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The optimum position for a tidal power barrage in the Severn estuary

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G. I. Taylor's approximate analytical solution for the tidal flow in the Severn estuary is extended to find the optimum location for a tidal power barrage, from the power point of view. It appears to be at the lowest point in the estuary, between Ilfracombe and Gower – contrary to earlier computations. The analytical solution shows that barrages radiate tidal waves out to sea, which highlights the important role of the far-field boundary condition in absorbing them. This appears to have been neglected in numerical models, which may explain the difference from the earlier results.

1. Introduction

Tidal power barrages in the Severn estuary were studied intensively 30 years ago, by a UK government committee chaired by Bondi (see Bondi *et al.* 1981). It was concluded from computer models that the optimum position for a barrage from the power point of view was approximately halfway down the estuary at Minehead. If the barrage was moved further downstream, no more power was obtained, because it was found that the barrage increasingly attenuated the incoming tides. Although tidal power barrages for the Severn have been studied on several more recent occasions, it appears that no more recent computer modelling has been undertaken on this point (see Burrows *et al.*, in press).

The problem can be investigated using G. I. Taylor's simple analytical model of the tidal flow in the Severn estuary (Taylor 1921). This has the advantage of revealing the fundamental features of the problem more clearly than a computer model.

Taylor's model is described in Lamb's account of the 'canal theory of the tides' (Lamb 1932, pp. 267–278), of which it is a special case. The canal theory considers tidal flow as a longitudinal gravity wave in a channel. Following Lamb's notation, if the width of the channel is $b(x)$ and its depth is $h(x)$, both varying with position x along the channel, then the equation for the surface elevation $\eta(x, t)$ at time t is (Lamb 1932, p. 274)

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b}\frac{\partial}{\partial x}\left(hb\frac{\partial \eta}{\partial x}\right), \tag{1}$$

where g is the acceleration due to gravity. In an estuary, high tide is assumed to occur at the same time, $t = 0$, everywhere, since the extent of the estuary, when measured in degrees of longitude, is small compared with the tidal cycle of approximately 180°. A

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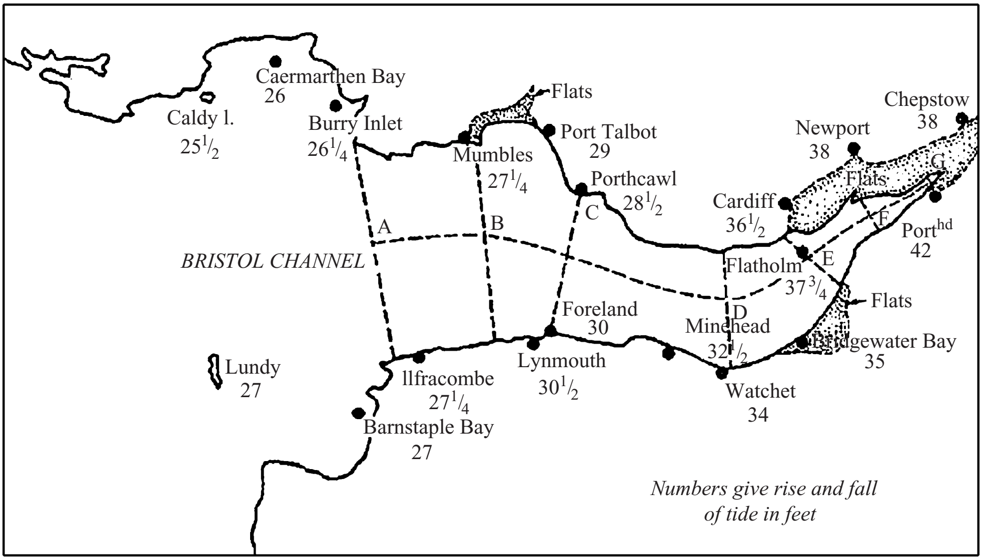

FIGURE 1. Taylor's model of the Bristol Channel. All the data in the table have been updated, from the latest admiralty charts. Breadths, depths and areas are at the mean sea level. The area upstream of Sharpness (17 km upstream of Chepstow) is excluded, since it is small and the tidal range is markedly reduced there. The time delays are mean values for neap tides (appropriate since we are considering a mean tidal range in figure 3, which will be reduced by the barrage as in figure 4) based on data for the north shore of the estuary (which appear more reliable than that from the south shore) in the 2009 admiralty tide tables. Parameters for $n = 7$ and 8 are defined to give the correct averages over the area upstream of section G, when used in (9)

solution is therefore sought of the form

$$\eta(x, t) = \eta_0(x)\cos(\omega t), \tag{2}$$

where $2\pi/\omega$ is the tidal period of approximately 12 h (half a lunar day). Thus (1) becomes

$$\frac{g}{b}\frac{\mathrm{d}}{\mathrm{d}x}\left(hb\frac{\mathrm{d}\eta_0}{\mathrm{d}x}\right) + \omega^2\eta_0 = 0. \tag{3}$$

In the case of the Severn estuary, Taylor observed that the width $b(x)$ and depth $h(x)$ both increase approximately linearly with distance x downstream (referred to henceforth as 'west') of the head of the estuary at Portishead (see figure 1, originally figure 1 and table 1 in Taylor 1921). He therefore took $x = 0$ at Portishead and put

$$b = \beta x \text{ and } h = \gamma x, \tag{4}$$

where β and γ are constants. This reduces (3) to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\eta_0}{\mathrm{d}x}\right) + k\eta_0 x = 0 \text{ with } k = \omega^2/(\gamma g), \tag{5}$$

which can be solved exactly as a Bessel function:

$$\eta_0 = \frac{K J_1\{2\sqrt{kx}\}}{\sqrt{kx}}, \tag{6}$$

Taylor's section	n	Distance x_n from section G (km)	Mean depth (m)	Breadth (km)	Area S_n to next section (sq. km)	Delay t_n of high tide, relative to section A (min)	Loss angle from (9) (deg.)
A	1	114.3	36.9	40.6	800	0	3.8
B	2	92.10	28.7	37.7	585	2	3.9
C	3	77.83	24.4	30.0	695	2	5.2
D	4	46.33	16.3	22.7	383	6	6.1
E	5	28.72	16.3	13.2	220	14	5.2
F	6	14.82	9.5	15.2	166	19	
G	7	0.0001	5.3	7.8	113	29	
	8	0.0001				53	

where K is a constant. Taylor took $\gamma = \{25 \text{ fathoms}\}/\{80 \text{ UK nautical miles}\} = 0.0003084$ (β is immaterial) and the tidal period $2\pi/\omega$ as 12.4 h, so that $k = 0.00655\,\text{km}^{-1}$, and found (6) to be a good approximation to the observed variation of tidal range in the Severn estuary, shown in figure 1 (close to modern values). This paper extends Taylor's analysis to the case of a tidal power barrage in the estuary.

2. Tidal power – the need for progressive waves

Considered as a function of time, the horizontal velocity in a tidal wave (and indeed in a water wave generally) is 90° out of phase with the surface slope $\partial\eta/\partial x$, since the latter is in phase with the horizontal acceleration. And the pressure variations are in phase with the surface elevation η. Thus for a standing-wave solution of the form (2), where the surface slope is in phase with the surface elevation, the velocity and pressure are 90° out of phase. Therefore the power flux (= velocity × pressure) has a mean value of zero everywhere. This is of course to be expected, since the tidal energy is nowhere being dissipated in the estuary in potential flow and only being stored. When we extract tidal power with a barrage, however, we require an equal mean power flux inwards at the mouth of the estuary. We thus reach the important conclusion that Taylor's solution (or any solution of form (2)) is 'inadmissible west of the barrage' because it transmits no mean power. What is required west of the barrage is a 'progressive wave', in which there is a power flux, because the surface slope is 90° out of phase with the surface elevation (and thus the velocity is in phase with the pressure). Rather than a solution of form (2) we can seek a solution of the more general form,

$$\eta(x,t) = \mathrm{Re}\{\eta_0(x)\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{7}$$

where $\eta_0(x)$ is now complex, and Re indicates the real part. This again leads to (5), which can be solved in the same way as

$$\eta_0 = \frac{K_1 H_1^{(1)}\{2\sqrt{kx}\} + K_2 H_1^{(2)}\{2\sqrt{kx}\}}{\sqrt{kx}}, \tag{8}$$

where $H_1^{(1)}$ and $H_1^{(2)}$ are a first and second Hankel functions of order one, and we now have two constants K_1 and K_2. The first term is a progressive wave travelling east, and the second is a progressive wave travelling west. Far to the west, both resemble tidal waves in open water of the same depth (since $H_1(x) \sim -\{\cos(x+\pi/4) \pm \mathrm{i}\sin(x+\pi/4)\}/\sqrt{x}$, for large x). East of the barrage, we can extend Taylor's solution empirically to include the observed delay times of the tide. These are caused by the need to transport energy into the estuary, to overcome natural energy losses from turbulence, and may therefore be important in the context of a tidal power barrage. (In fact they turn out to be of only minor importance; see figure 3).

3. An equivalent electric circuit

In his account of waves in channels, Lighthill (1978, p. 104) introduces the standard electrical analogy of voltage with pressure and electric current with volume flow rate. If the level variation of a reservoir of area S is written $\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\omega t}\}$, then its pressure variation is $\mathrm{Re}\{\rho g \mathrm{e}^{\mathrm{i}\omega t}\}$ (where ρ is the density of water), and the volume flow rate in and out of the reservoir is $S\mathrm{d}/\mathrm{d}t(\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\omega t}\})=\mathrm{Re}\{S\mathrm{i}\omega \mathrm{e}^{\mathrm{i}\omega t}\}$. Thus on the electrical analogy its impedance is $\rho g/(S\mathrm{i}\omega)$, so it is analogous to an electrical capacitance $S/\rho g$ (Lighthill 1978, p. 200, (3)).

A similar calculation applies in our case, for a reservoir formed by a barrage at one of Taylor's sections A–E in figure 1. The reservoir area can be can be discretized into the sub-areas S_n between the successive sections, given in figure 1. The level variation at the barrage is given by Taylor's formula (6) with his x-coordinate x_n given in figure 1, and this formula can also be used to find the average amplitude of the level variations of each sub-area. The phases of these level variations is given by the average delay times t_n in figure 1. Thus the reservoir impedance Z_1 of a barrages at the nth of Taylor's sections A–E can be written as

$$Z_1 = \frac{\dfrac{\rho g J_1\{2\sqrt{kx_n}\}}{\sqrt{kx_n}}}{\sum_{j=n}^{j=7}\left[\dfrac{J_1\{2\sqrt{k(x_j+x_{j+1})/2}\}}{\sqrt{k(x_j+x_{j+1})/2}} S_j \mathrm{i}\omega \mathrm{e}^{-\mathrm{i}\omega\{(t_j+t_{j+1})/2-t_n\}}\right]}. \tag{9}$$

Evidently (9) is no longer purely imaginary, but has a real part analogous to a resistance R_L as well as an imaginary part analogous to a capacitance C. The resistance R_L gives the natural energy dissipation in the reservoir – to continue the electrical analogy, it can be expressed as a 'loss angle' $\tan^{-1}(\omega C R_L)$, which is readily calculated from the argument of (9) and is given in figure 1. West of the barrage, it is convenient to consider the water pressure variation ($=\rho g\times$ level variation) as the sum of the pressure variation $\mathrm{Re}\{P\mathrm{e}^{\mathrm{i}\omega t}\}$ which would be seen in the absence of the barrage and the additional pressure variation $\mathrm{Re}\{P'\mathrm{e}^{\mathrm{i}\omega t}\}$ caused, immediately west of it, by the presence of the barrage. The additional pressure $\mathrm{Re}\{P'\mathrm{e}^{\mathrm{i}\omega t}\}$ at the barrage produces a tidal wave which propagates out to sea – as far as the flow to the west of the barrage is concerned, the barrage is acting like a wavemaker. We require its wavemaking impedance Z_2, i.e. the ratio of pressure to volume flow rate in the tidal wave it generates. A unit wave propagating west is described by the second term in (8), with $K_2=1$. The water acceleration in this wave, in the direction of propagation, is minus the surface slope times g, whence we can obtain the water velocity in a westward direction by integrating, as the real part of

$$\frac{-g}{\mathrm{i}\omega}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{H_1^{(2)}\{2\sqrt{kx}\}}{\sqrt{kx}}\right)\mathrm{e}^{\mathrm{i}\omega t}. \tag{10}$$

The volume flow rate in the direction of propagation is this velocity times bh, and the water pressure is $\rho g\eta$. We obtain the impedance Z_2 by dividing the latter by the

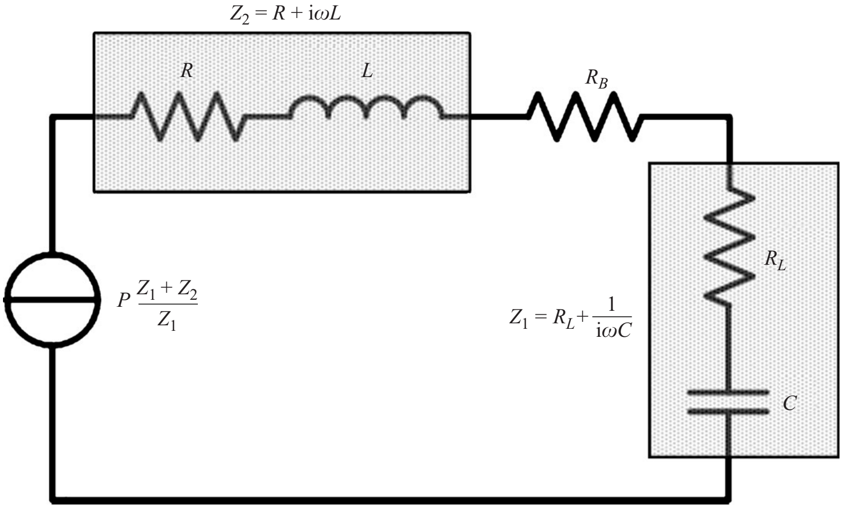

FIGURE 2. Equivalent electric circuit of barrage.

former, which gives this impedance as

$$Z_2 = \frac{-\mathrm{i}\rho\omega H_1^{(2)}\{2\sqrt{kx}\}}{bh\sqrt{kx}} \Bigg/ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{H_1^{(2)}\{2\sqrt{kx}\}}{\sqrt{kx}}\right) \tag{11}$$

which we can consider as a resistance R in series with an inductance L, giving a combined impedance of $R + \mathrm{i}\omega L$. For large x, the wave resembles a tidal wave in open water, for which the impedance is known to be purely a resistance of $\rho c/(bh)$ (Lighthill 1978, p. 104), where c is the open-water wave speed $\sqrt{(gh)}$. This gives a useful cross-check, when (11) is evaluated numerically. In the absence of the barrage, the (complex) volume flow rate at the barrage location is P/Z_1, in an eastward direction. The additional wavemaking volume flow immediately west of the barrage is P'/Z_2, in a westward direction. Thus the total (complex) volume flow rate at this location, in an eastward direction, can be written as follows:

$$\frac{P}{Z_1} - \frac{P'}{Z_2}. \tag{12}$$

If we write the total (complex) pressure at this location as $P'' = P + P'$, then (12) can be rearranged to

$$\frac{P\frac{Z_1+Z_2}{Z_1} - P''}{Z_2}. \tag{13}$$

On the electrical analogy, this is the same current as would be produced by a voltage generator $P(Z_1 + Z_2)/Z_1$ with a source impedance of Z_2. The flow in an eastward direction produced by this voltage generator passes first through the barrage and then into the reservoir beyond it. The impedance seen by the flow is thus the flow resistance of the turbines in the barrage, in series with the reservoir impedance Z_1. The turbines can be taken for simplicity as allowing flow in both directions. This is the most common arrangement (see Baker 1991, p. 31) and also the most efficient, before turbine losses (see Prandle 1984). Also for simplicity, the flow resistance of the turbines can be taken as a constant R_B, because very similar results have been obtained in simpler cases with linear and quadratic turbine characteristics (Garrett & Cummins 2004). Thus the equivalent circuit of the complete system is as shown in figure 2.

When $R_B = 0$, it may be seen that the pressure at the barrage is its undisturbed value P, as it should be.

4. Similarity to wave power

At first sight it may seem curious that to provide the inward power flux needed to power the barrage, we have introduced an additional tidal wave travelling in an *outward* direction. The reason is that from (8) Taylor's standing-wave solution (6) can be seen (by putting $K_1 = K_2 = K$ in (8) and noting that $H_1^{(1)} + H_1^{(2)} = 2J_1$) as the superposition of a tidal wave travelling east and an equal one travelling west. Our additional wave travelling west is cancelling part of his, giving a net inward wave. This situation is familiar in wave power (see for example Mei 1989, §7.9). Two-dimensional wave power devices likewise need to radiate waves out to sea, to cancel out wave reflections.

5. Power available at various locations in the Severn estuary

We can now calculate the power from the equivalent circuit of figure 2. The argument does not rely on the approximations above, but applies equally if accurate values for Z_1 and Z_2 are available. The (complex) volume flow rate through the barrage is

$$P\frac{Z_1 + Z_2}{Z_1(Z_1 + Z_2 + R_B)}, \tag{14}$$

and thus the average power is

$$\frac{1}{2}|P|^2\left|\frac{Z_1 + Z_2}{Z_1(Z_1 + Z_2 + R_B)}\right|^2 R_B. \tag{15}$$

This is readily calculated as a function of R_B, using expressions (9) and (11) for Z_1 and Z_2. It is given in figure 3 for Taylor's sections A–E of figure 1. The (complex) tidal pressure P in the absence of the barrage is taken as $4\rho g$ at Watchet, or 8 m tidal range, which is the approximate root-mean-square value between the mean spring range of 10 m and the mean neap range of 5 m, and thus gives the annual-average power. The values elsewhere are extrapolated from this 8 m figure, using Taylor's formula (6). Rather than being plotted against R_B, figure 3 is plotted against the pressure difference across the barrage (i.e (14) times R_B), expressed as a fraction of the tidal pressure variation $/P/$ in the absence of the barrage. Evidently the optimum value for this fraction is between 0.4 and 0.6, and the power increases steadily as the barrage is moved west. This is of course to be expected – as we move west, the reservoir area increases much more than the tidal range reduces (see figure 1).

6. Effect of the shape of the estuary west of Taylor's model

Taylor observed that the shape of the Severn estuary changes abruptly west of his outer boundary (section A in figure 2) and ceases to follow his formulae (4), even approximately. The width of the estuary approximately doubles immediately west of section A and thereafter follows another of Taylor's linearly tapering profiles, with both depth and width increasing approximately linearly with distance from a notional apex at Abergavenny, 100 km east of section A. The depth of 36.9 m at section A gives a new value of $\gamma^* = 36.9\text{ m}/100\text{ km} = 0.000369$ for γ, and thus a new value $k^* = 0.00547\text{ km}^{-1}$ for k. We wish to find the effect of this transition to a new profile on the barrage wavemaking impedance Z_2. The effect of the abrupt transition will be to reflect some of the wave travelling west considered in § 3, back up the channel. This reflection will be re-reflected from the barrage and then again from the abrupt transition after section A, in an infinite sequence. We can sum all the waves travelling west into a single wave travelling west between the barrage and Taylor's section A, and we can likewise sum all the waves travelling east into a single wave travelling east in this region. We can write the (complex) volume flow rates in the direction of wave propagation as

- V_O and V_B for the wave travelling west respectively at the outer boundary of the region at section A and at the barrage;
- V'_O and V'_B for the wave travelling east respectively at the outer boundary of the region at section A and at the barrage.

We can first find the ratio of V'_O to V_O, which we can express as a reflection coefficient r, where $V'_O = rV_O$. In the wave travelling west, the impedances at the two locations just considered are given by (11); we can write them as Z_O and Z_B. In the wave travelling east the impedances can be seen from (11) to be the complex conjugates of Z_O and Z_B. (The Hankel function $H_1^{(2)}$ from (8) becomes $H_1^{(1)} = \overline{H_1^{(2)}}$ and the −i from (10) becomes +i because the acceleration in the direction of wave propagation is now plus the surface slope times g.) In the region west of section A, we have only a wave travelling west, and the impedance is given by (11) with the new parameter k^* instead of k, and with $x = 100$ km. We can write this impedance as Z^* The sum of the pressures in the two waves immediately east of the transition at section A can now be equated to that in the single wave immediately west of it. The latter is obtained from the volume flow rate $V_O - V'_O$ in the westward direction:

$$V_O Z_O + V'_O \overline{Z_O} = (V_O - V'_O) Z^*, \text{ i.e. } V'_O = \frac{Z^* - Z_O}{Z^* + \overline{Z_O}} V_O \text{ so that } r = \frac{Z^* - Z_O}{Z^* + \overline{Z_O}}. \tag{16}$$

When $Z^* = Z_O$ there is no reflection from the outer boundary, and (16) accordingly predicts that $V'_O = 0$, as expected. We can now find the required wavemaking impedance Z_2 of the barrage, in terms of the reflection coefficient r given by (16). From (8),

$$\frac{V_B Z_B}{V_O Z_O} = \frac{H_1^{(2)}(2\sqrt{kx_B})/\sqrt{kx_B}}{H_1^{(2)}(2\sqrt{kx_O})/\sqrt{kx_O}} \text{ and } \frac{V'_B \overline{Z_B}}{rV_O \overline{Z_O}} = \frac{H_1^{(1)}(2\sqrt{kx_B})/\sqrt{kx_B}}{H_1^{(1)}(2\sqrt{kx_O})/\sqrt{kx_O}}, \tag{17}$$

where x_O and x_B are the x-coordinates of section A and the barrage. Since $H_1^{(1)} = \overline{H_1^{(2)}}$ the right-hand sides of these two equations are complex conjugates of each other. Thus

$$\overline{\left(\frac{V_B Z_B}{V_O Z_O}\right)} = \frac{V'_B \overline{Z_B}}{rV_O \overline{Z_O}}, \quad \text{i.e.} \quad V'_B = V_B r \frac{V_O \overline{V_B}}{\overline{(V_O \overline{V_B})}} = V_B r e^{-i2\omega T}, \tag{18}$$

in which we are noting that the argument of $V_O \overline{V_B}$ is $-\omega T$, where T is the wave transit time between the barrage and section A (readily calculated from (8)). We can thus obtain the wavemaking impedance at the barrage, as the sum of the pressures divided by the sum of the volume flow rates:

$$\frac{V_B Z_B + V_B r e^{-i2\omega T} \overline{Z_B}}{V_B - V_B r e^{-i2\omega T}} = \frac{Z_B + \overline{Z_B} r e^{-i2\omega T}}{1 - r e^{-i2\omega T}}. \tag{19}$$

When $r = 0$, there is no reflection at the outer boundary, and (19) then predicts that the wavemaking impedance of the barrage is Z_B, as expected. The barrage powers can be recalculated using this new wavemaking barrage impedance Z_2 – the results are shown in figure 3. Evidently the changed shape of the estuary west of Taylor's original model increases the power considerably, which is to be expected, since the increased width of the estuary will lower Z_2 and thus, from figure 2, increase the power. The closer the barrage to this increased width, the more pronounced the effect. Thus the conclusion remains that the power increases steadily as the barrage is moved west – indeed it now increases more. The question thus arises of the boundary condition even further out, where the second Taylor profile stops abruptly at the western extremities of England and Wales. This transition can be treated exactly like the transition at section A. If the impedance is assumed to halve at this transition, for example, and the calculations are repeated, the maximum powers in figure 3 all increase, by 1 % (barrage at section E) to 11 % (barrage at section A). So again the effect is more pronounced for barrages closer to the transition – it appears that features beyond the United Kingdom are relevant to the barrages furthest down the Severn estuary. This supports the practice in the most recent studies (e.g. Burrows *et al.*, in press) of extending computer models out to the limits of the continental shelf, although the type of boundary conditions applied there are very important. (Recent studies appear to be subject to the criticism that the boundary conditions are zero impedance; see the next section.) The calculations can also be repeated with the delay times t_n in figure 1 set to zero, which will remove natural energy dissipation. This is done in figure 3 and reveals that natural energy dissipation is only of minor importance. Finally, the changes in tidal range produced by the barrage are important. They are readily calculated from the equivalent circuit in figure 2, using the full expression (19) for Z_2, and are shown in figure 4, on the same horizontal axis as figure 3. Taking into account the fact that the power peak in figure 3 is further to the left for section C, the changes to the tidal range are very similar for all barrage locations. With barrages operated at maximum power, the tidal range is cut to 70 % of its former value east of the barrage and 90 % of its former value immediately west of the barrage. A very simple view of the barrage is that (from (9) and (11)) Z_2 is small compared with Z_1 and R_L is small compared with C. From figure 2, the optimum power, as a matter of elementary electrical engineering, is when R_B has the same impedance as C. This is an existing result in the tidal power literature, due to Garrett and Cummins (2004). It would reduce the tidal range east of the barrage by a factor $\sqrt{2}$ and leave the range immediately west of it unaffected because Z_2 is small.

7. Previous computations

The question of the optimum position for a barrage in the Severn estuary, from the power point of view, was studied 30 years ago (see Bondi *et al.* 1981). The power was computed with various finite-difference numerical models, some of which extended out into the Irish Sea. They showed the average power rising strongly from 0.5 to 2.3 GW as the barrage was moved west from Taylor's section F to section D (Bondi *et al.* 1981, vol. 1, p. 18). This is similar to the results in figure 3, allowing for conversion losses. However, very little increase was found for positions further west. By Taylor's section C, the power was starting to decline, in marked contrast to the increase seen in figure 3 – although significant discrepancies were found between computer models (Bondi *et al.* 1981, vol. 2, p. 57). We now explore a possible reason for this decline, which is that all the models simply held the tidal range fixed on the model boundary, at the same value it would have if the were barrage absent. This was then, and apparently still is, the usual assumption in tidal modelling (see e.g. Prandle 1980), although it has been recognized as wrong in principle (Garrett & Greenberg 1977). It

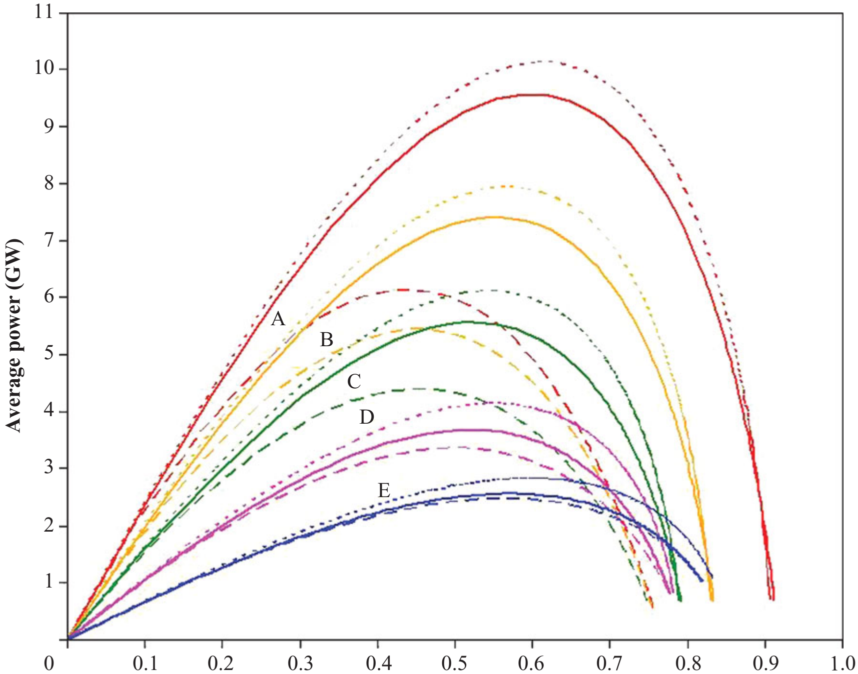

FIGURE 3. Average power (GW) for barrages at locations A–E of figure 1. The horizontal axis is the peak water level difference across the barrage, divided by the tidal amplitude (=range/2) in the absence of the barrage. The solid lines are with the outer estuary model (§ 6) included. The dashed lines are without it. The dotted lines are with it included, but with the delay times t_n in figure 1 set to zero, to remove natural energy losses.

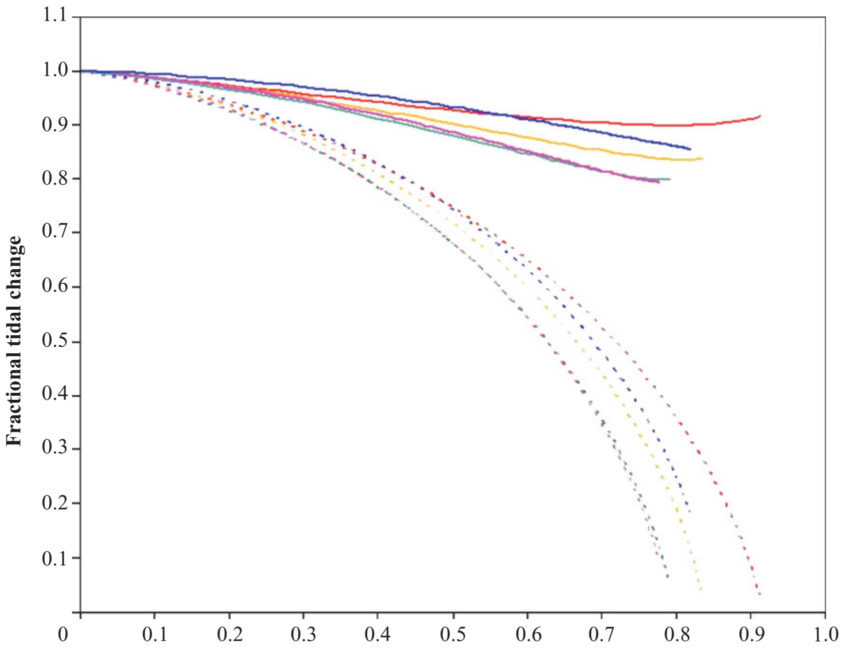

FIGURE 4. Fractional tidal change for barrages at locations A–E of figure 1. The horizontal axis and colour coding are the same as figure 3. The dashed lines are east of the barrage, and the solid lines are just west of it.

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will produce a total reflection of the outgoing tidal wave – it is equivalent to setting Z^* in (16) equal to zero. This leads to

$$V'_O = \frac{-Z_O}{\overline{Z_O}} V_O, \quad \text{i.e.} \quad r = \frac{-Z_O}{\overline{Z_O}} = -\mathrm{e}^{\mathrm{i}2\varphi}, \tag{20}$$

where $\varphi = \arg(Z_O)$ is the phase advance of pressure over volume flow rate, in an outward-propagating tidal wave, at the model boundary. For a model boundary at section A, for example, it can be calculated from (8) as 45.3°. If we similarly write $\theta = \arg(Z_B)$, then $Z_B = \zeta \mathrm{e}^{\mathrm{i}\theta}$, where ζ is real and θ is the phase advance of pressure over volume flow rate, in an outward-propagating tidal wave, at the barrage. For a barrage at section E, for example, it can be calculated from (8) as 68.9°. The wavemaking impedance Z_2 of the barrage (19) thus becomes

$$\frac{\zeta \mathrm{e}^{\mathrm{i}\theta} - \zeta \mathrm{e}^{-\mathrm{i}\theta} \mathrm{e}^{\mathrm{i}2\varphi} \mathrm{e}^{-\mathrm{i}2\omega T}}{1 + \mathrm{e}^{\mathrm{i}2\varphi} \mathrm{e}^{-\mathrm{i}2\omega T}} = \frac{\zeta \mathrm{e}^{\mathrm{i}(\pi - \theta + 2\varphi - 2\omega T)} + \zeta \mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}(2\varphi - 2\omega T)} + 1}. \tag{21}$$

Since $\mathrm{e}^{\mathrm{i}X} + \mathrm{e}^{\mathrm{i}\psi} = \{\mathrm{e}^{\mathrm{i}(X-\psi)/2} + \mathrm{e}^{-\mathrm{i}(X-\psi)/2}\}\mathrm{e}^{\mathrm{i}(X+\psi)/2} = 2\cos\{(X - \psi)/2\}\mathrm{e}^{\mathrm{i}(X+\psi)/2}$ this impedance can be written as

$$\frac{\zeta \cos\{\pi/2 + (\varphi - \theta - \omega T)\}\mathrm{e}^{\mathrm{i}(\pi/2 + \varphi - \omega T)}}{\cos(\varphi - \omega T)\mathrm{e}^{\mathrm{i}(\varphi - \omega T)}} = \mathrm{i}\zeta \frac{\sin(\omega T + \theta - \varphi)}{\cos(\omega T - \varphi)}. \tag{22}$$

Thus the wavemaking impedance at the barrage is purely imaginary (i.e. reactive), as we would expect – the barrage can radiate no wave power because the waves it sends west are perfectly reflected back by the model boundary. Its amplitude is small if the model boundary is close to the barrage because then θ and φ are nearly equal, and the phase delay ωT of a tidal wave between the barrage and the model boundary is then also small. Thus the change in the results will be small because Z_2 is small anyway, as noted at the end of the previous section. However, when the model boundary is a long way from the barrage, φ will be small because the tidal wave at the model boundary will resemble an open-water wave. Thus when the phase delay ωT reaches 90°, the denominator in (22) will drop to zero, and the wavemaking impedance of the barrage will become very large. The power from the barrage will accordingly drop. This condition requires the transit time T of a tidal wave between the barrage and the model boundary to be a quarter of the tidal period, or $12.4/4 = 3.1$ h. This is a resonant condition, with the natural sloshing period of the basin between the barrage and the outer boundary equal to the tidal period. With a mean tidal wave speed of $25\,\mathrm{m\,s^{-1}}$, say, it corresponds to a distance from the barrage to the outer boundary of $25 \times 3600 \times 4 = 360$ km. This is comparable with the size of the larger models used by Bondi *et al.* (1981). It is thus possible that the models used by Bondi *et al.* (1981) were giving spurious results due to internal resonances, caused by the incorrect model boundary condition, in which the tidal range was held at the same value it would have if the barrage were absent. The appropriate boundary condition is an 'absorbing' one, which does not reflect waves – these are standard in naval architecture and familiar in physical model testing too, as the beach in a wave tank.

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