Can. J. Math. Vol. 43 (4), 1991 pp. 825–831

POINTS OF SPHERICAL MAXIMA AND SOLVABILITY OF SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We give mild sufficient conditions on a nonlinear functional to have eigenvalues. These results are intended for the study of boundary value problems for semilinear elliptic equations.

1. **Introduction.** Let $g(u)$ be a differentiable functional on a real Hilbert space H. We are interested in finding eigenvalues and eigenelements of g, i.e., solutions (ρ, u) of

$$g'(u) = \rho u \tag{1.1}$$

where $\rho \in \mathbb{R}$ and $u \in H$ (for the applications we are considering, it is important that $\rho \neq 0$ and $u \neq 0$). Following an idea used in [4], we make use of the fact that an element $u_0 \in H$ which satisfies

$$\|u_0\|^2 = t_0 > 0, \quad g(u_0) = \max_{\|v\|^2 = t_0} g(v) \tag{1.2}$$

is a solution of (1.1) with

$$\rho = \big(g'(u_0), u_0\big) / t_0. \tag{1.3}$$

Our goal is to locate such elements u_0. In the present paper we assume as little on the the functional $g(u)$ as necessary to obtain the existence of these elements. Our only regularity assumption on g is that it be weakly continuous, i.e., that $u_k \longrightarrow u$ weakly in H implies $g(u_k) \longrightarrow g(u)$. This allows us to obtain solutions of semilinear partial differential equations under weaker conditions than normally assumed.

We have shown elsewhere [6] that the function

$$\gamma(t) = \sup_{\|v\|^2 = t} g(v) \tag{1.4}$$

plays an important role in the study of (1.1). We have shown that it is a continuous nondecreasing function of t. In Section 2 we show that if $\gamma(t_0) < \gamma(t_1)$, then there is an infinite number of solution of (1.1) satisfying $t_0 < \|u\|^2 < t_1$ with at least one of these solutions satisfying

$$\rho \geq 2[\gamma(t_1) - \gamma(t_0)] / (t_1 - t_0). \tag{1.5}$$

Research supported in part by an NSF grant.
Received by the editors January 23, 1990.
AMS subject classification: Primary: 35P30, 35T65, 47H12; Secondary: 47H15.

https://doi.org/10.4153/CJM-1991-047-0 Published online by Cambridge University Press

Moreover, we show that if $0 \leq t_0 < t_1 < t_2$ and ρ is any number satisfying

$$[\gamma(t_2) - \gamma(t_0)]/(t_2 - t_1) < \rho/2 < [\gamma(t_1) - \gamma(t_0)]/(t_1 - t_0) \tag{1.6}$$

then (1.1) has a solution u satisfying $t_0 < \|u\|^2 < t_2$. As a corollary we see that if

$$\alpha_\infty := \liminf_{t \to \infty} \gamma(t)/t < \alpha := \sup_{t_1, t_2} [\gamma(t_2) - \gamma(t_1)]/(t_2 - t_1) \tag{1.7}$$

then for any $\rho \in (2\alpha_\infty, 2\alpha)$ there is a solution $u \in H \setminus \{0\}$ of (1.1). Further results are given in Section 2.

Application of these results to boundary value problems for semilinear elliptic equations might be inferred from [6]. If A is a linear elliptic operator of order $2m$ and $f(x,t)$ is a Caratheodory function, (1.1) has a meaning of

$$Au = \lambda f(x,u), \quad u \in H_0^m(\Omega), \tag{1.8}$$

holding in a semi-strong sense. Here

$$g(u) = \int_\Omega F\big(x, u(x)\big)\, dx, \tag{1.9}$$

with

$$F(x,s) = \int_0^s f(x,\sigma)\, d\sigma. \tag{1.10}$$

Weak continuity of g and estimates on γ involved in theorems of this paper result in milder conditions on F rather than on f.

2. **Existence of Eigenfunctions.**

In this section we shall be concerned with proving the existence of eigenvalues assuming only

(i) $g(u)$ is a weakly continuous Frechet differentiable map from an infinite dimensional real Hilbert space H to $\mathbb{R}$.

We define for $t \geq 0$

$$S_t = \{x \in H \mid \|x\|^2 = t\} \tag{2.1}$$

$$\gamma(t) = \sup_{u \in S_t} g(u) \tag{2.2}$$

$$\Sigma_t = \{u \in S_t \mid g(u) = \gamma(t)\}. \tag{2.3}$$

It was shown in [6] that $\gamma(t)$ is a continuous nondecreasing function of t. First we have

THEOREM 2.1. *If $\gamma(t) > \gamma(t_0)$ for $t > t_0$, then there are sequences $\{s_k\} \subset \mathbb{R}$, $\{u_k\} \subset H$, $\{\rho_k\} \subset \mathbb{R}$ such that $s_k \searrow t_0$, $u_k \in \Sigma_{s_k}$, $\rho_k > 0$ and*

$$g'(u_k) = \rho_k u_k. \tag{2.4}$$

COROLLARY 2.2. *If $\gamma(t_0) < \gamma(t_1)$, then there is an infinite number of solutions (u, ρ) of*

$$g'(u) = \rho u \tag{2.5}$$

satisfying

$$t_0 < \|u\|^2 < t_1, \quad \rho > 0 \tag{2.6}$$

In proving Theorem 2.1 we shall make use of the following results from [6].

LEMMA 2.3. *If $\Sigma_t = \phi$, then there is a $t_- < t$ such that $\gamma(s) = \gamma(t)$ for $t_- \leq s \leq t$.*

LEMMA 2.4. *If $\varphi(t) \in C^1(0, \infty)$ is such that $\varphi(t) - \gamma(t)$ has a local minimum at t_0 and $\varphi'(t_0) \neq 0$, then there is a $u \in \Sigma_{t_0}$ such that*

$$g'(u) = 2\varphi'(t_0)u. \tag{2.7}$$

PROOF OF THEOREM 2.1. Let t_1 be any number $> t_0$, and $\hat{t} = (t_0 + t_1)/2$. For $t_0 < s < \hat{t}$ let $\varphi_s(t) = A(t-s)^2 + B$, where the constants A, B are chosen so that $\varphi_s(t_i) = \gamma(t_i), i = 0, 1$. Note that $A > 0$ as long as $s < \hat{t}$. Let $\psi_s(t) = \varphi_s(t) - \gamma(t)$. Then $\psi_s(t_i) = 0, i = 0, 1$. Since $\varphi_s(t)$ is decreasing for $t_0 < t < s$ and $\gamma(t)$ is nondecreasing, $\psi_s(t)$ has a negative minimum in $[t_0, t_1]$ for every s. Let t_s be a point where

$$\psi_s(t_s) = \min_{t_0 \leq t \leq t_1} \psi_s(t).$$

Clearly $t_s \geq s$ for every $s < \hat{t}$. We claim that there cannot be a $\delta > 0$ such that $t_s = s$ for $t_0 < s < t_0 + \delta$. For then

$$\varphi_s(t) - \gamma(t) \geq \varphi_s(s) - \gamma(s), \quad t_0 \leq t \leq t_1, \quad t_0 < s < t_0 + \delta.$$

This implies

$$\gamma(t) - \gamma(s) \leq \varphi_s(t) - \varphi_s(s) = A(t-s)^2$$

for such s, t. In turn this implies that $\gamma'(s)$ exists and vanishes for $t_0 < s < t_0 + \delta$. This would mean that $\gamma(t) = \gamma(t_0)$ for $t_0 \leq t < t_0 + \delta$, contrary to assumption. Hence for each $\delta > 0$ there is an s such that $t_0 < s < t_0 + \delta$ and $t_s > s$. Consequently $\psi_s(t)$ has a local minimum at t_s while $\varphi_s'(t_s) = 2A(t_s - s) > 0$. Thus there is a $u \in \Sigma_{t_s}$ such that $g'(u) = 2\varphi_s'(t_s)u$ (Lemma 2.4). This means that (2.5) has a solution satisfying (2.6). Since $t_1 > t_0$ was arbitrary, the result follows. ∎

PROOF OF COROLLARY 2.2. Let $\tilde{t}_0$ be the largest number such that $\gamma(t) = \gamma(t_0)$ for $t_0 \leq t \leq \tilde{t}_0 < t_1$ and $\gamma(t) > \gamma(\tilde{t}_0)$ for $t > \tilde{t}_0$. Apply Theorem 2.1. ∎

THEOREM 2.5. *If $\gamma(t_0) < \gamma(t_1)$, then for each ρ_0 satisfying $0 < \rho_0 < \sigma_0 := 2[\gamma(t_1) - \gamma(t_0)]/(t_1 - t_0)$ the following alternative holds: either*

a) (2.5) has a solution $u \in S_{t_1}$ with $\rho \geq \rho_0$ or

b) (2.5) has a solution u with $t_0 < \|u\|^2 < t_1$ and $\rho = \rho_0$.

In proving Theorem 2.5 we shall make use of

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LEMMA 2.6. *Assume that* $\varphi(t) \in C^1[t_0, t_1]$, $\varphi'(t_1) > 0$ *and*

$$\varphi(t) - \gamma(t) \geq \varphi(t_1) - \gamma(t_1), \quad t_0 \leq t \leq t_1 \tag{2.8}$$

Then $\Sigma_{t_1} \neq \emptyset$ *and every* $u \in \Sigma_{t_1}$ *is a solution of* (2.5) *with* $\rho \geq 2\varphi'(t_1)$.

PROOF. Clearly $\gamma(t) < \gamma(t_1)$ for $t < t_1$. For otherwise (2.9) will imply that $\varphi'(t_1) \leq 0$. By Lemma 2.3 we see that $\Sigma_{t_1} \neq \phi$. For $u \in S_t, t_0 \leq t \leq t_1$, let

$$I(u) = \varphi(\|u\|^2) - g(u).$$

Then if $u_1 \in \Sigma_{t_1}, u \in S_t$

$$\begin{aligned} I(u_1) &= \varphi(t_1) - g(u_1) = \varphi(t_1) - \gamma(t_1) \\ &\leq \varphi(t) - \gamma(t) \leq \varphi(t) - g(u) = I(u), \end{aligned}$$

Thus $I(u_1)$ is a minimum of $I(u)$ for u satisfying $t_0 \leq \|u\|^2 \leq t_1$. Consequently there is a $\beta \geq 0$ such that

$$I'(u_1) = -\beta u_1.$$

Thus

$$2\varphi'(t_1)u_1 - g'(u_1) = -\beta u_1$$

or

$$g'(u_1) = [2\varphi'(t_1) + \beta]u_1.$$

This gives the desired result. ∎

PROOF OF THEOREM 2.5. Let $\psi(t) = \frac{1}{2}\rho_0(t - t_0) + \gamma(t_0) - \gamma(t)$. Then $\psi(t_0) = 0$ and $\psi(t_1) \leq 0$. Assume first that $\psi(t) \geq \psi(t_1)$ for $t_0 \leq t \leq t_1$. Then by Lemma 2.6, $\Sigma_{t_1} \neq \emptyset$ and every $u \in \Sigma_{t_1}$ is a solution of (2.5) with $\rho \geq \rho_0$. This is alternative (a). Otherwise there is a t between t_0 and t_1 such that $\psi(t) < \psi(t_1)$. This means that ψ has a minimum in (t_0, t_1). We can now apply Lemma 2.4 to conclude that (2.5) has a solution satisfying $t_0 < \|u\|^2 < t_1$ and $\rho = \rho_0$. ∎

COROLLARY 2.7. *If* $\gamma(t_0) < \gamma(t_1)$, *then* (2.5) *has a solution satisfying*

$$t_0 < \|u\|^2 \leq t_1, \quad \rho \leq 2[\gamma(t_1) - \gamma(t_0)]/(t_1 - t_0). \tag{2.9}$$

PROOF. Let ρ_0 equal the right hand side in (2.9). ∎

THEOREM 2.8. *If* $0 \leq t_0 < t_1 < t_2$ *and*

$$[\gamma(t_2) - \gamma(t_0)]/(t_2 - t_0) < \rho/2 < [\gamma(t_1) - \gamma(t_0)]/(t_1 - t_0) \tag{2.10}$$

then (2.5) *has a solution satisfying* $t_0 < \|u\|^2 < t_2$.

PROOF. Let $\varphi(t) = \frac{1}{2}\rho(t - t_0) + \gamma(t_0)$ and $\psi(t) = \varphi(t) - \gamma(t)$. Then $\psi(t_1) < 0$ while $\psi(t_2) > 0$. Thus there is a point t_3 such that $t_1 < t_3 < t_2$ and $\psi(t_3) = 0$. Since $\psi(t_0) = 0$ and $\psi(t_1) < 0, \psi(t)$ must have a negative minimum in the interval (t_0, t_2). We can now apply Lemma 2.4 to conclude that (2.5) has a solution for $u \in \Sigma_t$ for some t satisfying $t_0 < t < t_2$. ∎

COROLLARY 2.9. *If* $0 < t_1 < t_2$ *and*

$$[\gamma(t_2) - \gamma(t_1)]/(t_2 - t_1) < \rho/2 < D^-\gamma(t_1), \tag{2.11}$$

then for each $\epsilon > 0$ *there is a solution* $u \in H$ *of* (2.5) *satisfying* $t_1 - \epsilon < \|u\|^2 < t_2$.

PROOF. By (2.11) there is a $t_0 < t_1$ such that $t_1 - t_0 < \epsilon$ and (2.10) holds. Apply Theorem 2.8. ∎

COROLLARY 2.10. *If* $0 \leq t_0 < t_1$ *and*

$$D^+\gamma(t_1) < \rho/2 < [\gamma(t_1) - \gamma(t_0)]/(t_1 - t_0) \tag{2.12}$$

then for every $\epsilon > 0$ *there is a solution* $u \in H$ *of* (2.5) *such that* $t_0 < \|u\|^2 < t_1 + \epsilon$.

PROOF. By (2.12), there is a $t_2 > t_1$ such that $t_2 - t_1 < \epsilon$ and (2.10) holds. Apply Theorem 2.8. ∎

COROLLARY 2.11. *If*

$$\alpha_\infty := \liminf_{t\to\infty} \gamma(t)/t < \alpha := \sup_{t_1,t_2}[\gamma(t_2) - \gamma(t_1)]/(t_2 - t_1) \tag{2.13}$$

then for any $\rho \in (2\alpha_\infty, 2\alpha)$ *there is a solution* $u \in H \setminus \{0\}$ *of* (2.5).

PROOF. By (2.13) one can pick t_0, t_1, t_2 such that (2.10) holds and apply Theorem 2.8. ∎

COROLLARY 2.12. *If* $\alpha \neq 0, \infty$, *then for every* $\epsilon > 0$ *there is a solution* (u, ρ) *of (2.5) such that* $u \neq 0$ *and* $\rho \geq \alpha - \epsilon$.

PROOF. Apply Corollary 2.7. ∎

COROLLARY 2.13. *If either* $D^-\gamma(t_1)$ *or* $D^+\gamma(t_1)$ *is positive, then there are sequences* $\{\rho_k\} \subset \mathbb{R}, \{u_k\} \subset H \setminus \{0\}$ *such that* $\rho_k > 0, \|u_k\|^2 \to t_1$ *and* (2.4) *holds.*

PROOF. If $D^-\gamma(t_1) > 0$, then for every $\epsilon > 0$ there is a $t_0 < t_1$ such that $t_1 - t_0 < \epsilon$ and $\gamma(t_0) < \gamma(t_1)$. By Corollary 2.2 there is an infinite number of solutions of (2.5), (2.6). A similar argument works if $D^+\gamma(t_1) > 0$. ∎

LEMMA 2.14. *If* $u_0 \in \Sigma_{t_0}$, *then*

$$D^-\gamma(t_0) \leq \big(g'(u_0), u_0\big)/2t_0 \leq D^+\gamma(t_0). \tag{2.14}$$

PROOF. We have, modulo $o(t - t_0)$,

$$\begin{aligned} \gamma(t) - \gamma(t_0) &\geq g(t^{1/2}t_0^{-1/2}u_0) - g(u_0) \\ &= \big(g'(u_0), u_0\big)(t^{1/2}t_0^{-1/2} - 1) \\ &= \big(g'(u_0), u_0\big)(t - t_0)/2t_0. \end{aligned} \tag{2.15}$$

If $t < t_0$, this gives the first inequality in (2.14); if $t > t_0$, it gives the second. ∎

COROLLARY 2.15. *If $D^-\gamma(t_0) > 0$, then $\Sigma_{t_0} \neq \emptyset$. Thus* (2.5) *has a solution* (u,ρ) *with* $u \in \Sigma_{t_0}$ *and* $D^-\gamma(t_0) \leq \rho/2$.

PROOF. If $\Sigma_{t_0} = \emptyset$, then $D^-\gamma(t_0) = 0$ by Lemma 2.3. If $u \in \Sigma_{t_0}$, then it is a solution of (2.5) for some $\rho \geq 0$.

COROLLARY 2.16. *If $D^+\gamma(t_0) > 0$, then the conclusion of Theorem 2.1 holds with*

$$\liminf \rho_k \geq D^+\gamma(t_0). \tag{2.16}$$

PROOF. For each k there is a point $t_k > t_0$ such that $t_k - t_0 < 1/k$ and

$$[\gamma(t_k) - \gamma(t_0)]/(t_k - t_0) \geq D^+\gamma(t_0) - (1/k).$$

By Corollary 2.7 there is a solution (u_k, ρ_k) of (2.4) satisfying

$$t_0 < \|u_k\|^2 \leq t_k, \quad \rho_k \geq D^+(t_0) - (1/k).$$

This gives the desired result. ∎

COROLLARY 2.17. *If $\gamma(t_0) < \gamma(t_1)$, then one of the following alternatives holds: either*

(a) *there is a solution (u,ρ) of* (2.5) *such that $u \in S_{t_1}$ and $\rho \geq \sigma_0 = 2[\gamma(t_1) - \gamma(t_0)]/(t_1 - t_0)$ or*

(b) *there is a sequence (u_k, ρ_k) of solutions of* (2.4) *such that $u_k \in S_{t_1}$ and $\rho_k \nearrow \sigma_0$, or*

(c) *there is a $\tilde{\rho} < \sigma_0$ such that for each ρ satisfying $\tilde{\rho} < \rho < \sigma_0$ there is a solution (u,ρ) of* (2.5) *with $t_0 < \|u\|^2 < t_1$.*

PROOF. Suppose (a) does not hold, and let $\tilde{\rho}$ be the supremum of all ρ such that (2.5) has a solution (u,ρ) with $u \in S_{t_1}$. If $\tilde{\rho} = \sigma_0$, then (b) must hold. If $\tilde{\rho} < \sigma_0$, then (c) must hold by Theorem 2.5. ∎

COROLLARY 2.18. *If $\gamma(t_0) < \gamma(t_1)$, then the following alternative holds: either*

(a) *there is a $\tilde{t} < t_1$ such that for every t satisfying $\tilde{t} < t < t_1$ there is a $u \in S_t$ and a $\rho > 0$ such that* (2.5) *holds, or*

(b) *there is a $\rho_0 > 0$ such that for each ρ satisfying $0 < \rho < \rho_0$ there is a u such that $t_0 < \|u\|^2 < t_1$ and* (2.5) *holds.*

PROOF. Let $\tilde{t}$ be the infimum of all t in $[t_0, t_1]$ such that $\gamma(t_0) < \gamma(t)$. If for each $t > \tilde{t}$ there is a $u \in S_t$ and a $\rho > 0$ such that (2.5) holds, then option (a) is true. Otherwise there is a $\hat{t}$ in $(\tilde{t}, t_1)$ such that (2.5) has no solution in $S_{\hat{t}}$ for any $\rho > 0$. In this case Theorem 2.5 tells us that for any ρ satisfying $0 < \rho < \rho_0 := 2[\gamma(\hat{t} - \gamma(t_0)]/(\hat{t} - t_0)$ there is a solution u of (2.5) satisfying $t_0 < \|u\|^2 < \hat{t}$. ∎

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