\mathcal{M}}(G, b, \mu)^+ \right] / J_b(\mathbb{Q}_p)^+$$
$$\simeq \coprod_{J_b(\mathbb{Q}_p)/J_b(\mathbb{Q}_p)^+} \check{\mathcal{M}}(G, b, \mu)^+,$$

where $J_b(\mathbb{Q}_p)^+ \subset J_b(\mathbb{Q}_p)$ is the stabilizer of $X_\mu^G(b)^+$ under the action of $J_b(\mathbb{Q}_p)$ on $X_\mu^G(b)$.

The above construction does not depend on the choice of the unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ as in the statement of the theorem, since if $(G_2, [b_2], \{\mu_2\})$ is another such one, then we have a canonical isomorphism

$$\check{\mathcal{M}}(G_1, b_1, \mu_1)^+ \simeq \check{\mathcal{M}}(G_2, b_2, \mu_2)^+.$$

This follows from the bijection $X_{\mu_1}^{G_1}(b_1)^+ \simeq X_{\mu_2}^{G_2}(b_2)^+$, the isomorphism of deformation rings $R_{G_1, x_1} \simeq R_{G_2, x_2}$, where $X_{\mu_1}^{G_1}(b_1)^+ \ni x_1 \mapsto x_2 \in X_{\mu_2}^{G_2}(b_2)^+$, *cf.* [38, 1.5.4]

(from the description there, R_G depends only on the adjoint group G^{ad}), and the constructions in [36, section 6]. ∎

By construction, we have a map of étale sheaves $\check{\mathcal{M}}(G, b, \mu) \to c_{b,\mu}\pi_1(G)^\Gamma$ over W, lifting the map $\omega_G\colon X^G_{\le\mu}(b) \to c_{b,\mu}\pi_1(G)^\Gamma$. As in [78, Corollary 3.12], we have the following dimension formula for the special fibers by applying [78, Theorem 3.1].

Corollary 4.7   *Let the notations be as in Theorem 4.6. We have* $\dim \mathcal{M}_{\mathrm{red}} = \langle \rho, \mu - \nu_{[b]} \rangle - \frac{1}{2}\mathrm{def}_G(b)$, *where ρ is the half-sum of (absolute) positive roots of G, and* $\mathrm{def}_G(b) = \mathrm{rank}_{\mathbb{Q}_p} G - \mathrm{rank}_{\mathbb{Q}_p} J_b$.

Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type. Take an embedding $G \hookrightarrow \mathrm{GL}_n$. Then we get an induced triple $(\mathrm{GL}_n, [b'], \{\mu'\})$. If $(G, [b], \{\mu\})$ is not of Hodge type, then $\{\mu'\}$ is not minuscule. In any case, we have the embedding

$$\overline{\mathcal{M}}(G, b, \mu)^{\mathrm{perf}} \simeq X^G_\mu(b) \hookrightarrow X^{\mathrm{GL}_n}_{\le\mu'}(b').$$

Remark 4.8   In this paper we do not study the Weil descent data on Rapoport–Zink spaces. To define the Weil descent datum on the abelian type Rapoport–Zink space $\check{\mathcal{M}}(G, b, \mu)$, we just mention that it should be possible to develop a similar theory as that in [11] by dividing the desired Weil descent datum into two parts, one part for $\check{\mathcal{M}}(G, b, \mu)^+$ and one part for $\pi_1(G)^\Gamma$ so that the morphism $\check{\mathcal{M}}(G, b, \mu) \to c_{b,\mu}\pi_1(G)^\Gamma$ is equivariant for the Weil descent data on two sides. The part for $\check{\mathcal{M}}(G, b, \mu)^+$ is inherited from the Weil descent datum for any associated Hodge type Rapoport–Zink space $\check{\mathcal{M}}(G_1, b_1, \mu_1)$. For the spaces as in the following Proposition 4.9(i) or Theorem 4.11, the Weil descent datum can be defined quite easily: by quotient from that for $\check{\mathcal{M}}(G_1, b_1, \mu_1)$ or by moduli methods as in [56, 3.48].

4.3   A Moduli Interpretation

Let $(G, [b], \{\mu\})$ be as in Theorem 4.6. Then by construction, locally the formal scheme $\check{\mathcal{M}}(G, b, \mu)$ admits a moduli interpretation. More precisely, take $(G_1, [b_1], \{\mu_1\})$ as in Definition 4.3. Then the formal scheme $\check{\mathcal{M}}(G_1, b_1, \mu_1)$ is a moduli space of p-divisible groups with crystalline Tate tensors. In particular, $\check{\mathcal{M}}(G, b, \mu)^+$ is a moduli space of p-divisible groups with crystalline Tate tensors such that under the map ω_{G_1} the image is fixed.

Suppose now that there exists a triple $(G_1, [b_1], \{\mu_1\})$ as in Definition 4.3 such that the map

$$\pi_1(G_1)^\Gamma \longrightarrow \pi_1(G_1^{\mathrm{ad}})^\Gamma$$

is surjective. Then the formal scheme $\check{\mathcal{M}}(G, b, \mu)$ admits a global moduli interpretation as follows.

Proposition 4.9   *Under the above assumption,*

(i)   *we have an isomorphism of formal schemes*

$$\check{\mathcal{M}}(G_1^{\mathrm{ad}}, b_1^{\mathrm{ad}}, \mu_1^{\mathrm{ad}}) \simeq \check{\mathcal{M}}(G_1, b_1, \mu_1)/X_*(Z_{G_1})^\Gamma;$$

(ii) $\check{\mathcal{M}}(G, b, \mu)$ *is the pullback of* $\check{\mathcal{M}}(G_1, b_1, \mu_1)/X_*(Z_{G_1})^\Gamma$ *under the morphism* $\pi_1(G)^\Gamma \to \pi_1(G^{\mathrm{ad}})^\Gamma$.

Proof We have the following cartesian diagrams (of morphisms between étale sheaves over $\overline{\mathbb{F}}_p$):

$$
\begin{array}{ccccc}
X^{G_1}_{\mu_1}(b_1) & \longrightarrow & X^{G_1^{\mathrm{ad}}}_{\mu_1^{\mathrm{ad}}}(b_1^{\mathrm{ad}}) & \longleftarrow & X^G_\mu(b) \\
\downarrow & & \downarrow & & \downarrow \\
c_{b_1,\mu_1}\pi_1(G_1)^\Gamma & \longrightarrow & c_{b_1^{\mathrm{ad}},\mu_1^{\mathrm{ad}}}\pi_1(G_1^{\mathrm{ad}})^\Gamma & \longleftarrow & c_{b,\mu}\pi_1(G)^\Gamma
\end{array}
$$

inducing the corresponding cartesian diagrams for Rapoport–Zink spaces (as étale sheaves over W). All the vertical maps in the above diagram are surjective by Proposition 2.4. The assertions follow by the assumption $\pi_1(G_1)^\Gamma \to \pi_1(G_1^{\mathrm{ad}})^\Gamma$ is surjective. ∎

Example 4.10 Consider Example 4.2 again. As the exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ induces a surjection

$$\pi_1(\mathrm{GL}_n)^\Gamma = \pi_1(\mathrm{GL}_n) \to \pi_1(\mathrm{PGL}_n)^\Gamma,$$

we have

$$\check{\mathcal{M}}(\mathrm{PGL}_n, b, \mu) \simeq \check{\mathcal{M}}(\mathrm{GL}_n, b_1, \mu_1)/p^{\mathbb{Z}}.$$

Another example will be given in Section 8.

By construction, both the above local moduli interpretation for $\check{\mathcal{M}}(G, b, \mu)^+$ and the global moduli interpretation in Proposition 4.9 are not canonical. Moreover, the formal scheme $\check{\mathcal{M}}(G, b, \mu)$ associated with a general unramified local Shimura datum of abelian type does not admit a moduli interpretation by p-divisible groups with additional structures. Nevertheless, we have the following theorem.

Theorem 4.11 *Let* $(G, [b], \{\mu\})$ *be an unramified local Shimura datum of abelian type. Assume that there exists an unramified local Shimura datum of Hodge type* $(G_1, [b_1], \{\mu_1\})$ *with a local Hodge embedding* $\iota\colon G_1 \hookrightarrow \mathrm{GL}_n$ *such that* $\iota(b_1)$ *has no slope 0, and such that* $(G_1^{\mathrm{ad}}, [b_1^{\mathrm{ad}}], \{\mu_1^{\mathrm{ad}}\}) \simeq (G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\})$. *Then the formal scheme* $\check{\mathcal{M}}(G, b, \mu)$ *represents the moduli functor of* (G, μ)-*displays* $\mathrm{RZ}_{G,\mu,b}$ *defined in* [4, 4.2].

Proof We just briefly sketch the arguments: by the proof of Proposition 4.9, we have the cartesian diagrams of étale sheaves:

$$
\begin{array}{ccccc}
\check{\mathcal{M}}(G_1, b_1, \mu_1) & \longrightarrow & \check{\mathcal{M}}(G^{\mathrm{ad}}, b^{\mathrm{ad}}, \mu^{\mathrm{ad}}) & \longleftarrow & \check{\mathcal{M}}(G, b, \mu) \\
\downarrow & & \downarrow & & \downarrow \\
c_{b_1,\mu_1}\pi_1(G_1)^\Gamma & \longrightarrow & c_{b_1^{\mathrm{ad}},\mu_1^{\mathrm{ad}}}\pi_1(G_1^{\mathrm{ad}})^\Gamma & \longleftarrow & c_{b,\mu}\pi_1(G)^\Gamma.
\end{array}
$$

Consider the Bültel–Pappas functors $\mathrm{RZ}_{G_1,\mu_1,b_1}$, $\mathrm{RZ}_{G_1^{\mathrm{ad}},\mu_1^{\mathrm{ad}},b_1^{\mathrm{ad}}}$, $\mathrm{RZ}_{G,\mu,b}$ as étale sheaves over W. By construction, their restrictions over $\overline{\mathbb{F}}_p$ are isomorphic to the étale sheaves

given by the corresponding affine Deligne–Lusztig varieties, *cf.* [4, Proposition 4.2.5 and Remark 4.2.6]. Thus, we also have the cartesian diagrams of étale sheaves

$$
\begin{array}{ccccc}
\mathrm{RZ}_{G_1,\mu_1,b_1} & \longrightarrow & \mathrm{RZ}_{G^{\mathrm{ad}},\mu^{\mathrm{ad}},b^{\mathrm{ad}}} & \longleftarrow & \mathrm{RZ}_{G,\mu,b} \\
\downarrow & & \downarrow & & \downarrow \\
c_{b_1,\mu_1}\pi_1(G_1)^\Gamma & \longrightarrow & c_{b_1^{\mathrm{ad}},\mu_1^{\mathrm{ad}}}\pi_1(G_1^{\mathrm{ad}})^\Gamma & \longleftarrow & c_{b,\mu}\pi_1(G)^\Gamma.
\end{array}
$$

By [4, Remark 5.2.7], $\check{\mathcal{M}}(G_1,b_1,\mu_1)$ represents the functor $\mathrm{RZ}_{G_1,\mu_1,b_1}$. Therefore, $\check{\mathcal{M}}(G,b,\mu)$ represents $\mathrm{RZ}_{G,\mu,b}$. ∎

When passing to the generic fibers, Rapoport–Zink spaces of abelian type are indeed canonical moduli spaces of some objects (local G-shtukas in the sense of Scholze): see the next section.

4.4 Generic Fibers and Local Shimura Varieties of Abelian Type

Let $(G,[b],\{\mu\})$ and $\check{\mathcal{M}} = \check{\mathcal{M}}(G,b,\mu)$ be as in Theorem 4.6. We consider the rigid analytic fiber $\mathcal{M} = \mathcal{M}(G,b,\mu)$ over L, regarded as an adic space. For any open compact subgroup $K \subset G(\mathbb{Z}_p)$, we construct a finite étale cover $\mathcal{M}_K$ of $\mathcal{M}$ as follows. If $(G,[b],\{\mu\})$ is of Hodge type, then this is known by [36, 7.4] (see also our subsection 3.3). Now consider the general case.

First, assume that $K = K_n$ for some $n \geq 1$, where $K_n = \ker(G(\mathbb{Z}_p) \to G(\mathbb{Z}_p/p^n\mathbb{Z}_p))$. On the component $\mathcal{M}^+ = (\check{\mathcal{M}}(G,b,\mu)^+)^{\mathrm{ad}}_\eta$, we can construct a finite étale cover $\mathcal{M}_n^+$ by taking some unramified local Shimura datum of Hodge type $(G_1,[b_1],\{\mu_1\})$ as in Definition 4.3 and using the moduli interpretation of $\mathcal{M}(G_1,b_1,\mu_1)$. We can take

$$
\mathcal{M}_n = [J_b(\mathbb{Q}_p) \times \mathcal{M}_n^+]/J_b(\mathbb{Q}_p)^+.
$$

In this way, we get a tower $(\mathcal{M}_n)_n$ on which $G(\mathbb{Z}_p)$ acts. Set $\mathcal{M}_0 = \mathcal{M}$. The action of $G(\mathbb{Z}_p)$ on $\mathcal{M}_n$ factors through $G(\mathbb{Z}_p)/K_n = G(\mathbb{Z}_p/p^n\mathbb{Z}_p)$. Now let $K \subset G(\mathbb{Z}_p)$ be arbitrary. Take some sufficiently large n such that $K_n \subset K$. Set $\mathcal{M}_K = \mathcal{M}_n/K$. Then $\mathcal{M}_K$ is a finite étale cover of $\mathcal{M}$, and it does not depend on the choice of n. When $K \subset G(\mathbb{Z}_p)$ is normal, $\mathcal{M}_K$ is a Galois cover of $\mathcal{M}$, with Galois group $G(\mathbb{Z}_p)/K$. For any $g \in G(\mathbb{Q}_p)$ and any open compact subgroup $K \subset G(\mathbb{Z}_p)$, we have a natural isomorphism

$$
\mathcal{M}_K \xrightarrow{\sim} \mathcal{M}_{gKg^{-1}}.
$$

As a result, the group $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ by Hecke correspondences.

As before, for any open compact $K \subset G(\mathbb{Z}_p)$, let $\mathcal{M}_K^+ \subset \mathcal{M}_K$ be the pullback of $\mathcal{M}^+ \subset \mathcal{M}$. In this way, we get a sub-tower $(\mathcal{M}_K^+)_K \subset (\mathcal{M}_K)_K$. Let $G(\mathbb{Q}_p)^+ \subset G(\mathbb{Q}_p)$ be the subgroup which is the stabilizer of the subtower $(\mathcal{M}_K^+)_K \subset (\mathcal{M}_K)_K$. By Lemma 2.3(i) the map $\omega_G: G(\mathbb{Q}_p) \longrightarrow \pi_1(G)^\Gamma$ is surjective. By construction, we have the induced bijection

$$
\omega_G: G(\mathbb{Q}_p)/G(\mathbb{Q}_p)^+ \xrightarrow{\sim} \pi_1(G)^\Gamma,
$$

and moreover,

$$
\mathcal{M}_K = J_b(\mathbb{Q}_p)\mathcal{M}_K^+, \quad (\mathcal{M}_K)_K = G(\mathbb{Q}_p)(\mathcal{M}_K^+)_K.
$$

Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type. In the sequel we want to construct a period map $\pi_{dR} \colon \mathcal{M} \to \mathscr{F}\ell_{G,\mu}$ on the generic fiber $\mathcal{M}$ of the associated Rapoport–Zink space and study some of its properties. Take any $(G_1, [b_1], \{\mu_1\})$ as in Definition 4.3. Then we have the canonical identification of the associated p-adic flag varieties over L:

$$\mathscr{F}\ell_{G,\mu} = G/P_\mu = \mathscr{F}\ell_{G_1,\mu_1} = G_1/P_{\mu_1}.$$

Sometimes we will simply write them as $\mathscr{F}\ell_\mu$. By [36, 7.5], we have a period map

$$\pi_{G_1,dR} \colon \mathcal{M}(G_1, b_1, \mu_1) \longrightarrow \mathscr{F}\ell_\mu,$$

which is $J_{b_1}(\mathbb{Q}_p)$-equivalent. If $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$ is an embedding of unramified local Shimura data of Hodge type, we get an induced embedding of flag varieties $\mathscr{F}\ell_{G_1,\mu_1} \hookrightarrow \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}$ over L. By construction, we have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}(G_1, b_1, \mu_1) & \lhook\joinrel\longrightarrow & \mathcal{M}(\mathrm{GL}_n, b', \mu') \\
\downarrow {\scriptstyle \pi_{G_1,dR}} & & \downarrow {\scriptstyle \pi_{\mathrm{GL}_n,dR}} \\
\mathscr{F}\ell_{G_1,\mu_1} & \lhook\joinrel\longrightarrow & \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}.
\end{array}
$$

Let us briefly review the construction of $\pi_{G_1,dR}$. Let $(s_\alpha) \subset \Lambda^{\otimes}$ be a finite collection of tensors (rank $\Lambda = n$) such that $G_1 \subset \mathrm{GL}(\Lambda)$ is the schematic stabilizer of (s_α). Then the closed embedding $\mathscr{F}\ell_{G_1,\mu_1} \hookrightarrow \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}$ classifies $\{\mu_1\}$-filtrations of Λ with respect to (s_α); *cf.* [36, Definition 2.2.3 and Lemma 2.2.8]. By [36, 7.5], the period morphism $\pi_{G_1,dR} \colon \mathcal{M}(G_1, b_1, \mu_1) \to \mathscr{F}\ell_{G_1,\mu_1}$ is given by $(\mathrm{Fil}^\bullet \mathbb{D}(X^{\mathrm{univ}})^{rig}_{\breve{\mathcal{M}}_1}, (t_\alpha^{rig}))$ using the induced isomorphism $\rho \colon \mathbb{D}(X^{\mathrm{univ}})^{rig}_{\breve{\mathcal{M}}_1} \simeq \mathcal{O}_{\mathcal{M}_1} \otimes \Lambda$, which matches (t_α^{rig}) with $1 \otimes s_\alpha$, where $\breve{\mathcal{M}}_1 = \breve{\mathcal{M}}(G_1, b_1, \mu_1)$, $\mathcal{M}_1 = \mathcal{M}(G_1, b_1, \mu_1)$, and $(X^{\mathrm{univ}}, (t_\alpha), \rho)$ is the universal p-divisible group with crystalline Tate tensors and quasi-isogeny over $\breve{\mathcal{M}}_1$. Thus, the above diagram is commutative.

Restricting the map $\pi_{G_1,dR}$ to $\mathcal{M}(G_1, b_1, \mu_1)^+ = \mathcal{M}^+$, we get a map

$$\pi_{dR}^+ \colon \mathcal{M}^+ = \mathcal{M}(G_1, b_1, \mu_1)^+ \longrightarrow \mathscr{F}\ell_\mu.$$

Then applying the group action of $J_b(\mathbb{Q}_p)$, we can define a $J_b(\mathbb{Q}_p)$-equivariant period map for $\mathcal{M}$:

$$\pi_{dR} = \pi_{G,dR} \colon \mathcal{M} = \mathcal{M}(G, b, \mu) \to \mathscr{F}\ell_\mu.$$

Let $\mathscr{F}\ell^{\mathrm{adm}}_{G_1,\mu_1} \subset \mathscr{F}\ell_\mu$ be the open subspace defined by Hartl (using Robba rings) in [27, section 6], which can be defined equivalently by using the crystalline period ring B_{cris} (*cf.* [14]). In [54, 65], the subspace $\mathscr{F}\ell^{\mathrm{adm}}_{G,\mu}$ is described using the Fargues–Fontaine curve, which applies to an arbitrary local Shimura datum $(G, [b], \{\mu\})$. See also Proposition 5.13.

Proposition 4.12 $\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}}$ *is the image of* $\pi_{G_1,dR}$. *And we have the commutative diagram*

$$
\begin{array}{ccc}
\mathcal{M}(G_1, b_1, \mu_1) & \lhook\joinrel\longrightarrow & \mathcal{M}(\mathrm{GL}_n, b', \mu') \\
\downarrow {\scriptstyle \pi_{G_1,dR}} & & \downarrow {\scriptstyle \pi_{\mathrm{GL}_n,dR}} \\
\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}} & \lhook\joinrel\longrightarrow & \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}^{\mathrm{adm}}.
\end{array}
$$

Proof By the above construction, the composition

$$
\mathcal{M}(G_1, b_1, \mu_1) \hookrightarrow \mathcal{M}(\mathrm{GL}_n, b', \mu') \xrightarrow{\pi_{\mathrm{GL}_n,dR}} \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}
$$

factors through $\mathscr{F}\ell_{G_1,\mu_1}$. By [27, Theorem 7.3] and [14, section 4], we have $\mathrm{Im}\,\pi_{\mathrm{GL}_n,dR} = \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}^{\mathrm{adm}}$. On the other hand, by [27, Proposition 6.2], we have

$$
\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}} = \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}^{\mathrm{adm}} \cap \mathscr{F}\ell_{G_1,\mu_1}.
$$

Thus, the above diagram commutes. To show $\mathrm{Im}\,\pi_{G_1,dR} = \mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}}$, it suffices to show that for any algebraically closed complete extension $C|L$, the induced map on C-valued points is surjective. Let $(x, (s_{x\alpha})) \in \mathscr{F}\ell_{G_1,\mu_1}(C, \mathcal{O}_C)$ with image $x \in \mathscr{F}\ell_{\mathrm{GL}_n,\mu'}(C, \mathcal{O}_C)$, such that there exists a point $(X/\mathcal{O}_C, \rho) \in \mathcal{M}(\mathrm{GL}_n, b', \mu')(C, \mathcal{O}_C)$ maps to x under $\pi_{\mathrm{GL}_n,dR}$. By definition, we have the isomorphism

$$
\rho : \mathbb{D}(X_{k_C})_{\mathbb{Q}} \simeq \mathcal{O}_{\mathcal{M}(\mathrm{GL}_n,b',\mu')} \otimes \Lambda,
$$

and $x = \rho(\mathrm{Fil}^1 \mathbb{D}(X_{k_C})_{\mathbb{Q}})$ considered as a filtration on the right-hand side. Via the rigidification ρ, there exists an element $g \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \subset \mathrm{GL}_n(\mathbb{Q}_p)/\mathrm{GL}_n(\mathbb{Z}_p)$ such that $\mathbb{D}(X_{k_C}) \simeq (g\Lambda \otimes W, g^{-1}b'\sigma(g)\sigma)$. Therefore, each tensor s_α on Λ induces a crystalline Tate tensor t_α on X. We get a point $(X/\mathcal{O}_C, (t_\alpha), \rho) \in \mathcal{M}(G_1, b_1, \mu_1)(C, \mathcal{O}_C)$, which, by construction, maps to $(X/\mathcal{O}_C, \rho) \in \mathcal{M}(\mathrm{GL}_n, b', \mu')(C, \mathcal{O}_C)$ and $(x, (s_{x\alpha})) \in \mathscr{F}\ell_{G_1,\mu_1}(C, \mathcal{O}_C)$ under the embedding $\mathcal{M}(G_1, b_1, \mu_1) \hookrightarrow \mathcal{M}(\mathrm{GL}_n, b', \mu')$ and the period map $\pi_{G_1,dR}$, respectively. ∎

For any open compact subgroup $K \subset G(\mathbb{Z}_p)$, we have the finite étale map $\mathcal{M}_K^+ = \mathcal{M}(G_1, b_1, \mu_1)_K^+ \to \mathcal{M}^+ = \mathcal{M}(G_1, b_1, \mu_1)^+$; thus, we get a morphism $\mathcal{M}_K^+ \longrightarrow \mathscr{F}\ell_\mu$. From this we can define a $J_b(\mathbb{Q}_p)$-equivalent period map for $\mathcal{M}_K$ $\pi_{G,dR} : \mathcal{M}_K \longrightarrow \mathscr{F}\ell_\mu$. When K varies, these period maps are compatible with the Hecke action of $G(\mathbb{Q}_p)$ on $(\mathcal{M}_K)_K$. Thus, we can think that there exists a $G(\mathbb{Q}_p)$-invariant map $(\mathcal{M}_K)_K \to \mathscr{F}\ell_\mu$.

Recall that we also have $\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{wa}}$ and $\mathscr{F}\ell_{G,\mu}^{\mathrm{wa}}$. By construction, we have $\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}} \subset \mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{wa}}$, and, similarly, $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}} \subset \mathscr{F}\ell_{G,\mu}^{\mathrm{wa}}$.

Lemma 4.13 *We have*

$$
\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{wa}} = \mathscr{F}\ell_{G,\mu}^{\mathrm{wa}}, \qquad \mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}} = \mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}.
$$

Proof The equality $\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{wa}} = \mathscr{F}\ell_{G,\mu}^{\mathrm{wa}}$ follows by [10, Proposition 9.5.3(iv)]. The second equality follows by the definition using G-bundles on the Fargues–Fontaine curve. ∎

Corollary 4.14 $\mathscr{F}\ell^{\mathrm{adm}}_{G,\mu} \subset \mathscr{F}\ell_\mu$ is the image of the above period map $\pi_{G,dR}$.

Proof Let $\mathscr{F}\ell^{\mathrm{adm},+}_{G_1,\mu_1} \subset \mathscr{F}\ell_{G_1,\mu_1}$ be the image of π^+_{dR}.

Since
$$\left(\mathcal{M}(G_1, b_1, \mu_1)_K\right)_K = G_1(\mathbb{Q}_p)\left(\mathcal{M}(G_1, b_1, \mu_1)^+_K\right)_K,$$
and the map $\mathcal{M}(G_1, b_1, \mu_1)_K \to \mathscr{F}\ell^{\mathrm{adm}}_{G_1,\mu_1}$ is $G_1(\mathbb{Q}_p)$-invariant, we get
$$\mathrm{Im}\,\pi_{G_1,dR} = \mathscr{F}\ell^{\mathrm{adm},+}_{G_1,\mu_1}.$$

We also have $(\mathcal{M}(G, b, \mu)_K)_K = G(\mathbb{Q}_p)(\mathcal{M}(G, b, \mu)^+_K)_K$, and by our construction, the map $(\mathcal{M}(G, b, \mu)_K)_K \to \mathscr{F}\ell_\mu$ is $G(\mathbb{Q}_p)$-invariant, we also get
$$\mathrm{Im}\,\pi_{G,dR} = \mathscr{F}\ell^{\mathrm{adm},+}_{G_1,\mu_1}.$$

Thus, $\mathrm{Im}\,\pi_{G_1,dR} = \mathrm{Im}\,\pi_{G,dR}$. By Proposition 4.12 and Lemma 4.13, we have
$$\mathrm{Im}\,\pi_{G,dR} = \mathrm{Im}\,\pi_{G_1,dR} = \mathscr{F}\ell^{\mathrm{adm}}_{G_1,\mu_1} = \mathscr{F}\ell^{\mathrm{adm}}_{G,\mu}. \qquad \blacksquare$$

Remark 4.15 We always have $\mathscr{F}\ell^{\mathrm{adm}}_{G,\mu} \subset \mathscr{F}\ell^{\mathrm{wa}}_{G,\mu}$. In [27, section 9] and [54, Question A.20], Hartl and Rapoport asked: when is $\mathscr{F}\ell^{\mathrm{adm}}_{G,\mu} = \mathscr{F}\ell^{\mathrm{wa}}_{G,\mu}$? For $G = \mathrm{GL}_n$, Hartl [27, Theorem 9.3] gave a complete solution of this question. For arbitrary G and minuscule μ, Fargues and Rapoport conjecture that this holds true with $[b]$ basic if and only if $(G, \{\mu\})$ is fully Hodge–Newton decomposable in the sense of [22, Definition 2.1 (2)]; *cf.* [22, Conjecture 0.1]. In the appendix we will see that $\mathscr{F}\ell^{\mathrm{adm}}_{G,\mu} = \mathscr{F}\ell^{\mathrm{wa}}_{G,\mu}$ in the case $[b]$ is basic and G is the special orthogonal group. For a solution of the Fargues–Rapoport conjecture, see [9].

Recall that by Lemma 2.3(i), the map $\omega_G \colon G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \longrightarrow \pi_1(G)^\Gamma$ is surjective.

Lemma 4.16 (i) *The following diagram is cartesian:*

$$\begin{array}{ccc}
G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \xrightarrow{\ \omega_G\ } & \pi_1(G)^\Gamma \\
\downarrow & & \downarrow \\
G^{\mathrm{ad}}(\mathbb{Q}_p)/G^{\mathrm{ad}}(\mathbb{Z}_p) & \xrightarrow[\ \omega_{G^{\mathrm{ad}}}\ ]{} & \pi_1(G^{\mathrm{ad}})^\Gamma.
\end{array}$$

(ii) *In particular, for G and G_1 as above we have $G(\mathbb{Q}_p)^+ \simeq G_1(\mathbb{Q}_p)^+$.*

Proof Note that non-empty fibers of both vertical maps are torsors under $X_*(Z_G)^\Gamma$. By [39, Lemma 1.2.4], if $g^{\mathrm{ad}} \in G^{\mathrm{ad}}(\mathbb{Q}_p)/G^{\mathrm{ad}}(\mathbb{Z}_p)$ and $\omega_{G^{\mathrm{ad}}}(g^{\mathrm{ad}})$ lifts to an element of $\pi_1(G)^\Gamma$, then g^{ad} lies in the image of $G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to G^{\mathrm{ad}}(\mathbb{Q}_p)/G^{\mathrm{ad}}(\mathbb{Z}_p)$. Therefore, the above diagram is cartesian.

In particular, we have the bijection $G(\mathbb{Q}_p)^+ \simeq G_1(\mathbb{Q}_p)^+$ from (i) for G and G_1 as above. $\qquad \blacksquare$

Let X be a rigid analytic space over a local field $k|\mathbb{Q}_p$. By [33, section 5] and [35, 8.4], we have the categories of $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems on X. Denote them by $\mathbb{Z}_p - Loc_X$ and $\mathbb{Q}_p - Loc_X$, respectively. Let G be a reductive group

over $\mathbb{Q}_p$. Denote by $\mathrm{Rep}\,G$ the category of rational representations of G. Recall that a $\mathbb{Q}_p$-G-local system on X is an exact tensor functor $\mathrm{Rep}\,G \to \mathbb{Q}_p - Loc_X$ (see [26], 4.3], for example). If G is, moreover, unramified, and we fix a reductive model $G_{\mathbb{Z}_p}$ of G over $\mathbb{Z}_p$, then we can define similarly $\mathbb{Z}_p$-G-local systems (or better notion: $G_{\mathbb{Z}_p}$-local systems) on X. In the following we will take $X = \mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$ or $X = \mathcal{M}$. By construction, we have the following proposition.

Proposition 4.17 *There exists a $J_b(\mathbb{Q}_p)$-equivariant $\mathbb{Q}_p$-G-local system $\mathbb{V}$ on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$ such that for any affinoid algebra (R, R^+) over $(L, \mathcal{O}_L)$, $\mathcal{M}(R, R^+)$ is the set of $G_{\mathbb{Z}_p}$-local systems in $\mathbb{V}_{\mathrm{Spa}(R,R^+)}$. In particular, there exists a $J_b(\mathbb{Q}_p)$-equivariant $G_{\mathbb{Z}_p}$-local system $\mathbb{L}$ on $\mathcal{M}$, and the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ is obtained by trivializing $\mathbb{L}$.*

Proof Under the identity $\mathscr{F}\ell_{G_1,\mu_1}^{\mathrm{adm}} = \mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$, we have a $\mathbb{Q}_p - G_1$-local system $\mathbb{V}_1$ on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$. Indeed, let $V_p(X^{\mathrm{univ}})$ be the rational Tate module of the universal p-divisible group X^{univ} over $\mathcal{M}_1$. We have the étale Tate tensors $t_{\alpha,et} \colon 1 \longrightarrow V_p(X^{\mathrm{univ}})^{\otimes}$ corresponding to each t_α under the comparison theorem, *cf.* [36], Theorem 7.1.6]. Then $V_p(X^{\mathrm{univ}})$ descends to a $\mathbb{Q}_p$-local system $\mathcal{V}_1$ on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$, equipped with the induced étale Tate tensors $t_{\alpha,et}$. Fix any geometric point $\overline{x} \to \mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$. Let $\rho_{\mathcal{V}_1,\overline{x}} \colon \pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x}) \to \mathrm{GL}_n(\mathbb{Q}_p)$ be the p-adic representation of the (de Jong's) fundamental group $\pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x})$ corresponding to $\mathcal{V}_1$, *cf.* [33], Theorem 4.2]. Then as $t_{\alpha,et}$ is invariant under $\pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x})$ (*cf.* [36], Theorem 7.1.6]), we get a morphism

$$\rho_{\mathbb{V}_1,\overline{x}} \colon \pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x}) \longrightarrow G_1(\mathbb{Q}_p)$$

which thus defines a $\mathbb{Q}_p - G_1$-local system $\mathbb{V}_1$ on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$. Moreover, as in the proof of Proposition [4.12], $\mathcal{M}_1(R, R^+)$ can be identified with the set of $\mathbb{Z}_p$-lattices together with tensors (t_α) in $\mathcal{V}_{1\mathrm{Spa}(R,R^+)}$, or equivalently, $[\mathrm{Isom}_{\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}}(\underline{G_1}, \mathbb{V}_1)/G_1(\mathbb{Z}_p)](R, R^+)$, where $\underline{G_1}$ is the trivial $\mathbb{Q}_p$-G_1-local system on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$. The tower $(\mathcal{M}_{1K})_{K \subset G_1(\mathbb{Z}_p)}$ is the geometric realization of $\mathbb{Q}_p$-G_1-local system $\mathbb{V}_1$ on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$ in the sense that $\mathcal{M}_{1K} \simeq \mathrm{Isom}_{\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}}(\underline{G_1}, \mathbb{V}_1)/K$. This identification preserves the Hecke actions of $G_1(\mathbb{Q}_p)$ and the actions of $J_{b_1}(\mathbb{Q}_p)$, *cf.* [27], Remark 2.7] and the proof of [27] Theorem 7.3 (c) and (d).

The group $\pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x})$ acts on $G_1(\mathbb{Q}_p)$ through $\rho_{\mathbb{V}_1,\overline{x}}$. The group $J_{b_1}(\mathbb{Q}_p)$ acts on $G_1(\mathbb{Q}_p)$ as the $\mathbb{Q}_p$-local system $\mathbb{V}_1$ on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$ is $J_{b_1}(\mathbb{Q}_p)$-equivariant.

Fix a point $x_0 \in \pi_1(G_1)^\Gamma$. Then we have the associated $\breve{\mathcal{M}}_1^+$ and $(\mathcal{M}_{1K}^+)_K$. The tower $(\mathcal{M}_{1K}^+)_K$ defines a subgroup $G_1(\mathbb{Q}_p)^+ \subset G_1(\mathbb{Q}_p)$ and a morphism

$$\rho_{\mathbb{V}_1,\overline{x}}^+ \colon \pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x}) \longrightarrow G_1(\mathbb{Q}_p)^+.$$

By Lemma [4.16](ii), we have $G(\mathbb{Q}_p)^+ \simeq G_1(\mathbb{Q}_p)^+$. Therefore, we can define an action of $\pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x})$ on $G(\mathbb{Q}_p)$ that commutes with the natural action of $J_b(\mathbb{Q}_p)$. Thus, we get a p-adic representation

$$\rho_{\overline{x}} \colon \pi_1(\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}, \overline{x}) \longrightarrow G(\mathbb{Q}_p),$$

which defines the desired $\mathbb{Q}_p$-G-local system $\mathbb{V}$ on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$. Moreover, for any $K \subset G(\mathbb{Z}_p)$, we have the identification

$$\mathcal{M}_K \simeq \mathrm{Isom}_{\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}}(\underline{G}, \mathbb{V})/K,$$

where $\underline{G}$ is the trivial $\mathbb{Q}_p$-G-local system on $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$. As above, this identification preserves the Hecke actions of $G(\mathbb{Q}_p)$ and the actions of $J_b(\mathbb{Q}_p)$. ∎

We note that Corollary 4.14 and Proposition 4.17 generalize [27, Theorem 8.4] (EL/PEL type case, but there one can allow ramification) to the abelian type case.

Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type. For each open compact subgroup $K \subset G(\mathbb{Z}_p)$, we get the associated Rapoport–Zink space

$$\mathcal{M}_K \simeq \coprod_{\pi_1(G)^\Gamma} \mathcal{M}_K^+.$$

Let Δ_G be the image of $\pi_1(G)^\Gamma \to \pi_1(G^{\mathrm{ad}})^\Gamma$. This is a finite group. We have an exact sequence

$$1 \longrightarrow X_*(Z_G)^\Gamma \longrightarrow \pi_1(G)^\Gamma \longrightarrow \Delta_G \longrightarrow 1.$$

We have the Hecke action of $G(\mathbb{Q}_p)$ on the tower $(\mathcal{M}_K)_K$. The Hecke action of the central subgroup $Z_G(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ stabilizes each $\mathcal{M}_K$. This action of $Z_G(\mathbb{Q}_p)$ is the same of that induced from $J_b(\mathbb{Q}_p)$ when we view $Z_G(\mathbb{Q}_p) \subset J_b(\mathbb{Q}_p)$. This action on

$$\mathcal{M}_K \simeq \coprod_{\pi_1(G)^\Gamma} \mathcal{M}_K^+ = \coprod_{\Delta_G} \coprod_{X_*(Z_G)^\Gamma} \mathcal{M}_K^+$$

is through the map $Z_G(\mathbb{Q}_p) \to X_*(Z_G)^\Gamma$ and the injection $X_*(Z_G)^\Gamma \to \pi_1(G)^\Gamma$.

In summary, the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ associated with an unramified local Shimura datum of abelian type can be viewed as the local Shimura varieties thought of in Conjecture 3.2. In the next section, we will put these spaces in a more general framework to get some moduli interpretation for each $\mathcal{M}_K$.

4.5 Infinite Level and the Hodge–Tate Period Map

Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type, and let $(\mathcal{M}_K)_K$ be associated tower of Rapoport–Zink spaces of abelian type. Let $\mathscr{F}\ell_{G,\mu^{-1}}$ be the p-adic flag variety over L associated with $(G, \{\mu^{-1}\})$.

Proposition 4.18 *There exists a pre-perfectoid space $\mathcal{M}_\infty$ over L such that*

$$\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K,$$

cf. [65, Definition 2.4.1] *for the precise meaning of such formula. Moreover, there exists a Hodge–Tate period map $\pi_{HT} \colon \mathcal{M}_\infty \longrightarrow \mathscr{F}\ell_{G,\mu^{-1}}$ that agrees with the period map previously defined in the EL/PEL cases in* [5, 65].

Proof If $(G, [b], \{\mu\})$ is of Hodge type, the existence of the preperfectoid space $\mathcal{M}_\infty$ over L such that $\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K$ is proved in [36, Proposition 7.6.1]. Fix an

embedding $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$ with $\{\mu'\}$ minuscule. We have the associated preperfectoid space $\mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty$ over L such that $\mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty \sim \varprojlim_{K'} \mathcal{M}(\mathrm{GL}_n, b', \mu')_{K'}$. The Hodge–Tate period map

$$\pi_{HT}\colon \mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty \longrightarrow \mathscr{F}\ell_{\mathrm{GL}_n, (\mu')^{-1}}$$

is defined in [65, 7.1]. Arguing as [5, section 2], we get that the composition

$$\mathcal{M}_\infty \hookrightarrow \mathcal{M}(\mathrm{GL}_n, b', \mu')_\infty \to \mathscr{F}\ell_{\mathrm{GL}_n, (\mu')^{-1}}$$

factors through $\mathscr{F}\ell_{G, \mu^{-1}}$. In particular we get $\pi_{HT}\colon \mathcal{M}_\infty \longrightarrow \mathscr{F}\ell_{G, \mu^{-1}}$. Now assume that we are in the general case. As $J_b(\mathbb{Q}_p)$ acts on $|\mathcal{M}_\infty| := \varprojlim_K |\mathcal{M}_K|$, it suffices to prove that there exist a preperfectoid space $\mathcal{M}_\infty^+$ over L such that

$$\mathcal{M}_\infty^+ \sim \varprojlim_K \mathcal{M}_K^+,$$

and a Hodge–Tate period map

$$\pi_{HT}^+\colon \mathcal{M}_\infty^+ \longrightarrow \mathscr{F}\ell_{G, \mu^{-1}}.$$

This follows from the Hodge type case. ∎

The following corollary is now clear.

Corollary 4.19 *There exists a subpreperfectoid space $\mathcal{M}_\infty^+ \subset \mathcal{M}_\infty$ over L that is stable under $G(\mathbb{Q}_p)^+$, such that*

$$\mathcal{M}_\infty^+ \sim \varprojlim_K \mathcal{M}_K^+, \quad \mathcal{M}_\infty = G(\mathbb{Q}_p)\mathcal{M}_\infty^+.$$

5 Generic Fibers of Rapoport–Zink Spaces as Moduli of Local G-shtukas

In this section, we work mainly on generic fibers. We want to explain that the generic fibers of the formal schemes $\breve{\mathcal{M}}(G, b, \mu)$, associated with unramified local Shimura data of abelian type $(G, [b], \{\mu\})$, can be viewed as moduli spaces fo local G-shtukas in mixed characteristic in the sense of Scholze[6], *cf.* [61]. We will work in the more general context of Conjecture 3.2. The first few subsections will be a brief review of works of Fargues [17, 20] and Scholze [61]. The reader familiar with these can go directly to Subsection 5.5.

5.1 The Fargues–Fontaine Curve and G-bundles

The Fargues–Fontaine curve $X_{F,E}$ is associated with a datum (F, E), where E is a local field with finite residue field $\mathbb{F}_q$ and $F|\mathbb{F}_q$ is a perfectoid field of characteristic p. For our purpose, we set $E = \mathbb{Q}_p$ and denote simply $X_{F, \mathbb{Q}_p}$ as X_F. It has several incarnations.

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[6]By [63, 64], the same should be true even for the formal schemes $\breve{\mathcal{M}}(G, b, \mu)$!

5.1.1 The Adic Curve

The adic curve X_F admits the adic uniformization $X_F = Y_F/\phi^{\mathbb{Z}}$, where $Y_F = \mathrm{Spa}(W(\mathcal{O}_F))\backslash V(p[\varpi_F])$, with $\varpi_F \in F$ satisfying $0 < |\varpi_F| < 1$. The action of the Frobenius ϕ on the Witt vectors is given by

$$\phi\left(\sum_n [x_n]p^n\right) = \sum_n [x_n^p]p^n \quad \text{for all } \sum_n [x_n]p^n \in W(\mathcal{O}_F).$$

It induces a totally discontinuous action on Y_F.

Suppose now that F is algebraically closed. Then there is a unique non-analytic point $x_k \in \mathrm{Spa}(W(\mathcal{O}_F))$. Set $\mathcal{Y} = \mathcal{Y}_F = \mathrm{Spa}(W(\mathcal{O}_F))\backslash\{x_k\}$. There exists a surjective continuous map $\kappa\colon \mathcal{Y} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by

$$\kappa(x) = \frac{\log|[\varpi_F](\tilde{x})|}{\log|p(\tilde{x})|},$$

where $\tilde{x}$ is the unique maximal generalization of x; *cf.* [61, 12.2]. For any $I \subset \mathbb{R}_{\geq 0}\cup\{\infty\}$, we denote $\mathcal{Y}_I = \kappa^{-1}(I)$. Then $Y := Y_F = \mathcal{Y}_{(0,\infty)}$.

Let $I \subset [0,\infty]$ be an interval of the form $[r,\infty)$ or $[r,\infty]$. Recall that a ϕ-module over $\mathcal{Y}_I$ is a pair $(\mathcal{E}, \phi_{\mathcal{E}})$, where $\mathcal{E}$ is a vector bundle over $\mathcal{Y}_I$ and $\phi_{\mathcal{E}}\colon \phi^*\mathcal{E}|_{\mathcal{Y}_I} \to \mathcal{E}$ is an isomorphism; *cf.* [61, Definition 13.2.1]. It follows that ϕ-modules over $\mathcal{Y}_{(0,\infty)}$ are the same as vector bundles over $X := X_F$.

5.1.2 The Algebraic Curve

There is a natural line bundle $\mathcal{O}(1)$ on X, corresponding to the ϕ-module on $\mathcal{Y}_{(0,\infty)}$ whose underlying line bundle is trivial and for which ϕ is $p^{-1}\phi$. Set $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ and

$$P = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(n)).$$

We have

$$H^0(X, \mathcal{O}(n)) = \mathcal{O}(Y)^{\phi = p^n}.$$

Let $X^{\mathrm{sch}} = \mathrm{Proj}(P)$. By [20], this is a one dimensional noetherian regular scheme over $\mathbb{Q}_p$. There exists a morphism of ringed spaces $X \to X^{\mathrm{sch}}$, and X can be viewed as the analytification of X^{sch} in some generalized sense.

Remark 5.1 Using the theory of diamond developed in [61], the curve admits yet another version: the diamond curve

$$X^{\diamond} = \left(\mathrm{Spa}(F) \times \mathrm{Spa}(\mathbb{Q}_p)^{\diamond}\right)/\phi^{\mathbb{Z}},$$

where $\phi = \mathrm{Frob}_F \times \mathrm{Id}$. We will not use this version in the sequel.

Let $\mathrm{Bun}_{X^{\mathrm{sch}}}$ and Bun_X be the categories of vector bundles on X^{sch} and X respectively. The morphism $X \to X^{\mathrm{sch}}$ induces a GAGA functor

$$\mathrm{Bun}_{X^{\mathrm{sch}}} \longrightarrow \mathrm{Bun}_X.$$

Theorem 5.2 ([16, 35]) *The GAGA functor induces an equivalence of categories*

$$\mathrm{Bun}_{X^{\mathrm{sch}}} \xrightarrow{\sim} \mathrm{Bun}_X.$$

There is another way to describe vector bundles on X. Consider the Robba ring

$$\widetilde{\mathcal{R}}_F = \varinjlim_r H^0(\mathcal{Y}_{(0,r]}, \mathcal{O}_{\mathcal{Y}_{(0,r]}}).$$

The Frobenius ϕ induces an action on $\widetilde{\mathcal{R}}_F$. Recall a ϕ-module over $\widetilde{\mathcal{R}}_F$ is a finite free $\widetilde{\mathcal{R}}_F$-module M equipped with a ϕ-linear automorphism.

Theorem 5.3 ([35, Theorem 6.3.12])   *There is an equivalence of categories*

$$\mathrm{Bun}_X \simeq \{\phi - \text{modules over } \widetilde{\mathcal{R}}_F\}.$$

The idea for the proof is that any ϕ-module over $\widetilde{\mathcal{R}}_F$ is defined over

$$\widetilde{\mathcal{R}}_F^r := H^0(\mathcal{Y}_{(0,r]}, \mathcal{O}_{\mathcal{Y}_{(0,r]}})$$

for some r small enough. This can be spread to a ϕ-module over $Y_F = \mathcal{Y}_{(0,\infty)}$ via pullback under Frobenius. Giving a ϕ-module over $\mathcal{Y}_{(0,\infty)}$ is the same as giving a vector bundle over X_F by the uniformization $X_F = \mathcal{Y}_{(0,\infty)}/\phi^{\mathbb{Z}}$.

Let $\phi-\mathrm{Mod}_L$ be the category of F-isocrystals over $\overline{\mathbb{F}}_p$, where as before $L = W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$. For any $(D, \phi) \in \phi - \mathrm{Mod}_L$, we can construct a vector bundle $\mathcal{E}(D, \phi)$ on X^{sch} by

$$\mathcal{E}(D, \phi) = \mathrm{Proj}\Big(\bigoplus_{n \geq 0} (D \otimes_L \mathcal{O}(Y))^{\phi \otimes \phi = p^n} \Big).$$

Theorem 5.4 ([20])   *The functor* $\mathcal{E}(-) \colon \phi - \mathrm{Mod}_L \to \mathrm{Bun}_{X^{\mathrm{sch}}}$ *is essentially surjective.*

Therefore, the composite $\mathcal{E}(-) \colon \phi - \mathrm{Mod}_L \to \mathrm{Bun}_{X^{\mathrm{sch}}} \to \mathrm{Bun}_X$ is also essentially surjective.

Let G be a connected reductive group over $\mathbb{Q}_p$. We have the following equivalent definitions of a G-bundle on X (or equivalently on X^{sch}):

(1)   an exact tensor functor $\mathrm{Rep}G \to \mathrm{Bun}_X$, where as before $\mathrm{Rep}G$ is the category of rational algebraic representations of G;
(2)   a G-torsor on X locally trivial for the étale topology.

Recall that an F-isocrystal with G-structure over $\overline{\mathbb{F}}_p$ is an exact tensor functor

$$\mathrm{Rep}G \longrightarrow \phi - \mathrm{Mod}_L.$$

If $b \in G(L)$, it then defines an F-isocystal with a G-structure

$$M_b \colon \mathrm{Rep}G \longrightarrow \phi - \mathrm{Mod}_L$$
$$V \longmapsto (V_L, b\sigma).$$

Its isomorphism class only depends on the σ-conjugacy class $[b] \in B(G)$ of b. Conversely, by Steinberg's theorem any F-isocrystal with G-structure arises in this way. Thus $B(G)$ is the set of isomorphism classes of F-isocrystals with G-structure; *cf.* [55, Remarks 3.4 (i)]. For $b \in G(L)$, let $\mathcal{E}_b$ be the composition of the above functor M_b and

$$\mathcal{E}(-) \colon \phi - \mathrm{Mod}_L \longrightarrow \mathrm{Bun}_{X^{\mathrm{sch}}} \simeq \mathrm{Bun}_X.$$

In this way, the set $B(G)$ also classifies G-bundles on X. In fact, we have the following theorem.

Theorem 5.5 ([17]) *Assume that F is algebraically closed. Then there is a bijection of sets*

$$B(G) \xrightarrow{\sim} H^1_{et}(X, G)$$
$$[b] \longmapsto [\mathcal{E}_b].$$

We discuss briefly the relative version of the above theory. Let (R, R^+) be a perfectoid affinoid $\mathbb{F}_p$-algebra, and let $S = \mathrm{Spa}(R, R^+)$ be the associated perfectoid space. We have an adic space over $\mathbb{Q}_p$:

$$X_S = Y_S / \phi^{\mathbb{Z}},$$

with $Y_S = Y_{R,R^+} = \mathrm{Spa}(A, A^+) \setminus V(p[\varpi_R])$, where

$$A = W(R^\circ) = \Big\{ \sum_{n \geq 0} [x_n] p^n \ \Big| \ x_n \in R^\circ \Big\}, \quad A^+ = \Big\{ \sum_{n \geq 0} [x_n] p^n \in A \ \Big| \ x_0 \in R^+ \Big\},$$

and ϖ_R is a pseudo-uniformizer of R. The adic space X_S is the relative version of the Fargues–Fontaine curve. We can also define the scheme

$$X_S^{\mathrm{sch}} = \mathrm{Proj}\Big(\bigoplus_{d \geq 0} H^0(X_S, \mathcal{O}_{X_S}(d)) \Big).$$

Then there exists a map of locally ringed spaces $X_S \to X_S^{\mathrm{sch}}$. We can define vector bundles on X_S, X_S^{sch} as above and the relative Robba ring $\widetilde{\mathcal{R}}_R$. Moreover, we have the following theorem.

Theorem 5.6 ([16, 35])

$$\mathrm{Bun}_{X_S^{\mathrm{sch}}} \simeq \mathrm{Bun}_{X_S} \simeq \{\phi - \text{modules over } \widetilde{\mathcal{R}}_R\}.$$

Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathbb{F}_p$, and let ϖ_R be a pseudo-uniformizer of R. We denote

$$\mathcal{Y}_{[0,\infty)}(R, R^+) = \mathrm{Spa}\, W(R^+) \setminus V([\varpi_R]).$$

Then we have a continuous map

$$\kappa \colon \mathcal{Y}_{[0,\infty)}(R, R^+) \longrightarrow [0, \infty),$$

the relative version of the map defined previously. With the same notation as there, we have

$$Y_S = \mathcal{Y}_{(0,\infty)}(R, R^+).$$

Let G be a connected reductive group over $\mathbb{Q}_p$. Then, as above, we can define G-bundles on $X_S, Y_S = \mathcal{Y}_{(0,\infty)}(R, R^+)$. If G is unramified over $\mathbb{Q}_p$, after fixing a reductive model $G_{\mathbb{Z}_p}$ of G over $\mathbb{Z}_p$, we can further define G-bundles on $\mathcal{Y}_{[0,\infty)}(R, R^+)$.

If we start with a perfectoid space S over $\mathbb{Q}_p$, then there exits a canonical closed embedding $x_S \colon S \hookrightarrow Y_{S^\flat}$, which in turn induces a closed embedding $x_S \colon S \hookrightarrow X_{S^\flat}$; *cf.* [15, 1.4]. Here $S^\flat$ is the tilt of S over $\mathbb{F}_p$ in the sense of [60]. Thus, we can view S as a Cartier divisor on $X_{S^\flat}$. If $S = \mathrm{Spa}(R, R^+)$ is perfectoid affinoid over $\mathbb{Q}_p$, by [17, 1.6] we have a corresponding Cartier divisor D on $X_{S^\flat}^{\mathrm{sch}}$. The formal completion of $X_{S^\flat}^{\mathrm{sch}}$ along D is $\mathrm{Spf}\, B_{\mathrm{dR},R}^+$; *cf.* [15, Proposition 1.33].

5.2 Local G-shtukas in Mixed Characteristic

Let the notation be as above. From now on, we assume that G is unramified[7] over $\mathbb{Q}_p$ and fix a reductive model $G_{\mathbb{Z}_p}$ of G over $\mathbb{Z}_p$. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathbb{F}_p$, with an untilt $S^\sharp$ of S. Then there exists a closed embedding $S^\sharp \hookrightarrow \mathcal{Y}_{[0,\infty)}(R, R^+)$.

Definition 5.7 ([61, Definition 11.4.1])   A local G-shtuka over S with one paw $x \colon S^\sharp \to \mathcal{Y}_{[0,\infty)}(R, R^+)$ is a pair $(\mathcal{E}, \phi_{\mathcal{E}})$, where

- $\mathcal{E}$ is a G-bundle over $\mathcal{Y}_{[0,\infty)}(R, R^+)$;
- $\phi_{\mathcal{E}} \colon \phi^* \mathcal{E} \to \mathcal{E}$ is an isomorphism over $\mathcal{Y}_{[0,\infty)}(R, R^+) \backslash \Gamma_x$, such that along Γ_x it is meromorphic. Here, Γ_x is the image of x.

One can then generalize the above notion to define a local G-shtuka over a general perfectoid space over $\mathbb{F}_p$.

Let C be a complete algebraically closed extension of $\mathbb{Q}_p$. We have the associated de Rham period ring $B_{\mathrm{dR}}^+ := B_{\mathrm{dR},C}^+$ with a fixed uniformizer $\xi \in B_{\mathrm{dR}}^+$. Let $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[\frac{1}{\xi}]$, $A_{\inf} = W(\mathcal{O}_{C^\flat})$. We have the following various descriptions of local G-shtukas with one paw at C, in the case $G = \mathrm{GL}_n$.

Theorem 5.8 ([61, Proposition 20.1.1]; see also [19])   *The following categories are equivalent:*

(i)   *Shtukas over* $\mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat})$ *with one paw at* C.
(ii)   *Pairs* (T, Ξ), *where* T *is a finite free* $\mathbb{Z}_p$-*module, and* $\Xi \subset T \otimes B_{\mathrm{dR}}$ *is a* B_{dR}^+-*lattice.*
(iii)   *Breuil–Kisin–Fargues modules over* $A_{\inf}$.
(iv)   *Quadruples* $(\mathcal{F}, \mathcal{F}', \beta, T)$, *where* $\mathcal{F}$ *and* $\mathcal{F}'$ *are vector bundles on the Fargues–Fontaine curve* $X = X_{C^\flat}$, *and* $\beta \colon \mathcal{F}|_{X \backslash \{\infty\}} \xrightarrow{\sim} \mathcal{F}'_{X \backslash \{\infty\}}$ *is an isomorphism, where* $\mathcal{F}$ *is trivial, and* $T \subset H^0(X, \mathcal{F})$ *is a* $\mathbb{Z}_p$-*lattice.*

If the paw is minuscule, i.e., *we have*

$$\xi(T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+) \subset \Xi \subset T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+,$$

then these categories are equivalent to the category of p-divisible groups over $\mathcal{O}_C$.

Recall that a Breuil–Kisin–Fargues module over $A_{\inf}$ is a pair (M, ϕ_M), where M is a finite free $A_{\inf}$-module and $\phi_M \colon (\phi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}]$ is an isomorphism; *cf.* [61, Definition 11.4.2].

5.3 Moduli of Local G-shtukas in Mixed Characteristic

We have the following generalizations of Definitions 3.1 and 3.3.

Definition 5.9   (i)   A local shtuka datum is a triple $(G, [b], \{\mu\})$, where

- G is a connected reductive group over $\mathbb{Q}_p$;

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[7]This is not necessary by the methods of [61–64]. Here, we restrict to the unramified case to simplify the exposition, which is also sufficient for our purpose.

- $\{\mu\}$ is a conjugacy class of cocharacters $\mu\colon \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ over $\overline{\mathbb{Q}}_p$;
- $[b] \in B(G,\mu) \subset B(G)$.

(ii) Let $(G_1,[b_1],\{\mu_1\}), (G_2,[b_2],\{\mu_2\})$ be two local shtuka data. A morphism

$$(G_1,[b_1],\{\mu_1\}) \longrightarrow (G_2,[b_2],\{\mu_2\})$$

is a homomorphism $f\colon G_1 \to G_2$ of algebraic groups sending $([b_1],\{\mu_1\})$ to $([b_2],\{\mu_2\})$.

Remark 5.10   (i) By definition, a local Shimura datum $(G,[b],\{\mu\})$ is a local shtuka datum with $\{\mu\}$ minuscule. For a local shtuka datum $(G,[b],\{\mu\})$, the simple factors of G^{ad} can be groups of arbitrary type.

(ii) In [61], several $\{\mu\}$'s can be allowed, as in the classical function field case; *cf.* [70].

(iii) In particular, if $(G,[b],\{\mu\})$ is a local shtuka datum, and $G \to G'$ is a homomorphism of reductive groups over $\mathbb{Q}_p$, we get the induced $[b'],\{\mu'\}$ such that $(G',[b'],\{\mu'\})$ is also a local shtuka datum.

(iv) We refer the reader to [28] for local function field case, where $\{\mu\}$ is replaced by a bound $\widehat{Z}$ in the sense of [28] Definition 2.1(b).

Let $(G,[b],\{\mu\})$ be a local shtuka datum. As before, we have the associated local reflex field E, and the reductive group J_b over $\mathbb{Q}_p$. Let F be an algebraically closed perfectoid field of characteristic p. By Theorem 5.5, we have a G-bundle on X_F, which is the same as a ϕ-G-module $(\mathcal{E}_b,\phi_{\mathcal{E}_b})$ on Y_F, well defined up to isomorphism. We will use freely the notion of diamond in the following; *cf.* [61] for basic definitions and properties. We define a functor on the category of perfectoid affinoid algebras over $\overline{\mathbb{F}}_p$ as follows.

Definition 5.11   ([61, Definition 19.3.3]) Let (R,R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_p$-algebra together with a map $x\colon \mathrm{Spa}(R,R^+)^{\diamond} \to \mathrm{Spa}(\breve{E})^{\diamond}$ (which is the same as giving an untilt of R over $\breve{E}$). Let $\mathrm{Sht}(G,b,\mu) \to \mathrm{Spa}(\breve{E})^{\diamond}$ be the functor such that for any $((R,R^+),x)$,

$$\mathrm{Sht}(G,b,\mu)\big((R,R^+),x\big) = \{((\mathcal{E},\phi_{\mathcal{E}}),\iota)\}/\simeq,$$

where

- $(\mathcal{E},\phi_{\mathcal{E}})$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}(R,R^+)$ with one paw at x, such that $(\mathcal{E},\phi_{\mathcal{E}})$ is bounded by $\{\mu\}$;
- $\iota\colon (\mathcal{E},\phi_{\mathcal{E}})|_{[\rho,\infty)} \xrightarrow{\sim} (\mathcal{E}_b,\phi_{\mathcal{E}_b})|_{[\rho,\infty)}$ is an isomorphism for some sufficiently large ρ.

The following is the main theorem of [61].

Theorem 5.12 (Scholze [61, Theorem 20.3.1])   *The functor* $\mathrm{Sht}(G,b,\mu)$ *is represented by a diamond over* $\mathrm{Spa}(\breve{E})^{\diamond}$.

(In [61] the theorem is proved for the case $G = \mathrm{GL}_n$, but one sees immediately that the proof given there also works for the general case. See also [64].)

We want to discuss period maps in this setting. Consider the B_{dR}^+-affine Grassmannian $Gr_G^{B_{\mathrm{dR}}^+}$ over $\mathbb{Q}_p$. This is the functor associating with any perfectoid affinoid $\mathbb{Q}_p$-algebra (R, R^+) the set

$$Gr_G^{B_{\mathrm{dR}}^+}(R, R^+) = \{(\mathcal{E}, \beta)\}/{\simeq},$$

where $\mathcal{E}$ is a G-torsor over $\mathrm{Spec} B_{\mathrm{dR},R}^+$, and β is a trivialization of $\mathcal{E} \otimes_{B_{\mathrm{dR},R}^+} B_{\mathrm{dR},R}$. One can check that $Gr_G^{B_{\mathrm{dR}}^+}$ is the étale sheaf associated with the presheaf

$$(R, R^+) \longmapsto G(B_{\mathrm{dR},R})/G(B_{\mathrm{dR},R}^+).$$

Consider the case (C, C^+) with $C|\mathbb{Q}_p$ an algebraically closed perfectoid field. Then we have the Cartan decomposition

$$G(B_{\mathrm{dR},C}) = \coprod_{\mu \in X_*(T)_+} G(B_{\mathrm{dR},C}^+)\mu(\xi)^{-1}G(B_{\mathrm{dR},C}^+),$$

where $T \subset B \subset G$ is a fixed choice of maximal torus inside a Borel subgroup B of G, and $X_*(T)_+ \subset X_*(T)$ is the associated set of dominant cocharacters. Fix a conjugacy class of cocharacters $\{\mu\}$ with the dominant representative μ. Let E be the field of definition of $\{\mu\}$. Consider $Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+} \subset Gr_G^{B_{\mathrm{dR}}^+} \otimes E$ the subfunctor such that

$$Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+}(R, R^+) =$$
$$\left\{(\mathcal{E}, \xi) \in Gr_G^{B_{\mathrm{dR}}^+}(R, R^+) \;\middle|\; \mathrm{Inv}(\mathcal{E}_x, \mathcal{E}_{0x}) \leq \mu^{-1}, \; \forall x \in \mathrm{Spa}(R, R^+)\right\}.$$

This is the analogue of the classical Schubert variety associated with $\{\mu\}$ in the setting of B_{dR}^+-affine Grassmannian $Gr_G^{B_{\mathrm{dR}}^+}$. There is an action of $J_b(\mathbb{Q}_p)$ on $Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+}$. By abuse of notation, we still denote by $Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+} \to \mathrm{Spa}(\check{E})^\diamond$ the sheaf base changed over $\mathrm{Spa}(\check{E})^\diamond$. By [61, Theorem 21.3.6], this is a diamond.

There exists an étale morphism of diamonds over $\mathrm{Spa}(\check{E})^\diamond$ (*cf.* [61, 20.4])

$$\pi_{\mathrm{dR}} \colon \mathrm{Sht}(G, b, \mu) \longrightarrow Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+}.$$

When $G = \mathrm{GL}_n$, this morphism can be defined by using Theorem 5.8(iv). Let

$$Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}} \subset Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+}$$

be the image of π_{dR}. This is an open sub-diamond, and we call it the admissible locus. We have the following description of the admissible locus.

Proposition 5.13 ([61, 20.5], [35])   *Let* $(\mathcal{E}, \beta) \in Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+}(R, R^+)$. *Then*

$$(\mathcal{E}, \beta) \in Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}}(R, R^+)$$

if and only if one of the following equivalent conditions holds: for any representation $V \in \mathrm{Rep} G$ *such that the center of* G *is mapped into the center of* $\mathrm{GL}(V)$, *with the associated vector bundle* $(\mathcal{E}_V, \beta_V)$,

  (i) $\forall x \in \mathrm{Spa}(R, R^+)$ *the vector bundle* $\mathcal{E}_{V,x}$ *is semi-stable of slope* 0;
  (ii) ϕ-*module of* $\mathcal{E}_V$ *is trivial*;

(iii) $\mathcal{E}_V$ extends to a ϕ-module over $\widetilde{\mathcal{R}}_R^{int}$, where $\widetilde{\mathcal{R}}_R^{int} = \varinjlim_r H^0(\mathcal{Y}_{[0,r]}, \mathcal{O}_{\mathcal{Y}})$ is the integral Robba ring.

The action of $J_b(\mathbb{Q}_p)$ on $Gr_{G,\leq\mu}^{B_{dR}^+}$ stabilizes the open sub-diamond $Gr_{G,\leq\mu}^{B_{dR}^+,\mathrm{adm}}$. The period morphism

$$\pi_{\mathrm{dR}} \colon \mathrm{Sht}(G,b,\mu) \longrightarrow Gr_{G,\leq\mu}^{B_{dR}^+,\mathrm{adm}}$$

is then $J_b(\mathbb{Q}_p)$-equivariant.

We have the following definition of local systems with additional structures on the diamond $Gr_{G,\leq\mu}^{B_{dR}^+,\mathrm{adm}}$, similar to the classical situation.

Definition 5.14 Let X be a diamond, and let G be a reductive group over $\mathbb{Q}_p$. Denote by $\mathrm{Rep}\,G$ the category of rational representations of G, and by $\mathbb{Q}_p - Loc_X$ the category of $\mathbb{Q}_p$-local systems on X. Then a $\mathbb{Q}_p$-G-local system on X is a tensor functor $\mathrm{Rep}\,G \to \mathbb{Q}_p - Loc_X$. If G is, moreover, unramified, and fix a reductive model (over $\mathbb{Z}_p$) $G_{\mathbb{Z}_p}$ of G, then we can define similarly $\mathbb{Z}_p$-G-local systems (or better notion: $G_{\mathbb{Z}_p}$-local systems) on X.

By [35, Corollary 8.7.10], there exists a $J_b(\mathbb{Q}_p)$-equivariant $\mathbb{Q}_p$-G-local system $\mathbb{V}$ over $Gr_{G,\leq\mu}^{B_{dR}^+,\mathrm{adm}}$, which realizes $\mathrm{Sht}(G,b,\mu)$ as the functor of the set of $\mathbb{Z}_p$-G-local systems in $\mathbb{V}$. In particular, there exists a $J_b(\mathbb{Q}_p)$-equivariant $\mathbb{Z}_p$-G-local system $\mathbb{L}$ over $\mathrm{Sht}(G,b,\mu)$.

Scholze's theorem above (Theorem 5.12) in fact gives us more information. More precisely, we get a tower of diamonds

$$\left(\mathrm{Sht}(G,b,\mu)_K\right)_{K \subset G(\mathbb{Z}_p)}$$

indexed by open compact subgroups $K \subset G(\mathbb{Z}_p)$ with $\mathrm{Sht}(G,b,\mu)_{G(\mathbb{Z}_p)} = \mathrm{Sht}(G,b,\mu)$, and the group $G(\mathbb{Q}_p)$ acts on this tower $(\mathrm{Sht}(G,b,\mu)_K)_{K \subset G(\mathbb{Z}_p)}$ by Hecke correspondences. Let (R,R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_p$-algebra together with a map $x \colon \mathrm{Spa}(R,R^+)^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$. Then

$$\mathrm{Sht}(G,b,\mu)_K\left((R,R^+),x\right) = \{((\mathcal{E},\phi_{\mathcal{E}}),\iota,\alpha)\}/\simeq,$$

where

- $(\mathcal{E},\phi_{\mathcal{E}})$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}(R,R^+)$ with one paw at x, such that $(\mathcal{E},\phi_{\mathcal{E}})$ is bounded by $\{\mu\}$;
- $\iota \colon (\mathcal{E},\phi_{\mathcal{E}})|_{[\rho,\infty)} \xrightarrow{\sim} (\mathcal{E}_b,\phi_{\mathcal{E}_b})|_{[\rho,\infty)}$ is an isomorphism for some sufficiently large ρ;
- α is a K-orbit of an isomorphism $\mathbb{L}(\mathcal{E},\phi_{\mathcal{E}}) \simeq \mathbb{L}_0$, where $\mathbb{L}(\mathcal{E},\phi_{\mathcal{E}})$ is the G-local system associated with $(\mathcal{E},\phi_{\mathcal{E}})$, $\mathbb{L}_0$ is the trivial G-local system over $\mathcal{Y}_{[0,\infty)}(R,R^+)$.

As $J_b(\mathbb{Q}_p) \subset \mathrm{Aut}(\mathcal{E}_b,\phi_{\mathcal{E}_b})$, (*cf.* [15, 2.5]), $J_b(\mathbb{Q}_p)$ acts each $\mathrm{Sht}(G,b,\mu)_K$ by modifying ι, and these actions are compatible when K varies. When the context is clear, we will simply denote $\mathrm{Sht}(G,b,\mu)_K$ by Sht_K. The cover

$$\pi_K \colon \mathrm{Sht}(G,b,\mu)_K \longrightarrow \mathrm{Sht}(G,b,\mu)$$

is obtained by trivializing K-level structures, which is finite étale. By trivializing all of $\mathbb{L}$, we get a pro-étale cover

$$\pi_\infty : \mathrm{Sht}(G, b, \mu)_\infty \longrightarrow \mathrm{Sht}(G, b, \mu).$$

We have the following moduli interpretation for $\mathrm{Sht}(G, b, \mu)_\infty$. Let (R, R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_p$-algebra together with a map $x : \mathrm{Spa}(R, R^+)^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$. Then

$$\mathrm{Sht}(G, b, \mu)_\infty\big((R, R^+), x\big) = \{\big((\mathcal{E}, \phi_\mathcal{E}), \iota, \alpha\big)\}/\simeq,$$

where

- $(\mathcal{E}, \phi_\mathcal{E})$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}(R, R^+)$ with one paw at x, such that $(\mathcal{E}, \phi_\mathcal{E})$ is bounded by $\{\mu\}$;
- $\iota : (\mathcal{E}, \phi_\mathcal{E})|_{[\rho,\infty)} \overset{\sim}{\to} (\mathcal{E}_b, \phi_{\mathcal{E}_b})|_{[\rho,\infty)}$ is an isomorphism for some sufficiently large ρ;
- $\alpha : \mathbb{L}(\mathcal{E}, \phi_\mathcal{E}) \simeq \mathbb{L}_0$ is an isomorphism, where, as before, $\mathbb{L}_0$ is the trivial G-local system over $\mathcal{Y}_{[0,\infty)}(R, R^+)$.

By construction, we have an isomorphism of diamonds over $\mathrm{Spa}(\breve{E})^\diamond$:

$$\mathrm{Sht}(G, b, \mu)_\infty/K \simeq \mathrm{Sht}(G, b, \mu)_K, \quad \mathrm{Sht}(G, b, \mu)_\infty = \varprojlim_K \mathrm{Sht}(G, b, \mu)_K.$$

For any open compact subgroup $K \subset G(\mathbb{Q}_p)$, we know that the fibers of

$$\mathrm{Sht}(G, b, \mu)_K(C, \mathcal{O}_C) \longrightarrow Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}}(C, \mathcal{O}_C)$$

are in bijection with $G(\mathbb{Q}_p)/K$. We remark that it should be possible to define a notion of étale fundamental group for the diamond $Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}}$ as [33], so that the $\mathbb{Q}_p$-G-local system $\mathbb{V}$ on $Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}}$ can be described in term of a collection of representations

$$\pi_1\big(Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}}, \overline{x}\big) \longrightarrow G(\mathbb{Q}_p),$$

for the geometric point $\overline{x}$ runs through each connected component of $Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}}$; see [62].

At the infinite level, there exists a Hodge–Tate period map (*cf.* [15, p. 38]; see also [26, Theorem 5.4])

$$\pi_{HT} : \mathrm{Sht}(G, b, \mu)_\infty \longrightarrow Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+},$$

where $Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+} \subset Gr_G^{B_{\mathrm{dR}}^+} \otimes E$ is the Schubert diamond associated with $\{\mu^{-1}\}$. We can be a little precise on the image of π_{HT}. By [5, Corollary 3.5.2], there is a natural map

$$\mathcal{E} : Gr_G^{B_{\mathrm{dR}}^+}(R, R^+) \longrightarrow \mathrm{Bun}_{G, X_{R^\flat, R^{+\flat}}}.$$

Take $(R, R^+) = (C, \mathcal{O}_C)$ with $C|\mathbb{Q}_p$ complete and algebraically closed. By Theorem 5.5, we get a map $b(-) : Gr_G^{B_{\mathrm{dR}}^+}(C, \mathcal{O}_C) \to B(G)$. By [5, Proposition 3.5.3], when restricting to $x \in Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+}(C, \mathcal{O}_C)$, one has $b(x) \in B(G, \mu)$. Then for any $[b'] \in B(G, \mu)$, we get a locally closed sub-diamond
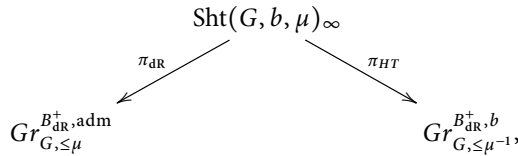
$$Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+,b'} \subset Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+},$$

such that the underling topological space $|Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+,b'}|$ is the fiber over $[b']$ under the above map $b(\,\cdot\,)$. Consider $[b'] = [b]$ as in the local shtuka datum. Then by construction, one has

$$\pi_{HT} \colon \mathrm{Sht}(G, b, \mu)_\infty(C, \mathcal{O}_C) \longrightarrow Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+,b}(C, \mathcal{O}_C),$$

for any $(C, \mathcal{O}_C)$ with $C|\mathbb{Q}_p$ complete and algebraically closed. That is, π_{HT} factors through $Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+,b}$.

In summary, we get two period morphisms

$$\begin{array}{ccc}
 & \mathrm{Sht}(G, b, \mu)_\infty & \\
 \swarrow{\scriptstyle\pi_{\mathrm{dR}}} & & \searrow{\scriptstyle\pi_{HT}} \\
 Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}} & & Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+,b},
\end{array}$$

and the period morphism π_{dR} factors through $\mathrm{Sht}(G, b, \mu)$.

Remark 5.15 In [15, 8.2.1], there is an alternative construction of the diamond $\mathrm{Sht}(G, b, \mu)_\infty$.

By construction, a morphism $(G_1, [b_1], \{\mu_1\}) \to (G_2, [b_2], \{\mu_2\})$ of local shtuka data induces a morphism of diamonds

$$\mathrm{Sht}(G_1, b_1, \mu_1) \longrightarrow \mathrm{Sht}(G_2, b_2, \mu_2).$$

More generally, we have morphisms

$$\mathrm{Sht}(G_1, b_1, \mu_1)_{K_1} \longrightarrow \mathrm{Sht}(G_2, b_2, \mu_2)_{K_2}$$

if K_1 is mapped into K_2 under $G_1 \to G_2$.

The above functoriality enables us to apply the Tannakian formalism. As before, we assume that G is unramified over $\mathbb{Q}_p$. Now consider an embedding $G \hookrightarrow \mathrm{GL}_n$; then $([b], \{\mu\})$ induces $([b'], \{\mu'\})$, so that $(\mathrm{GL}_n, [b'], \{\mu'\})$ forms a local shtuka datum, and we get a morphism of local shtuka data $(G, [b], \{\mu\}) \to (\mathrm{GL}_n, [b'], \{\mu'\})$. The following proposition is the local analogue of Deligne's theorem for Shimura varieties.

Proposition 5.16 *In the above setting, for any $K \subset G(\mathbb{Z}_p)$, there exists a $K' \subset \mathrm{GL}_n(\mathbb{Z}_p)$ such that there exists a natural closed embedding of diamonds*

$$\mathrm{Sht}(G, b, \mu)_K \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n, b', \mu')_{K'}.$$

The induced embedding of diamonds

$$\mathrm{Sht}(G, b, \mu)_\infty \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n, b', \mu')_\infty$$

is compatible with the de Rham and Hodge–Tate period morphisms on both sides.

Proof It suffices to prove that we have a closed embedding of diamonds

$$\mathrm{Sht}(G, b, \mu)_\infty \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n, b', \mu')_\infty.$$

This is clear from the construction above. Moreover, we have a closed embedding

$$Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}} \hookrightarrow Gr_{\mathrm{GL}_n,\leq\mu'}^{B_{\mathrm{dR}}^+,\mathrm{adm}},$$

and the following diagram on de Rham period maps is commutative

$$
\begin{array}{ccc}
\mathrm{Sht}(G,b,\mu)_\infty & \hookrightarrow & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')_\infty \\
\downarrow{\scriptstyle \pi_{\mathrm{dR}}} & & \downarrow{\scriptstyle \pi_{\mathrm{dR}}} \\
Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}} & \hookrightarrow & Gr_{\mathrm{GL}_n,\leq\mu'}^{B_{\mathrm{dR}}^+,\mathrm{adm}}.
\end{array}
$$

We also have the following commutative diagram on Hodge–Tate period maps

$$
\begin{array}{ccc}
\mathrm{Sht}(G,b,\mu)_\infty & \hookrightarrow & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')_\infty \\
\downarrow{\scriptstyle \pi_{HT}} & & \downarrow{\scriptstyle \pi_{HT}} \\
Gr_{G,\leq\mu^{-1}}^{B_{\mathrm{dR}}^+,b} & \hookrightarrow & Gr_{\mathrm{GL}_n,\leq(\mu')^{-1}}^{B_{\mathrm{dR}}^+,b'}.
\end{array}
$$ ∎

5.4 Moduli of Local G-shtukas and Affine Deligne–Lusztig Varieties

Let $(G,[b],\{\mu\})$ be a local shtuka datum. Recall that we assume G is unramified. We want to compare the moduli space of local G-shtukas $\mathrm{Sht}(G,b,\mu)$ and the affine Deligne–Lusztig variety $X_{\leq\mu}^G(b)$ associated with $(G,[b],\{\mu\})$ as in Section 2.

Let $(C,\mathcal{O}_C)$ be an affinoid perfectoid field of characteristic p with an untilt $C^\sharp$ of C. Let k be the residue field of $\mathcal{O}_C$. We have a $J_b(\mathbb{Q}_p)$-equivariant morphism of sets

$$\mathrm{sp} = \mathrm{sp}_{(G,b,\mu)}\colon \mathrm{Sht}(G,b,\mu)(C,\mathcal{O}_C) \longrightarrow X_{\leq\mu}^G(b)(k).$$

Indeed, first consider the case $G = \mathrm{GL}_n$; we have

$$\mathrm{Sht}(G,b,\mu)(C,\mathcal{O}_C) = \left\{\left((\mathcal{E},\phi_{\mathcal{E}}),\iota\right)\right\}/\simeq$$

with $((\mathcal{E},\phi_{\mathcal{E}}),\iota)$ a shtuka over $\mathrm{Spa}(C,\mathcal{O}_C)$ with one paw at $C^\sharp$. By Theorem 5.8, there exists a Breuil–Kisin module (M,ϕ) over $A_{\inf} = W(\mathcal{O}_C)$. Let $(M \otimes_{A_{\inf}} W(k),\phi)$ be the associated Dieudonné module. This defines a point in $X_{\leq\mu}^G(b)(k)$. This construction is compatible with the $J_b(\mathbb{Q}_p)$ actions on both sides. For the general case, we apply the Tannakian formalism: take any embedding $(G,[b],\{\mu\}) \to (\mathrm{GL}_n,[b'],\{\mu'\})$; then we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sht}(G,b,\mu)(C,\mathcal{O}_C) & \hookrightarrow & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')(C,\mathcal{O}_C) \\
\downarrow{\scriptstyle \mathrm{sp}_{G,b,\mu}} & & \downarrow{\scriptstyle \mathrm{sp}_{\mathrm{GL}_n,b',\mu'}} \\
X_{\leq\mu}^G(b)(k) & \hookrightarrow & X_{\leq\mu'}^{\mathrm{GL}_n}(b')(k).
\end{array}
$$

Recall that we have the map $\omega_G\colon G(L)/G(W) \to \pi_1(G)$. In the rest of this subsection we will only consider $X_{\leq\mu}^G(b)$ as a subset of $G(L)/G(W)$. Restricting ω_G to $X_{\leq\mu}^G(b)$, it gives

$$\omega_G\colon X_{\leq\mu}^G(b) \longrightarrow c_{b,\mu}\pi_1(G)^\Gamma.$$

Recall that as in Subsection 2.2, after replacing b by b' we can assume $c_{b,\mu} = 1$. On the other hand, restricting ω_G to $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ we get

$$\omega_G \colon G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \longrightarrow \pi_1(G)^\Gamma.$$

We can regard the quotient set $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ as a certain subset of $\mathrm{Sht}(G, b, \mu)$ $(C, \mathcal{O}_C)$; see below. The above two maps are related by the following reduction map. Recall that we have the period map

$$\pi_{\mathrm{dR}} \colon \mathrm{Sht}(G, b, \mu) \to Gr_{G,\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}},$$

which is $J_b(\mathbb{Q}_p)$-equivariant. Take any point $y \in Gr_{\leq\mu}^{B_{\mathrm{dR}}^+,\mathrm{adm}}(C, \mathcal{O}_C)$; then the fiber $\pi_{\mathrm{dR}}^{-1}(y)$ is in bijection with $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ once we fix a point $x \in \pi_{\mathrm{dR}}^{-1}(y)$. For $g \in J_b(\mathbb{Q}_p)$, we take the point $gx \in \pi_{\mathrm{dR}}^{-1}(gy)$ to identify $\pi_{\mathrm{dR}}^{-1}(gy)$ with $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$. In this way we can define an action of $J_b(\mathbb{Q}_p)$ on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$.

Lemma 5.17 *There is a $J_b(\mathbb{Q}_p)$-equivariant map*

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \longrightarrow X_{\leq\mu}^G(b), \quad g \longmapsto g_0,$$

such that $\omega_G(g) = \omega_G(g_0)$.

Proof Fix any point $x \in \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C)$. Then we have an injection

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C)$$

that identifies $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ with the Hecke orbit $\pi_{\mathrm{dR}}^{-1}(\pi_{\mathrm{dR}}(x))$ of x. For any $g \in J_b(\mathbb{Q}_p)$, the choice of the point gx to identify $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ with $\pi_{\mathrm{dR}}^{-1}(\pi_{\mathrm{dR}}(gx))$ shows that the injection $G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C)$ is $J_b(\mathbb{Q}_p)$-equivariant. The composite

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) \to X_{\leq\mu}^G(b)$$

gives the desired map, which is $J_b(\mathbb{Q}_p)$-equivariant, since, by the above construction, the specialization map $\mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) \to X_{\leq\mu}^G(b)$ is $J_b(\mathbb{Q}_p)$-equivariant. The second assertion follows by the same argument as that in the proof of [39, Lemma 1.2.18], by applying Theorem 5.8 (and Tannakian formalism) instead of [39, subsection 1.1]. ∎

Remark 5.18 Consider the composite map $G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to X_{\leq\mu}^G(b) \to \pi_1(G)^\Gamma$. Then this is surjective by Lemma 2.3(i). In [39, Proposition 1.2.23], Kisin proved a stronger result: the map

$$G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \longrightarrow \pi_0\bigl(X_{\leq\mu}^G(b)\bigr)$$

is surjective if $(G, [b], \{\mu\})$ is an unramified local Shimura datum of Hodge type.

The following is an analogue of [7, Lemma 2.4.1 and Corollary 2.4.2]; see also Proposition 2.5.

Proposition 5.19 *Let $Z \subset Z_G$ be a central subgroup and $G' = G/Z$, with the induced $[b']$ and $\{\mu'\}$ such that $(G', [b'], \{\mu'\})$ is a local shtuka datum. Then we have a cartesian diagram*

$$
\begin{array}{ccc}
\mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) & \longrightarrow & \mathrm{Sht}(G', b', \mu')(C, \mathcal{O}_C) \\
\downarrow & & \downarrow \\
X^G_{\leq \mu}(b) & \longrightarrow & X^{G'}_{\leq \mu'}(b').
\end{array}
$$

In particular, the induced diagram

$$
\begin{array}{ccc}
\mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) & \longrightarrow & \mathrm{Sht}(G', b', \mu')(C, \mathcal{O}_C) \\
\downarrow & & \downarrow \\
\pi_1(G)^\Gamma & \longrightarrow & \pi_1(G')^\Gamma
\end{array}
$$

is also cartesian.

Proof First, we have the natural identification $Gr^{B^+_{\mathrm{dR}}, \mathrm{adm}}_{G, \leq \mu} = Gr^{B^+_{\mathrm{dR}}, \mathrm{adm}}_{G', \leq \mu'}$, since, by construction, $Gr^{B^+_{\mathrm{dR}}, \mathrm{adm}}_{G, \leq \mu}$ depends only on the adjoint local shtuka datum $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\})$. Now consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sht}(G, b, \mu)(C, \mathcal{O}_C) & \longrightarrow & \mathrm{Sht}(G', b', \mu')(C, \mathcal{O}_C) \\
 & \searrow \scriptstyle{\pi_{G, dR}} \quad \scriptstyle{\pi_{G', dR}} \swarrow & \\
 & Gr^{B^+_{\mathrm{dR}}, \mathrm{adm}}_{G, \leq \mu}(C, \mathcal{O}_C). &
\end{array}
$$

For any point $x \in Gr^{B^+_{\mathrm{dR}}, \mathrm{adm}}_{G, \leq \mu}(C, \mathcal{O}_C)$, the above horizontal map induces a map on fibers

$$
G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \longrightarrow G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p);
$$

thus, it suffices to show that the following diagram is cartesian:

$$
\begin{array}{ccc}
G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \longrightarrow & G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
X^G_{\leq \mu}(b) & \longrightarrow & X^{G'}_{\leq \mu'}(b'),
\end{array}
$$

where the vertical maps are those constructed in Lemma 5.17. Consider the following diagram:

$$
\begin{array}{ccc}
G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \longrightarrow & G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
X_{\leq\mu}^{G}(b) & \longrightarrow & X_{\leq\mu'}^{G'}(b') \\
\downarrow{\scriptstyle\omega_G} & & \downarrow{\scriptstyle\omega_{G'}} \\
\pi_1(G)^{\Gamma} & \longrightarrow & \pi_1(G')^{\Gamma}.
\end{array}
$$

We know that the lower square is cartesian (*cf.* Proposition 2.5), and by Lemma 4.16 ,

$$
\begin{array}{ccc}
G(\mathbb{Q}_p)/G(\mathbb{Z}_p) & \longrightarrow & G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
\pi_1(G)^{\Gamma} & \longrightarrow & \pi_1(G')^{\Gamma}
\end{array}
$$

is also cartesian. Therefore, the upper square is cartesian. ∎

5.5 Local Shimura Varieties as Moduli of Local G-shtukas

We return to the setting of Definition 3.1. The following strengthened version of Theorem 5.12, which can be viewed as a partial solution of Conjecture 3.2 (as we do not give information on the desired Weil descent datum), is implied by the results in [5,61,62]. See also [64, Lecture 24] and [9, section 3]. Recall that by [61, Proposition 10.2.8], there is a fully faithful functor $X \mapsto X^{\diamond}$ from the category of normal rigid analytic spaces over k to the category of diamonds over $\mathrm{Spa}(k)^{\diamond}$ for any non-archimedean field k of characteristic 0.

Theorem 5.20 *Let* $(G,[b],\{\mu\})$ *be a local Shimura datum. Assume that G is unramified. Then there exists a tower of rigid analytic spaces over* $\mathrm{Sp}\breve{E}$, $(\mathcal{M}_K)_K$, *where K runs through all open compact subgroups of $G(\mathbb{Z}_p)$, with the following properties:*

(i) *the group $J_b(\mathbb{Q}_p)$ acts on each space $\mathcal{M}_K$;*
(ii) *the group $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_K$ as Hecke correspondences;*
(iii) *there exists a compatible system of étale and partially proper period maps*

$$
\pi_K \colon \mathcal{M}_K \to \mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}
$$

that is equivariant for the action of $J_b(\mathbb{Q}_p)$, where $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}} \subset \mathscr{F}\ell_{G,\mu}$ is the open subspace defined in [54, A.6] (see also Proposition 5.13 and [9]);
(iv) *for any K, we have an isomorphism of diamonds $\mathcal{M}_K^{\diamond} \simeq \mathrm{Sht}_K$.*

Proof Consider the Bialynicki–Birula morphism

$$
Gr_{G,\mu}^{B_{\mathrm{dR}}^+} \longrightarrow \mathscr{F}\ell_{G,\mu};
$$

cf. [5, Proposition 3.4.3]. Since μ is minuscule, it is an isomorphism, *cf.* [5, Theorem 3.4.5] that induces an isomorphism

$$Gr_{G,\mu}^{B_{\mathrm{dR}}^{+},\mathrm{adm}} \xrightarrow{\sim} \mathscr{F}\ell_{G,\mu}^{\mathrm{adm},\diamond}.$$

The tower $(\mathrm{Sht}_K)_K$ is constructed out of a $J_b(\mathbb{Q}_p)$-equivariant $\mathbb{Q}_p - G$-local system $\mathbb{V}$ over $Gr_{G,\mu}^{B_{\mathrm{dR}}^{+},\mathrm{adm}}$, which realizes $\mathrm{Sht}(G,b,\mu)$ as the functor of the set of $G_{\mathbb{Z}_p}$-local systems in $\mathbb{V}$. Since $Gr_{G,\mu}^{B_{\mathrm{dR}}^{+},\mathrm{adm}} \simeq \mathscr{F}\ell_{G,\mu}^{\mathrm{adm},\diamond}$, there exists a corresponding $J_b(\mathbb{Q}_p)$-equivariant $\mathbb{Q}_p$-G-local system over $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$, which we still denote by $\mathbb{V}$. Here we use the fact that the categories of étale $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems on an adic space X are equivalent to the corresponding categories on the pro-étale site $X_{\mathrm{proét}}$; *cf.* [35, Lemma 9.1.11]. Therefore, we get a tower of rigid analytic spaces $(\mathcal{M}_K)_K$ with the properties listed as in the theorem. ∎

Remark 5.21 In the above situation it is natural to conjecture that there exists a preperfectoid space $\mathcal{M}_\infty$ over $\breve{E}$ such that $\mathcal{M}_\infty \sim \varprojlim_K \mathcal{M}_K$ and $\mathcal{M}_\infty^\diamond = \mathrm{Sht}_\infty$. We will see that this is true if $(G,[b],\{\mu\})$ is unramified of abelian type; *cf.* Corollary 5.22. This is the local analogue of the fact that Shimura varieties of abelian type with infinite level at p are perfectoid; *cf.* [67].

Finally, we return to Rapoport–Zink spaces of abelian type. In particular, we assume $p > 2$ in the rest of this section.

Corollary 5.22 *Let $(G,[b],\{\mu\})$ be an unramified local Shimura datum of abelian type. For any open compact subgroup $K \subset G(\mathbb{Z}_p)$, let $\mathcal{M}_K$ and $\mathcal{M}_K'$ be the rigid analytic spaces over $\breve{E}$ constructed in Subsection 4.4 and Theorem 5.20 respectively. Then we have an isomorphism of rigid analytic spaces over $\breve{E}$ $\mathcal{M}_K \simeq \mathcal{M}_K'$.*

In particular, we get isomorphisms of diamonds over $\mathrm{Spa}(\breve{E})^\diamond$, $\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$, and $\mathcal{M}_\infty^\diamond \simeq \mathrm{Sht}_\infty$, with compatible period morphisms on both sides. In particular, the Hodge–Tate period map π_{HT} in Proposition 4.18 factors through $\pi_{HT} \colon \mathcal{M}_\infty \to \mathscr{F}\ell_{G,\mu^{-1}}^b$.

Proof We first prove the case where $(G,[b],\{\mu\})$ is of Hodge type. This follows exactly as the proof of [61, Theorem 19.4.5]. Moreover, we have the following cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{M}(G,b,\mu)_K^\diamond & \xrightarrow{\ \sim\ } & \mathrm{Sht}(G,b,\mu)_K \\
\big\uparrow & & \big\uparrow \\
\mathcal{M}(\mathrm{GL}_n,b',\mu')_{K'}^\diamond & \xrightarrow{\ \sim\ } & \mathrm{Sht}(\mathrm{GL}_n,b',\mu')_{K'}.
\end{array}
$$

Now assume that $(G,[b],\{\mu\})$ is of abelian type. We can apply Propositions 4.17 and 5.19, and compare the construction of $\mathcal{M}(G,b,\mu)_K$ with that of $\mathrm{Sht}(G,b,\mu)_K$. Here, as above, we use the fact that the categories of étale $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems on an adic space X are equivalent to the corresponding categories on the pro-étale site $X_{\mathrm{proét}}$, *cf.* [35, Lemma 9.1.11]. ∎

Let $(G, [b], \{\mu\})$ be a local Shimura datum with G unramified. By Theorem 5.20, there exists a tower of local Shimura varieties $(\mathcal{M}(G, b, \mu)_K)_K$ over $\mathrm{Sp}\breve{E}$ as conjectured by Rapoport and Viehmann. Take an embedding $G \hookrightarrow \mathrm{GL}_n$. Then we get an induced triple $(\mathrm{GL}_n, [b'], \{\mu'\})$, which is a local shtuka datum. The following corollary is now a consequence of Proposition 5.16 and Theorem 5.20.

Corollary 5.23   For any $K \subset G(\mathbb{Z}_p)$, there exists a $K' \subset \mathrm{GL}_n(\mathbb{Z}_p)$ such that there exists a natural closed embedding of diamonds

$$\mathcal{M}(G, b, \mu)_K^\diamond \hookrightarrow \mathrm{Sht}(\mathrm{GL}_n, b', \mu')_{K'}.$$

Remark 5.24   (i) Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of Hodge type, with the associated Rapoport–Zink spaces $\mathcal{M}_K$ and the moduli spaces of local G-shtukas Sht_K. The isomorphism $\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$ of diamonds over $\mathrm{Spa}(\breve{E})^\diamond$ indicates the magic "switching p-divisible groups with additional structures to local G-shtukas".

(ii) If $(G, [b], \{\mu\})$ is a general local Shimura datum, *e.g.*, an unramified local Shimura datum of abelian type but not of Hodge type, then we do not have p-divisible groups any more. However, via $\mathcal{M}_K^\diamond \simeq \mathrm{Sht}_K$, the local Shimura varieties $\mathcal{M}_K$ can be viewed as moduli of local G-shtukas.

(iii) Corollary 5.22 should be upgraded to the integral level as [64, Theorem 25.1.2 and Corollary 25.1.3]. Namely, we should get an isomorphisms of v-sheaves over $\mathrm{Spd}W$, $\breve{\mathcal{M}}^\diamond \simeq \mathcal{M}^{\mathrm{int}}$, where $\mathcal{M}^{\mathrm{int}}$ is the moduli functor introduced in [64, Definition 25.1.1] for the unramified local Shimura datum $(G, [b], \{\mu\})$. This could be done similarly to the methods of [64, Lecture XXV], and we will leave the details to the reader. Therefore, at the end we would have a canonical moduli interpretation for general $\breve{\mathcal{M}}$, compared with Subsection 4.3.

Remark 5.25   We refer to [57, sections 6,7,8] and [15, section 8] for the discussions on the conjectures on the realizations of local Langlands correspondences and local Jacquet–Langlands correspondences in the ℓ-adic cohomology of the tower $(\mathcal{M}_K)_K$ or $(\mathrm{Sht}_K)_K$.

6 Rapoport–Zink Uniformization for Shimura Varieties of Abelian Type

We discuss some global applications in this section. As [56, Chapter 6] and [37], we apply our construction of the formal schemes $\breve{\mathcal{M}}(G, b, \mu)$ to prove a uniformization theorem for Kisin's integral canonical models of Shimura varieties of abelian type [38]. Throughout this section, we assume $p > 2$.

6.1 Integral Canonical Models for Shimura Varieties of Abelian Type

Let (G, X) be a Shimura datum of abelian type, *i.e.*, there exists a Shimura datum of Hodge type (G_1, X_1) together with a central isogeny, $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$, such that it induces an isomorphism of the associated adjoint Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Fix a prime $p > 2$. Assume that G is unramified at p from now on. By [38, Lemma 3.4.13],

we can find a Shimura datum of Hodge type (G_1, X_1) satisfying the above and G_1 is unramified at p. Let E be the local reflex field of (G, X) for some place over p. In the sequel, we will only consider the open compact subgroups $K \subset G(\mathbb{A}_f)$ in the form $K = K_p K^p$ with $K_p = G(\mathbb{Z}_p)$.

Theorem 6.1 ([38, Theorem 3.4.10, Corollary 3.4.14])   *With the above notation and assumption, for any sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, there exists an integral canonical smooth model $S_K(G, X)$ of $\mathrm{Sh}_K(G, X)$ over $\mathcal{O}_E$. When K^p varies, the prime to p Hecke action on $(\mathrm{Sh}_K(G, X))_K$ extends to $(S_K(G, X))_K$.*

It will be useful to review how these integral models are constructed; *cf.* [38, 2.3 and 3.4].

6.1.1 Case (G, X) of Hodge Type

Take an embedding of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}, S^\pm)$. Let $K = K_p K^p \subset G(\mathbb{A}_f)$ be an open compact subgroup with $K_p = G(\mathbb{Z}_p)$. Take an open compact subgroup $K' = K'_p K'^p$ with $K'_p = \mathrm{GSp}(\mathbb{Z}_p)$, such that $K \subset K'$ and we have an closed immersion

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}, S^\pm)_E,$$

where E is the local reflex field for (G, X). For $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^\pm)$, we have the integral canonical model $S_{K'}(\mathrm{GSp}, S^\pm)$. Consider the Zariski closure $S_K^-(G, X)$ of $\mathrm{Sh}_K(G, X)_E$ in $S_{K'}(\mathrm{GSp}, S^\pm)_{\mathcal{O}_E}$. Then $S_K(G, X)$ is defined as the normalization of $S_K^-(G, X)$. In particular we have a finite morphism

$$S_K(G, X) \longrightarrow S_K^-(G, X) \subset S_{K'}(\mathrm{GSp}, S^\pm)_{\mathcal{O}_E}.$$

It will be useful to review some further structures for the integral canonical model $S_K(G, X)$. Let T be a scheme over $\mathcal{O}_E$. Attached to each point $x \in S_K(G, X)(T)$ we have a triple $(A_x, \lambda_x, \varepsilon_{x,K}^p)$, where (A_x, λ_x) is the polarized abelian scheme up to prime to p isogeny coming from pullback of the universal polarized abelian scheme over $S_{K'}(\mathrm{GSp}, S^\pm)$, and

$$\varepsilon_{x,K}^p \in \Gamma(T, \mathrm{Isom}(V_{\mathbb{A}_f^p}, \widehat{V}^p(A_x)_{\mathbb{Q}})/K^p)$$

is the (promoted) K-level structure coming from the K'-level structure $\varepsilon_{x,K'}^p$ on A_x; *cf.* [38, 3.4.2]. The triple $(A_x, \lambda_x, \varepsilon_{x,K'}^p)$ can be viewed as the polarized abelian scheme with level structure attached to the T-point of $S_{K'}(\mathrm{GSp}, S^\pm)$ induced by x. Let (s_α) be a finite collection of tensors that cut off the inclusion $G \subset \mathrm{GL}(V)$. As explained in [39, 1.3.6], there exist de Rham tensors $s_{\alpha, dR, x}$ and ℓ-adic étale tensors $(s_{\alpha, l, x})_{l \neq p}$ on the first relative de Rham cohomology and the first ℓ-adic cohomology of A_x, respectively. The level $\varepsilon_{x,K}^p$ takes s_α to $(s_{\alpha, l, x})_{l \neq p}$.

If $T = \mathrm{Spec}\, k$ where $k \subset \overline{\mathbb{F}}_p$ is a subfield containing the residue field k_E of $\mathcal{O}_E$, then there exists crystalline Tate tensors $(s_{\alpha, 0, x})$ on the first crystalline cohomology of A_x. If x is the specialization of a point $\widetilde{x}$ over F with $F|E$ an extension, then there exist p-adic étale tensors $(s_{\alpha, p, \widetilde{x}})$ on the first p-adic étale cohomology of $A_{\widetilde{x}}$, and $(s_{\alpha, 0, x})$ and $(s_{\alpha, p, \widetilde{x}})$ are related by the p-adic comparison theorem; *cf.* [39, Proposition 1.3.7].

By [39, Corollary 1.3.11] the datum

$$\left(A_x, \lambda_x, \varepsilon_{x,K}^p, (s_{\alpha,0,x})\right)$$

uniquely determines the point $x \in S_K(G, X)(k)$. Sometimes we will write $s_{\alpha,0,x}$ as $t_{\alpha,x}$ to be compatible with our previous notation on crystalline Tate tensors on p-divisible groups.

6.1.2 Case (G, X) of Abelian Type

Take a Shimura datum of Hodge type (G_1, X_1) that is unramified at p, together with a central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Let $K = K_p K^p \subset G(\mathbb{A}_f)$ be an open compact subgroup with $K_p = G(\mathbb{Z}_p)$. The integral model $S_K(G, X)$ is constructed as the quotient

$$S_{K_p}(G, X)/K^p,$$

where $S_{K_p}(G, X)$ is an integral model over $\mathcal{O}_E$ of the pro-scheme

$$\mathrm{Sh}_{K_p}(G, X) = \varprojlim_{K^p} \mathrm{Sh}_{K_p K^p}(G, X).$$

The scheme $S_{K_p}(G, X)$ is constructed as follows. Fix a connected component $X^+ \subset X$. We get the induced connected component $\mathrm{Sh}_K(G, X)_{\mathbb{C}}^+$ of the complex Shimura variety as usual. By [38, Proposition 2.2.4], it is defined over L. Consider the connected component

$$\mathrm{Sh}_{K_{1p}}(G_1, X_1)^+ = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1p} K_1^p}(G_1, X_1)^+$$

of $\mathrm{Sh}_{K_{1p}}(G_1, X_1) = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1p} K_1^p}(G_1, X_1)$, where $K_{1p} = G_1(\mathbb{Z}_p)$. Let $S_{K_{1p}}(G_1, X_1)^+$ be the Zariski closure of $\mathrm{Sh}_{K_{1p}}(G_1, X_1)^+$ in $S_{K_{1p}}(G_1, X_1)$ over $W = W(\overline{\mathbb{F}}_p)$. Write $Z = Z_G$. The above integral model $S_{K_p}(G, X)$ of $\mathrm{Sh}_{K_p}(G, X)$ over W is given by

$$S_{K_p}(G, X) = \left[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S_{K_{1p}}(G_1, X_1)^+\right]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{\circ},$$

where

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}}) = G(\mathbb{A}_f^p)/Z(\mathbb{Z}_{(p)})^- \ast_{G(\mathbb{Z}_{(p)})_+/Z(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$$

and

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{\circ} = G(\mathbb{Z}_{(p)})_+^-/Z(\mathbb{Z}_{(p)})^- \ast_{G(\mathbb{Z}_{(p)})_+/Z(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+;$$

similarly, we have $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})$ and $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{\circ}$; see [38, 3.3.2]. The scheme $S_{K_p}(G, X)$ descends to $\mathcal{O}_E$ and gives the integral canonical model of $\mathrm{Sh}_{K_p}(G, X) = \varprojlim_{K^p} \mathrm{Sh}_{K_p K^p}(G, X)$; see the proof of loc. cit. Theorem 3.4.10.

6.2 Newton Stratification of the Special Fibers

We keep the notations as above. We will work over $\overline{\mathbb{F}}_p$ in this subsection. By abuse of notation, denote the special fiber of $S_K = S_K(G, X)$ over $\overline{\mathbb{F}}_p$ by $\overline{S}_K$ for simplicity. In this subsection, we will write an element of $B(G_{\mathbb{Q}_p})$ simply by b, and $B(G, \mu) = B(G_{\mathbb{Q}_p}, \mu)$, as usual. In [68], we proved the following results.

Theorem 6.2   (i) *For any $b \in B(G, \mu)$, there exists a non-empty locally closed subset $\overline{S}_K^b \subset \overline{S}_K$, which we view as a subscheme of $\overline{S}_K$ with its reduced structure such that set theoretically, we have*

$$\overline{S}_K = \coprod_{b \in B(G, \mu)} \overline{S}_K^b.$$

(ii) *For any $b \in B(G, \mu)$, the Zariski closure of $\overline{S}_K^b$ in $\overline{S}_K$ is $\coprod_{b' \leq b} \overline{S}_K^{b'}$.*

For $b \in B(G, \mu)$, we call the subschemes $\overline{S}_K^b$ as the Newton strata of $\overline{S}_K$. If (G, X) is of Hodge type, then the existence of the Newton stratification is implied by [55]; see also [25, 2.3] and [74, 5.2].

For later use, we briefly review the construction of the Newton stratification. If (G, X) is of Hodge type, it is constructed by the associated p-divisible groups with crystalline Tate tensors. We now assume that (G, X) is of abelian type. In this case, let (G_1, X_1) be an unramified Shimura datum of Hodge type (G_1, X_1), together with a central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$, such that it induces an isomorphism of the associated adjoint Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Then we have a canonical bijection $B(G_1, \mu_1) \simeq B(G, \mu)$. Consider the Newton stratification at level K_1^p,

$$\overline{S}_{K_{1p} K_1^p}(G_1, X_1) = \coprod_{b \in B(G_1, \mu_1)} \overline{S}_{K_{1p} K_1^p}(G_1, X_1)^b.$$

When the level K_1^p varies, the Newton stratifications are compatible. Therefore, we get a Newton stratification

$$\overline{S}_{K_{1p}}(G_1, X_1) = \coprod_{b \in B(G_1, \mu_1)} \overline{S}_{K_{1p}}(G_1, X_1)^b$$

by taking inverse limit over K_1^p. As [39, 3.5.8], consider

$$\pi(G_1) := G_1(\mathbb{Q})_+^- \backslash G_1(\mathbb{A}_f)/G_1(\mathbb{Z}_p) = G_1(\mathbb{Z}_{(p)})_+^- \backslash G_1(\mathbb{A}_f^p),$$

which is the set of geometric connected components of $S_{K_{1p}}(G_1, X_1)$. By [68],

$$\overline{S}_{K_{1p}}(G_1, X_1)^b \subset \overline{S}_{K_{1p}}(G_1, X_1)$$

is stable under the action of $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})$, and we have a surjective $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})$-equivariant map

$$\overline{S}_{K_{1p}}(G_1, X_1)^b \longrightarrow \pi(G_1).$$

Let $\overline{S}_{K_{1p}}(G_1, X_1)^{b,+}$ be the pullback of $\overline{S}_{K_{1p}}(G_1, X_1)^b$ under the inclusion $\overline{S}_{K_{1p}}(G_1, X_1)^+ \hookrightarrow \overline{S}_{K_{1p}}(G_1, X_1)$. In other words, we consider the following commutative diagrams:

$$
\begin{array}{ccc}
\overline{S}_{K_{1p}}(G_1, X_1)^{b,+} & \hookrightarrow & \overline{S}_{K_{1p}}(G_1, X_1)^+ \\
\uparrow & & \uparrow \\
\overline{S}_{K_{1p}}(G_1, X_1)^b & \hookrightarrow & \overline{S}_{K_{1p}}(G_1, X_1) \\
\downarrow & & \downarrow \\
\pi(G_1) & = & \pi(G_1),
\end{array}
$$

where the upper square is cartesian. The stabilizer of $\overline{S}_{K_{1p}}(G_1, X_1)^{b,+} \subset \overline{S}_{K_{1p}}(G_1, X_1)^b$ is $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ$, and we have the identity

$$\overline{S}_{K_{1p}}(G_1, X_1)^b = \left[\mathscr{A}(G_{1\mathbb{Z}_{(p)}}) \times \overline{S}_{K_{1p}}(G_1, X_1)^{b,+}\right] / \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

For more details, we refer the reader to [68]. Now, as

$$\overline{S}_{K_p}(G, X) = \left[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \overline{S}_{K_{1p}}(G_1, X_1)^+\right] / \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ,$$

we get the Newton stratification

$$\overline{S}_{K_p}(G, X) = \coprod_{b \in B(G, \mu)} \overline{S}_{K_p}(G, X)^b,$$

where for any $b \in B(G, \mu)$, the associated stratum

$$\overline{S}_{K_p}(G, X)^b = \left[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \overline{S}_{K_{1p}}(G_1, X_1)^{b,+}\right] / \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ \hookrightarrow \overline{S}_{K_p}(G, X).$$

For any sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, we define

$$\overline{S}_{K_p K^p}(G, X)^b = \overline{S}_{K_p}(G, X)^b / K^p.$$

Therefore, we get the Newton stratification at the finite level

$$\overline{S}_{K_p K^p}(G, X) = \coprod_{b \in B(G, \mu)} \overline{S}_{K_p K^p}(G, X)^b.$$

6.3 Rapoport–Zink Uniformization

The notation will be the same as in the previous subsection. We will work over W in the rest of this section. For simplicity, denote the base change of $S_K = S_K(G, X)$ over W by the same notation. Let $b \in B(G, \mu)$ (the same convention as the last subsection). We get an unramified local Shimura datum of abelian type $(G_{\mathbb{Q}_p}, b, \{\mu\})$, thus a formal scheme $\breve{\mathcal{M}} = \breve{\mathcal{M}}(G, b, \mu)$ over W. Fix a point $x \in \overline{S}_K^b(\overline{\mathbb{F}}_p)$.

6.3.1 Case (G, X) of Hodge Type

We want to construct a morphism of formal schemes over $\mathrm{Spf}\, W$:

$$\Theta = \Theta_x \colon \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \longrightarrow \widehat{S}_K,$$

where $\widehat{S}_K$ is the formal completion of S_K along its special fiber. The morphism Θ is constructed in [37, Proposition 4.3 and Corollary 4.3.2]. Let $(A_x, \lambda_x, \varepsilon_{x,K}^p, (t_{\alpha,x}))$ be the abelian variety with additional structures attached to x, and let $I_\phi(\mathbb{Q})$ be the group of quasi-isogenies of A_x preserving $(t_{\alpha,x})$. Then $I_\phi(\mathbb{Q})$ is the group of $\mathbb{Q}$-points of a reductive group I_ϕ over $\mathbb{Q}$ (cf. [39, Corollary 2.3.1]), which depends only on the isogeny class of x ([39, 1.4.14]). In this case, Θ factors through the quotient by $I_\phi(\mathbb{Q})$,

$$\Theta \colon I_\phi(\mathbb{Q}) \backslash \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \longrightarrow \widehat{S}_K,$$

and the image $\mathcal{Z}_{\phi, K^p}$ is contained in the stratum $\overline{S}_K^b$.

6.3.2 Case (G, X) of Abelian Type

We first work on the level of sets. By [39, Theorem 4.6.7], we have the bijection

$$\overline{S}_{K_p}(G, X)^b(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi], b(\phi)=b} S(G, \phi),$$

where $\phi \colon \mathfrak{Q} \to \mathfrak{G}_G$ runs through the set of admissible morphisms of Galois gerbs, $[\phi]$ is the associated equivalence class (*cf.* [39, 3.3]), and

$$S(G, \phi) = \varprojlim_{K^p} I_\phi(\mathbb{Q}) \backslash \mathcal{M}_{\mathrm{red}}(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p,$$

where $\mathcal{M}_{\mathrm{red}}$ is the reduced special fiber of the Rapoport–Zink space $\breve{\mathcal{M}}$ associated with $(G_{\mathbb{Q}_p}, b(\phi), \{\mu\})$.

Remark 6.3 In [39, 3.3], in fact, one considers the set

$$S(G, \phi) = \varprojlim_{K^p} I_\phi(\mathbb{Q}) \backslash X_p(\phi) \times X^p(\phi)/K^p,$$

where $X_p(\phi)$ and $X^p(\phi)$ are certain sets canonically associated with ϕ, such that (*cf.* [39, Lemma 3.3.4])

$$X_p(\phi) \simeq X_\mu^G(b) \simeq \mathcal{M}_{\mathrm{red}}(\overline{\mathbb{F}}_p)$$

and $X^p(\phi)$ is a $G(\mathbb{A}_f^p)$-torsor.

Take an unramified Shimura datum of Hodge type (G_1, X_1), together with a central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$, such that it induces an isomorphism of the associated adjoint Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Let $\phi_1 \colon \mathfrak{Q} \to \mathfrak{G}_{G_1}$ be an admissible morphism of Galois gerbs. We note that

$$S(G_1, \phi_1) = \varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \backslash \mathcal{M}_{1\,\mathrm{red}}(\overline{\mathbb{F}}_p) \times G_1(\mathbb{A}_f^p)/K_1^p$$

$$= I_{\phi_1}(\mathbb{Q}) \backslash \mathcal{M}_{1\,\mathrm{red}}(\overline{\mathbb{F}}_p) \times G_1(\mathbb{A}_f^p),$$

where $\mathcal{M}_{1\,\mathrm{red}}$ is the reduced special fiber of the Rapoport–Zink space $\breve{\mathcal{M}}_1$ associated with $(G_{1\mathbb{Q}_p}, b(\phi_1), \{\mu_1\})$.

Fix an admissible morphism $\phi_0 \colon \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$. Consider

$$S(G, \phi_0) = \coprod_{[\phi], \phi^{\mathrm{ad}}=\phi_0} S(G, \phi)$$

$$= \coprod_{[\phi], \phi^{\mathrm{ad}}=\phi_0} \varprojlim_{K^p} I_\phi(\mathbb{Q}) \backslash \mathcal{M}_{\mathrm{red}}(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)/K^p.$$

By [39, Lemmas 3.7.2 and 3.7.4], there is an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ on $S(G, \phi_0)$, together with an $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$-equivariant surjective map

$$c_G \colon S(G, \phi_0) \longrightarrow \pi(G).$$

Recall that we have fixed a point $x \in \overline{S}_{K_p}(G, X)^b(\overline{\mathbb{F}}_p)$. We choose ϕ_0 such that $x \in S(G, \phi_0)$ under the bijection

$$\overline{S}_{K_p}(G, X)^b(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi], b(\phi)=b} S(G, \phi).$$

For the identity class $e \in \pi(G)$, consider the fiber

$$S(G, \phi_0)^+ = c_G^{-1}(e).$$

Let (G_1, X_1) be the unramified Shimura datum of Hodge type as above. Similarly, we have $S(G_1, \phi_0) = \coprod_{[\phi_1], \phi_1^{ad} = \phi_0} S(G_1, \phi_1)$ and $S(G_1, \phi_0)^+$.

Proposition 6.4 *We have the following isomorphism of sets with $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \langle \Phi \rangle$-action:*

$$S(G, \phi_0) \simeq \left[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times S(G_1, \phi_0)^+ \right] / \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

Proof This follows from of [39, Corollary 3.8.12]. ∎

Now we come back to Rapoport–Zink spaces. If $K_1^{p'} \subset K_1^p$ is another open compact subgroup of $G_1(\mathbb{A}_f^p)$, then we have the following commutative diagram:

$$
\begin{array}{ccc}
I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^{p'} & \longrightarrow & I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p \\
\downarrow{\scriptstyle \Theta_{1K_1^{p'}}} & & \downarrow{\scriptstyle \Theta_{1K_1^p}} \\
\widehat{S}_{K_{1p}K_1^{p'}}(G_1, X_1) & \longrightarrow & \widehat{S}_{K_{1p}K_1^p}(G_1, X_1)
\end{array}
$$

with horizontal maps finite. Therefore, if we set

$$\widehat{S}(G_1, \phi_1) := \varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p,$$

then we get

$$\Theta_1 = \varprojlim_{K_1^p} \Theta_{1K_1^p} \colon \widehat{S}(G_1, \phi_1) \longrightarrow \varprojlim_{K_1^p} \widehat{S}_{K_{1p}K_1^p}(G_1, X_1),$$

where both limits are taken in the category of formal schemes.

Lemma 6.5 *Let $\widehat{S}_{K_{1p}}(G_1, X_1)$ be the formal completion of $S_{K_{1p}}(G_1, X_1)$ along its special fiber. Then we have a canonical isomorphism of formal schemes*

$$\varprojlim_{K_1^p} \widehat{S}_{K_{1p}K_1^p}(G_1, X_1) = \widehat{S}_{K_{1p}}(G_1, X_1).$$

Proof This follows from the definition of inverse limit of formal schemes. ∎

We have thus $\Theta_1 \colon \widehat{S}(G_1, \phi_1) \longrightarrow \widehat{S}_{K_{1p}}(G_1, X_1)$. On the other hand, we have a surjective map

$$c_{G_1} \colon S_{K_{1p}}(G_1, X_1) \longrightarrow \pi(G_1).$$

Consider the fiber over e of this map c_{G_1}, $S_{K_{1p}}(G_1, X_1)^+ \subset S_{K_{1p}}(G_1, X_1)$, and let $\widehat{S}_{K_{1p}}(G_1, X_1)^+$ be formal completion of $S_{K_{1p}}(G_1, X_1)^+$ along its special fiber. Let

$$\Theta_1^+ \colon \widehat{S}(G_1, \phi_1)^+ := \left(\varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p)/K_1^p \right)^+ \longrightarrow \widehat{S}_{K_{1p}}(G_1, X_1)^+$$

be the pullback of

$$\Theta_1 : \widehat{S}(G_1, \phi_1) = \varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p) / K_1^p \longrightarrow \widehat{S}_{K_{1p}}(G_1, X_1)$$

under the inclusion $\widehat{S}_{K_{1p}}(G_1, X_1)^+ \hookrightarrow \widehat{S}_{K_{1p}}(G_1, X_1)$. The morphism Θ_1^+ can be written as $\Theta_1^+ = \varprojlim_{K_1^p} \Theta_{1K_1^p}^+$, with

$$
\begin{array}{ccc}
\left(I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p) / K_1^p \right)^+ & \longrightarrow & I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p) / K_1^p \\
\Big\downarrow {\scriptstyle \Theta_{1K_1^p}^+} & & \Big\downarrow {\scriptstyle \Theta_{1K_1^p}} \\
\widehat{S}_{K_{1p} K_1^p}(G_1, X_1)^+ & \longrightarrow & \widehat{S}_{K_{1p} K_1^p}(G_1, X_1).
\end{array}
$$

Define formal schemes

$$\widehat{S}(G_1, \phi_0)^+ = \coprod_{[\phi_1], \phi_1^{\mathrm{ad}} = \phi_0} \widehat{S}(G_1, \phi_1)^+$$

$$= \coprod_{[\phi_1], \phi_1^{\mathrm{ad}} = \phi_0} \left(\varprojlim_{K_1^p} I_{\phi_1}(\mathbb{Q}) \backslash \breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p) / K_1^p \right)^+$$

and

$$\widehat{S}(G, \phi_0) = \coprod_{[\phi], \phi^{\mathrm{ad}} = \phi_0} \varprojlim_{K^p} I_{\phi}(\mathbb{Q}) \backslash \breve{\mathcal{M}} \times G(\mathbb{A}_f^p) / K^p.$$

Proposition 6.6   *In the above situation, we have*

$$\widehat{S}(G, \phi_0) \simeq \left[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \widehat{S}(G_1, \phi_0)^+ \right] / \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

Proof   This is identical to the proof of Proposition 6.4.   ∎

Let $\mathcal{Z}_{\phi_1, K_1^p}$ (resp. $\mathcal{Z}_{\phi_1, K_1^p}^+$) be the image of $\Theta_{1K_1^p}$ (resp. $\Theta_{1K_1^p}^+$). This exists a geometric structure on $\mathcal{Z}_{\phi_1, K_1^p}$ as follows. We can write

$$\mathcal{Z}_{\phi_1, K_1^p} = \bigcup_{j \in J_{K_1^p}} Z_{\phi_1, K_1^p}^j,$$

where $J_{K_1^p}$ is the $I_{\phi_1}(\mathbb{Q})$-orbits of irreducible components of $\breve{\mathcal{M}}_1 \times G_1(\mathbb{A}_f^p) / K^p$, and $Z_{\phi_1, K_1^p}^j$ is the image of the irreducible components under $\Theta_{1K_1^p}$ corresponding to $j \in J_{K_1^p}$. For each $j \in J_{K_1^p}$, there exists only finitely many $j' \in J_{K_1^p}$ such that

$$Z_{\phi_1, K_1^p}^j \cap Z_{\phi_1, K_1^p}^{j'} \neq \varnothing.$$

Thus, we get an induced geometric structure on $\mathcal{Z}_{\phi_1, K_1^p}^+$ as

$$\mathcal{Z}_{\phi_1, K_1^p}^+ = \bigcup_{j \in J_{K_1^p}} Z_{\phi_1, K_1^p}^{j,+},$$

where $Z^{j,+}_{\phi_1, K^p_1}$ is the pullback of $Z^j_{\phi_1 K^p_1}$ to $\widehat{S}_{K_{1p} K^p_1}(G_1, X_1)^+$. When K^p_1 varies, $J_{K^p_1}$, $\mathcal{Z}_{\phi_1, K^p_1}$, and $\mathcal{Z}^+_{\phi_1, K^p_1}$ form inverse systems, and we set

$$\mathcal{Z}_{\phi_1} = \varprojlim_{K^p_1} \mathcal{Z}_{\phi_1, K^p_1}, \quad \mathcal{Z}^+_{\phi_1} = \varprojlim_{K^p_1} \mathcal{Z}^+_{\phi_1, K^p_1}.$$

Let J_1 be the $I_{\phi_1}(\mathbb{Q})$-orbits of irreducible components of $\check{\mathcal{M}}_1 \times G_1(\mathbb{A}^p_f)$. For any $j \in J_1$, let $Z^j_{\phi_1}$ be the image of the irreducible components under Θ_1 corresponding to j; then we can write

$$\mathcal{Z}_{\phi_1} = \bigcup_{j \in J_1} Z^j_{\phi_1} \quad \text{and} \quad Z^j_{\phi_1} = \varprojlim_{K^p_1} Z^j_{\phi_1, K^p_1},$$

where $Z^j_{\phi_1 K^p_1}$ is the image of the irreducible components corresponding to j under the composition

$$\check{\mathcal{M}}_1 \times G_1(\mathbb{A}^p_f) \longrightarrow \check{\mathcal{M}}_1 \times G_1(\mathbb{A}^p_f)/K^p_1 \longrightarrow \widehat{S}_{K_{1p} K^p_1}(G_1, X_1).$$

This is similar for $\mathcal{Z}^+_{\phi_1}$. By the proof of [39, Proposition 4.6.2], we have $\langle \Phi \rangle \times Z_1(\mathbb{Q}_p) \times \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$-equivariant bijection of sets (*cf.* Remark 6.8)

$$\mathcal{Z}_{\phi_1}(\overline{\mathbb{F}}_p) \simeq S(G_1, \phi_1), \quad \mathcal{Z}^+_{\phi_1}(\overline{\mathbb{F}}_p) \simeq S(G_1, \phi_1)^+.$$

We have (*cf.* [38, 3.4.11])

$$[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \widehat{S}_{K_{1p}}(G_1, X_1)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ = \widehat{S}_{K_p}(G, X).$$

Recall that we fixed an admissible morphism $\phi_0 \colon \mathfrak{Q} \to \mathfrak{G}_{G^{ad}}$. Set

$$\mathcal{Z}^+_{G_1, \phi_0} = \coprod_{[\phi_1], \phi^{ad}_1 = \phi_0} \mathcal{Z}^+_{\phi_1}.$$

Applying the functor $[\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times -]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ$ to $\mathcal{Z}^+_{G_1, \phi_0}$, we get a subset $\mathcal{Z}_{\phi_0}(= \mathcal{Z}_{G, \phi_0}) \subset \widehat{S}_{K_p} = \widehat{S}_{K_p}(G, X)$. Let $\mathcal{Z}_{\phi_0, K^p}$ be the image of $\mathcal{Z}_{\phi_0}$ under the projection $\widehat{S}_{K_p} \to \widehat{S}_K = \widehat{S}_{K_p K^p}$. Then we can define the formal completion of $\widehat{S}_K$ along $\mathcal{Z}_{\phi_0, K^p}$ as [56, Chapter 6] and [37, Definition 4.6].

Theorem 6.7 *We have an isomorphism of formal schemes over W,*

$$\Theta \colon \coprod_{[\phi], \phi^{ad} = \phi_0} I_\phi(\mathbb{Q}) \backslash \check{\mathcal{M}} \times G(\mathbb{A}^p_f)/K^p \xrightarrow{\sim} \widehat{S}_{K/\mathcal{Z}_{\phi_0, K^p}}.$$

Proof If (G, X) is of Hodge type, this is proved in [37, Theorem 4.7]. Assume that we are in the general case. By the above notation, it suffices to prove that

$$\coprod_{[\phi], \phi^{ad} = \phi_0} \varprojlim_{K^p} I_\phi(\mathbb{Q}) \backslash \check{\mathcal{M}} \times G(\mathbb{A}^p_f)/K^p \simeq [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \widehat{S}(G_1, \phi_0)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

This is given by Proposition 6.6. ∎

Remark 6.8 Denote by $G^{ad}_1(\mathbb{Z}_{(p)})^{+, I_{\phi_1}}$ the kernel of the composite of

$$G^{ad}_1(\mathbb{Z}_{(p)})^+ \hookrightarrow G^{ad}_1(\mathbb{Z}_{(p)}) \to H^1(\mathbb{Q}, Z_1) \to H^1(\mathbb{Q}, I_{\phi_1}),$$

where Z_1 is the center of G_1. Similarly, we define $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I_\phi}$. Following [39, 4.3.4], we define

$$\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}} = G_1(\mathbb{A}_f^p)/Z_1(\mathbb{Z}_{(p)})^- *_{G_1(\mathbb{Z}_{(p)})_+/Z_1(\mathbb{Z}_{(p)})} G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I_{\phi_1}},$$

$$\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1},\circ} = G_1(\mathbb{Z}_{(p)})_+^-/Z_1(\mathbb{Z}_{(p)})^- *_{G_1(\mathbb{Z}_{(p)})_+/Z_1(\mathbb{Z}_{(p)})} G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+,I_{\phi_1}}.$$

Similarly, we define $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi}$ and $\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi,\circ}$. The group $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$ acts on $S(G_1,\phi_1)$, *cf.* [39, Lemma 4.5.9]. By construction, we have an $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$-equivariant map $c_{G_1}: S(G_1,\phi_1) \longrightarrow \pi(G_1)$, which is surjective, since $G_1(\mathbb{A}_f^p)$ (and thus $\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}$) acts transitively on $\pi(G_1)$. For the identity class $e \in \pi(G_1)$, consider the fiber $S(G_1,\phi_1)^+ = c_{G_1}^{-1}(e)$. Then we have $S(G_1,\phi_0)^+ = \coprod_{[\phi_1],\phi_1^{\mathrm{ad}}=\phi_0} S(G_1,\phi_1)^+$. The stabilizer of $S(G_1,\phi_1)^+ \subset S(G_1,\phi_1)$ is

$$\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1},\circ} \subset \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}}.$$

We have

$$S(G_1,\phi_1) = \left[\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}} \times S(G_1,\phi_1)^+\right]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1},\circ}.$$

Take any $\phi_1: \mathfrak{Q} \to \mathfrak{G}_{G_1}$, such that

$$\phi^{\mathrm{ad}} = \phi_1^{\mathrm{ad}}: \mathfrak{Q} \longrightarrow \mathfrak{G}_{G^{\mathrm{ad}}}.$$

It should be possible that the strategy of [39, 3.8] enables us to prove the following refinement of Proposition 6.4:

$$S(G,\phi) = \left[\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}} \times S(G_1,\phi_1)^+\right]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi,\circ}.$$

Once this is done, the same argument as above shows that there is an isomorphism of formal schemes over W,

$$I_\phi(\mathbb{Q})\backslash \breve{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \widehat{S_K}_{/\mathcal{Z}_{\phi,K^p}},$$

where $\mathcal{Z}_{\phi,K^p}$ is the image under the projection $\widehat{S}_{K_p} \to \widehat{S}_{K_p K^p}$ of

$$\mathcal{Z}_\phi := \left[\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{I_{\phi_1}} \times \mathcal{Z}_{\phi_1}^+\right]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{I_\phi,\circ}.$$

Remark 6.9 In the special cases of Shimura curves associated with quaternion algebras over a totally real field, see [3] for a construction of the uniformization by Drinfeld spaces.

Let $\mathrm{Sh}_K(\phi_0) = (\widehat{S_K}_{/\mathcal{Z}_{\phi_0,K^p}})^{\mathrm{ad}}_\eta$. We get a natural morphism of adic spaces $\mathrm{Sh}_K(\phi_0) \to \mathrm{Sh}_K^{\mathrm{ad}}$. For any open compact subgroup $K_p' \subset G(\mathbb{Q}_p)$, let $\mathrm{Sh}_{K_p'K^p}(\phi_0) \to \mathrm{Sh}_{K_p'K^p}^{\mathrm{ad}}$ be the pullback of $\mathrm{Sh}_K(\phi_0) \to \mathrm{Sh}_K^{\mathrm{ad}}$ under the projection $\mathrm{Sh}_{K_p'K^p}^{\mathrm{ad}} \to \mathrm{Sh}_{K_pK^p}^{\mathrm{ad}}$. We get the following corollary from Theorem 6.7.

Corollary 6.10 *With the above notations, Θ induces an isomorphism of rigid analytic spaces over L,*

$$\Theta: \coprod_{[\phi],\phi^{\mathrm{ad}}=\phi_0} I_\phi(\mathbb{Q})\backslash \mathcal{M}_{K_p'} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \mathrm{Sh}_{K_p'K^p}(\phi_0).$$

We fix a morphism $\pi\colon \mathcal{M} \longrightarrow \mathrm{Sh}_K^{\mathrm{ad}}$ coming from the above uniformization isomorphism, which factors through the good reduction locus $(\widehat{S}_K)_\eta^{\mathrm{ad}} \subset \mathrm{Sh}_K^{\mathrm{ad}}$. By [46], the universal $\mathbb{Q}_p - G$-local system $\mathcal{L}_K$ on $\mathrm{Sh}_K^{\mathrm{ad}}$ is de Rham (which can be proved directly for the abelian type case; moreover we assume that $G = G^c$ for the notation G^c of [46]). When restricting to $(\widehat{S}_K)_\eta^{\mathrm{ad}}$, it is even crystalline. Recall by Proposition 4.17, we have the universal $\mathbb{Q}_p$-G-local system $\mathbb{V}$ on $\mathcal{M}$. We have the natural local-global compatibility identity $\mathbb{V} = \pi^* \mathcal{L}_K$.

Recall that in [67] we have proved that there exists a perfectoid space $\mathcal{S}_{K^p}$ over $\mathbb{C}_p$ such that

$$\mathcal{S}_{K^p} \sim \varprojlim_{K_p'} \mathrm{Sh}_{K_p' K^p}(G, X)^{\mathrm{ad}}.$$

On the other hand, by Proposition 4.18, we get a perfectoid space $\mathcal{M}_\infty$ over $\mathbb{C}_p$ such that

$$\mathcal{M}_\infty \sim \varprojlim_{K_p'} \mathcal{M}_{K_p', \mathbb{C}_p}.$$

From Corollary 6.10 we get the following corollary.

Corollary 6.11 *There exists a perfectoid space $\mathcal{S}_{K^p}(\phi_0)$ together with a map $\mathcal{S}_{K^p}(\phi_0) \to \mathcal{S}_{K^p}$, such that*

$$\mathcal{S}_{K^p}(\phi_0) \simeq \coprod_{[\phi], \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \mathcal{M}_\infty \times G(\mathbb{A}_f^p)/K^p.$$

Remark 6.12 For the $b \in B(G, \mu)$ we fixed in this subsection, we can define the Newton stratum $\mathcal{S}_{K^p}^b \subset \mathcal{S}_{K^p}$, which is a locally closed subspace; *cf.* [5, subsection 4.3]. Then we have $\mathcal{S}_{K^p}(\phi_0) \to \mathcal{S}_{K^p}$ factors through $\mathcal{S}_{K^p}^b$. In the case where b is basic, we will have $\mathcal{S}_{K^p}(\phi_0) = \mathcal{S}_{K^p}^b$; *cf.* the next subsection. In the general case, the image of $\mathcal{S}_{K^p}(\phi_0) \to \mathcal{S}_{K^p}^b$ is a strict subspace, and to understand the whole stratum $\mathcal{S}_{K^p}^b$, one should introduce Igusa varieties; *cf.* [5, section 4] in the PEL case.

6.4 The Case of Basic Strata

Let the notations be as in the last subsection. Assume now that $b = b_0$ is the basic element. Note that up to equivalence there is only one ϕ such that $b(\phi) = b_0$.

Theorem 6.13 *In the setting above, $\mathcal{Z}_{\phi_0, K^p} = \overline{S}_K^b$. Thus, we have an isomorphism of formal schemes over W*

$$\Theta\colon I_\phi(\mathbb{Q}) \backslash \check{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \widehat{S_K}_{/\overline{S}_K^b}.$$

Proof In the case where (G, X) is of Hodge type, this is proved in [37, Theorem 4.11]. We note that by [75, Lemma 7.2.14], I_ϕ is an inner form of G with $I_{\phi, p} = J_b$. The general case follows from the Hodge type case by construction. ∎

Corresponding to Corollaries 6.10 and 6.11, we have the following corollary.

Corollary 6.14 For any open compact subgroup $K_p' \subset G(\mathbb{Q}_p)$, Θ *induces an isomorphism of rigid analytic spaces over L,*

$$\Theta \colon I_\phi(\mathbb{Q}) \backslash \mathcal{M}_{K_p'} \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \mathrm{Sh}_{K_p' K^p}^b,$$

and an isomorphism of perfectoid spaces over $\mathbb{C}_p$,

$$\Theta \colon I_\phi(\mathbb{Q}) \backslash \mathcal{M}_\infty \times G(\mathbb{A}_f^p)/K^p \xrightarrow{\sim} \mathcal{S}_{K^p}^b.$$

7 Ekedahl–Oort Stratifications for Good Reductions of Rapoport–Zink Spaces

In this section, we will construct and study Ekedahl–Oort stratifications for the special fibers of our Rapoport–Zink spaces of abelian type, motivated by the study of Artin invariants of K3 surfaces. Then we will discuss some special interesting cases, namely, the fully Hodge–Newton decomposable cases; *cf.* [22]. In the next section we will further specialize to orthogonal groups. As before, we assume $p > 2$ in this section.

7.1 Ekedahl–Oort Stratifications for Special Fibers of Rapoport–Zink Spaces

Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type, $b \in G(L)$ be a representative of $[b]$, and let $\check{\mathcal{M}} = \check{\mathcal{M}}(G, b, \mu)$ be the associated Rapoport–Zink space by Theorem 4.6. Consider the special fiber $\overline{\mathcal{M}}$ over $\overline{\mathbb{F}}_p$ of $\check{\mathcal{M}}$ and the associated reduced special fiber $\mathcal{M}_{\mathrm{red}}$ of $\check{\mathcal{M}}$, which is by definition the reduced subscheme of $\overline{\mathcal{M}}$.

Since our local Shimura datum is unramified, we can consider $G\text{–Zip}^\mu$, the stack of G-zips of type μ over $\overline{\mathbb{F}}_p$ (we refer the reader to [51] and [77, 1.2] for some basic facts about G-zips and the stack $G\text{–Zip}^\mu$). The underling set of geometric points of $G\text{–Zip}^\mu$ is in canonical bijection with a subset $^J\mathcal{W}$ of the Weyl group $\mathcal{W}$ of G (for a fixed choice of maximal torus). More precisely, let J be the type of the parabolic subgroup of G attached to $\{\mu\}$ in the usual way, and let $\mathcal{W}_J$ be the associated subgroup of $\mathcal{W}$; then $^J\mathcal{W} \subset \mathcal{W}$ is the set of elements $w \in \mathcal{W}$ that are of minimal length in their coset $\mathcal{W}_J w$. There is a partial order $\preceq$ on $^J\mathcal{W}$ making it into a topological space; *cf.* [77, 3.1] or [74, 5.3]. In fact we have isomorphisms of topological spaces

$$|G\text{–Zip}^\mu| \simeq {}^J\mathcal{W};$$

cf. [77, Theorem 3.1.5] and [74, Proposition 5.12].

Theorem 7.1 There exists a formally smooth morphism

$$\zeta \colon \overline{\mathcal{M}} \longrightarrow G\text{–Zip}^\mu,$$

which induces a decomposition

$$\overline{\mathcal{M}} = \coprod_{w \in {}^J\mathcal{W}} \overline{\mathcal{M}}_w,$$

where $\overline{\mathcal{M}}_w \subset \overline{\mathcal{M}}$ is locally closed (could be empty).

Proof Assume first that $(G, [b], \{\mu\})$ is of Hodge type. Then the universal p-divisible group with crystalline Tate tensors gives rise to a G-zip of type μ (I, I_+, I_-, ι) on $\overline{\mathcal{M}}$: the

arguments of [77, Theorem 2.4.1] apply to our local setting. Thus, we get a morphism

$$\zeta \colon \overline{\mathcal{M}} \longrightarrow G\text{--}\mathrm{Zip}^\mu.$$

This morphism is $J_b(\mathbb{Q}_p)$-invariant. To show this morphism is formally smooth, one can apply the arguments for the proof of [77, Theorem 3.1.2]. More precisely, from the datum (I_+, I_-, ι) one can construct an $E_{G,\mu}$-torsor $\overline{\mathcal{M}}^\sharp$ over $\overline{\mathcal{M}}$, which sits in the following cartesian diagram:

$$
\begin{array}{ccc}
\overline{\mathcal{M}}^\sharp & \xrightarrow{\ \zeta^\sharp\ } & G_{\overline{\mathbb{F}}_p} \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}} & \longrightarrow & [E_{G,\mu} \backslash G_{\overline{\mathbb{F}}_p}] = G\text{--}\mathrm{Zip}^\mu.
\end{array}
$$

Here the algebraic group $E_{G,\mu}$ is as that in [51, section 3] (see also [77, subsection 1.2]), considered as a group over $\overline{\mathbb{F}}_p$. It suffices to show that $\zeta^\sharp$ is formally smooth. By [24, Prop. 17.1.6], formal smoothness is local in both the sources and targets; thus, we can reduce to the algebra side: taking any affine subschemes $\mathrm{Spec}\, B \subset \overline{\mathcal{M}}^\sharp$, $\mathrm{Spec}\, A \subset G_{\overline{\mathbb{F}}_p}$ over $\overline{\mathbb{F}}_p$ with induced $\zeta^\sharp \colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, we need to prove that B is formally smooth over A. Since $\overline{\mathcal{M}}^\sharp$ is formally smooth (and formally locally of finite type) over $\overline{\mathbb{F}}_p$, $G_{\overline{\mathbb{F}}_p}$ is smooth over $\overline{\mathbb{F}}_p$. By [48, Theorems 65 and 66], the fact that B is formally smooth over A is equivalent to the map

$$\Omega_{A/\overline{\mathbb{F}}_p} \otimes_A B/J \longrightarrow \Omega_{B/\overline{\mathbb{F}}_p} \otimes B/J$$

being left invertible, where $J \subset B$ is an ideal of definition. Note that $\Omega_{B/\overline{\mathbb{F}}_p} \otimes B/J$ is a projective B/J-module. By [23, Cor. 19.1.12], the last statement is equivalent to the following: for any closed points $x \in \mathrm{Spec}\, B$, $y = \zeta^\sharp(x) \in \mathrm{Spec}\, A$, the induced maps on the tangent spaces $T_x \mathrm{Spec}\, B \to T_y \mathrm{Spec}\, A$ are surjective. One can then show the surjectivity on tangent spaces by the arguments in the proof of [77, Theorem 3.1.2].

Now assume that $(G, [b], \{\mu\})$ is unramified of abelian type. Take any unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\}) \simeq (G_1^{\mathrm{ad}}, [b_1^{\mathrm{ad}}], \{\mu_1^{\mathrm{ad}}\})$. Let $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_1$ be the special fibers of Rapoport–Zink spaces attached to $(G, [b], \{\mu\})$ and $(G_1, [b_1], \{\mu_1\})$, respectively. By construction, after fixing $x_0 \in \pi_1(G_1)^\Gamma$, we have $\overline{\mathcal{M}}^+ = \overline{\mathcal{M}}_1^+$. Consider the restriction $\zeta_1^+ \colon \overline{\mathcal{M}}_1^+ \to G_1\text{--}\mathrm{Zip}^{\mu_1}$. As $|G_1\text{--}\mathrm{Zip}^{\mu_1}| \simeq |G\text{--}\mathrm{Zip}^\mu|$, we get a formally smooth morphism $\zeta^+ \colon \overline{\mathcal{M}}^+ \to G\text{--}\mathrm{Zip}^\mu$. Applying the $J_b(\mathbb{Q}_p)$-action, we get a formally smooth $J_b(\mathbb{Q}_p)$-invariant morphism

$$\zeta \colon \overline{\mathcal{M}} \longrightarrow G\text{--}\mathrm{Zip}^\mu,$$

as desired. ∎

We note that in the EL/PEL cases, Wedhorn and Lau proved the above proposition; *cf.* [72], [51, Example 9.21], and [69, Theorem 3.2]. If $(G, [b], \{\mu\}) \hookrightarrow (\mathrm{GL}_n, [b'], \{\mu'\})$ is an embedding of unramified local Shimura data of Hodge type, by construction we have the commutative diagram

$$\overline{\mathcal{M}}(G, b, \mu) \lhook\joinrel\longrightarrow \overline{\mathcal{M}}(\mathrm{GL}_n, b', \mu')$$

$$\Big\downarrow \zeta_G \qquad\qquad\qquad \Big\downarrow \zeta_{\mathrm{GL}_n}$$

$$G\text{-Zip}^\mu \longrightarrow \mathrm{GL}_n\text{-Zip}^{\mu'}.$$

Let $^J\mathcal{W}^b \subset {}^J\mathcal{W}$ be the subset defined by the image of ζ. For each $w \in {}^J\mathcal{W}^b$, we call $\overline{\mathcal{M}}_w$ the Ekedahl–Oort stratum of $\overline{\mathcal{M}}$ attached to w. We get a stratification

$$\overline{\mathcal{M}} = \coprod_{w \in {}^J\mathcal{W}^b} \overline{\mathcal{M}}_w.$$

We also get an induced stratification

$$\mathcal{M}_{\mathrm{red}} = \coprod_{w \in {}^J\mathcal{W}^b} \mathcal{M}_w,$$

where $\mathcal{M}_w \subset \mathcal{M}_{\mathrm{red}}$ is a locally closed subscheme of $\mathcal{M}_{\mathrm{red}}$, which we call the Ekedahl–Oort stratum of $\mathcal{M}_{\mathrm{red}}$ associated with w. For a locally closed subscheme $X \subset Y$, we write X^{cl} the (Zariski) closure of X in Y. By construction, we have the closure relations

$$\overline{\mathcal{M}}_w^{cl} = \coprod_{w' \le w, w' \in {}^J\mathcal{W}^b} \overline{\mathcal{M}}_{w'} \quad \text{and} \quad \mathcal{M}_w^{cl} = \coprod_{w' \le w, w' \in {}^J\mathcal{W}^b} \mathcal{M}_{w'}.$$

In [22, 1.4] (and [21, 3.4]), there exists a decomposition

$$X_\mu^G(b) = \coprod_{w \in \mathrm{Adm}(\mu) \cap {}^K\widetilde{W}} X_{K,w}(b),$$

where in our case $K = G(\mathbb{Z}_p)$ is the fixed hyperspecial group. Here are some explanations about the notation: $\widetilde{W}$ is the Iwahori Weyl group of G; $\mathrm{Adm}(\mu) \subset \widetilde{W}$ is the (finite) subset defined as [30, (2.1)]; $^K\widetilde{W} \subset \widetilde{W}$ is the set of minimal length elements in the coset $W_K \backslash \widetilde{W}$, with $W_K \subset \widetilde{W}$ the subgroup corresponding to $K = G(\mathbb{Z}_p)$, our fixed hyperspecial group. The above decomposition holds in the general setting of arbitrary local Shimura data. On the other hand, if we set $\mathrm{Adm}^K(\mu) = W_K \mathrm{Adm}(\mu) W_K$, then we have

$$\mathrm{Adm}(\mu) \cap {}^K\widetilde{W} = \mathrm{Adm}^K(\mu) \cap {}^K\widetilde{W} \xrightarrow{\sim} {}^J\mathcal{W},$$

where the first equality follows from [30, Theorem 6.10], and the second bijection is induced by the projection $\widetilde{W} \to \mathcal{W}$ by [71, Theorem 1.1 (1)]. Moreover, this bijection preserves the order $\le_{K,\sigma}$ on $^K\widetilde{W}$ (cf. [30, 6.5] and [21, 3.3]) and the order $\le$ on $^J\mathcal{W}$. Therefore, we can rewrite the above decomposition in the hyperspecial level as

$$X_\mu^G(b) = \coprod_{w \in {}^J\mathcal{W}} X_{K,w}(b).$$

Recall that by Theorem 4.6, we have

$$\overline{\mathcal{M}}^{\mathrm{perf}} \simeq X_\mu^G(b).$$

Proposition 7.2 *For $w \in {}^J\mathcal{W}$, the strata $X_{K,w}(b) \ne \varnothing$ if and only if $w \in {}^J\mathcal{W}^b$, in which case, we have*

$$\overline{\mathcal{M}}_w^{\mathrm{perf}} \simeq X_{K,w}(b).$$

In particular, $\overline{\mathcal{M}}_w$ is of dimension $\dim X_{K,w}(b)$ if it is non-empty.

Proof It suffices to prove $\overline{\mathcal{M}}_w^{\mathrm{perf}} \simeq X_{K,w}(b)$ for any $w \in {}^J\mathcal{W}$. We first consider the Hodge type case. By the proof of [78, Proposition 3.11], we have two morphisms $f \colon \overline{\mathcal{M}}^{\mathrm{perf}} \to X_\mu^G(b)$ and $f^{-1} \colon X_\mu^G(b) \to \overline{\mathcal{M}}^{\mathrm{perf}}$, inverse to each other, by using Dieudonné theory over perfect rings. It suffices to check that f (resp. f^{-1}) induces $f \colon \overline{\mathcal{M}}_w^{\mathrm{perf}} \to X_{K,w}(b)$ (resp. $f^{-1} \colon X_{K,w}(b) \to \overline{\mathcal{M}}_w^{\mathrm{perf}}$). This follows from [71, Theorem 1.1] (see [71] Section 7 for some discussion in the global setting of Shimura varieties), see also [30, Remark 6.5 (2)]. Indeed, the above decomposition $X_\mu^G(b) = \coprod_{w \in \mathrm{Adm}(\mu) \cap {}^K \widetilde{W}} X_{K,w}(b)$ is given by truncations of level 1 of elements in the Witt loop group. Since G is unramified, by [71, Theorem 1.1], the set of $\sigma - \check{K}$-conjugacy classes in $\check{K}_1 \backslash \check{K} \mu(p) \check{K} / \check{K}_1$ (together with the partial order $\leq_{K,\sigma}$) is identified with the underlying topological space of the algebraic stack $G\text{–Zip}^\mu$. Here, $\check{K}_1$ is the kernel of the projection $\check{K} = G(W) \to G(\overline{\mathbb{F}}_p)$. Thus, the above decomposition is equivalent to a morphism of algebraic stacks

$$X_\mu^G(b) \longrightarrow G\text{–Zip}^{\mu,perf},$$

which is given by $\zeta^{\mathrm{perf}} \circ f^{-1}$, since by construction f and f^{-1} preserve the universal G-Zips.

Now assume that $(G, [b], \{\mu\})$ is unramified of abelian type. Take any unramified local Shimura datum of Hodge type $(G_1, [b_1], \{\mu_1\})$ such that $(G^{\mathrm{ad}}, [b^{\mathrm{ad}}], \{\mu^{\mathrm{ad}}\}) \simeq (G_1^{\mathrm{ad}}, [b_1^{\mathrm{ad}}], \{\mu_1^{\mathrm{ad}}\})$. Let $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_1$ be the special fibers of Rapoport–Zink spaces attached to $(G, [b], \{\mu\})$ and $(G_1, [b_1], \{\mu_1\})$, respectively. As always, after fixing a point $x_0 \in \pi_1(G_1)^\Gamma$ we have $\overline{\mathcal{M}}^+ = \overline{\mathcal{M}}_1^+$. The restriction of f induces an isomorphism $f_1^+ \colon \overline{\mathcal{M}}_{1w}^{+,perf} \simeq X_{K_1,w}(b_1)^+$. Applying the $J_b(\mathbb{Q}_p)$ action, we get $\overline{\mathcal{M}}_w^{\mathrm{perf}} \simeq X_{K,w}(b)$, as desired. ∎

Remark 7.3 Let $(G, [b], \{\mu\})$ be a local Shimura datum such that G is unramified. Fix a representative $b \in G(L)$ of $[b]$.

(i) The closure relation for the decomposition $X_\mu^G(b) = \coprod_{w \in {}^J\mathcal{W}} X_{K,w}(b)$ can be proved similarly as [30] Theorem 6.15. See also [22, 4.11] and [21, section 7].

(ii) If we were working in the equal characteristic setting, then a formula for $\dim X_{K,w}(b)$ is known by combining [21, Theorem 4.1.2 (2)] and [29, Theorem 6.1]. In our mixed characteristic setting, it should be possible to prove that the same formula holds by applying [30, Proposition 6.20], the Witt vector affine flag varieties in [78, 1.4], and the method in 3.1 of [78].

7.2 Rapoport–Zink Spaces for a Fully Hodge–Newton Decomposable Pair $(G, \{\mu\})$

We discuss some special Rapoport–Zink spaces in this subsection. Our motivation here is the observation that the list in the classification of [22, Theorem 2.5] (a posteriori) lies in our class of local abelian type (for minuscule μ).

Let G be a connected reductive group over $\mathbb{Q}_p$ and let $\{\mu\}$ be a conjugacy class of cocharacters $\mu \colon \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$. Assume that G is quasi-split. Recall the following definition.

Definition 7.4 ([22, Definition 2.1], [9, Lemma 4.11]) The pair $(G, \{\mu\})$ (or the set $B(G, \mu)$) is called fully Hodge–Newton decomposable if for any non-basic $[b'] \in B(G, \mu)$, the pair $([b'], \{\mu\})$ is Hodge–Newton decomposable, *i.e.*, there exists a proper Levi subgroup $G \supsetneq M \supset M_{b'}$ such that $\kappa_M(b') = \mu^\natural$ in $\pi_1(M)_\Gamma$.

Recall that $M_{b'} = M_{\nu_{b'}}$ is the Levi subgroup of G defined as the centralizer of $\nu_{[b']}$. In [22, Theorem 2.5] there is a purely group theoretical classification of all the fully Hodge–Newton decomposable pairs $(G, \{\mu\})$, and in loc. cit. Theorem 2.3 one can find further equivalent conditions (those in (ii)–(vi) of the statement of the theorem) for $(G, \{\mu\})$ being fully Hodge–Newton decomposable.

Theorem 7.5 *Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type and let $\mathfrak{M}_{\mathrm{red}}$ be the reduced Rapoport–Zink space associated to $(G, [b], \{\mu\})$. Suppose that $(G, \{\mu\})$ is fully Hodge–Newton decomposable.*

(i) *If $[b]$ is non-basic, then $\dim \mathfrak{M}_{\mathrm{red}} = 0$.*
(ii) *If $[b]$ is basic, then the perfection of each Ekedahl–Oort stratum $\overline{\mathcal{M}}_w \subset \overline{\mathcal{M}}$ is a disjoint union of perfections of classical Deligne–Lusztig varieties.*
(iii) *For each $w \in {}^J\mathcal{W}$, there exists a unique $[b'] \in B(G, \mu)$ such that $\mathcal{M}_w \neq \varnothing$, where $\mathcal{M}_w$ is an Ekedahl–Oort stratum of $\mathcal{M}'_{\mathrm{red}}$, the reduced Rapoport–Zink space associated with $(G, [b'], \{\mu\})$. In particular, we get a decomposition*

$$ {}^J\mathcal{W} = \coprod_{[b'] \in B(G, \mu)} {}^J\mathcal{W}^{b'}. $$

Conversely, if $(G, \{\mu\})$ is part of any unramified local Shimura datum of abelian type with one of the above three conditions holds, then $(G, \{\mu\})$ is fully Hodge–Newton decomposable.

Proof This follows from [22, Theorem 2.3] (in the hyperspecial level case), our Theorem 4.6, and Proposition 7.2. ∎

Remark 7.6 Let $(G, [b], \{\mu\})$ be an unramified local Shimura datum of abelian type, with $\overline{\mathcal{M}}$ the special fiber of the associated Rapoport–Zink space. Suppose that $[b]$ is non-basic and the pair $(b, \{\mu\})$ is Hodge–Newton decomposable. With our Theorem 4.6 at hand, we refer the reader to [22, Theorems 3.16 and 6.2] (in the unramified case) to write down an isomorphism between $\overline{\mathcal{M}}^{\mathrm{perf}}$ and $\overline{\mathcal{M}}(M, b_M, \mu_M)^{\mathrm{perf}}$, the perfection of the special fiber of some Rapoport–Zink space of abelian type attached to a Levi subgroup M of G.

8 Applications to Moduli Spaces of K3 Surfaces in Mixed Characteristic

In this section, we discuss some applications to K3 surfaces and their moduli in mixed characteristic. We will first discuss some examples of Rapoport–Zink spaces of orthogonal type, following the line of the previous section. Then we will move to orthogonal Shimura varieties and moduli spaces of K3 surfaces. Finally, we will apply our

constructions of Rapoport–Zink uniformization and Ekedahl–Oort stratifications to moduli spaces of K3 surfaces. Again, we assume that $p > 2$ in this section.

8.1 Rapoport–Zink Spaces of Orthogonal Type

In this and the next subsection, we will discuss an example of Rapoport–Zink space for a fully Hodge–Newton decomposable pair $(G, \{\mu\})$.

Let (L, Q) be a non-degenerate self dual quadratic lattice of rank $n + 2$ over $\mathbb{Z}_p$, where $n \geq 1$ is an integer. We write (V, Q) as the induced quadratic space over $\mathbb{Q}_p$. Let $G = SO(V, Q), G_1 = GSpin(V, Q)$ be the associated special orthogonal and spinor similitudes groups over $\mathbb{Q}_p$. By our assumption that L is self dual, both G and G_1 are unramified. We have an exact sequence of groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G_1 \longrightarrow G \longrightarrow 1,$$

which is, in fact, defined over $\mathbb{Z}_p$.

As in [32, subsection 4.2], there is a natural choice of minuscule cocharacter μ_1 of G_1. Take any $[b_1] \in B(G_1, \mu_1)$. Then $(G_1, [b_1], \{\mu_1\})$ is a local Shimura datum of Hodge type. We get a local Shimura datum $(G, [b], \{\mu\})$ by taking $[b], \{\mu\}$ as the image of $[b_1], \{\mu_1\}$ under the map $G_1 \to G$. By construction $(G, [b], \{\mu\})$ is unramified of abelian type. We get the associated Rapoport–Zink spaces $\breve{\mathcal{M}}_1 = \breve{\mathcal{M}}(G_1, b_1, \mu_1)$ and $\breve{\mathcal{M}} = \breve{\mathcal{M}}(G, b, \mu)$. The pairs $(G_1, \{\mu_1\})$ and $(G, \{\mu\})$ are fully Hodge–Newton decomposable by [22, Theorem 2.5] (or one can compute the sets $B(G_1, \mu_1)$ and $B(G, \mu)$ directly to see they are fully Hodge–Newton decomposable).

Let X_0 be the p-divisible group over $\overline{\mathbb{F}}_p$ with contravariant Dieudonné module $(C(V)^\vee \otimes W, b_1\sigma)$, where $C(V)$ is the Clifford algebra attached to V and $C(V)^\vee$ is its dual. Fix any $\delta \in C(V)^\times$ with $\delta^* = \delta$ where $*$ is the canonical involution on $C(V)$. Then $\psi_\delta(c_1, c_2) = \mathrm{Tr}(c_1 \delta c_2^*)$ is a perfect symplectic form on $C(V)$. Here $\mathrm{Tr}: C(V) \to \mathbb{Z}_p$ is the reduced trace map. The perfect symplectic form ψ_δ on $C(V)$ induces a principal polarization $\lambda_0: X_0 \to X_0^\vee$. There exists a finite collection tensors $(s_\alpha)_{\alpha \in I}$ that includes ψ_δ, such that $G_1 \subset GL(C(V))$ is cut out by $(s_\alpha)_{\alpha \in I}$. Recall that $\breve{\mathcal{M}}_1$ has the following moduli interpretation. For any $R \in \mathrm{Nilp}_W^{s\,m}$, $\breve{\mathcal{M}}_1(R) = \{(X, (t_\alpha)_{\alpha \in I}, \rho)\}/\simeq$, where

- X is a p-divisible group on $\mathrm{Spec}R$;
- $(t_\alpha)_{\alpha \in I}$ is a collection of crystalline Tate tensors of X;
- $\rho: X_0 \otimes R/J \to X \otimes R/J$ is a quasi-isogeny that sends $s_\alpha \otimes 1$ to t_α for $\alpha \in I$, where J is some ideal of definition of R,

such that the following condition holds: the R-scheme

$$\mathrm{Isom}\big((\mathbb{D}(X)_R, (t_\alpha), \mathrm{Fil}^\bullet(\mathbb{D}(X)_R)), (\Lambda \otimes R, (s_\alpha \otimes 1), \mathrm{Fil}^\bullet \Lambda \otimes R)\big)$$

that classifies the isomorphisms between locally free sheaves $\mathbb{D}(X)_R$ and $\Lambda \otimes R$ on $\mathrm{Spec}R$ preserving the tensors and the filtrations is a $P_{\mu_1} \otimes R$-torsor.

The exact sequence $1 \to \mathbb{G}_m \to G_1 \to G \to 1$ induces a long exact sequence (*cf.* [2, Lemma 1.5])

$$1 \longrightarrow \pi_1(\mathbb{G}_m)^\Gamma \longrightarrow \pi_1(G_1)^\Gamma \longrightarrow \pi_1(G)^\Gamma \longrightarrow H^1(\Gamma, \pi_1(\mathbb{G}_m)) \longrightarrow \cdots.$$

We have the following isomorphisms:

$$\pi_1(\mathbb{G}_m)^\Gamma = \pi_1(\mathbb{G}_m) \simeq X_*(\mathbb{G}_m) \simeq \mathbb{Z}.$$

Since $G_1^{\mathrm{der}} = \mathrm{Spin}(V)$ is simply connected, and we have the exact sequence

$$1 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathbb{G}_m \longrightarrow 1,$$

we get ([2, 1.6])

$$\pi_1(G_1)^\Gamma \simeq \pi_1(\mathbb{G}_m)^\Gamma \simeq \mathbb{Z}.$$

On the other hand, since

$$1 \longrightarrow \mu_2 \to \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1$$

is exact, we get

$$\pi_1(G) = \mu_2(-1) = \mathbb{Z}/2\mathbb{Z}.$$

Lemma 8.1   *We have $\pi_1(G)^\Gamma \simeq \mathbb{Z}/2\mathbb{Z}$, and the map $\pi_1(G_1)^\Gamma \to \pi_1(G)^\Gamma$ is surjective.*

Proof   As $\mu_2 \subset \mathbb{G}_m$, and Γ acts trivially on the latter, we get $\pi_1(G)^\Gamma = \mu_2(-1)^\Gamma = \mu_2(-1) = \mathbb{Z}/2\mathbb{Z}$. For the second assertion, note that

$$\pi_1(G_1)^\Gamma / 2\pi_1(G_1)^\Gamma = \mathbb{Z}/2\mathbb{Z} \subset \mathrm{Im}\big(\pi_1(G_1)^\Gamma \longrightarrow \pi_1(G)^\Gamma\big).$$

Thus, the image is $\pi_1(G)^\Gamma$.   ∎

Corollary 8.2   *We have an isomorphism of formal schemes $\breve{\mathcal{M}} \simeq \breve{\mathcal{M}}_1 / p^{\mathbb{Z}}$.*

Proof   By the above lemma $\pi_1(G_1)^\Gamma \to \pi_1(G)^\Gamma$ is surjective. Thus, $\breve{\mathcal{M}} \simeq \breve{\mathcal{M}}_1 / p^{\mathbb{Z}}$ as in the proof of Proposition 4.9(i).   ∎

As the pairs $(G_1, \{\mu_1\})$ and $(G, \{\mu\})$ are fully Hodge–Newton decomposable, by Theorem 7.5 we get the following corollary.

Corollary 8.3   *Assume that $[b_1]$ (and hence $[b]$) is non-basic. Then we have* $\dim \mathcal{M}_{\mathrm{red}} = \dim \mathcal{M}_{1\,\mathrm{red}} = 0$.

8.2 Ekedahl–Oort and Howard–Pappas Stratifications for Basic Rapoport–Zink Spaces of Orthogonal Type

Let the notations be as in the previous subsection. Now assume that $[b_1]$ (thus $[b]$) is basic.

In [32], Howard and Pappas introduced a stratification[8] for the reduced special fiber $\mathcal{M}_{1\,\mathrm{red}}$ of $\breve{\mathcal{M}}_1$:

$$\mathcal{M}_{1\,\mathrm{red}} = \coprod_\Lambda \mathcal{M}_{1\Lambda}^\circ,$$

where Λ runs through the set of vertex lattices. By definition (*cf.* [32, section 5]), a vertex Λ lattice is a $\mathbb{Z}_p$-lattice in V_L^Φ such that $p\Lambda \subset \Lambda^\vee \subset \Lambda \subset V_L^\Phi$. Here, $L = W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$, $\Phi = b\sigma$ is the Frobenius and V_L^Φ admits a quadratic form induced from V_L, so that this quadratic space V_L^Φ has the same dimension and determinant as V, but

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[8]In [32, 6.5], it is called the Bruhat–Tits stratification, and our $\mathcal{M}_{1\Lambda}^\circ$ is denoted there by BT_Λ.

has Hasse invariant -1. Associated with a vertex, we have the type $t(\Lambda) := \dim \Lambda/\Lambda^{\vee}$, which is an even integer with $2 \le t(\Lambda) \le t_{\max}$, where

$$t_{\max} = \begin{cases} n+1, & n \text{ odd,} \\ n+2, & n \text{ even}, \det V \neq (-1)^{\frac{n}{2}}, \\ n, & n \text{ even}, \det V = (-1)^{\frac{n}{2}}. \end{cases}$$

Recall that we have the inclusion $V_L^{\Phi} \subset \operatorname{End}(X_0)_{\mathbb{Q}}$, so each vertex lattice $\Lambda \subset V_L^{\Phi}$ can be viewed as a set of self quasi-isogenies of X_0. For each vertex lattice Λ, the associated Howard–Pappas stratum[9] $\mathcal{M}_{1\Lambda}^{\circ} \subset \mathcal{M}_{1\,\mathrm{red}}$ is the locus $(X, (t_\alpha), \rho)$, where $\rho \circ \Lambda^{\vee} \circ \rho \subset \operatorname{End}(X)$ and this does not hold for any smaller vertex lattice $\Lambda' \subsetneq \Lambda$. Let $\mathcal{M}_{1\Lambda} \subset \mathcal{M}_{1\,\mathrm{red}}$ be its Zariski closure. In [32, 4.3.3 and 6.4.1], Howard and Pappas proved that there exists a decomposition

$$\mathcal{M}_{1\,\mathrm{red}} = \coprod_{j \in \mathbb{Z}} \mathcal{M}_1^{(j)},$$

such that each $\mathcal{M}_1^{(j)}$ is a connected component of $\mathcal{M}_{1\,\mathrm{red}}$. Accordingly, we get a decomposition for each stratum:

$$\mathcal{M}_{1\Lambda}^{\circ} = \coprod_{j \in \mathbb{Z}} \mathcal{M}_{1\Lambda}^{(j),\circ}.$$

By [32, Theorem 6.5.6], each connected stratum $\mathcal{M}_{1\Lambda}^{(j),\circ}$ is isomorphic to a Deligne–Lusztig variety $X_B(w)$ for the group $\mathrm{SO}(\Lambda_W/\Lambda_W^{\vee})$.

As

$$\check{\mathcal{M}} \simeq \check{\mathcal{M}}_1/p^{\mathbb{Z}} \simeq \check{\mathcal{M}}^{(0)} \coprod \check{\mathcal{M}}^{(1)},$$

we get an induced Howard–Pappas stratification for $\mathcal{M}_{\mathrm{red}}$:

$$\mathcal{M}_{\mathrm{red}} = \coprod_{\Lambda} \mathcal{M}_{\Lambda}^{\circ}.$$

In fact, in [32, sections 5 and 6], Howard and Pappas studied the geometric structures of $\mathcal{M}_{1\,\mathrm{red}}$ by passing to the quotient space $\mathcal{M}_{\mathrm{red}} = \mathcal{M}_{1\,\mathrm{red}}/p^{\mathbb{Z}}$ first.

Recall that $W = W(\overline{\mathbb{F}}_p)$, $L = W_{\mathbb{Q}}$. Following [32], we can describe the sets $\mathcal{M}_{\mathrm{red}}(\overline{\mathbb{F}}_p)$, $\mathcal{M}_{\Lambda}(\overline{\mathbb{F}}_p)$, and $\mathcal{M}_{\Lambda}^{\circ}(\overline{\mathbb{F}}_p)$ in terms of special lattices of V_L as follows. By definition ([32, Definition 5.2.1]) a special lattice $\mathcal{L} \subset V_L$ is a self-dual W-lattice such that $(\mathcal{L} + \Phi_*(\mathcal{L}))/\mathcal{L} \simeq W/pW$, where $\Phi_*(\mathcal{L})$ is the W-submodule generated by $\Phi(\mathcal{L})$. By [32, Proposition 6.2.2], we have a bijection $\mathcal{M}_{\mathrm{red}}(\overline{\mathbb{F}}_p) \simeq \{$special lattices $\mathcal{L} \subset V_L\}$. By loc. cit. 5.3.1 and Theorem 6.3.1, we have bijections

$$\begin{aligned} \mathcal{M}_{\Lambda}(\overline{\mathbb{F}}_p) &\simeq \{\text{Lagrangians } \mathcal{L} \subset \Omega : \dim(\mathcal{L} + \Phi(\mathcal{L})) = d+1\} \\ &\simeq \{\text{special lattices } \mathcal{L} \subset V_L : \Lambda_W^{\vee} \subset \mathcal{L} \subset \Lambda_W\} \\ &= \{\text{special lattices } \mathcal{L} \subset V_L : \Lambda(\mathcal{L}) \subset \Lambda\}, \end{aligned}$$

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[9]We refer the reader to [44, Definition 2.9.1 and Theorem 4.2.11] for a variant.

where $\Omega = \Lambda_W / \Lambda_W^\vee$, $\Lambda(\mathcal{L}) = (\mathcal{L}^{(d)})^\Phi$, $d = \frac{t(\Lambda)}{2}$, and $\mathcal{L}^{(d)} = \mathcal{L} + \Phi(\mathcal{L}) + \cdots + \Phi^d(\mathcal{L})$. Under the above description, we have the bijection

$$\mathcal{M}_\Lambda^\circ(\overline{\mathbb{F}}_p) \simeq \{\text{special lattices } \mathcal{L} \subset V_L : \mathcal{L}^{(d)} = \Lambda_W\}$$
$$= \{\text{special lattices } \mathcal{L} \subset V_L : \Lambda(\mathcal{L}) = \Lambda\}.$$

In fact, the above descriptions are true for any finitely generated field extension $k' | \overline{\mathbb{F}}_p$ (*cf.* [32]).

Let $G_1\text{–Zip}^{\mu_1}$ be the stack of G_1-zips of type μ_1 over $\overline{\mathbb{F}}_p$. The universal p-divisible group with crystalline Tate tensors on $\breve{\mathcal{M}}_1$ defines a morphism

$$\zeta \colon \overline{\mathcal{M}}_1 \longrightarrow G_1\text{–Zip}^{\mu_1}.$$

The underlying set of geometric points of $G_1\text{–Zip}^{\mu_1}$ is in canonical bijection with the subset ${}^J\mathcal{W}$ of the Weyl group of G_1. In fact, we have isomorphisms of topological spaces

$$|G_1\text{–Zip}^{\mu_1}| \simeq |G\text{–Zip}^\mu| \simeq {}^J\mathcal{W}.$$

Let ${}^J\mathcal{W}^b \subset {}^J\mathcal{W}$ be the subset defined by the image of ζ. For each $w \in {}^J\mathcal{W}^b$, recall we have the Ekedahl–Oort stratum of $\mathcal{M}_{1\,\mathrm{red}}$ associated with w:

$$\mathcal{M}_{1w} = \zeta^{-1}(w)_{\mathrm{red}}.$$

We get the Ekedahl–Oort stratification

$$\mathcal{M}_{1\,\mathrm{red}} = \coprod_{w \in {}^J\mathcal{W}^b} \mathcal{M}_{1w}.$$

We get also the induced Ekedahl–Oort stratification for $\mathcal{M}_{\mathrm{red}}$.

Let $m \geq 1$ be such that $2m = n + 1$ if n is odd, and $2m = n + 2$ if n is even. Then there is a bijection (*cf.* [77, subsection 4.4])

$${}^J\mathcal{W} \xrightarrow{\sim} \begin{cases} \{0, 1, \ldots, 2m-1\}, & n = 2m-1 \text{ odd,} \\ \{0, 1, \ldots, m-2, m-1, m-1', m, \ldots, 2m-2\}, & n = 2m-2 \text{ even,} \end{cases}$$

induced by the length function $w \mapsto \ell(w)$, where we use the symbols $m-1'$, $m-1$ to distinguish the two elements with the same length $m-1$. Under the above bijection, the subset ${}^J\mathcal{W}^b \subset {}^J\mathcal{W}$ can be described as

$${}^J\mathcal{W}^b \xrightarrow{\sim} \begin{cases} \{m, \ldots, 2m-1\}, & n = 2m-1 \text{ odd,} \\ \{m, \ldots, 2m-2\}, & n = 2m-2 \text{ even, } \det V = (-1)^{\frac{n}{2}}, \\ \{m-1, m-1', m, \ldots, 2m-2\}, & n = 2m-2 \text{ even, } \det V \neq (-1)^{\frac{n}{2}}. \end{cases}$$

For each $i \neq m-1'$ on the right-hand side, we denote the corresponding element of the left-hand side as w_i. The element corresponding to $m-1'$ will be denoted by w'_{m-1}.

We can describe the map $i \mapsto w_i$ in more detail. Assume first that n is odd. The simple reflections are

$$\begin{cases} s_i = (i, i+1)(2m+1-i, 2m+2-i), & 1 \leq i \leq m-1, \\ s_m = (m, m+2), & i = m, \end{cases}$$

and we have

$$w_i = \begin{cases} s_1 \cdots s_i, & 0 \le i \le m, \\ s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_{2m-i}, & m+1 \le i \le 2m-1. \end{cases}$$

Now assume that n is even. The simple reflections are

$$\begin{cases} s_i = (i, i+1)(2m-i, 2m+1-i), & 1 \le i \le m-1, \\ s_m = (m-1, m+1)(m, m+2), & i = m, \end{cases}$$

and we have

$$w_i = \begin{cases} s_1 \cdots s_i, & 0 \le i \le m, \\ s_1 \cdots s_m s_{m-2} \cdots s_{2m-1-i}, & m+1 \le i \le 2m-2, \end{cases}$$

and

$$w'_{m-1} = s_1 \cdots s_{m-2} s_m.$$

Let $\overline{V} = L_W \otimes \overline{\mathbb{F}}_p$ be the quadratic space over $\overline{\mathbb{F}}_p$. For each $w_i \in {}^J\mathcal{W}$, we will attach to it an orthogonal F-zip (also called a SO(V)-zip) as follows. Fix a basis $e_1, \ldots, e_{n+2}$ of L such that the quadratic form Q has the form $x_1 x_{n+2} + x_2 x_{n+1} + \cdots + x_m x_{m+2} + x_{m+1}^2$ (*cf.* the proof of [77, Proposition 4.4.1]). By abuse of notation, we still denote by $e_1, \ldots, e_{n+2}$ the induced basis of $(\overline{V}, Q)$. For each $w \in {}^J\mathcal{W}$, let M_w be the orthogonal F-zip $(\overline{V}, Q, C^\bullet, D_\bullet, \phi_\bullet)$ where

- $C^\bullet$ is the descending filtration $\overline{V} \supset \langle e_2, e_3, \ldots, e_{n+2} \rangle \supset \langle e_{n+2} \rangle \supset 0$, denoted by $C^0 \supset C^1 \supset C^2 \supset C^3$,
- $D_\bullet$ is the ascending filtration

$$0 \subset \langle w(e_1) \rangle \subset \langle w(e_1), w(e_2), \ldots, w(e_{n+1}) \rangle \subset \overline{V},$$

  denoted by $D_0 \subset D_1 \subset D_2 \subset D_3$,
- $\varphi_\bullet$ is the collections of isomorphisms

$$\varphi_0 \colon (C^0/C^1)^{(p)} \xrightarrow{\sim} D_1, \quad \varphi_1 \colon (C^1/C^2)^{(p)} \xrightarrow{\sim} D_2/D_1, \quad \varphi_2 \colon (C^2/C^3)^{(p)} \xrightarrow{\sim} D_3/D_2.$$

We remark that the above construction is not the standard isomorphism ${}^J\mathcal{W} \simeq |G - \mathrm{Zip}^\mu|$ of Pink, Wedhorn, and Ziegler (for example as in [77, Theorem 3.1.5]). The standard association is the twist $w \longmapsto M_{w_0 w}$ of ours, where w_0 is the maximal length element of ${}^J\mathcal{W}$. In particular, $\ell(w_0 w) = n - \ell(w)$.

Theorem 8.4 *Each stratum $\mathcal{M}_{1w}$ is some union of Howard–Pappas strata of $\mathcal{M}_{1\,\mathrm{red}}$.*

Proof By the methods of [32], it suffices to prove the following assertion first. ∎

Corollary 8.5 *Each stratum $\mathcal{M}_w$ is some union of Howard–Pappas strata of $\mathcal{M}_{\mathrm{red}}$.*

Proof We first prove the equalities for the sets of k-points, where k is an algebraically closed field of characteristic p. This follows from [32, Theorem 6.5.6] and [21, Corollary 4.1.3].

Indeed, by [32, Theorem 6.5.6], we have an isomorphism

$$\mathcal{M}_\Lambda^\circ \simeq X_B(w^+) \coprod X_B(w^-),$$

where $X_B(w^+)$ and $X_B(w^-)$ are the Deligne–Lusztig varieties associated with the elements w^+ and w^- of $\mathcal{W}_\Omega$, the Weyl group of $\mathrm{SO}(\Omega)$, where as before, $\Omega = \Lambda_W/\Lambda_W^\vee$. As in [32, 6.5.4], $w^\pm$ are Coxeter elements. Write $w(\Lambda) = w^+$, and consider it as an element in $\mathcal{W}$ under the inclusion $\mathcal{W}_\Omega \hookrightarrow \mathcal{W}$. Then by [21, Corollary 4.1.3], we have

$$\mathcal{M}_w(k) = \coprod_{\Lambda, w(\Lambda)=w} \mathcal{M}_\Lambda^\circ(k).$$

To prove the identities on the level of schemes, we argue as in the proof of [69, Corollary 4.10]. That is, it suffices to show that $\mathcal{M}_\Lambda^\circ$ is open and closed in $\mathcal{M}_w$. This follows from the facts that $\mathcal{M}_\Lambda^\circ$ is open in $\mathcal{M}_\Lambda$, $\mathcal{M}_\Lambda \cap \mathcal{M}_w = \mathcal{M}_\Lambda^\circ$, and the above identities on the level of points.   ∎

Consider the case $k = \overline{\mathbb{F}}_p$. For any vertex lattice Λ and any point $x \in \mathcal{M}_\Lambda^\circ(\overline{\mathbb{F}}_p)$, we have the associated special lattice $\mathcal{L}_x$. By reduction modulo p, we get an orthogonal F-zip M_x, which we write it as $M_{w_0 w_x}$ attached to $w_0 w_x \in {}^J\mathcal{W}^b$ for some $w_x \in {}^J\mathcal{W}^b$. Then by definition, $x \in \mathcal{M}_{w_0 w_x}$. By the above corollary, we have the equality $d - 1 = \ell(w_0 w_x)$, where $d = \frac{t(\Lambda)}{2}$. The following corollaries are coarser versions of Theorem 8.4 and Corollary 8.5. However, they are more explicit in terms of types.

Corollary 8.6 (i) *If n is odd, or n is even with $\det(V) = (-1)^{\frac{n}{2}}$, then we have the identity*

$$\mathcal{M}_{1w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}_{1\Lambda}^\circ.$$

(ii) *If n is even with $\det(V) \neq (-1)^{\frac{n}{2}}$, then*
    (a) *if $m \leq i \leq 2m - 1$,*

$$\mathcal{M}_{1w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}_{1\Lambda}^\circ;$$

    (b) *if $i = m - 1$,*

$$\mathcal{M}_{1w_{m-1}} \coprod \mathcal{M}_{1w'_{m-1}} = \coprod_{\Lambda, t(\Lambda)=2m} \mathcal{M}_{1\Lambda}^\circ.$$

Corollary 8.7 (i) *If n is odd, or n is even with $\det(V) = (-1)^{\frac{n}{2}}$, then we have the identity*

$$\mathcal{M}_{w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}_\Lambda^\circ.$$

(ii) *If n is even with $\det(V) \neq (-1)^{\frac{n}{2}}$, then*
    (a) *if $m \leq i \leq 2m - 1$,*

$$\mathcal{M}_{w_i} = \coprod_{\Lambda, t(\Lambda)=2(n-i+1)} \mathcal{M}_\Lambda^\circ;$$

    (b) *if $i = m - 1$,*

$$\mathcal{M}_{w_{m-1}} \coprod \mathcal{M}_{w'_{m-1}} = \coprod_{\Lambda, t(\Lambda)=2m} \mathcal{M}_\Lambda^\circ.$$

8.3 Moduli Spaces of Polarized K3 Surfaces with Level Structures and the Integral Kuga–Satake Map

In this and the next subsection, we will turn to moduli spaces of polarized K3 surfaces, with the involved Shimura varieties; cf. [47, Sections 2 and 4], [58, Section 6].

Let U be the hyperbolic lattice over $\mathbb{Z}$ of rank 2, and let E_8 be the positive quadratic lattice associated with the Dynkin diagram of type E_8. Set $N = U^{\oplus 3} \oplus E_8^{\oplus 2}$, which is a self-dual lattice. Let $d \geq 1$ be an integer. Choose a basis e, f for the first copy of U in N and set $L_d = \langle e - df \rangle^{\perp} \subset N$. This is a quadratic lattice over $\mathbb{Z}$ of discriminant $2d$ and rank 21 (in [58] it is denoted by L_{2d}). Let $V_d = L_d \otimes \mathbb{Q}$ and $L_d^{\vee} \subset V_d$ be the dual lattice. Set $G = \mathrm{SO}(V_d)$, which is isomorphic to the special orthogonal group over $\mathbb{Q}$ of signature $(2, 19)$. Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup that stabilizes $L_{d, \widehat{\mathbb{Z}}}$ and acts trivially on $L_d^{\vee}/L_d$. Such compact opens are called *admissible*. We fix a prime $p > 2$ such that $p \nmid d$ from now on. Then as L is self dual at p, the local reductive group $G_{\mathbb{Q}_p}$ is unramified. Let $K_p = G(\mathbb{Z}_p)$ be the hyperspecial group. We only consider open compact subgroups $K^p \subset G(\mathbb{A}_f^p)$ that are contained in the discriminant kernel of $L_{d, \widehat{\mathbb{Z}}^p}$ with finite index. In particular, $K = K_p K^p$ is admissible; cf. [58, 5.3]. For the reductive group G, we have the associated Shimura varieties $\mathrm{Sh}_{K_p K^p}$, which are defined over $\mathbb{Q}$. By [38], there exists an integral smooth canonical model $S_{K_p K^p}$ of $\mathrm{Sh}_{K_p K^p}$ over $\mathbb{Z}_p$.

Let M_{2d} (resp. M_{2d}^*) be the moduli spaces of K3 surfaces $f: X \to S$ together with a primitive polarization ξ (resp. quasi-polarization) of degree $2d$ over $\mathbb{Z}_p$ (in [47, Section 2], these spaces are denoted by M_{2d}° and M_{2d} respectively). These are Deligne–Mumford stacks of finite type over $\mathbb{Z}_p$. The natural map $\mathrm{M}_{2d} \to \mathrm{M}_{2d}^*$ is an open immersion. Moreover, M_{2d} is separated and smooth of dimension 19 over $\mathbb{Z}_p$; cf. [58, Theorem 4.3.3, Proposition 4.3.11], and [47, Proposition 2.2].

Let $(f: \mathcal{X} \to \mathrm{M}_{2d}, \xi)$ be the universal object over M_{2d}. For any prime ℓ, we consider the second relative étale cohomology H_ℓ^2 of $\mathcal{X}$ over M_{2d}. This is a lisse $\mathbb{Z}_\ell$-sheaf of rank 22 equipped with a perfect symmetric Poincaré pairing $\langle \cdot, \cdot \rangle: H_\ell^2 \times H_\ell^2 \to \mathbb{Z}_\ell(-2)$. The ℓ-adic Chern class $\mathrm{ch}_\ell(\xi)$ of ξ is a global section of the Tate twist $H_\ell^2(1)$ that satisfies $\langle \mathrm{ch}_\ell(\xi), \mathrm{ch}_\ell(\xi) \rangle = 2d$. The product

$$H_{\widehat{\mathbb{Z}}}^2 = \prod_\ell H_\ell^2$$

is a lisse $\widehat{\mathbb{Z}}$-sheaf, and the Chern classes of ξ can be put together to get the Chern class $\mathrm{ch}_{\widehat{\mathbb{Z}}}(\xi)$ in $H_{\widehat{\mathbb{Z}}}^2(1)$. Recall that we have the quadratic lattice N of rank 22 over $\mathbb{Z}$.

Definition 8.8 Consider the étale sheaf over M_{2d} whose sections over any scheme $T \to \mathrm{M}_{2d}$ are given by

$$I(T) = \left\{ \eta: N \otimes \widehat{\mathbb{Z}} \xrightarrow{\sim} H_{\widehat{\mathbb{Z}}, T}^2(1) \text{ isometries, with } \eta(e - df) = \mathrm{ch}_{\widehat{\mathbb{Z}}}(\xi) \right\}.$$

Let $K = K_p K^p \subset K_{L_{\widehat{\mathbb{Z}}^p}}$ be an admissible open compact subgroup. Then I admits a natural action by the constant sheaf of groups K. A section $\overline{\eta} \in H^0(T, I/K)$ is called a K-level structure over T (in [58, 5.3] it is called a full K-level structure).

Let $M_{2d,K}$ (resp. $M_{2d,K}^*$) be the relative moduli problem over M_{2d} (resp. M_{2d}^*) that parametrizes K-level structures. For K^p (thus K) small enough, these are smooth algebraic spaces. Moreover, the maps

$$M_{2d,K} \longrightarrow M_{2d} \quad \text{and} \quad M_{2d,K}^* \longrightarrow M_{2d}^*$$

are finite étale. For another admissible $K' = K_p K^{p'} \subset K = K_p K^p$, we have natural finite étale projections

$$M_{2d,K'} \longrightarrow M_{2d,K} \quad \text{and} \quad M_{2d,K'}^* \longrightarrow M_{2d,K}^*$$

as algebraic spaces over M_{2d}, M_{2d}^* respectively. When $K^{p'}$ is a normal subgroup of K^p, these projections are Galois with Galois group $K^p/K^{p'}$.

For any prime ℓ, we have the primitive cohomology sheaf

$$P_\ell = \langle \mathrm{ch}_\ell(\xi) \rangle^\perp \subset H_\ell^2.$$

Let H_B^2 and H_{dR}^2 be the second relative Betti and de Rham cohomology respectively of the universal K3 surface $\mathcal{X} \to M_{2d,K,\mathbb{C}}^*$. We have also the primitive cohomology sheaves

$$P_B = \langle \mathrm{ch}_B(\xi) \rangle^\perp \subset H_B^2, \quad P_{dR} = \langle \mathrm{ch}_{dR}(\xi) \rangle^\perp \subset H_{dR}^2.$$

Consider $\widetilde{M}_{2d,K}^* \to M_{2d,K}^*$, the two-fold finite étale cover parameterizing isometric trivializations $\det(L_d) \otimes \mathbb{Z}_2 \xrightarrow{\sim} \det(P_2)$ of the determinant of the primitive 2-adic cohomology of the universal quasi-polarized K3 surface. We can identify $\widetilde{M}_{2d,K}^*$ with the the space of isometric trivializations $\det(L_d) \xrightarrow{\sim} \det(P_B)$ of the determinant of the primitive Betti cohomology. There is a Hodge-de Rham filtration $F^\bullet P_{dR}$ on P_{dR}, and we have a natural isometric trivialization $\eta \colon \mathrm{disc}(L_d) \xrightarrow{\sim} \mathrm{disc}(P_B)$ and the the tautological trivialization $\beta \colon \det(L_d) \xrightarrow{\sim} \det(P_B)$. The tuple $(P_B, F^\bullet P_{dR}, \eta, \beta)$ gives rise to a natural period map

$$\widetilde{M}_{2d,K,\mathbb{C}}^* \longrightarrow \mathrm{Sh}_{K,\mathbb{C}};$$

cf. [47, Propositions 4.2 and 3.3]. There is a section map $M_{2d,K,\mathbb{C}} \subset M_{2d,K,\mathbb{C}}^* \to \widetilde{M}_{2d,K,\mathbb{C}}^*$, whose composition with the above period map gives us the Kuga–Satake period map

$$\iota_\mathbb{C} \colon M_{2d,K,\mathbb{C}} \longrightarrow \mathrm{Sh}_{K,\mathbb{C}}.$$

It follows from [59, Theorem 3.9.1], this map is defined over $\mathbb{Q}$. Therefore we get the map over $\mathbb{Q}_p$:

$$\iota_{\mathbb{Q}_p} \colon M_{2d,K,\mathbb{Q}_p} \longrightarrow \mathrm{Sh}_{K,\mathbb{Q}_p}.$$

As S_K is the integral canonical model of Sh_K, by extension property of S_K, the Kuga–Satake map extends to a map over $\mathbb{Z}_p$:

$$\iota \colon M_{2d,K} \longrightarrow S_K.$$

Theorem 8.9 ([47, Corollary 5.15]) *The integral Kuga–Satake period map*

$$\iota \colon M_{2d,K} \longrightarrow S_K$$

is an open immersion.

When $K_1^p \subset K^p$ is another open compact subgroup, we note that the following diagram is cartesian:

$$
\begin{array}{ccc}
M_{2d,K_1} & \longrightarrow & S_{K_1} \\
\downarrow & & \downarrow \\
M_{2d,K} & \longrightarrow & S_K.
\end{array}
$$

As a corollary, we see that for K^p small enough, $M_{2d,K}$ is a scheme.

8.4 Newton and Ekedahl–Oort Stratifications of the Moduli Spaces of K3 Surfaces

In the rest of this section we will work over W. As before, we simply denote by the same notation for an object base changed to W. Let $\overline{M}_{2d,K}$ be the special fiber of $M_{2d,K}$, which can be viewed as an open subspace of the special fiber $\overline{S}_K$ of S_K by Theorem 8.9. For the good reduction of Shimura varieties of abelian type, in [68] we have introduced the Newton and Ekedahl–Oort stratifications for the special fibers. In Subsection 6.2, we have seen the Newton stratification. In the cases of GSpin and SO Shimura varieties, we can compare the Newton and Ekedahl–Oort stratifications as follows. These are in the list of Shimura varieties of coxeter type studied in [21] (comp. [22]).

Theorem 8.10 ([68])   *Assume that n is odd.*[10]

(i)   *We have*

$$
\overline{S}_K = \coprod_{b \in B(G,\mu)} \overline{S}_K^b, \quad \overline{S}_K = \coprod_{w \in {}^J\mathcal{W}} \overline{S}_K^w,
$$

   with each stratum in the two stratifications non-empty.

(ii)   *Let b_0 be the unique basic element in $B(G, \mu)$.*

   • *For $b \neq b_0$, there exists a unique $w_b \in {}^J\mathcal{W}$ such that $\overline{S}_K^b = \overline{S}_K^{w_b}$;*
   • *For b_0, $\overline{S}_K^{b_0} = \coprod_{w \in {}^J\mathcal{W}^{b_0}} \overline{S}_K^w$.*

Note that the subset ${}^J\mathcal{W}^b = \{w_b\}$ for any $b \neq b_0$. In fact these statements are just the global analogue of Theorem 7.5 in the setting of Shimura varieties of abelian type; *cf.* [68, Section 7] (see also [22, Section 6], where the authors there assume that the axioms of [32] are verified).

We return to the case $n = 19$. Consider the Kuga–Satake map $\overline{\iota}: \overline{M}_{2d,K} \hookrightarrow \overline{S}_K$, which is an open immersion by Theorem 8.9. The above stratifications of $\overline{S}_K$ in turn induce stratifications of $\overline{M}_{2d,K}$:

$$
\overline{M}_{2d,K} = \coprod_{b \in B(G,\mu)} \overline{M}_{2d,K}^b, \quad \overline{M}_{2d,K} = \coprod_{w \in {}^J\mathcal{W}} \overline{M}_{2d,K}^w,
$$

where $\overline{M}_{2d,K}^b$ and $\overline{M}_{2d,K}^w$ are the pullbacks of the corresponding strata $\overline{S}_K^b$ and $\overline{S}_K^w$ under the open immersion $\overline{\iota}: \overline{M}_{2d,K} \hookrightarrow \overline{S}_K$. We have the similar relation

---

[10]When n is even, we have a similar but more delicate statement that each Newton stratum is a disjoint union of some Ekedahl–Oort strata; *cf.* [68].

- for $b \neq b_0$, there exists a unique $w_b \in {}^J\mathcal{W}$ such that $\overline{\mathrm{M}}_{2d,K}^b = \overline{\mathrm{M}}_{2d,K}^{w_b}$,
- for b_0, $\overline{\mathrm{M}}_{2d,K}^{b_0} = \coprod_{w \in {}^J\mathcal{W}^{b_0}} \overline{\mathrm{M}}_{2d,K}^w$. We will also write $\overline{\mathrm{M}}_{2d,K}^{b_0}$ as $\overline{\mathrm{M}}_{2d,K}^{ss}$ to indicate that it is the supersingular locus of $\overline{\mathrm{M}}_{2d,K}$.

We will investigate these stratifications in some more classical terms, which appeared in the literature; see [50], for example.

8.4.1 Newton Stratification vs. Height Stratification

Let X be a K3 surface over a field k of characteristic p. Consider the functor on local Artinian k-algebras with residue field k defined by

$$\Phi_{X/k}^2 \colon (Art/k) \longrightarrow (\text{Abelian groups})$$
$$R \longmapsto \ker\left(H_{et}^2(X \times \mathrm{Spec}R, \mathbb{G}_m) \to H_{et}^2(X, \mathbb{G}_m)\right).$$

It is pro-representable by a one-dimensional formal group $\widehat{\mathrm{Br}}(X)$, the so-called formal Brauer group. The height h of this formal Brauer group of the K3 surface X satisfies $1 \leq h \leq 10$ or $h = \infty$.

The Newton slopes of the F-crystal $H_{cris}^2(X/W)$ are equal to $(1 - \frac{1}{h}, 1, 1 + \frac{1}{h})$. Thus, the set $B(G, \mu)$ is in bijection with the set $\{1, \ldots, 10, \infty\}$. The basic element b_0 corresponds to ∞. We write $B(G, \mu) = \{b_1, \ldots, b_{10}, b_{11} = b_0\}$. The Newton stratification of $\overline{\mathrm{M}}_{2d,K}$ is just the classical height stratification. By [13], for each $b \in B(G, \mu)$, the Newton stratum $\overline{\mathrm{M}}_{2d,K}^b$ is non-empty.

8.4.2 Ekedahl–Oort Stratification vs. Artin Invariant Stratification

Thanks to the recent proof of the Tate conjecture for K3 surfaces, we know that for a K3 surface X over $\overline{\mathbb{F}}_p$, $h = \infty$ if and only if its Picard rank $\rho = 22$; *i.e.*, it is Artin supersingular if and only if it is Shioda supersingular; *cf.* [45, Theorem 2.3]. We simply call X supersingular in this case. Let X be a supersingular K3 surface over $\overline{\mathbb{F}}_p$; then the discriminant of its Néron–Severi lattice is equal to $-p^{2\sigma_0(X)}$ for some integer $1 \leq \sigma_0(X) \leq 10$. The integer $\sigma_0(X)$ is called the Artin invariant of X.

By [13], we have an explicit description of the set ${}^J\mathcal{W}$ as $\{w_1, \ldots, w_{20}\}$, with w_i corresponds to b_i for $1 \leq i \leq 10$, and for $11 \leq i \leq 20$, the elements w_i are basic. The K3 surfaces in the stratum $\overline{\mathrm{M}}_{2d,K}^{w_i}$ have Artin invariant $21 - i$. In particular, we note that the index i in the description of the set ${}^J\mathcal{W}$ in subsection 8.2 (where $0 \leq i \leq 19$ in our case) is shifted to $i + 1$ here. By [13], for each $w \in {}^J\mathcal{W}$, the Ekedahl–Oort stratum $\overline{\mathrm{M}}_{2d,K}^w$ is non-empty.

8.5 Rapoport–Zink Type Uniformization and Artin Invariants

In this final subsection, we make the link between Rapoport–Zink spaces and moduli spaces of K3 surfaces.

Let $\widehat{\mathrm{M}}_{2d,K}$ and $\widehat{S}_K$ be the formal completion of $\mathrm{M}_{2d,K}$ and let S_K along their special fibers respectively. Then the integral Kuga–Satake period map in Theorem 8.9 induces an open immersion of formal schemes: $\widehat{\iota} \colon \widehat{\mathrm{M}}_{2d,K} \longrightarrow \widehat{S}_K$.

Let $x_0 \in \overline{M}_{2d,K}$ be any point in the special fiber $\overline{M}_{2d,K}$ of $M_{2d,K}$, and $x = \bar{\iota}(x_0)$ be its image in $\overline{S}_K$. Let $b \in B(G, \mu)$ be the Newton point associated with x and consider the corresponding formal Rapoport–Zink space $\check{\mathcal{M}} = \check{\mathcal{M}}_b$ for the group $SO(V)$. The choice of the point x determines a morphism of formal schemes: $\Theta_x \colon \check{\mathcal{M}} \longrightarrow \widehat{S}_K$. Denote by $\check{\mathcal{N}}$ the pullback of $\check{\mathcal{M}}$ under $\widehat{\iota} \colon \widehat{M}_{2d,K} \to \widehat{S}_K$. In other words, we get a cartesian diagram

$$
\begin{array}{ccc}
\check{\mathcal{N}} & \longrightarrow & \check{\mathcal{M}} \\
\downarrow & & \downarrow{\scriptstyle \Theta_x} \\
\widehat{M}_{2d,K} & \xrightarrow{\ \widehat{\iota}\ } & \widehat{S}_K,
\end{array}
$$

with the upper horizontal map $\check{\mathcal{N}} \to \check{\mathcal{M}}$ is an open immersion. By the moduli description of $\check{\mathcal{M}}$, we get the following description of $\check{\mathcal{N}}$. For any $R \in \mathrm{Nilp}_W^{\mathrm{s}\,m}$,

$$
\check{\mathcal{N}}(R) = \{(X, (t_\alpha), \bar{\rho}) \in \check{\mathcal{M}}(R)\},
$$

where the following hold:

- $(X, (t_\alpha), \rho) \in \check{\mathcal{M}}_1(R)$, with $X = KS(Y)[p^\infty]$, where Y is a K3 surface over R, $KS(Y)$ is the Kuga–Satake abelian scheme attached to Y (*cf.* Theorem 8.9 and [47, section 5]);
- $\bar{\rho}$ is the $p^{\mathbb{Z}}$-orbit of ρ.

In particular, $\check{\mathcal{N}}$ is stable under the action of $J_b(\mathbb{Q}_p)$ on $\check{\mathcal{M}}$.

Remark 8.11 By construction, we have an open subspace $\check{\mathcal{N}}_1 \subset \check{\mathcal{M}}_1$, such that for any $R \in \mathrm{Nilp}_W^{\mathrm{s}\,m}$,

$$
\check{\mathcal{N}}_1(R) = \{(X, (t_\alpha), \rho)\}
$$

with $(X, (t_\alpha), \rho) \in \check{\mathcal{M}}_1(R)$ as above. The space $\check{\mathcal{N}}$ is given by $\check{\mathcal{N}} = \check{\mathcal{N}}_1/p^{\mathbb{Z}}$. On the level of affine Deligne–Lusztig varieties, we get subsets

$$
\mathcal{N}_{\mathrm{red}}(\overline{\mathbb{F}}_p) \subset \mathcal{M}_{\mathrm{red}}(\overline{\mathbb{F}}_p) = X_\mu^G(b) \quad \text{and} \quad \mathcal{N}_{1\,\mathrm{red}}(\overline{\mathbb{F}}_p) \subset \mathcal{M}_{1\,\mathrm{red}}(\overline{\mathbb{F}}_p) = X_{\mu_1}^{G_1}(b_1).
$$

In the case that b is basic, it will be interesting to describe the above subsets by special lattices as in [32, Section 5].

We can apply the Rapoport–Zink uniformization theorem for S_K to deduce a similar uniformization for $M_{2d,K}$. Recall that as $\dim V = 21$ is odd, the group $G = SO(V)$ is adjoint.

Corollary 8.12 *Let J_ϕ be the pullback of $\mathcal{Z}_{\phi,K^p}$ under the open immersion $\bar{\iota} \colon \overline{M}_{2d,K} \hookrightarrow \overline{S}_K$. Then we have the identity*

$$
\widehat{M_{2d,K}}_{/J_\phi} = \coprod_{j \in I} \check{\mathcal{N}}/\Gamma_j,
$$

where I is certain countable set, and for each $j \in I$, $\Gamma_j \subset J_b(\mathbb{Q}_p)$ is some discrete subgroup (constructed as usual from the uniformization theorem of the last section). If, moreover, $b = b_0$ is basic, then $J_\phi = \overline{M}_{2d,K}^{ss}$, which is the supersingular locus in $\overline{M}_{2d,K}$, and the above disjoint union is finite.

Recently there has been a definition of isogeny between two K3 surfaces in characteristic p; *cf.* [76]. One can cheek that the locus J_ϕ parametrizes an isogeny class of polarized K3 surfaces.

Remark 8.13   If the open compact subgroup $K = K_p K^p \subset G(\mathbb{A}_f)$ ($K_p = G(\mathbb{Z}_p)$) is the image of some open compact subgroup $K_1 = K_{1p} K_1^p \subset G_1(\mathbb{A}_f)$ ($K_{1p} = G_1(\mathbb{Z}_p)$), then it will be much easier to prove the uniformization theorem for S_K. One can work directly on the finite level and take a finite étale quotient from the corresponding Rapoport–Zink uniformization for G_1; *cf.* [67, section 4], for example.

Assume that $b = b_0$ is basic. Let $\mathcal{N}_{\mathrm{red}}$ be the reduced special fiber of $\check{\mathcal{N}}$. Then the Howard–Pappas stratification of the reduced special fiber $\mathcal{M}_{\mathrm{red}}$ of $\check{\mathcal{M}}$ induces a similar stratification of the open subspace $\mathcal{N}_{\mathrm{red}}$:

$$\mathcal{N}_{\mathrm{red}} = \coprod_\Lambda \mathcal{N}_\Lambda^\circ,$$

where $\mathcal{N}_\Lambda^\circ \subset \mathcal{N}_{\mathrm{red}}$ is the pullback of the stratum $\mathcal{M}_\Lambda^\circ \subset \mathcal{M}_{\mathrm{red}}$. For each $w_i \in {}^J\mathcal{W}^b$, consider the corresponding Ekedahl–Oort stratum

$$\mathcal{M}_{w_i} = \coprod_{\Lambda, t(\Lambda) = 2(21-i)} \mathcal{M}_\Lambda^\circ \quad \text{and} \quad \mathcal{N}_{w_i} = \coprod_{\Lambda, t(\Lambda) = 2(21-i)} \mathcal{N}_\Lambda^\circ.$$

For each $11 \le i \le 20$, the image of $\mathcal{N}_{w_i}$ under the uniformization morphism gives us the corresponding Ekedahl–Oort stratum $\overline{\mathrm{M}}_{2d,K}^{w_i}$ in supersingular locus.

For $(X, \xi) \in \overline{\mathrm{M}}_{2d,K}^{ss}(\overline{\mathbb{F}}_p)$, consider $\mathcal{L} = \langle \mathrm{ch}_{\mathrm{cris}}(\xi) \rangle^\perp \subset H_{\mathrm{cris}}^2(X/W)$. This is a special lattice in the sense of [32, Definition 5.2.1]. Then we can apply Proposition 5.2.2 of loc. cit. to produce a vertex lattice $\Lambda(\mathcal{L})$. For any integer $r \ge 0$, define

$$\mathcal{L}^{(r)} = \mathcal{L} + \Phi(\mathcal{L}) + \cdots + \Phi^r(\mathcal{L}).$$

Then there is a unique integer $1 \le d \le 10$ such that

$$\mathcal{L} = \mathcal{L}^{(0)} \subsetneq \mathcal{L}^{(1)} \subsetneq \cdots \subsetneq \mathcal{L}^{(d)} = \mathcal{L}^{(d+1)}.$$

The vertex lattice $\Lambda(\mathcal{L})$ is defined by $\Lambda(\mathcal{L}) = (\mathcal{L}^{(d)})^\Phi$. It has type $t(\Lambda(\mathcal{L})) = 2d$ and $\Lambda(\mathcal{L})^\vee = \mathcal{L}^\Phi$. The following corollary follows from the above uniformization and Corollary 8.5.

Corollary 8.14   *Under the uniformization identity*

$$\overline{\mathrm{M}}_{2d,K}^{ss} = \coprod_{j \in I} \mathcal{N}_{\mathrm{red}} / \Gamma_j,$$

the Ekedahl–Oort stratum $\overline{\mathrm{M}}_{2d,K}^{w_i}$ for each $11 \le i \le 20$ is the image of $\mathcal{N}_{w_i}$. In particular, if $x \in \overline{\mathrm{M}}_{2d,K}^{ss}(\overline{\mathbb{F}}_p)$, let X_x be the associated supersingular K3 surface over $\overline{\mathbb{F}}_p$, then we have the identity between the Artin invariant $\sigma_0(X_x)$ and the type $t(\Lambda_x)$:

$$\sigma_0(X_x) = \frac{t(\Lambda_x)}{2},$$

where $\Lambda_x = \Lambda(\mathcal{L}_x)$ is the vertex lattice attached to the special lattice associated with (X_x, ξ_x) as above.

A Admissibility and Weakly Admissibility in the Basic Orthogonal Case

In this appendix, we investigate the p-adic period domains $\mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$ and $\mathscr{F}\ell_{G,\mu}^{\mathrm{wa}}$ in the case b is basic and $G = \mathrm{SO}$. Although Theorem A.2 appears as a special case of our more recent work [9], we present it here, since it provides some concrete computations, which can serve as a good example-based introduction to our proof for [9, Theorem 6.1] in the direction (i)$\Rightarrow$ (ii). All the following material is taken from [18]. We thank Fargues sincerely for kindly allowing us to include it here.

Let $V = \mathbb{Q}_p^n$ equipped with the quadratic form Q with matrix

$$\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

Let $G = \mathrm{SO}(V, Q)$ and consider the minuscule cocharacter $\mu\colon \mathbb{G}_m \to G_{\overline{\mathbb{Q}_p}}$ given by $\mu(z) = \mathrm{diag}(z, 1, \ldots, 1, z^{-1})$. Then the basic class in $B(G, \mu)$ is $[b] = [1]$ and thus $J_b = G$. One checks easily that any non-basic Newton polygon has a nontrivial contact point with the Hodge polygon, *i.e.*, $(G, \{\mu\})$ is fully Hodge–Newton decomposable in the sense of [22, Definition 2.1].

For simplicity, we write $\mathscr{F}\ell = \mathscr{F}\ell_{G,\mu}$ as the p-adic flag variety, $\mathscr{F}\ell^{\mathrm{wa}} = \mathscr{F}\ell_{G,\mu}^{\mathrm{wa}}$, and $\mathscr{F}\ell^{\mathrm{adm}} = \mathscr{F}\ell_{G,\mu}^{\mathrm{adm}}$. We first describe the weakly admissible locus $\mathscr{F}\ell^{\mathrm{wa}}$. The associated isocrystal is $\breve{\mathbb{Q}}_p^n$ with Frobenius $\sigma^{\oplus n}$. The sub-isocrystals are in bijection with the sub-$\mathbb{Q}_p$-vector space of V. Let C be a complete and algebraically closed extension of $\breve{\mathbb{Q}}_p$. Then we have

$$\mathscr{F}\ell(C, \mathcal{O}_C) = \{\text{Lagrangian lines } D \subset V_C\}.$$

It follows that $\mathscr{F}\ell \subset \mathbb{P}_{\mathbb{Q}_p}^n$ is the quadric defined by the equation $\sum_{i=1}^n x_i x_{n-i+1} = 0$. Let $\mathbb{Q}_p^{\left[\frac{n}{2}\right]} \oplus (0) \subset V$ be a Lagrangian subspace with associated parabolic subgroup $P \subset G$. We attach to it the Hodge filtration

$$0 \subset \mathrm{Fil}^1 = D \subset \mathrm{Fil}^0 = D^\perp \subset \mathrm{Fil}^{-1} = V_C$$

to any line $D \in \mathscr{F}\ell(C, \mathcal{O}_C)$. Then

$$\mathscr{F}\ell^{\mathrm{wa}}(C, \mathcal{O}_C) = \{D \in \mathscr{F}\ell(C, \mathcal{O}_C) \mid D \cap W_C = 0,$$
$$\text{for each totally isotropic subspace } W \subset V\}.$$

Therefore, we get the following proposition.

Proposition A.1

$$\mathscr{F}\ell^{\mathrm{wa}} = \mathscr{F}\ell \backslash G(\mathbb{Q}_p) S^{\mathrm{ad}},$$

where S^{ad} is the adic space associated with the Schubert variety attached to P (S is defined by the locus $x_{\left[\frac{n}{2}\right]+1} = \cdots = x_n = 0$ inside $\mathscr{F}\ell$).

Now we look at the admissible locus $\mathscr{F}\ell^{\mathrm{adm}}$ (*cf.* [54, Definition A.6] or [9, Definition 3.1]). We have the following theorem.

Theorem A.2 $\mathscr{F}\ell^{\mathrm{adm}} = \mathscr{F}\ell^{\mathrm{wa}}$.

Proof For any point $x \in \mathscr{F}\ell^{\mathrm{wa}}(C, \mathcal{O}_C)$, let $\mathcal{E}_x$ be the associated modification of $\mathcal{O}_X^n$ such that the relative position of $(B_{\mathrm{dR}}^+)^n$ and $\widehat{\mathcal{E}}_{x,\infty}$ is bounded by μ. Here X is the Fargues–Fontaine curve over $\mathbb{Q}_p$ associated with the perfectoid field $C^\flat$, and $\infty = x_C \in X$ is the point defined by C. We need to show that this weakly admissible modification is in fact an admissible modification (*i.e.*, $\mathcal{E}_x$ is semi-stable of slope 0).

By [54, Proposition A.9], we have either

$$\mathcal{E}_x \simeq \mathcal{O}_X\left(\frac{1}{r}\right) \oplus \mathcal{O}_X^{n-2r} \oplus \mathcal{O}_X\left(-\frac{1}{r}\right)$$

for some integer $1 \le r \le \left[\frac{n}{2}\right]$, or $\mathcal{E}_x \simeq \mathcal{O}_X^n$. The second case is admissible. We have to show this is always the case. Suppose that we are in the first case; we will find a contradiction. The perfect quadratic form on $\mathcal{E}_x$ is such that for any $\lambda \in \mathbb{Q}$, we have $(\mathcal{E}_x^{\ge\lambda})^\perp = \mathcal{E}_x^{>-\lambda}$, where $\mathcal{E}_x^\lambda \subset \mathcal{E}_x$ is a step in the Harder–Narasimhan filtration of $\mathcal{E}_x$. Therefore, we get

$$\mathcal{O}_X\left(\frac{1}{r}\right)^\perp = \mathcal{O}_X\left(\frac{1}{r}\right) \oplus \mathcal{O}_X^{n-2r},$$

and $\mathcal{O}_X\left(\frac{1}{r}\right)$ is totally isotropic. It follows that there exists a unique subvector bundle $\mathscr{F} \subset \mathcal{O}_X^n$ that is a locally direct summand, such that the modification $\mathcal{E}_x|_{X\setminus\infty} \xrightarrow{\sim} \mathcal{O}_{X\setminus\infty}^n$ induces a modification

$$\mathcal{O}_X\left(\frac{1}{r}\right)\Big|_{X\setminus\infty} \xrightarrow{\sim} \mathscr{F}|_{X\setminus\infty}.$$

In particular, $\mathscr{F}$ is totally isotropic in $\mathcal{O}_X^n$. Such a modification is necessarily of one of the following types:

(1) $(-1, 0, \ldots, 0)$,
(2) $(0, \ldots, 0, 1)$,
(3) $(0, \ldots, 0)$.

Indeed, it suffices to look at the relative positions of the lattices $E \cap (B_{\mathrm{dR}}^+)^n$ and $E \cap \langle te_1, e_2, \ldots, e_{n-1}, t^{-1}e_n \rangle$ for all the sub-B_{dR}-vector spaces E of B_{dR}^n, where $e_1, \ldots, e_n$ is a basis of V. As $\mathcal{O}_X^n$ is semi-stable, we have $\deg(\mathscr{F}) \le 0$. By looking at the above three cases, we get that $\mathscr{F}$ is a degree -1 modification of $\mathcal{O}_X\left(\frac{1}{r}\right)$. Thus,

$$\mathscr{F} \simeq \mathcal{O}_X^r;$$

that is, $\mathscr{F} = W \otimes \mathcal{O}_X$ for some totally isotropic subspace $W \subset \mathbb{Q}_p^n$ of dimension r. This implies that our modification $\mathcal{E}_x|_{X\setminus\infty} \xrightarrow{\sim} \mathcal{O}_{X\setminus\infty}^n$ is not weakly admissible. Thus, we get a contradiction. ∎

References

[1] B. Bhatt and P. Scholze, *Projectivity of the Witt vector affine Grassmannian.* Invent. Math. 209(2017), no. 2, 329–423. https://doi.org/10.1007/s00222-016-0710-4

[2] M. Borovoi, *Abelian Galois cohomology of reductive groups.* Mem. Amer. Math. Soc. 132(1998), no. 626, 1–50. https://doi.org/10.1090/memo/0626

[3] J.-F. Boutot and T. Zink, *The p-adic uniformization of Shimura curves*, Preprint, available at https://www.math.uni-bielefeld.de/~zink/p-adicuni.ps

[4] O. Bültel and G. Pappas, (G, μ)-*displays and Rapoport–Zink spaces.* J. Inst. Math. Jussieu, to appear. arxiv:1702.00291

[5] A. Caraiani and P. Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties.* Ann. of Math. 186(2017), no. 3, 649–766. https://doi.org/10.4007/annals.2017.186.3.1

[6] M. Chen, *Composantes connexes géométriques d'espaces de modules de groupes p-divisibles.* Ann. Sci. Éc. Norm. Sup. 47(2014), 723–764. https://doi.org/10.24033/asens.2225

[7] M. Chen, M. Kisin, and E. Viehmann, *Connected components of affine Deligne–Lusztig varieties in mixed characteristic.* Compositio. Math. 151(2015), 1697–1762. https://doi.org/10.1112/S0010437X15007253

[8] M. Chen, M. Kisin, and E. Viehmann, *Corrigendum Connected components of affine Deligne–Lusztig varieties in mixed characteristic.* Compositio. Math. 153(2017), 218–222. https://doi.org/10.1112/S0010437X1600782X

[9] M. Chen, L. Fargues, and X. Shen, *On the structure of some p-adic period domains.* arxiv:1710.06935

[10] J.-F. Dat, S. Orlik, and M. Rapoport, *Period domains over finite and p-adic fields.* Cambridge Tracts in Mathematics, 183, Cambridge University Press, Cambridge, 2010. https://doi.org/10.1017/CBO9780511762482

[11] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques.* In: *Automorphic forms, representations and L-functions (Corvallis 1977).* Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, RI, 1979, pp. 247–289.

[12] V. G. Drinfeld, *Coverings of p-adic symmetric regions.* Funkcional Anal. i Prilozen 10(1976), 29–40.

[13] T. Ekedahl and G. van der Geer, *Cycle classes on the moduli of K3 surfaces in positive characteristic.* Selecta Math. (N. S.) 21(2015), no. 1, 245–291. https://doi.org/10.1007/s00029-014-0156-8

[14] G. Faltings, *Coverings of p-adic period domains.* J. Reine Angew. Math. 643(2010), 111–139. https://doi.org/10.1515/CRELLE.2010.046

[15] L. Fargues, *Geometrization of the local Langlands correspondence: an overview.* arxiv:1602.00999

[16] L. Fargues, *Quelques résultats et conjectures concernant la courbe.* In: *De la géométrie algébrique aux formes automorphes (I) - (Une collection d'articles en l'honneur du soixantième anniversaire de Gérard Laumon).* Astérisque, 369, 2015.

[17] L. Fargues, *G-torseurs en théorie de Hodge p-adique.* Compositio Math., to appear.

[18] L. Fargues, *Lettre à Rapoport.* https://webusers.imj-prg.fr/~laurent.fargues/index.html

[19] L. Fargues, *p-adic Twistors and Shtukas*, Lecture notes. https://webusers.imj-prg.fr/~laurent.fargues/Notes.html

[20] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* Astérisque, 406, Soc. Math. France, 2018.

[21] U. Görtz and X. He, *Basic loci in Shimura varieties of coxeter type.* Camb. J. Math. 3(2015), 323–353. https://doi.org/10.4310/CJM.2015.v3.n3.a2

[22] U. Görtz, X. He, and S. Nie, *Fully Hodge–Newton decomposable Shimura varieties.* arxiv:1610.05381

[23] A. Grothendieck, *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): I. Étude locale des schémas et de morphismes de schémas, Première partie.* Inst. Hautes Études Publ. Math. 20(1964), 5–259.

[24] A. Grothendieck, *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): IV. Étude locale des schémas et de morphismes de schémas, Quatrième partie.* Inst. Hautes Études Publ. Math. 32(1967), 5–361.

[25] P. Hamacher, *On the Newton stratification in the good reduction of Shimura varieties.* J. Algebraic. Geom., to appear. arxiv:1605.05540

[26] D. Hansen, *Period morphisms and variation of p-adic Hodge structure (preliminary draft).* http://www.math.columbia.edu/~hansen/periodmapmod.pdf

[27] U. Hartl, *On a conjecture of Rapoport and Zink.* Invent. Math. 195(2013), 627–696. https://doi.org/10.1007/s00222-012-0437-9

[28] U. Hartl and E. Viehmann, *The generic fiber of moduli spaces of bounded local G-shtukas*.
      arxiv:1712.07936

[29] X. He, *Geometric and homological properties of affine Deligne–Lusztig varieties*. Ann. of Math.
      179(2014), 367–404.   https://doi.org/10.4007/annals.2014.179.1.6

[30] X. He and M. Rapoport, *Stratifications in the reduction of Shimura varieties*. Manuscripta Math.
      152(2017), no. 3–4, 317–343.   https://doi.org/10.1007/s00229-016-0863-x

[31] X. He and R. Zhou, *On the connected components of affine Deligne–Lusztig varieties*.
      arxiv:1610.06879

[32] B. Howard and G. Pappas, *Rapoport–Zink spaces for spinor groups*. Compos. Math. 153(2017), no. 5,
      1050–1118.   https://doi.org/10.1112/S0010437X17007011

[33] A. J. de Jong, *Étale fundamental groups of non-Archimedean analytic spaces*. Compositio Math.
      97(1995), 89–118.

[34] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*. Inst. Hautes
      Études Sci. Publ. Math. 82(1995), 5–96.

[35] K. S. Kedlaya and R. Liu, *Relative p-adic Hodge theory: foundations*. Astérisque, 371, Soc. Math.
      France, 2015.

[36] W. Kim, *Rapoport–Zink spaces of Hodge type*. Forum Math. Sigma 6(2018), e8, 110 pp.
      https://doi.org/10.1017/fms.2018.6

[37] W. Kim, *Rapoport–Zink uniformization of Hodge type Shimura varieties*. Forum Math. Sigma
      6(2018), e16, 36 pp.   https://doi.org/10.1017/fms.2018.18

[38] M. Kisin, *Integral models for Shimura varieties of abelian type*. J. Amer. Math. Soc. 23(2010),
      967–1012.   https://doi.org/10.1090/S0894-0347-10-00667-3

[39] M. Kisin, *Mod p points on Shimura varieties of abelian type*. J. Amer. Math. Soc. 30(2017),
      819–914.   https://doi.org/10.1090/jams/867

[40] R. E. Kottwitz, *Isocrystals with additional structure*. Compositio Math. 56(1985), 201–220.

[41] R. E. Kottwitz, *Isocrystals with additional structure. II*. Compositio Math. 109(1997), 255–339.
      https://doi.org/10.1023/A:1000102604688

[42] V. Lafforgue, *Introduction aux chtoucas pour les groupes réductifs et à la paramétrisation de
      Langlands globale*. J. Amer. Math. Soc. 31(2018), 719–891.   https://doi.org/10.1090/jams/897

[43] E. Lau, *Displays and formal p-divisible groups*. Invent. Math. 171(2008), no. 3, 617–628.
      https://doi.org/10.1007/s00222-007-0090-x

[44] C. Li and Y. Zhu, *Arithmetic intersection on GSpin Rapoport–Zink spaces*. Compositio Math.
      154(2018), no. 7, 1407–1440.   https://doi.org/10.1112/S0010437X18007108

[45] C. Liedtke, *Supersingular K3 surfaces are unirational*. Invent. Math. 200(2015), no. 3, 979–1014.
      https://doi.org/10.1007/s00222-014-0547-7

[46] R. Liu and X. Zhu, *Rigidity and a Riemann–Hilbert correspondence for p-adic local systems*. Invent.
      Math. 207(2017), no. 1, 291–343.   https://doi.org/10.1007/s00222-016-0671-7

[47] K. Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*. Invent. Math. 201(2015),
      625–668.   https://doi.org/10.1007/s00222-014-0557-5

[48] H. Matsumura, *Commutative algebra*, Second ed., Mathematics Lecture Note Series, 56,
      Benjamin/Cummings Publishing Co., Inc., Reading, MA, 1980.

[49] S. Nie, *Connected components of closed affine Deligne–Lusztig varieties in affine Grassmannians*.
      Amer. J. Math. 140(2018), no. 5, 1357–1397.   https://doi.org/10.1353/ajm.2018.0034

[50] A. Ogus, *Singularities of the height strata in the moduli of K3 surfaces*. In: *Moduli of abelian varieties
      (Texel Island, 1999)*. Progr. Math., 195, Birkhäuser, Basel, 2001, pp. 325–343.

[51] R. Pink, T. Wedhorn, and P. Ziegler, *F-zips with additional structure*. Pacific J. Math. 274(2015),
      no. 1, 183–236.   https://doi.org/10.2140/pjm.2015.274.183

[52] M. Rapoport, *Non-archimedean period domains*. In: *Proceedings of the International Congress of
      Mathematicians, vol. 1, 2 (Zürich, 1994)*. Birkhäuser, Basel, 1995, pp. 423–434.

[53] M. Rapoport, *A guide to the reduction modulo p of Shimura varieties. Automorphic forms. I.*
      Astérisque 298(2005), 271–318.

[54] M. Rapoport, *Accessible and weakly accessible period domains*. Appendix of P. Scholze, *On the
      p-adic cohomology of the Lubin–Tate tower*. Ann. Sci. École Norm. Sup. 51(2018), no. 4, 856–863.
      https://doi.org/10.24033/asens.2367

[55] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with
      additional structure*. Compositio Math. 103(1996), no. 2, 153–181.

[56] M. Rapoport and T. Zink, *Period spaces for p-divisible groups*. Ann. of Math. Stud., 141, Princeton
      University Press, Princeton, NJ, 1996.

[57] M. Rapoport and E. Viehmann, *Towards a theory of local Shimura varieties*. Münster J. Math.
      7(2014), 273–326.

[58] J. Rizov, *Moduli stacks of polarized K3 surfaces in mixed characteristic*. Serdica Math. J. 32(2006), no. 2–3, 131–178.

[59] J. Rizov, *Kuga–Satake abelian varieties of K3 surfaces in mixed characteristic*. J. Reine Angew. Math. 648(2010), 13–67. https://doi.org/10.1515/CRELLE.2010.078

[60] P. Scholze, *Perfectoid spaces*. Publ. Math. Inst. Hautes Études Sci. 116(2012), no. 1, 245–313. https://doi.org/10.1007/s10240-012-0042-x

[61] P. Scholze, *Lectures on p-adic geometry*. Notes taken by J. Weinstein, Fall 2014. http://math.bu.edu/people/jsweinst/Math274/ScholzeLectures.pdf

[62] P. Scholze, *Étale cohomology of diamonds*. arxiv:1709.07343

[63] P. Scholze, *p-adic geometry*. Proceedings of the ICM 2018.

[64] P. Scholze and J. Weinstein, *Berkeley lectures on p-adic geometry*. http://www.math.uni-bonn.de/people/scholze/Berkeley.pdf

[65] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*. Cambridge J. Math. 1(2013), 145–237. https://doi.org/10.4310/CJM.2013.v1.n2.a1

[66] J.-P. Serre, *Groupes algébriques associés aux modules de Hodge–Tate*. In: *Journées de Géométrie Algébrique de Rennes (Rennes 1978), Vol. III*. Astérisque, 65, Soc. Math. France, Paris, 1979, pp. 155–188.

[67] X. Shen, *Perfectoid Shimura varieties of abelian type*. Int. Math. Res. Not. IMRN 2017 no. 21, 6599–6653. https://doi.org/10.1093/imrn/rnw202

[68] X. Shen and C. Zhang, *Stratifications in good reductions of Shimura varieties of abelian type*. arxiv:1707.00439

[69] I. Vollaard and T. Wedhorn, *The supersingular locus of the Shimura variety of $GU(1, n-1)$. II*. Invent. Math. 184(2011), 591–627. https://doi.org/10.1007/s00222-010-0299-y

[70] Y. Varshavsky, *Moduli spaces of principal F-bundles*. Selecta Math. (N.S.) 10(2004), no. 1, 131–166. https://doi.org/10.1007/s00029-004-0343-0

[71] E. Viehmann, *Truncations of level 1 of elements in the loop group of a reductive group*. Ann. of Math. (2) 179(2014), 1009–1040. https://doi.org/10.4007/annals.2014.179.3.3

[72] T. Wedhorn, *Ekedahl–Oort strata of Shimura varieties*. Lecture notes in a Summer School on Shimura varieties, 2016, NCTS, Taiwan.

[73] J.-P. Wintenberger, *Existence de F-cristaux avec structures supplémentaires*. Adv. Math. 190(2005), 196–224. https://doi.org/10.1016/j.aim.2003.12.006

[74] D. Wortmann, *The μ-ordinary locus for Shimura varieties of Hodge type*. arxiv:1310.6444

[75] L. Xiao and X. Zhu, *Cycles on Shimura varieties via geometric Satake*. arxiv:1707.05700

[76] Z. Yang, *Isogenies between K3 surfaces over $\overline{\mathbb{F}}_p$*. arxiv:1810.08546

[77] C. Zhang, *Ekedahl–Oort strata for good reductions of Shimura varieties of Hodge type*. Canad. J. Math. 70(2018), no. 2, 451–480. https://doi.org/10.4153/CJM-2017-020-5

[78] X. Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*. Ann. of Math. 185(2017), no. 2, 403–492. https://doi.org/10.4007/annals.2017.185.2.2

[79] T. Zink, *The display of a formal p-divisible group. Cohomologies p-adiques et applications arithmétiques, I*. Astérisque 278(2002), 127–248.

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