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Global attraction for systems with almost C^1 vector fields

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Sufficient conditions are given for an autonomous differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \quad x \in D,$$

to have a single point global attractor (repeller) with f continuously differentiable almost everywhere. These results incorporate those of Hartman and Olech as a special case even when the condition $f \in C^1(D, \mathbb{R}^N)$ is fully met. Moreover, these criteria are simplified for a class of region-wise linear systems in $\mathbb{R}^N$.

1. Introduction

Region-wise linear systems have been used to model real-world phenomena. For example, Chua and Yang [2,3], Li and Dayan [10] and Li, Huang and Zhu [12] used region-wise linear differential systems to model neural networks. The interest in this type of systems is due to its applicability not only to neural networks but also to electrical circuits (see, for example, [5,9,13]). Besides, it can also be applied to the construction of a variety of mechanical models by connecting different masses with springs and stops. These show the practical need to investigate such systems. It is well known that linear autonomous systems (LASs) have the following two features: no limit cycles exist (in any two-dimensional plane) and the global dynamics is completely determined by local behaviour near a critical point. For region-wise LASs, however, neither of the two features survives. This is demonstrated by the system

$$x' = A(x_1)x + b, \quad x \in \mathbb{R}^2, \tag{1.1}$$

where $b = (1, 2)^{\mathrm{T}}$, the transpose of $(1, 2)$, and

$$A(x_1) = \begin{cases} \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}, & x_1 \geqslant 0, \\ \begin{pmatrix} 8 & 2 \\ -8 & -1 \end{pmatrix}, & x_1 < 0. \end{cases} \tag{1.2}$$

System (1.1) has a unique critical point $(1, 0)$, which is a stable focus but not globally asymptotically stable. Indeed, there is a $\delta \in (0, 1)$ such that the trajectory of (1.1) passing through the point $(0, -\delta)$ is a cycle and it is a global repeller in $\mathbb{R}^2 \setminus \{(1, 0)\}$. This example shows that region-wise LASs are globally nonlinear. Note that $A(x_1)x$ is continuous but not differentiable on the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$ and $\mathcal{M}(S)$, the Lebesgue measure of S, is zero. Thus, the right-hand side of (1.1) is C^1 almost everywhere in $\mathbb{R}^2$.

Recall that most available results for a nonlinear autonomous system

$$x' = f(x), \quad x \in D, \tag{1.3}$$

where $D \subset \mathbb{R}^N$ is a simply connected open set, are based on the assumption that

$$f \in C^1(D, \mathbb{R}^N). \tag{1.4}$$

Since region-wise LASs do not satisfy (1.4), none of the results for (1.3) with (1.4) can be applied to region-wise LASs directly without further clarification. Therefore, there is also a theoretical need to investigate the global behaviour of region-wise LASs in particular, and nonlinear systems with non-smooth vector fields in general. In this paper, however, we restrict our interest only to systems with continuous almost C^1 vector fields although the results obtained here might further be extended to some systems with discontinuous vector fields.

Under condition (1.4), the matrix

$$H(x) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial x}\right)^{\mathrm{T}}\right)$$

(where $\partial f/\partial x$ is the Jacobian matrix of f and $(\partial f/\partial x)^{\mathrm{T}}$ is the transpose of $\partial f/\partial x$) is continuous, and so are the eigenvalues of H arranged as

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N, \tag{1.5}$$

on D. Among many of the results obtained for (1.3) in terms of the λ_i, we highlight the following, which are relevant to this paper.

(I) If $\lambda_1 + \lambda_2 < 0$ on $D = \mathbb{R}^N$, then each bounded semi-orbit of (1.3) converges to a critical point [15].

(II) If $\lambda_1 + \lambda_2 < 0$ for all $x \in D = \mathbb{R}^N$ or $\lambda_{N-1} + \lambda_N > 0$ for all $x \in \mathbb{R}^N$, then there is no simple closed rectifiable curve that is invariant with respect to (1.3). In particular, (1.3) has no non-constant periodic solution [11, 14, 15].

(III) Assume that the origin is the only critical point, it is locally asymptotically stable, $\lambda_1 + \lambda_2 \leqslant 0$ in $\mathbb{R}^N$ and $\int^{\infty} p(u)\,\mathrm{d}u$ diverges, where $p(u) = \min_{|x|=u} |f(x)|$. Then the origin is globally asymptotically stable [6].

(IV) If $f(0) = 0$ and $\lambda_1 < 0$, then the origin is globally asymptotically stable (see [6] and the references therein).

Now suppose the existence of a closed set $D_0 \subset D$ with $\mathcal{M}(D_0) = 0$ such that

$$f \in C(D, \mathbb{R}^N) \quad \text{and} \quad f \in C^1(D \setminus D_0, \mathbb{R}^N) \tag{1.6}$$

instead of (1.4). Our obvious questions are whether (I)–(IV) are still valid for some, if not all, systems (1.3) with (1.6) and whether they can be extended further.

The answers for (II) are yes. It is known that (II) is an extension of Bendixson's criterion [1] from systems in $D = \mathbb{R}^2$ to systems (1.3) with (1.4) in $\mathbb{R}^N$. Smith [15] first gave the extension under the condition $\lambda_1 + \lambda_2 < 0$ in $\mathbb{R}^N$. Muldowney and Li [11, 14] gave more general conditions,

$$\mu\biggl(\biggl(\frac{\partial f}{\partial x}\biggr)^{[2]}\biggr) < 0 \quad \text{or} \quad \mu\biggl(-\biggl(\frac{\partial f}{\partial x}\biggr)^{[2]}\biggr) < 0, \tag{1.7}$$

where $(\partial f/\partial x)^{[2]}$ is the second additive compound $\binom{N}{2} \times \binom{N}{2}$ matrix of $\partial f/\partial x$ and μ is the Lozinskiĭ norm (to be defined later). When $|y| = (y^{\mathrm{T}}y)^{1/2}$ in $\mathbb{R}^N$, (1.7) becomes $\lambda_1 + \lambda_2 < 0$ or $\lambda_N + \lambda_{N-1} > 0$, incorporating Smith's extension as a particular instance. Hou [7] further extended (II) to some systems (1.3) with (1.6), including a class of region-wise LASs, under the conditions

$$\sup_{x\in B\setminus D_0} \mu\biggl(\biggl(\frac{\partial f}{\partial x}\biggr)^{[2]}\biggr) < 0 \quad \text{or} \quad \sup_{x\in B\setminus D_0} \mu\biggl(-\biggl(\frac{\partial f}{\partial x}\biggr)^{[2]}\biggr) < 0 \tag{1.8}$$

for every bounded set $B \subset D$.

The answers to the two questions above for (I) given by Smith [15] will be dealt with separately. In this paper, we are mainly concerned with the extension of (III) and (IV) to (1.3) with (1.6). Sufficient conditions in terms of

$$\mu\biggl(\frac{\partial f}{\partial x}\biggr) \quad \text{and} \quad \mu\biggl(\biggl(\frac{\partial f}{\partial x}\biggr)^{[2]}\biggr)$$

will be derived for (1.3) to have a single point global attractor (repeller). The main results for (1.3) with (1.6) will be given in § 2 and their proofs will be left to § 3. Finally, a class of region-wise LASs will be dealt with in § 4.

2. Global attraction for general systems

In this section, we are concerned with global attraction of a critical point $x^0 \in D$ of a system

$$x' = f(x), \quad x \in D, \tag{2.1}$$

where $D \subset \mathbb{R}^N$ is a simply connected open set and $f \in C(D, \mathbb{R}^N)$. By a *solution* of a differential equation, we mean an absolutely continuous function on an interval that satisfies the equation almost everywhere in the interval.

Let $I_k = \{1, 2, \dots, k\}$ for any integer $k > 0$ and let $|\cdot|$ be any norm in $\mathbb{R}^N$ with the property

$$|\bar{y}| = |\check{y}| \leqslant |y| \quad \text{for all } y \in \mathbb{R}^N, \tag{2.2}$$

where $\check{y} \in \mathbb{R}^N$ with $\check{y}_i = y_i$ or $\check{y}_i = 0$ for each $i \in I_N$ and $\bar{y} \in \mathbb{R}^m$ is composed of all the non-zero components of $\check{y}$. For an $N \times N$ matrix A, $|A|$ is derived from the norm $|\cdot|$ in $\mathbb{R}^N$ and $\mu(A)$ (the *Lozinskiĭ logarithmic norm* of A) is defined (see [4, pp. 41, 58] or [14] for this concept and lemma 2.4 given below) to be

$$\mu(A) = D_+|I + hA|_{h=0}, \tag{2.3}$$

where D_+ denotes the right-hand derivative.

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THEOREM 2.1. *Assume that the following hold.*

(a) $f \in \mathrm{Lip}(B, \mathbb{R}^N)$ *(i.e. f on B is Lipschizian) for every bounded set $B \subset D$ and there is a closed set $D_0 \subset D$ satisfying $\mathcal{M}(D_0) = 0$ and $f \in C^1(D \setminus D_0, \mathbb{R}^N)$.*

(b) *For every $x^0 \in D_0$, either x^0 is an isolated critical point or there is a $\delta > 0$ such that $x(t, x^0) \notin D_0$ for $0 < |t| < \delta$.*

(c) *No solution of (2.1) will approach ∂D (the boundary of D) in a positive (or negative) finite time.*

(d) *In D (2.1) has a bounded solution $x(t, x^1)$ for $t \geqslant 0$ (or $t \leqslant 0$).*

Then (2.1) has a single point global attractor (or repeller) if (2.4) (or (2.5)) holds for every bounded set $B \subset D$:

$$\rho_B = \sup_{x \in B \setminus D_0} \mu\left(\frac{\partial f}{\partial x}(x)\right) < 0, \tag{2.4}$$

$$\gamma_B = \sup_{x \in B \setminus D_0} \mu\left(-\frac{\partial f}{\partial x}(x)\right) < 0. \tag{2.5}$$

REMARK 2.2. Condition (a) guarantees the existence and uniqueness of a solution and the continuity of $\partial f/\partial x$ almost everywhere on D. Condition (b) implies that '$\partial f/\partial x(x(t, x^0))$ for any fixed solution $x(t, x^0)$ on an interval is continuous almost everywhere and locally integrable'. Indeed, the part in quotation marks is what we need in the proof, so it may replace (b) in theorem 2.1. Condition (c) is obviously met when $D = \mathbb{R}^N$ as $\partial D = \emptyset$. This remark applies to all the results given in this section. In § 4 we shall see that most of these conditions will become redundant for a class of region-wise LASs.

REMARK 2.3. The existence of a critical point is not a condition but a part of the conclusion of theorem 2.1. However, if we know that $x^0 \in D$ is a critical point, then condition (d) is fulfilled and the conclusion says that every solution has the limit x^0 as $t \to \infty$ (or $t \to -\infty$).

In some cases of $|\cdot|$, $\mu(A)$ has explicit expressions in terms of the entries or eigenvalues of $\frac{1}{2}(A + A^{\mathrm{T}})$.

LEMMA 2.4. *Corresponding to the three definitions of $|v|$ given by*

$$\sup_i |v_i|, \quad \sum_i |v_i| \quad \text{and} \quad \left(\sum_i v_i^2\right)^{1/2},$$

$\mu(A)$ is equal to

$$\sup_i \left(\mathrm{Re}\, a_{ii} + \sum_{j \neq i} |a_{ij}|\right), \quad \sup_i \left(\mathrm{Re}\, a_{ii} + \sum_{j \neq i} |a_{ji}|\right) \quad \text{and} \quad \lambda_1,$$

respectively, where the λ_i are the eigenvalues of $\frac{1}{2}(A + A^{\mathrm{T}})$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$.

According to lemma 2.4 we can replace (2.4) and (2.5) by inequalities involving the entries or eigenvalues of $H = \frac{1}{2}(\partial f/\partial x + (\partial f/\partial x)^{\mathrm{T}})$ and obtain the following.

COROLLARY 2.5. *Under conditions (a)–(d) of theorem 2.1, (2.1) has a single point global attractor (or repeller) if any one of (2.6)–(2.8) (or (2.9)–(2.11)) holds for every bounded set $B \subset D$,*

$$\sup_{x\in B\setminus D_0}\left\{\max_{1\leqslant i\leqslant N}\frac{\partial f_i}{\partial x_i}+\sum_{j\neq i}\left|\frac{\partial f_i}{\partial x_j}\right|\right\}<0, \tag{2.6}$$

$$\sup_{x\in B\setminus D_0}\left\{\max_{1\leqslant i\leqslant N}\frac{\partial f_i}{\partial x_i}+\sum_{j\neq i}\left|\frac{\partial f_j}{\partial x_i}\right|\right\}<0, \tag{2.7}$$

$$\sup_{x\in B\setminus D_0}\lambda_1<0, \tag{2.8}$$

$$\inf_{x\in B\setminus D_0}\left\{\min_{1\leqslant i\leqslant N}\frac{\partial f_i}{\partial x_i}-\sum_{j\neq i}\left|\frac{\partial f_i}{\partial x_j}\right|\right\}>0, \tag{2.9}$$

$$\inf_{x\in B\setminus D_0}\left\{\min_{1\leqslant i\leqslant N}\frac{\partial f_i}{\partial x_i}-\sum_{j\neq i}\left|\frac{\partial f_j}{\partial x_i}\right|\right\}>0, \tag{2.10}$$

$$\inf_{x\in B\setminus D_0}\lambda_N>0, \tag{2.11}$$

where the λ_i are the eigenvalues of H satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$.

REMARK 2.6. When $D_0 = \emptyset$ and $D = \mathbb{R}^N$, condition (a) becomes $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, (b) and (c) vanish by remark 2.2, and $\sup_{x\in B\setminus D_0}$ and $\inf_{x\in B\setminus D_0}$ in (2.6)–(2.11) are redundant. Therefore, Olech's result (IV) (see [6] and the references therein) given in § 1 is only a particular case of theorem 2.1.

For an $N \times N$ matrix $A = (a_{ij})$, the *second additive compound* $A^{[2]}$ is a matrix of $\binom{N}{2} \times \binom{N}{2}$, defined as follows. For any integer $i \in \{1, 2, \dots, \binom{N}{2}\}$, let $(i) = (i_1, i_2)$ be the ith member in the lexicographic ordering of the set $\{(i_1, i_2) : 1 \leqslant i_1 < i_2 \leqslant N\}$. The entries of $A^{[2]} = (\tilde{a}_{ij})$ are given by

$$\tilde{a}_{ij}=\begin{cases} a_{i_1i_1}+a_{i_2i_2} & \text{if } (i)=(j),\\ (-1)^{r+s}a_{i_rj_s} & \text{if } i_r\notin\{j_1,j_2\},\ j_s\notin\{i_1,i_2\}\\ & \text{but } \{i_1,i_2\}\setminus\{i_r\}=\{j_1,j_2\}\setminus\{j_s\},\\ 0 & \text{otherwise.}\end{cases}$$

For instance, when $N = 3$,

$$A^{[2]}=\begin{pmatrix} a_{11}+a_{22} & a_{23} & -a_{13}\\ a_{32} & a_{11}+a_{33} & a_{12}\\ -a_{31} & a_{21} & a_{22}+a_{33}\end{pmatrix}\quad \text{if } A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}.$$

For any $v^1, v^2 \in \mathbb{R}^N$, their *external product* $v^1 \wedge v^2$ is a vector in $\mathbb{R}^{\binom{N}{2}}$ defined by

$$(v^1\wedge v^2)_i=\begin{vmatrix} v^1_{i_1} & v^2_{i_1}\\ v^1_{i_2} & v^2_{i_2}\end{vmatrix}$$

for each $i \in \{1, 2, \dots, \binom{N}{2}\}$ with $(i) = (i_1, i_2)$.

THEOREM 2.7. *In addition to conditions (a)–(c) of theorem 2.1, we also assume that the following hold.*

(i) *For any bounded half orbit $\Gamma_+(x_0)$ (or $\Gamma_-(x_0)$), we have $\overline{\Gamma_+(x_0)} \subset D$ (or $\overline{\Gamma_-(x_0)} \subset D$), where the overbar denotes closure.*

(ii) *$x^0 \in D$ is the unique critical point and it is a local attractor (or repeller).*

(iii) *With $p(s) = \min_{|x|=s} |f(x)|$, $\int^{\infty} p(s)\,\mathrm{d}s$ diverges if D is unbounded.*

Then, for any $|\cdot|$ equivalent to the Euclidian norm $\|\cdot\|$, x^0 is a global attractor (or repeller) if (2.12) (or (2.13)) holds for $x \in D \setminus D_0$:

$$\mu\biggl(\frac{\partial f}{\partial x}^{[2]}(x)\biggr) \leqslant 0, \tag{2.12}$$

$$\mu\biggl(-\frac{\partial f}{\partial x}^{[2]}(x)\biggr) \leqslant 0. \tag{2.13}$$

The corollary below is an immediate consequence of theorem 2.7, lemma 2.4 and the definition of $(\partial f/\partial x)^{[2]}$.

COROLLARY 2.8. *Under the assumptions of theorem 2.7, x^0 is a global attractor (or repeller) if any one of (2.14)–(2.16) (or (2.17)–(2.19)) holds for $x \in D \setminus D_0$:*

$$\max_{1\leqslant r<s\leqslant N} \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\biggl(\biggl|\frac{\partial f_r}{\partial x_q}\biggr| + \biggl|\frac{\partial f_s}{\partial x_q}\biggr|\biggr) \leqslant 0, \tag{2.14}$$

$$\max_{1\leqslant r<s\leqslant N} \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\biggl(\biggl|\frac{\partial f_q}{\partial x_r}\biggr| + \biggl|\frac{\partial f_q}{\partial x_s}\biggr|\biggr) \leqslant 0, \tag{2.15}$$

$$\lambda_1 + \lambda_2 \leqslant 0, \tag{2.16}$$

$$\min_{1\leqslant r<s\leqslant N} \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s}\biggl(\biggl|\frac{\partial f_r}{\partial x_q}\biggr| + \biggl|\frac{\partial f_s}{\partial x_q}\biggr|\biggr) \geqslant 0, \tag{2.17}$$

$$\min_{1\leqslant r<s\leqslant N} \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q\neq r,s}\biggl(\biggl|\frac{\partial f_q}{\partial x_r}\biggr| + \biggl|\frac{\partial f_q}{\partial x_s}\biggr|\biggr) \geqslant 0, \tag{2.18}$$

$$\lambda_{N-1} + \lambda_N \geqslant 0, \tag{2.19}$$

where the λ_i are the eigenvalues of H satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$.

REMARK 2.9. When $D_0 = \emptyset$ and $D = \mathbb{R}^N$, conditions (a)–(c) in theorem 2.1 become $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and (i) is automatically met. If (2.16) holds, corollary 2.8 coincides with Hartman and Olech's result (III) [6] given in § 1.

EXAMPLE 2.10. Viewing the system

$$\left.\begin{aligned} x_1' &= x_1^3 + \tfrac{1}{4}|x_2 - x_3| + \tfrac{53}{8128},\\ x_2' &= \tfrac{1}{4}|x_1| + 2x_2 - \tfrac{1}{4}x_3,\\ x_3' &= \tfrac{1}{8}x_1 - \tfrac{1}{4}x_2 + 4x_3 \end{aligned}\right\} \tag{2.20}$$

as (2.1), we have $D = \mathbb{R}^3$, $D_0 = \{x \in \mathbb{R}^3 : x_1 = 0 \text{ or } x_2 = x_3\}$, $f \in C^1(\mathbb{R}^3 \setminus D_0, \mathbb{R}^3)$ and $f \in \mathrm{Lip}(B, \mathbb{R}^3)$ for every bounded set $B \subset \mathbb{R}^3$. There is only one critical point $x^0 = (-\frac{1}{4}, -\frac{31}{1016}, \frac{3}{508})$ and the linearized system at x^0 has the characteristic equation

$$\lambda^3 - \tfrac{99}{16}\lambda^2 + \tfrac{287}{32}\lambda - \tfrac{307}{256} = 0.$$

As $\frac{99}{16} \times \frac{287}{32} > \frac{307}{256}$, by the Routh–Hurwitz criterion, every eigenvalue has a positive real part, so x^0 is a local repeller. Since the conditions (a)–(c) of theorem 2.1, (i)–(iii) and (2.18), which becomes $2 - \frac{3}{8} > 0$, are satisfied, x^0 is a global repeller.

3. The proofs of theorems 2.1 and 2.7

In the proofs of the main results, an estimate of solutions of a linear system and the existence of $\partial x(t, x_0)/\partial x_0$ will be needed.

LEMMA 3.1 (see [4]). *Every solution of $x'(t) = A(t)x(t)$ satisfies*

$$|x(t_0)| \exp\biggl(-\int_{t_0}^{t} \mu(-A(s))\,\mathrm{d}s\biggr) \leqslant |x(t)| \leqslant |x(t_0)| \exp\biggl(\int_{t_0}^{t} \mu(A(s))\,\mathrm{d}s\biggr) \tag{3.1}$$

for $t \geqslant t_0$.

LEMMA 3.2 (see [7, lemma 2.9]). *Under conditions (a) and (b) of theorem 2.1, $\partial x(t, x_0)/\partial x_0$ is continuous in (t, x_0) and is a fundamental matrix solution of the variational equation*

$$y'(t) = \frac{\partial f}{\partial x}(x(t, x_0))y(t) \tag{3.2}$$

if x_0 is not a critical point of (2.1) in D_0.

Proof of theorem 2.1. We first assume that (2.4) holds and a solution $x(t, x^1)$ of (2.1) is bounded for $t \geqslant 0$.

(α) If $x(t, x^1) \equiv x^1$, i.e. x^1 is a critical point, then we need only show that

$$\lim_{t\to\infty} x(t, x^2) = x^1 \tag{3.3}$$

for every $x^2 \in D \setminus \{x^1\}$. Since D is connected, there is a one-to-one mapping $\psi \in \mathrm{Lip}([0, 1], D)$ such that $\psi(0) = x^1$ and $\psi(1) = x^2$. By (b), there are at most a finite number of $s \in [0, 1]$ such that $\psi(s) \in D_0$ is a critical point. Then, for any $t_0 > 0$, as long as $x(t_0, \psi(s))$ exists for all $s \in [0, 1]$, lemma 3.2 ensures that $\psi_t(\cdot) = x(t, \psi(\cdot)) \in \mathrm{Lip}([0, 1], D)$ and

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi_t(s) = \frac{\partial}{\partial x_0}x(t, \psi(s))\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \tag{3.4}$$

for each $t \in [0, t_0]$ and almost every $s \in [0, 1]$. Moreover, when (3.4) holds, $\mathrm{d}\psi_t(s)/\mathrm{d}s$ in t is a solution of (3.2) with x_0 replaced by $\psi(s)$. It then follows from lemma 3.1 that

$$\biggl|\frac{\mathrm{d}}{\mathrm{d}s}\psi_t(s)\biggr| \leqslant \biggl|\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}\biggr| \exp\biggl(\int_0^t \mu\biggl(\frac{\partial f}{\partial x}(x(\ell, \psi(s)))\biggr)\,\mathrm{d}\ell\biggr) \tag{3.5}$$

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for $t \in [0, t_0]$. Let ℓ_0 be the length of the curve given by $x = \psi(s)$ for $s \in [0, 1]$, i.e.

$$\ell_0 = \int_0^1 \left| \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \right| \mathrm{d}s,$$

and let $B = \{x \in D : |x - x^1| \leqslant \ell_0 + 1\}$. Now that $|\psi(s) - x^1| \leqslant \ell_0 < \ell_0 + 1$ for all $s \in [0, 1]$, we claim that

$$|\psi_t(s) - x^1| < \ell_0 + 1, \tag{3.6}$$

so that $\psi_t(s) \in B$, for all $s \in [0, 1]$ and $t \in [0, t_0]$. Indeed, if this is not true, by continuous dependence upon initial values there are $s_1 \in (0, 1]$ and $t_1 \in (0, t_0]$ such that (3.6) holds for $(s, t) \in [0, s_1] \times [0, t_1]$ with $(s, t) \neq (s_1, t_1)$ but $|\psi_{t_1}(s_1) - x^1| = \ell_0 + 1$. On the other hand, however, (2.4) and (3.5) yield

$$\begin{aligned} |\psi_{t_1}(s_1) - x^1| &\leqslant \int_0^{s_1} \left| \frac{\mathrm{d}}{\mathrm{d}s} \psi_{t_1}(s) \right| \mathrm{d}s \\ &\leqslant \int_0^{s_1} \left| \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \right| \mathrm{e}^{\rho_B t_1} \, \mathrm{d}s \\ &< \ell_0. \end{aligned}$$

The contradiction shows our claim. By (c), $x(t, \psi(s))$ exists and satisfies (3.6) for all $s \in [0, 1]$ and $t \geqslant 0$. Hence,

$$|x(t, x^2) - x^1| \leqslant \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}s} \psi_t(s) \right| \mathrm{d}s \leqslant \ell_0 \mathrm{e}^{\rho_B t} \to 0$$

as $t \to \infty$.

(β) Suppose that x^1 is not a critical point. By condition (d), there is a bounded set $B \subset D$ such that $x(t, x^1) \in B$ for all $t \geqslant 0$. Then, for $t \geqslant 0$ and $s \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s} x(t + s, x^1) = \frac{\mathrm{d}}{\mathrm{d}s} x(t, x(s, x^1)) = \frac{\partial}{\partial x_0} x(t, x(s, x^1)) f(x(s, x^1)).$$

By lemma 3.2, $\mathrm{d}x(t+s, x^1)/\mathrm{d}s$ in t is a solution of (3.2) with x_0 replaced by $x(s, x^1)$. Hence, by lemma 3.1 and (2.4),

$$\left| \frac{\mathrm{d}}{\mathrm{d}s} x(t + s, x^1) \right| \leqslant \mathrm{e}^{\rho_B t} |f(x(s, x^1))|.$$

From this we obtain

$$\begin{aligned} \int_0^\infty \left| \frac{\mathrm{d}x(t, x^1)}{\mathrm{d}t} \right| \mathrm{d}t &= \sum_{k=0}^\infty \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}s} x(k, x(s, x^1)) \right| \mathrm{d}s \\ &\leqslant \sum_{k=0}^\infty \mathrm{e}^{\rho_B k} \int_0^1 |f(x(s, x^1))| \, \mathrm{d}s < \infty. \end{aligned}$$

As

$$|x(t_2, x^1) - x(t_1, x^1)| \leqslant \int_{t_1}^{t_2} \left| \frac{\mathrm{d}x(s, x^1)}{\mathrm{d}s} \right| \mathrm{d}t$$

for any $t_2 > t_1 \geqslant 0$, by the Cauchy convergence principle there is an $x^0 \in D$ such that $\lim_{t \to \infty} x(t, x^1) = x^0$. From ($\alpha$) we know that x^0 is a global attractor.

If (2.5) holds for every bounded set $B \subset D$ and $x(\ell, x^1)$ is bounded for $\ell \leqslant 0$, by letting $\ell = -t$ the above reasoning shows the existence of a global repeller. □

LEMMA 3.3. *For any $y^1, y^2 \in \mathbb{R}^N$ and $A \in \mathbb{R}^{N\times N}$, we have*

$$(Ay^1) \wedge y^2 + y^1 \wedge (Ay^2) = A^{[2]}(y^1 \wedge y^2). \tag{3.7}$$

Proof. Let $(i) = (i_1, i_2)$ and $(j) = (j_1, j_2)$ with $1 \leqslant i_1 < i_2 \leqslant N$ and $1 \leqslant j_1 < j_2 \leqslant N$. Then, by the definitions of $\wedge$ and $A^{[2]}$, we have

$$\begin{aligned}
(A^{[2]}(y^1 \wedge y^2))_i &= \sum_{j=1}^{N(N-1)/2} a_{ij}^{[2]} \begin{vmatrix} y_{j_1}^1 & y_{j_1}^2 \\ y_{j_2}^1 & y_{j_2}^2 \end{vmatrix} \\
&= -\sum_{1\leqslant j_1<i_1} a_{i_2 j_1} \begin{vmatrix} y_{j_1}^1 & y_{j_1}^2 \\ y_{i_1}^1 & y_{i_1}^2 \end{vmatrix} + \sum_{i_1<j_2\leqslant N, j_2\neq i_2} a_{i_2 j_2} \begin{vmatrix} y_{i_1}^1 & y_{i_1}^2 \\ y_{j_2}^1 & y_{j_2}^2 \end{vmatrix} \\
&\quad + (a_{i_1 i_1} + a_{i_2 i_2}) \begin{vmatrix} y_{i_1}^1 & y_{i_1}^2 \\ y_{i_2}^1 & y_{i_2}^2 \end{vmatrix} + \sum_{1\leqslant j_1<i_2, j_1\neq i_1} a_{i_1 j_1} \begin{vmatrix} y_{j_1}^1 & y_{j_1}^2 \\ y_{i_2}^1 & y_{i_2}^2 \end{vmatrix} \\
&\quad - \sum_{i_2<j_2\leqslant N} a_{i_1 j_2} \begin{vmatrix} y_{i_2}^1 & y_{i_2}^2 \\ y_{j_2}^1 & y_{j_2}^2 \end{vmatrix} \\
&= \sum_{k=1}^{N} y_k^1 (a_{i_1 k} y_{i_2}^2 - a_{i_2 k} y_{i_1}^2) + \sum_{k=1}^{N} y_k^2 (a_{i_2 k} y_{i_1}^1 - a_{i_1 k} y_{i_2}^1) \\
&= ((Ay^1) \wedge y^2 + y^1 \wedge (Ay^2))_i.
\end{aligned}$$

Thus (3.7) holds. □

LEMMA 3.4. *For any $u, v \in \mathbb{R}^N$ with the Euclidean norm $\|\cdot\|$,*

$$\|u \wedge v\|^2 = \|u\|^2\|v\|^2 - (u^{\mathrm{T}}v)^2. \tag{3.8}$$

In particular, $\|u \wedge v\| = \|u\|\|v\|$ if u and v are perpendicular.

Proof. By the definition of $\wedge$, we have

$$\begin{aligned}
\|u \wedge v\|^2 &= \sum_{1\leqslant i<j\leqslant N} \begin{vmatrix} u_i & v_i \\ u_j & v_j \end{vmatrix}^2 = \sum_{1\leqslant i<j\leqslant N} (u_i^2 v_j^2 + u_j^2 v_i^2 - 2u_i v_i u_j v_j) \\
&= \sum_{i,j=1}^{N} u_i^2 v_j^2 - \sum_{i=1}^{N} u_i^2 v_i^2 - 2 \sum_{1\leqslant i<j\leqslant N} u_i v_i u_j v_j \\
&= \biggl(\sum_{i=1}^{N} u_i^2\biggr)\biggl(\sum_{j=1}^{N} v_j^2\biggr) - \biggl(\sum_{i=1}^{N} u_i v_i\biggr)^2 = \|u\|^2\|v\|^2 - (u^{\mathrm{T}}v)^2.
\end{aligned}$$

From this it follows that $\|u \wedge v\| = \|u\|\|v\|$ if $u^{\mathrm{T}}v = 0$. □

LEMMA 3.5. *For any $\varphi \in C^1([0,1], \mathbb{R}^N)$ and $g \in C(\mathbb{R}^N, \mathbb{R}^N)$, the inequality*

$$\int_0^1 |g(\varphi(t))| \left|\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right| \mathrm{d}t \geqslant \int_m^M \min_{|x|=s} |g(x)| \,\mathrm{d}s \tag{3.9}$$

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holds in any norm $|\cdot|$, *where*

$$m = \min_{0 \leqslant t \leqslant 1} |\varphi(t)| \quad \textit{and} \quad M = \max_{0 \leqslant t \leqslant 1} |\varphi(t)|.$$

Proof. On every subinterval $[s_1, s_2] \subset [m, M]$, there is at least one subinterval $I = [t_1, t_2] \subset [0, 1]$ such that

$$s_1 = \min_{t \in I} |\varphi(t)| = \min\{|\varphi(t_1)|, |\varphi(t_2)|\},$$

$$s_2 = \max_{t \in I} |\varphi(t)| = \max\{|\varphi(t_1)|, |\varphi(t_2)|\}.$$

Then

$$\int_{t_1}^{t_2} |g(\varphi(t))| \left|\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right| \mathrm{d}t \geqslant \min_{t \in I} |g(\varphi(t))| \int_{t_1}^{t_2} \left|\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right| \mathrm{d}t \geqslant \min_{s_1 \leqslant |x| \leqslant s_2} |g(x)| ||\varphi(t_2)| - |\varphi(t_1)||$$
$$= \min_{|x| = s_0} |g(x)|(s_2 - s_1)$$

for some $s_0 \in [s_1, s_2]$. Since g and $\mathrm{d}\varphi/\mathrm{d}t$ are continuous, (3.9) follows from the above inequality and the definition of a definite integral. □

In the proof of theorem 2.7, we shall use Hartman and Olech's idea of considering in $\mathbb{R}^N$ a two-dimensional surface composed of trajectories of (2.1) and curves on this surface perpendicular to those trajectories of (2.1). For a fixed solution $x(t, x_0) \in D$ on $[0, \omega)$ and any unit vector $u \in \mathbb{R}^N$ perpendicular to the trajectory of (2.1) at x_0, i.e. $u^{\mathrm{T}} f(x_0) = 0$, we consider the solution $x(t, x_0 + ru)$ for $r \geqslant 0$ and $t \geqslant 0$. By continuous dependence on initial values, the trajectories of these solutions for $r \in [0, r_0]$, $r_0 > 0$ small enough, and $t \geqslant 0$ form a two-dimensional surface S. For each $s \in [0, \omega)$, we look for a curve $\gamma(s)$ on S given by

$$y(r) = x(T, x_0 + ru) \quad \text{for } r \in [0, r_0], \tag{3.10}$$

$$y(0) = x(s, x_0), \tag{3.11}$$

such that γ is perpendicular to each trajectory in S, i.e.

$$f(x(T, x_0 + ru))^{\mathrm{T}} \frac{\mathrm{d}y(r)}{\mathrm{d}r} = 0. \tag{3.12}$$

Obviously, T must be a function of (r, s). From (3.10) we find that

$$\frac{\mathrm{d}y(r)}{\mathrm{d}r} = f(x(T, x_0 + ru)) \frac{\mathrm{d}T}{\mathrm{d}r} + \frac{\partial x}{\partial \tilde{x}_0}(T, x_0 + ru)u, \tag{3.13}$$

where $\tilde{x}_0$ denotes the initial value of x. From (3.12) and (3.13) we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{f(x(T, x_0 + ru))^{\mathrm{T}} (\partial x / \partial \tilde{x}_0)(T, x_0 + ru)u}{f(x(T, x_0 + ru))^{\mathrm{T}} f(x(T, x_0 + ru))}, \tag{3.14}$$

provided that $x_0 + ru$ is not a critical point. Then T is a solution of (3.14) with $T(0, s) = s$.

For any set S_0, a point p_0 and a number $\varepsilon > 0$, we denote the open ball with centre p_0 and a radius ε by $B(p_0, \varepsilon)$, the union of balls with a radius ε and centres in S_0 by $B(S_0, \varepsilon)$, and the distance between p_0 and S_0 by $\mathrm{dist}(p_0, S_0)$.

Proof of theorem 2.7. Assume that (2.12) holds and x^0 is a local attractor. Since the domain A of attraction of x^0 is open, there is a value $d > 0$ such that $B(x^0, d) \subset A$. Now suppose that x^0 is not globally attractive. Then ∂A is a closed non-empty set. We can always take an $x_0 \in \partial A$ satisfying $|x_0 - x^0| = \text{dist}(x^0, \partial A)$ and find a unit vector $u \in \mathbb{R}^N$ and a small $r_0 > 0$ such that $u^{\mathrm{T}} f(x_0) = 0$ and $x_0 + ru \in A$ for $r \in (0, r_0]$.

(α) If $x(t, x_0)$ is bounded, by conditions (c) and (i), $x(t, x_0)$ exists on $[0, \infty)$ and $\overline{B(\Gamma_+(x_0), \varepsilon)} \subset D \backslash \overline{B(x^0, d/2)}$ for some $\varepsilon > 0$. By condition (ii), there is a $\delta > 0$ such that $|f(x)| \geqslant \delta$ for $x \in \overline{B(\Gamma_+(x_0), \varepsilon)}$. We show that $x(t, x_0 + ru) \in \overline{B(\Gamma_+(x_0), \varepsilon)}$ for all $t \in [0, \infty)$ and $r \in [0, r_0]$ when $r_0 > 0$ is small enough.

By conditions (ii), (a), (b) and lemma 3.2, the right-hand side of (3.14) is continuous in (r, T) and locally Lipschitzian in T. Therefore, (3.14) has a unique solution T with $T(0) = s$ for each $s \geqslant 0$. Substituting T into (3.10), we obtain a unique curve $\gamma(s)$ perpendicular to the trajectories of (2.1) on S. Let

$$z(T) = f(x(T, x_0 + ru)) \wedge \frac{\mathrm{d}y(r)}{\mathrm{d}r}. \tag{3.15}$$

Note that

$$\frac{\mathrm{d}f}{\mathrm{d}T}(x(T, x_0 + ru)) = \frac{\partial f}{\partial x}(x(T, x_0 + ru)) f(x(T, x_0 + ru)) \tag{3.16}$$

and, from (3.13) with (3.14) and lemma 3.2,

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\mathrm{d}y(r)}{\mathrm{d}r}\right) = \frac{\partial f}{\partial x}(x(T, x_0 + ru))\frac{\mathrm{d}y(r)}{\mathrm{d}r} + f(x(T, x_0 + ru))\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right). \tag{3.17}$$

From the definition of $\wedge$, it is obvious that $f \wedge f = 0$. Then, from (3.15)–(3.17), lemma 3.3 and the definition of $\wedge$,

$$\begin{aligned}
\frac{\mathrm{d}z(T)}{\mathrm{d}T} &= \frac{\mathrm{d}f}{\mathrm{d}T}(x(T, x_0 + ru)) \wedge \frac{\mathrm{d}y(r)}{\mathrm{d}r} + f(x(T, x_0 + ru)) \wedge \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\mathrm{d}y(r)}{\mathrm{d}r}\right) \\
&= \left(\frac{\partial f}{\partial x}(x(T, x_0 + ru)) f(x(T, x_0 + ru))\right) \wedge \frac{\mathrm{d}y(r)}{\mathrm{d}r} \\
&\qquad + f(x(T, x_0 + ru)) \wedge \left(\frac{\partial f}{\partial x}\left(x(T, x_0 + ru)\right)\frac{\mathrm{d}y(r)}{\mathrm{d}r}\right) \\
&= \frac{\partial f}{\partial x}^{[2]}(x(T, x_0 + ru)) z(T).
\end{aligned}$$

This, together with lemma 3.1 and the assumption (2.12), gives

$$|z(T_1)| \leqslant |z(T_0)| \exp\left(\int_{T_0}^{T_1} \mu\left(\frac{\partial f}{\partial x}^{[2]}(x(T, x_0 + ru))\right) \mathrm{d}T\right) \leqslant |z(T_0)|$$

for $T_1 \geqslant T_0$. Since the solution T of (3.14) is a function of (r, s) and $s_1 > s_2$ implies $T(r, s_1) > T(r, s_2)$ by uniqueness, we have

$$|z(T(r, s))| \leqslant |z(T(r, 0))| \tag{3.18}$$

for $s \geqslant 0$ and $r \in [0, r_0]$. As $|\cdot|$ is equivalent to $\|\cdot\|$, by lemma 3.4 and (3.15) there is a constant $K > 0$ such that

$$|z(T(r,s))| \geqslant K|f(x(T, x_0 + ru))|\left|\frac{\mathrm{d}y(r)}{\mathrm{d}r}\right| \geqslant \delta K\left|\frac{\mathrm{d}y(r)}{\mathrm{d}r}\right|$$

as long as $x(T, x_0 + ru) \in \overline{B(\Gamma_+(x_0), \varepsilon)}$. Choose $r_0 > 0$ such that

$$2\int_0^{r_0} |z(T(r,0))|\,\mathrm{d}r < \varepsilon\delta K \quad \text{and} \quad |x(T(r,0), x_0 + ru) - x_0| < \tfrac{1}{2}\varepsilon$$

for all $r \in [0, r_0]$. We show that

$$x(T(r,s), x_0 + ru) \in B(\Gamma_+(x_0), \varepsilon), \quad \text{for } s \geqslant 0,\ r \in [0, r_0], \tag{3.19}$$

by contradiction. Suppose (3.19) is not true. Then there is an $s_1 > 0$ and $r_1 \in (0, r_0]$ such that

$$|x(T(r,s), x_0 + ru) - x(s, x_0)| < \varepsilon$$

for all $(r,s) \in [0, r_1] \times [0, s_1]$ with $(r,s) \neq (r_1, s_1)$ but

$$|x(T(r_1, s_1), x_0 + r_1 u) - x(s_1, x_0)| = \varepsilon. \tag{3.20}$$

But, from (3.18) and the inequalities following it, and the choice of r_0,

$$|y(r_1) - y(0)| \leqslant \int_0^{r_1} \left|\frac{\mathrm{d}y(r)}{\mathrm{d}r}\right| \mathrm{d}r \leqslant \frac{1}{\delta K}\int_0^{r_0} |z(T(r,0))|\,\mathrm{d}r < \tfrac{1}{2}\varepsilon$$

for $s = s_1$, i.e.

$$|x(T(r_1, s_1), x_0 + r_1 u) - x(s_1, x_0)| < \tfrac{1}{2}\varepsilon.$$

This contradiction to (3.20) shows (3.19). It can be shown that $T(r,s) \to \infty$ as $s \to \infty$ for all $r \in [0, r_0]$ (see [8, lemma 6]). Therefore, $x(t, x_0 + ru) \in B(\Gamma_+(x_0), \varepsilon)$ for all $r \in [0, r_0]$ and $t \geqslant 0$. Since $B(\Gamma_+(x_0), \varepsilon) \cap B(x^0, d/2) = \emptyset$, this shows that $x_0 + ru \notin A$ for $r \in [0, r_0]$, a contradiction to the choice of x_0 and u.

(β) Suppose $x(t, x_0)$ is unbounded on $[0, \omega)$. We can then choose a sequence $\{t_n\} \subset [0, \omega)$ such that $t_n < t_{n+1}$,

$$|x(t_n, x_0)| = \max_{0 \leqslant t \leqslant t_n} |x(t, x_0)|, \quad t_n \to \omega \quad \text{and} \quad |x(t_n, x_0)| \to \infty \quad \text{as } n \to \infty.$$

By (3.18) and the inequalities following it, we have

$$\int_0^{r_0} |f(x(T(r,s), x_0 + ru))|\left|\frac{\mathrm{d}y(r)}{\mathrm{d}r}\right| \mathrm{d}r \leqslant \frac{1}{K}\int_0^{r_0} |z(T(r,0))|\,\mathrm{d}r.$$

Let

$$m_n = \min_{0 \leqslant r \leqslant r_0} |x(T(r, t_n), x_0 + ru)| \quad \text{and} \quad M_n = \max_{0 \leqslant r \leqslant r_0} |x(T(r, t_n), x_0 + ru)|.$$

Then, by lemma 3.5,

$$\int_{m_n}^{M_n} \min_{|x| = \ell} |f(x)|\,\mathrm{d}\ell \leqslant \frac{1}{K}\int_0^{r_0} |z(T(r,0))|\,\mathrm{d}r.$$

As $M_n \geqslant |x(t_n, x_0)|$, by the choice of t_n we must have $\lim_{n\to\infty} M_n = \infty$ and, by condition (iii), $\lim_{n\to\infty} m_n = \infty$. This implies the unboundedness of $x(t, x_0 + ru)$ on $[0, \infty)$ for each $r \in (0, r_0]$: a contradiction to $\lim_{t\to\infty} x(t, x_0 + ru) = x^0$.

The contradictions in (α) and (β) have shown that x^0 is a global attractor.

If (2.13) holds and x^0 is a local repeller, then the above reasoning with the replacement of t by $t' = -t$ shows that x^0 is a global repeller. □

4. Global attraction for region-wise linear systems

In this section, we investigate global asymptotic behaviour of solutions of the system

$$x' = L(x) + b, \quad x \in \mathbb{R}^N, \tag{4.1}$$

where $b \in \mathbb{R}^N$ is a constant, $L \in C(\mathbb{R}^N, \mathbb{R}^N)$ and L is linear in each of the 2^N closed cones bounded by the coordinate planes $\{x \in \mathbb{R}^N : x_i = 0\}$, $i \in I_N$. In other words, in each such cone $\partial L(x)/\partial x$ is a constant matrix and

$$L(x) = \frac{\partial L(x)}{\partial x} x \tag{4.2}$$

(see [7, § 3] for a more detailed description of (4.1)). On viewing (4.1) as (2.1), we have $D = \mathbb{R}^N$ and $D_0 = \{x \in \mathbb{R}^N : x_i = 0 \text{ for some } i \in I_N\}$, though a smaller set might be taken as D_0 for each individual system (e.g. see the example given in § 1). For any $x_0 \in \mathbb{R}^N$, the solution $x(t, x_0)$ of (4.1) with $x(0, x_0) = x_0$ exists on $\mathbb{R}$ due to the region-wise linearity of L. Moreover, by [7, lemma 3.1], for any $T > 0$, the zero set of each component of $x(t, x_0)$ on $[0, T]$ is either finite or the union of a finite set and some interval(s). Thus, since it is possible for $x(t, x_0)$ to stay in D_0 on an interval, (4.1) satisfies conditions (a) and (c), but not (b), of theorem 2.1. Therefore, superficially, the results given in § 2 may not be applicable to every system (4.1). In reality, however, less restrictive conditions should be expected for (4.1) due to its region-wise linearity. We shall see that some requirements of theorems 2.1 and 2.7 will become redundant for (4.1).

THEOREM 4.1. *System (4.1) has a single point global attractor (or repeller) if (4.3) (or (4.4)) holds,*

$$\rho = \max_{x\in P} \mu\left(\frac{\partial L(x)}{\partial x}\right) < 0, \tag{4.3}$$

$$\eta = \max_{x\in P} \mu\left(-\frac{\partial L(x)}{\partial x}\right) < 0, \tag{4.4}$$

where $P = \{x \in \mathbb{R}^N : |x_i| = 1$ *for all* $i \in I_N\}$.

Proof. Assume that (4.3) holds and suppose that (4.1) has a critical point x^0. Then, for any $x^1 \in \mathbb{R}^N \setminus \{x^0\}$, we have

$$(x(t, x^1) - x^0)' = L(x(t, x^1)) - L(x^0),$$

so that

$$\begin{aligned}|x(t+h,x^1)-x^0| &= \left|x(t,x^1)-x^0+\int_0^h (L(x(t+s,x^1))-L(x^0))\,\mathrm{d}s\right| \\ &\leqslant |x(t,x^1)-x^0+h(L(x(t,x^1))-L(x^0))| \\ &\quad + \int_0^h |L(x(t+s,x^1))-L(x(t,x^1))|\,\mathrm{d}s\end{aligned}$$

for $t \geqslant 0$ and $h \geqslant 0$. If $x^0 \in \mathbb{R}_0$ and $x(t,x^1) \in \mathbb{R}_0$ for a fixed t, where $\mathbb{R}_0$ is a closed cone bounded by the N coordinate planes, then

$$L(x(t,x^1)) - L(x^0) = L(x(t,x^1)-x^0) = \frac{\partial L}{\partial x}(x(t,x^1)-x^0).$$

By (2.3) and (4.3),

$$\begin{aligned}&D_+|x(t,x^1)-x^0| \\ &\quad = \lim_{h\to 0+}\frac{1}{h}(|x(t+h,x^1)-x^0| - |x(t,x^1)-x^0|) \\ &\quad \leqslant \lim_{h\to 0+}\frac{1}{h}\left(\left|x(t,x^1)-x^0+h\frac{\partial L}{\partial x}(x(t,x^1)-x^0)\right| - |x(t,x^1)-x^0|\right) \\ &\quad \leqslant \lim_{h\to 0+}\frac{1}{h}\left(\left|I+h\frac{\partial L}{\partial x}\right|-1\right)|x(t,x^1)-x^0| \\ &\quad \leqslant \rho|x(t,x^1)-x^0|.\end{aligned}$$

Suppose $x(t,x^1)$ and x^0 are not in the same closed cone $\mathbb{R}_0$. Then there are $y^1, y^2, \dots, y^k$, $k \in I_N$, in D_0 such that, with $y^0 = x^0$ and $y^{k+1} = x(t,x^1)$, the y^i are on a straight line and y^{i-1}, y^i for each i are in one closed cone. Hence,

$$|x(t,x^1)-x^0| = \sum_{i=1}^{k+1}|y^i - y^{i-1}|,$$

$$L(y^i) - L(y^{i-1}) = L(y^i - y^{i-1}) = \frac{\partial L}{\partial x}(y^i - y^{i-1})$$

for $i \in I_{k+1}$, and

$$L(x(t,x^1)) - L(x^0) = \sum_{i=1}^{k+1} L(y^i - y^{i-1}) = \sum_{i=1}^{k+1}\frac{\partial L}{\partial x}(y^i - y^{i-1}).$$

Again, by (2.3) and (4.3),

$$D_+|x(t,x^1)-x^0| \leqslant \sum_{i=1}^{k+1}\mu\left(\frac{\partial L}{\partial x}\right)|y^i - y^{i-1}| \leqslant \rho|x(t,x^1)-x^0|.$$

Therefore, for all $t \geqslant 0$,

$$|x(t,x^1)-x^0| \leqslant \mathrm{e}^{\rho t}|x^1 - x^0|$$

so $\lim_{t\to\infty} x(t,x^1) = x^0$.

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Next we show that (4.1) must have a critical point. Suppose that $x^1 \in \mathbb{R}^N$ is not a critical point. Then, for each $s \geqslant 0$ and every $t \in [0,1]$, $x(t+s+h, x^1)$ and $x(t+s, x^1)$ are in one closed cone $\mathbb{R}_0$ for all sufficiently small $h > 0$. Hence, as

$$\lim_{h\to 0} \frac{1}{h}(x(t+s+h, x^1) - x(t+s, x^1)) = L(x(t+s, x^1)) + b,$$

we have

$$\begin{aligned} D_+\left|\frac{\mathrm{d}x(t+s, x^1)}{\mathrm{d}s}\right| &= \lim_{h\to 0+} \frac{1}{h}(|L(x(t+s+h, x^1)) + b| - |L(x(t+s, x^1)) + b|) \\ &\leqslant \lim_{h\to 0+} \frac{1}{h}\left(\left|\left(I + h\frac{\partial L}{\partial x}\right)(L(x(t+s, x^1)) + b)\right| - |L(x(t+s, x^1)) + b|\right) \\ &\leqslant \mu\left(\frac{\partial L}{\partial x}\right)|L(x(t+s, x^1)) + b| \\ &\leqslant \rho\left|\frac{\mathrm{d}x(t+s, x^1)}{\mathrm{d}s}\right|. \end{aligned}$$

Thus,

$$\left|\frac{\mathrm{d}x(t+s, x^1)}{\mathrm{d}s}\right| \leqslant \mathrm{e}^{\rho t}\left|\frac{\mathrm{d}x(s, x^1)}{\mathrm{d}s}\right|$$

for all $t \in [0,1]$. By integration,

$$\begin{aligned} \int_0^\infty \left|\frac{\mathrm{d}x(t, x^1)}{\mathrm{d}t}\right| \mathrm{d}t &= \sum_{k=0}^\infty \int_0^1 \left|\frac{\mathrm{d}x(t+k, x^1)}{\mathrm{d}t}\right| \mathrm{d}t \\ &\leqslant \sum_{k=0}^\infty \int_0^1 \mathrm{e}^{\rho k}\left|\frac{\mathrm{d}x(t, x^1)}{\mathrm{d}t}\right| \mathrm{d}t < \infty. \end{aligned}$$

As

$$|x(t_2, x^1) - x(t_1, x^1)| \leqslant \int_{t_1}^{t_2} \left|\frac{\mathrm{d}x(t, x^1)}{\mathrm{d}t}\right| \mathrm{d}t$$

for any $t_2 \geqslant t_1 \geqslant 0$, by the Cauchy convergence principle $\lim_{t\to\infty} x(t, x^1) = x^0$ for some $x^0 \in \mathbb{R}^N$. Clearly, x^0 is a critical point.

The above reasoning shows that (4.3) implies the existence of a critical point x^0 that is globally attractive. If (4.4) holds, by putting $t = -t'$ and following the same argument as above, we obtain a critical point that is a global repeller. □

The corollary below is an immediate consequence of theorem 4.1 and lemma 2.4.

COROLLARY 4.2. *System (4.1) has a single point global attractor (or repeller) if any one of (4.5)–(4.7) (or (4.8)–(4.10)) holds:*

$$\max_{x\in P, 1\leqslant i\leqslant N}\left\{\frac{\partial L_i}{\partial x_i} + \sum_{j\neq i}\left|\frac{\partial L_i}{\partial x_j}\right|\right\} < 0, \tag{4.5}$$

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$$\max_{x\in P,1\leqslant i\leqslant N}\left\{\frac{\partial L_i}{\partial x_i}+\sum_{j\neq i}\left|\frac{\partial L_j}{\partial x_i}\right|\right\}<0, \tag{4.6}$$

$$\max_{x\in P}\lambda_1<0, \tag{4.7}$$

$$\min_{x\in P,1\leqslant i\leqslant N}\left\{\frac{\partial L_i}{\partial x_i}-\sum_{j\neq i}\left|\frac{\partial L_i}{\partial x_j}\right|\right\}>0, \tag{4.8}$$

$$\min_{x\in P,1\leqslant i\leqslant N}\left\{\frac{\partial L_i}{\partial x_i}-\sum_{j\neq i}\left|\frac{\partial L_j}{\partial x_i}\right|\right\}>0, \tag{4.9}$$

$$\min_{x\in P}\lambda_N>0, \tag{4.10}$$

where the λ_i *are the eigenvalues of* $\frac{1}{2}((\partial L/\partial x)^{\mathrm{T}}+\partial L/\partial x)$ *satisfying* $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_N$ *and* $P=\{x\in\mathbb{R}^N:|x_i|=1$ *for all* $i\in I_N\}$.

Theorem 4.1 and corollary 4.2 are particularly useful when it is not obvious whether or not (4.1) has a critical point or when there is a critical point $x^0\in D_0$. In the latter case, it is not a trivial matter to find out the local behaviour near x^0 if $N>2$. The following example demonstrates that theorem 4.1 and corollary 4.2 are sufficient conditions for (4.1) to have a single point global attractor (or repeller).

Example 4.3. Consider the system

$$x'=A(x_1)x,\quad x=(x_1,x_2)^{\mathrm{T}}\in\mathbb{R}^2, \tag{4.11}$$

where, with $a\in\mathbb{R}$,

$$A(x_1)=\begin{cases}\begin{pmatrix}-1 & 2\\ -2 & -1\end{pmatrix}, & x_1\geqslant 0,\\[2ex] \begin{pmatrix}a & 2\\ -2 & -1\end{pmatrix}, & x_1<0.\end{cases} \tag{4.12}$$

Clearly, the origin ‘O’ is a critical point. If $a<0$, then (4.7) is satisfied so O is a global attractor. If $a\geqslant 0$, none of the conditions (4.5)–(4.10) are met. However, by solving (4.11) explicitly, we know that O is a global attractor for $a<\frac{11}{5}$, a centre for $a=\frac{11}{5}$, and a global repeller for $a>\frac{11}{5}$.

Remark 4.4. Consider the system (4.1) with $b=0$ and assume that every eigenvalue of $\partial L/\partial x$ has a negative (or positive) real part for all $x\in P$, where P is given in theorem 4.1. Is the origin a global attractor (or repeller)? Further investigation is needed on this.

Example 4.5. Consider the system

$$x'=(AD(x)-I)x+b, \tag{4.13}$$

where $A=(a_{ij})\in\mathbb{R}^{N\times N}$, $b\in\mathbb{R}^N$, I is the identity matrix and $D:\mathbb{R}^N\to\mathbb{R}^{N\times N}$ is given by

$$D(x)=\operatorname{diag}\left[d(x_1),d(x_2),\dots,d(x_N)\right]$$

with $d(s) = 1$ for $s > 0$ and $d(s) = 0$ for $s \leqslant 0$. Li and Dayan [10] used (4.13) as a neural network model and briefly discussed the relationship on local stability between (4.13) and another region-wise linear system. Now, applying (4.5)–(4.7) to (4.13), we find that (4.13) has a single point global attractor if

$$\max_{1\leqslant i\leqslant N}\left(\sum_{j\neq i}|a_{ij}| - \alpha_i\right) < 0,$$

$$\max_{1\leqslant i\leqslant N}\left(\sum_{j\neq i}|a_{ji}| - \alpha_i\right) < 0$$

or

$$\max_{1\leqslant i\leqslant N}\left(\frac{1}{2}\sum_{j\neq i}(|a_{ij}| + |a_{ji}|) - \alpha_i\right) < 0$$

holds, where $\alpha_i = \min\{1, 1 - a_{ii}\}$.

Theorem 4.6. *Assume that (4.1) satisfies the following:*

(i) $x^0 \in \mathbb{R}^N$ *is the unique critical point and it is a local attractor (or repeller);*

(ii) *no non-trivial solution will stay in* D_0 *for* t *in any interval.*

Then x^0 *is a global attractor (or repeller) if (4.14) (or (4.15)) holds for* $x \in P$*:*

$$\mu\left(\frac{\partial L}{\partial x}^{[2]}(x)\right) \leqslant 0, \tag{4.14}$$

$$\mu\left(-\frac{\partial L}{\partial x}^{[2]}(x)\right) \leqslant 0, \tag{4.15}$$

where $P = \{x \in \mathbb{R}^N : |x_i| = 1$ *for all* $i \in I_N\}$.

Remark 4.7. Theorem 4.6 is a direct translation from theorem 2.7. It is tempting to assume that the condition (ii) is not essential. However, it is difficult to clarify this, and further investigation is needed.

The next result is an analogue of corollary 2.8.

Corollary 4.8. *Under the assumptions of theorem 4.6,* x^0 *is a global attractor (or repeller) if any one of (4.16)–(4.18) (or (4.19)–(4.21)) holds for* $x \in P$*:*

$$\max_{1\leqslant r<s\leqslant N}\frac{\partial L_r}{\partial x_r} + \frac{\partial L_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial L_r}{\partial x_q}\right| + \left|\frac{\partial L_s}{\partial x_q}\right|\right) \leqslant 0, \tag{4.16}$$

$$\max_{1\leqslant r<s\leqslant N}\frac{\partial L_r}{\partial x_r} + \frac{\partial L_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial L_q}{\partial x_r}\right| + \left|\frac{\partial L_q}{\partial x_s}\right|\right) \leqslant 0, \tag{4.17}$$

$$\lambda_1 + \lambda_2 \leqslant 0, \tag{4.18}$$

$$\min_{1\leqslant r<s\leqslant N} \frac{\partial L_r}{\partial x_r} + \frac{\partial L_s}{\partial x_s} - \sum_{q\neq r,s}\left(\left|\frac{\partial L_r}{\partial x_q}\right| + \left|\frac{\partial L_s}{\partial x_q}\right|\right) \geqslant 0, \tag{4.19}$$

$$\min_{1\leqslant r<s\leqslant N} \frac{\partial L_r}{\partial x_r} + \frac{\partial L_s}{\partial x_s} - \sum_{q\neq r,s}\left(\left|\frac{\partial L_q}{\partial x_r}\right| + \left|\frac{\partial L_q}{\partial x_s}\right|\right) \geqslant 0, \tag{4.20}$$

$$\lambda_{N-1} + \lambda_N \geqslant 0, \tag{4.21}$$

where the λ_i *are the eigenvalues of*

$$\frac{1}{2}\left(\left(\frac{\partial L}{\partial x}\right)^{\mathrm{T}} + \frac{\partial L}{\partial x}\right)$$

satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ *and* $P = \{x \in \mathbb{R}^N : |x_i| = 1$ *for all* $i \in I_N\}$.

EXAMPLE 4.9. Consider the system

$$x' = A(x_1)x + b, \quad x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2, \tag{4.22}$$

where $b = (b_1, b_2)^{\mathrm{T}} \in \mathbb{R}^2$ and $A(x_1)$ is given by (4.12). Then, for any b satisfying $b_1 + 2b_2 > 0$ and any $a \leqslant 1$, (4.22) has a local stable focus $x^0 = \frac{1}{5}(b_1 + 2b_2, b_2 - 2b_1)^{\mathrm{T}}$ and, by (4.18), x^0 is a global attractor.

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