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Pulsating travelling fronts: Asymptotics and homogenization regimes

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This paper is concerned with some non-linear propagation phenomena for reaction–advection–diffusion equations with Kolmogrov–Petrovsky–Piskunov (KPP)-type non-linearities in general periodic domains or in infinite cylinders with oscillating boundaries. Having a variational formula for the minimal speed of propagation involving eigenvalue problems (proved in Berestycki, H., Hamel, F. & Nadirashvili, N. (2005) The speed of propagation for KPP type problems (periodic framework). *J. Eur. Math. Soc.* **7**, 173–213), we consider the minimal speed of propagation as a function of diffusion factors, reaction factors and periodicity parameters. There we study the limits, the asymptotic behaviours and the variations of the considered functions with respect to these parameters. One of the sections deals with homogenization problem as an application of the results in the previous sections in order to find the limit of the minimal speed when the periodicity cell is very small.

1 Introduction

This paper is a continuation in the study of the propagation phenomena of pulsating travelling fronts in a periodic framework corresponding to reaction–advection–diffusion equations with heterogenous KPP (Kolmogrov–Petrovsky–Piskunov) non-linearities. We will precisely describe the *heterogeneous-periodic* setting, recall the extended notion of *pulsating travelling fronts* and then we move to announce the main results. Let us first recall some of the basic features of the *homogeneous* KPP equations.

Consider the Fisher–KPP equation:

$$u_t - \Delta u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

It was introduced in the celebrated papers of Fisher (1937) and in [19] originally motivated by models in biology. Here, the main assumption is that f is, say, a C^1 function satisfying

$$\begin{cases} f(0) = f(1) = 0,\ f'(1) < 0,\ f'(0) > 0, \\ f > 0 \text{ in } (0,1),\ f < 0 \text{ in } (1,+\infty), \end{cases} \tag{1.2}$$

$$f(s) \leqslant f'(0)s, \forall s \in [0,1]. \tag{1.3}$$

As examples of such non-linearities, we have $f(s) = s(1-s)$ and $f(s) = s(1-s^2)$.

The important feature in (1.1) is that this equation has a family of planar travelling fronts. These are solutions of the form

$$\begin{cases} \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ u(t,x) = \phi(x \cdot e + ct), \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{1.4}$$

where $e \in \mathbb{R}^N$ is a fixed vector of unit norm which is the direction of propagation, and $c > 0$ is the speed of the front. The function $\phi : \mathbb{R} \mapsto \mathbb{R}$ satisfies

$$\begin{cases} -\phi'' + c\,\phi = f(\phi), \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{1.5}$$

In the original paper of Kolmogorov, Petrovsky and Piskunov, it was proved that, under the above assumptions, there is a threshold value $c^* = 2\sqrt{f'(0)} > 0$ for the speed c. Namely, no fronts exist for $c < c^*$, and, for each $c \geqslant c^*$, there is a unique front of the type (1.4 and 1.5). Uniqueness is up to shift in space or time variables.

Later, the homogeneous setting was extended to a general heterogeneous periodic one. The heterogeneous character appeared both in the reaction–advection–diffusion equation and in the underlying domain. The general form of these equations is

$$\begin{cases} u_t = \nabla \cdot (A(z)\nabla u) + q(z) \cdot \nabla u + f(z,u),\ t \in \mathbb{R},\ z \in \Omega, \\ \nu \cdot A\,\nabla u(t,z) = 0,\ t \in \mathbb{R},\ z \in \partial\Omega, \end{cases} \tag{1.6}$$

where $\nu(z)$ is the unit outward normal on $\partial\Omega$ at the point z.

The propagation phenomena attached with equation (1.6) has been widely studied in many papers. Several properties of pulsating fronts in periodic media and their speed of propagation were given in several papers [2, 3, 5, 6, 36]. In Section 2, we will recall the periodic framework and some known results which motivate our study. The main results of this paper are presented in Sections 3–6.

2 The periodic framework

2.1 Pulsating travelling fronts in periodic domains

In this section, we introduce the general setting with the precise assumptions. Concerning the domain, let $N \geqslant 1$ be the space dimension, and let d be an integer so that $1 \leqslant d \leqslant N$. For an element $z = (x_1, x_2, \ldots, x_d, x_{d+1}, \ldots, x_N) \in \mathbb{R}^N$, we call $x = (x_1, x_2, \ldots, x_d)$ and $y = (x_{d+1}, \ldots, x_N)$, so that $z = (x, y)$. Let $L_1, \ldots, L_d$ be d positive real numbers, and let Ω be a C^3 non–empty connected open subset of $\mathbb{R}^N$ satisfying

$$\begin{cases} \exists\, R \geqslant 0\,; \forall\, (x,y) \in \Omega,\ |y| \leqslant R, \\ \forall\, (k_1, \ldots, k_d) \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z}, \quad \Omega = \Omega + \displaystyle\sum_{k=1}^{d} k_i e_i, \end{cases} \tag{2.1}$$

where $(e_i)_{1 \leqslant i \leqslant N}$ is the canonical basis of $\mathbb{R}^N$. In particular, since $d \geqslant 1$, the set Ω is *unbounded*.

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In this periodic situation, we give the following definitions:

Definition 2.1 (Periodicity cell) The set $C = \{(x, y) \in \Omega;\ x_1 \in (0, L_1), \ldots, x_d \in (0, L_d)\}$ is called the periodicity cell of Ω.

Definition 2.2 (L-periodic flows) A field $w : \Omega \to \mathbb{R}^N$ is said to be L-periodic with respect to x if $w(x_1 + k_1, \ldots, x_d + k_d, y) = w(x_1, \ldots, x_d, y)$ almost everywhere in Ω, and for all $k = (k_1, \ldots, k_d) \in \prod_{i=1}^{d} L_i\mathbb{Z}$.

Before going further, we point out that this framework includes several types of simpler geometrical configurations. The case of the whole space $\mathbb{R}^N$ corresponds to $d = N$, where $L_1, \ldots, L_N$ are any positive numbers. The case of the whole space $\mathbb{R}^N$ with a periodic array of holes can also be considered. The case $d = 1$ corresponds to domains which have only one unbounded dimension, namely infinite cylinders which may be straight or have oscillating periodic boundaries, and which may or may not have periodic holes. The case $2 \leqslant d \leqslant N - 1$ corresponds to infinite slabs.

We are concerned with propagation phenomena for the reaction–advection–diffusion equation (1.6) set in the periodic domain Ω. Such equations arise in combustion models for flame propagation (see [27], [31] and [37]), as well as in models in biology and for population dynamics of a species (see [14], [18], [20] and [28]). These equations are used in modelling the propagation of a flame or of an epidemics in a periodic heterogeneous medium. The passive quantity u typically stands for the temperature or a concentration which diffuses in a periodic excitable medium. However, in some sections we will ignore the advection and deal only with reaction–diffusion equations.

Let us now detail the assumptions concerning the coefficients in (1.6). First, the diffusion matrix $A(x, y) = (A_{ij}(x, y))_{1 \leqslant i,j \leqslant N}$ is a *symmetric* $C^{2,\delta}(\overline{\Omega})$ (with $\delta > 0$) matrix field satisfying

$$\begin{cases} A \text{ is } L\text{-periodic with respect to } x, \\ \exists\, 0 < \alpha_1 \leqslant \alpha_2, \forall (x, y) \in \Omega, \forall\, \xi \in \mathbb{R}^N, \\ \alpha_1 |\xi|^2 \leqslant \displaystyle\sum_{1 \leqslant i,j \leqslant N} A_{ij}(x, y)\xi_i\xi_j \leqslant \alpha_2 |\xi|^2. \end{cases} \tag{2.2}$$

The boundary condition $v \cdot A\nabla u(x, y) = 0$ stands for $\sum_{1 \leqslant i,j \leqslant N} v_i(x, y) A_{ij}(x, y) \partial_{x_j} u(t, x, y)$, and v stands for the unit outward normal on $\partial\Omega$. We note that when A is the identity matrix, then this boundary condition reduces to the usual Neumann condition $\partial_v u = 0$.

The underlying advection $q(x, y) = (q_1(x, y), \ldots, q_N(x, y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) vector field satisfying

$$\begin{cases} q \text{ is } L\text{-periodic with respect to } x, \\ \nabla \cdot q = 0 \quad \text{in } \overline{\Omega}, \\ q \cdot v = 0 \quad \text{on } \partial\Omega, \\ \forall\, 1 \leqslant i \leqslant d, \quad \displaystyle\int_C q_i \, dx\, dy = 0. \end{cases} \tag{2.3}$$

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Concerning the non-linearity, let $f = f(x,y,u)$ be a non-negative function defined in $\overline{\Omega} \times [0,1]$, such that

$$\begin{cases} f \geqslant 0, f \text{ is } L\text{-periodic with respect to } x, \text{ and of class } C^{1,\delta}(\overline{\Omega} \times [0,1]), \\ \forall\, (x,y) \in \overline{\Omega}, \quad f(x,y,0) = f(x,y,1) = 0, \\ \exists\, \rho \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall\, 1-\rho \leqslant s \leqslant s' \leqslant 1, f(x,y,s) \geqslant f(x,y,s'), \\ \forall\, s \in (0,1),\ \exists\, (x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0, \\ \forall\, (x,y) \in \overline{\Omega}, \quad f'_u(x,y,0) = \lim\limits_{u \to 0^+} \dfrac{f(x,y,u)}{u} > 0, \end{cases} \tag{2.4}$$

with the additional assumption

$$\forall\, (x,y,s) \in \overline{\Omega} \times (0,1), \quad 0 < f(x,y,s) \leqslant f'_u(x,y,0) \times s. \tag{2.5}$$

We denote by $\zeta(x,y) := f'_u(x,y,0)$, for each $(x,y) \in \overline{\Omega}$.
The set of such non-linearities contains two particular types of functions:

- The homogeneous (KPP) type: $f(x,y,u) = g(u)$, where g is a $C^{1,\delta}$ function that satisfies $g(0) = g(1) = 0,\ g > 0$ on $(0,1),\ g'(0) > 0,\ g'(1) < 0$ and $0 < g(s) \leqslant g'(0)s$ in $(0,1)$.
- Another type of such non-linearities consists of functions $f(x,y,u) = h(x,y).\tilde{f}(u)$, such that $\tilde{f}$ is of the previous type, while h lies in $C^{1,\delta}(\overline{\Omega}), L$-periodic with respect to x, and positive in $\overline{\Omega}$.

Having this periodic framework, the notions of travelling fronts and propagation were extended in [2, 3, 18, 26, 28, 29] and [34] as follows:

Definition 2.3 Let $e = (e^1, \ldots, e^d)$ be an arbitrarily given vector in $\mathbb{R}^d$. A function $u = u(t,x,y)$ is called a pulsating travelling front propagating in the direction of e with an effective speed $c \neq 0$, if u is a classical solution of

$$\begin{cases} u_t = \nabla \cdot (A(x,y)\nabla u) + q(x,y) \cdot \nabla u + f(x,y,u),\ t \in \mathbb{R},\ (x,y) \in \Omega, \\ v \cdot A\, \nabla u(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega, \\ \forall\, k \in \prod\limits_{i=1}^{d} L_i\mathbb{Z},\ \forall\, (t,x,y) \in \mathbb{R} \times \overline{\Omega}, \quad u\left(t - \dfrac{k \cdot e}{c}, x, y\right) = u(t, x+k, y), \\ \lim\limits_{x \cdot e \to -\infty} u(t,x,y) = 0, \text{ and } \lim\limits_{x \cdot e \to +\infty} u(t,x,y) = 1, \\ 0 \leqslant u \leqslant 1, \end{cases} \tag{2.6}$$

where the above limits hold locally in t and uniformly in y and in the directions of $\mathbb{R}^d$ which are orthogonal to e.

2.2 Some important known results concerning the propagation phenomena in a periodic framework

Under the assumptions (2.1), (2.2), (2.3), (2.4) and (2.5) set in the previous subsection, Berestycki and Hamel [2] proved that having a pre-fixed unit vector $e \in \mathbb{R}^d$, there exists

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$c^*(e) > 0$ such that pulsating travelling fronts propagating in the direction e (i.e satisfying (2.6)) with a speed of propagation c exist if and only if $c \geqslant c^*(e)$; moreover, the pulsating fronts (within a speed $c \geqslant c^*(e)$) are increasing in the time t. The value $c^*(e) = c^*_{\Omega,A,q,f}(e)$ is called the *minimal speed of propagation in the direction of* e. Other non-linearities have been considered in the cases of the whole space $\mathbb{R}^N$ or in the general periodic framework (see [2, 28, 29, 32, 33, 34] and [35]).

Having the threshold value $c^*_{\Omega,A,q,f}(e)$, our paper aims to study the limits, the asymptotic behaviours and the variations of some parametric quantities. These parametric quantities involve the parametric speeds of propagation of different reaction–advection–diffusion problems within a diffusion factor $\varepsilon > 0$, a reaction factor $B > 0$ or a periodicity parameter L. Thus, it is important to have a variational characterization which shows the dependance of the minimal speed of propagation on the coefficients A, q and f and on the geometry of the domain Ω. In this context, Berestycki, Hamel and Nadirashvili [3] gave such a formulation for $c^*_{\Omega,A,q,f}(e)$ involving elliptic eigenvalue problems. We recall this variational characterization in the following theorem:

Theorem 2.4 (*Berestycki, Hamel and Nadirashvili* [3]) *Let e be a fixed unit vector in $\mathbb{R}^d$. Let $\tilde{e} = (e, 0, \ldots, 0) \in \mathbb{R}^N$. Assume that Ω, A and f satisfy* (2.1), (2.2), (2.4) *and* (2.5). *The minimal speed $c^*(e) = c^*_{\Omega,A,q,f}(e)$ of pulsating fronts solving* (2.6) *and propagating in the direction of e is given by*

$$c^*(e) = c^*_{\Omega,A,q,f}(e) = \min_{\lambda>0} \frac{k(\lambda)}{\lambda}, \tag{2.7}$$

where $k(\lambda) = k_{\Omega,e,A,q,\zeta}(\lambda)$ is the principal eigenvalue of the operator $L_{\Omega,e,A,q,\zeta,\lambda}$ which is defined by

$$\begin{aligned} L_{\Omega,e,A,q,\zeta,\lambda}\psi \;:=\; & \nabla\cdot(A\nabla\psi) - 2\lambda\tilde{e}\cdot A\nabla\psi + q\cdot\nabla\psi \\ & + [\lambda^2\tilde{e}A\tilde{e} - \lambda\nabla\cdot(A\tilde{e}) - \lambda q\cdot\tilde{e} + \zeta]\psi \end{aligned} \tag{2.8}$$

acting on the set

$$E = \{\psi \in C^2(\overline{\Omega}), \psi \text{ is } L\text{-periodic with respect to } x \text{ and } \nu\cdot A\nabla\psi = \lambda(\nu A\tilde{e}\psi) \text{ on } \partial\Omega\}.$$

The proof of formula (2.7) is based on methods developed in [2, 7, 9]. These are techniques of sub- and super-solutions, regularizing and approximations in bounded domains.

We note that in formula (2.7), the value of the minimal speed $c^*(e)$ is given in terms of the direction e, the domain Ω and the coefficients A, q and $f'_u(.,.,0)$. Moreover, it is important to notice that the dependence of $c^*(e)$ on the non-linearity f is only through the derivative of f with respect to u at $u = 0$.

Before going further, let us mention that formula (2.7) extends some earlier results about front propagation. When $\Omega = \mathbb{R}^N$, $A = Id$ and $f = f(u)$ (with $f(u) \leqslant f'(0)u$ in $[0,1]$), formula (2.7) then reduces to the well-known KPP formula $c^*(e) = 2\sqrt{f'(0)}$. That

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is the value of the minimal speed of propagation of planar fronts for the *homogeneous* reaction–diffusion equation: $u_t - \Delta u = f(u)$ in $\mathbb{R}^N$.[1]

The above variational characterization of the minimal speed of propagation of pulsating fronts in general periodic excitable media will play the main role in studying the dependence of the minimal speed $c^*(e) = c^*_{\Omega,A,q,f}(e)$ on the coefficients of reaction, diffusion, advection and on the geometry of the domain. In this context, we have the following theorem.

Theorem 2.5 (*Berestycki, Hamel, Nadirashvili* [3]) *Under the assumptions* (2.1), (2.2) *and* (2.3) *on* Ω, A *and* q, *let* $f = f(x,y,u)$ [*respectively* $g = g(x,y,u)$] *be a non-negative nonlinearity satisfying* (2.4) *and* (2.5). *Let* e *be a fixed unit vector in* $\mathbb{R}^d$, *where* $1 \leqslant d \leqslant N$,

(*a*) *If* $f'_u(x,y,0) \leqslant g'_u(x,y,0)$ *for all* $(x,y) \in \overline{\Omega}$, *then*

$$c^*_{\Omega,A,q,f}(e) \leqslant c^*_{\Omega,A,q,g}(e).$$

Moreover, if $f'_u(x,y,0) \leqslant, \not\equiv g'_u(x,y,0)$ *in* $\overline{\Omega}$, *then* $c^*_{\Omega,A,q,f}(e) < c^*_{\Omega,A,q,g}(e)$.

(*b*) *The map* $B \mapsto c^*_{\Omega,A,q,Bf}(e)$ *is increasing in* $B > 0$ *and*

$$\limsup_{B \to +\infty} \frac{c^*_{\Omega,A,q,Bf}(e)}{\sqrt{B}} < +\infty.$$

Furthermore, if $\Omega = \mathbb{R}^N$ *or if* $\nu A\tilde{e} \equiv 0$ *on* $\partial\Omega$, *then* $\liminf_{B \to +\infty} \frac{c^*_{\Omega,A,q,Bf}(e)}{\sqrt{B}} > 0$.

(*c*)

$$c^*_{\Omega,A,q,f}(e) \leqslant ||(q.\tilde{e})^-||_\infty + 2\sqrt{\max_{(x,y)\in\overline{\Omega}} \zeta(x,y)}\sqrt{\max_{(x,y)\in\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}}, \tag{2.9}$$

where $||(q.\tilde{e})^-||_\infty = \max_{(x,y)\in\overline{\Omega}}(q(x,y).\tilde{e}^-$ *and* $s^- = \max(-s,0)$ *for each* $s \in \mathbb{R}$. *Furthermore, the equality holds in* (2.9) *if and only if* $\tilde{e}A\tilde{e}$ *and* ζ *are constant,* $q.\tilde{e} \equiv \nabla.(A\tilde{e}) \equiv 0$ *in* $\overline{\Omega}$ *and* $\nu.A\tilde{e} = 0$ *on* $\partial\Omega$ (*in the case when* $\partial\Omega \neq \emptyset$).

(*d*) *Assume furthermore that* $f = f(u)$ *and* $q \equiv 0$ *in* $\overline{\Omega}$, *then the map* $\beta \mapsto c^*_{\Omega,\beta A,0,f}(e)$ *is increasing in* $\beta > 0$.

As a corollary of (2.9), we see that $\limsup_{M\to+\infty} \frac{c^*_{\Omega,MA,q,f}(e)}{\sqrt{M}} \leqslant C$ where C is a positive constant. Furthermore, part (d) implies that a larger diffusion speeds up the propagation in the absence of the advection field.

We mention that the existence of pulsating travelling fronts in space-time periodic media was proved in [23–25] and recently in [21, 22]. In [22], Nadin characterized the minimal speed of propagation and he studied the influence of the diffusion, the amplitude of the reaction term and the drift on the characterized speed.

[1] In fact, the uniqueness, up to multiplication by a non-zero real number, of the first eigenvalue function of $L_{\mathbb{R}^N,e,Id,f'(0),\lambda}\psi = k(\lambda)\psi$ together with this particular situation, yield that the principal eigenfunction ψ is constant and $k(\lambda) = \lambda^2 + f'(0)$ for all $\lambda > 0$. Therefore by (2.7), we have $c^*(e) = \min_{\lambda>0}(\lambda + \frac{f'(0)}{\lambda}) = 2\sqrt{f'(0)}$.

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After reviewing some results in the study of the KPP propagation phenomena in a periodic framework, we pass now to announce new results concerning the limiting behaviour of the minimal speed of propagation within a small (resp. large) diffusion and reaction coefficients (in some particular situations of the general periodic framework) and we will study the minimal speed as a function of the period of the coefficients in the KPP reaction–diffusion–advection (or reaction–diffusion) equation in the case where $\Omega = \mathbb{R}^N$. The proofs will be shown in details in Section 7. The announced results will be applied to find the homogenization limit of the minimal speeds of propagation. We believe that this limit might help to find the homogenized equation in the "KPP" periodic framework (see Section 8 for more details).

3 The minimal speed within small diffusion factors or within large period coefficients

In this section, our problem is a reaction–diffusion equation with absence of advection terms

$$\begin{cases} u_t = \beta \nabla \cdot (A(x,y)\nabla u) + f(x,y,u),\ t \in \mathbb{R},\ (x,y) \in \Omega, \\ \nu \cdot A\, \nabla u(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega, \end{cases} \tag{3.1}$$

where $\beta > 0$.

We mention that (3.1) is a reaction–diffusion problem within a diffusion matrix βA. Let e be a unit direction in $\mathbb{R}^d$. Under the assumptions (2.1), (2.2), (2.4) and (2.5), for each $\beta > 0$, there corresponds a minimal speed of propagation $c^*_{\Omega,\beta A,0,f}(e)$ so that a pulsating front with a speed c and satisfying (3.1) exists if and only if $c \geqslant c^*_{\Omega,\beta A,0,f}(e)$.

Referring to part (c) of Theorem 2.5, one gets $0 < c^*_{\Omega,\beta A,0,f}(e) \leqslant 2\sqrt{\beta}\sqrt{M_0 M}$, for any $\beta > 0$, where $M_0 = \max_{(x,y)\in\overline{\Omega}} \zeta(x,y)$ and $M = \max_{(x,y)\in\overline{\Omega}} \tilde{e}A(x,y)\tilde{e}$.

Consequently, there exists $C > 0$ and independent of β such that

$$\forall\, \beta > 0,\ 0 < \frac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}} \leqslant C. \tag{3.2}$$

The inequality (3.2) leads us to investigate the limits of $\frac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}}$ as $\beta \to 0$ and as $\beta \to +\infty$. The following theorem gives the precise limit when the diffusion factor tends to zero. However, it will not be announced in the most general periodic setting. We will describe the situation before the statement of the theorem.

The domain will be in the form $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$, where $\omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-d-1}$ $(d \geqslant 0)$. If $d = 0$, then ω is a C^3 connected, open bounded subset of $\mathbb{R}^{N-1}$. While, in the case where $1 \leqslant d \leqslant N-1$, ω is a $(L_1,\ldots,L_d)$-periodic open domain of $\mathbb{R}^{N-1}$ which satisfies (2.1); and hence, Ω is a $(l, L_1,\ldots,L_d)$-periodic subset of $\mathbb{R}^N$ that satisfies (2.1) with $l > 0$ and arbitrary. An element of $\Omega = \mathbb{R} \times \omega$ will be represented as $z = (x,y)$ where $x \in \mathbb{R}$ and $y \in \omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-1-d}$.

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The non-linearity $f = f(x, y, u)$, in this section, is a KPP non-linearity defined on $\overline{\Omega} \times [0,1]$ that satisfies

$$\begin{cases} f \geqslant 0, \text{ and of class } C^{1,\delta}(\mathbb{R} \times \overline{\omega} \times [0,1]), \\ f \text{ is } (l, L_1, \dots, L_d)\text{-periodic with respect to } (x, y_1, \dots, y_d), \text{ when } d \geqslant 1, \\ f \text{ is } l\text{-periodic in } x, \text{ when } d = 0, \\ \forall\, (x,y) \in \overline{\Omega} = \mathbb{R} \times \overline{\omega},\ f(x,y,0) = f(x,y,1) = 0, \\ \exists\, \rho \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall\, 1-\rho \leqslant s \leqslant s' \leqslant 1,\ f(x,y,s) \geqslant f(x,y,s'), \\ \forall\, s \in (0,1), \quad \exists\, (x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0, \end{cases} \tag{3.3}$$

together with the assumptions

$$\begin{cases} f'_u(x,y,0) \text{ depends only on } y; \text{ we denote by } \zeta(y) = f'_u(x,y,0),\ \forall (x,y) \in \overline{\Omega}. \\ \forall\, (x,y) \in \overline{\Omega} = \mathbb{R} \times \overline{\omega}, \quad f'_u(x,y,0) = \zeta(y) > 0, \\ \forall\, (x,y,s) \in \overline{\Omega} \times (0,1), \quad 0 < f(x,y,s) \leqslant \zeta(y)\, s. \end{cases} \tag{3.4}$$

Notice that $f'_u(x,y,u)$ is assumed to depend only on y, but $f(x,y,u)$ may depend on x.

Lastly, concerning the diffusion matrix, $A(x,y) = A(y) = (A_{ij}(y))_{1 \leqslant i,j \leqslant N}$ is a $C^{2,\delta}(\overline{\Omega})$ (with $\delta > 0$) symmetric matrix field whose entries are depending only on y, and satisfying

$$\begin{cases} A \text{ is } (L_1, \dots, L_d)\text{-periodic with respect to } (y_1, \dots, y_d), \\ \exists\, 0 < \alpha_1 \leqslant \alpha_2,\ \forall\, y \in \omega, \forall\, \xi \in \mathbb{R}^N, \\ \alpha_1 |\xi|^2 \leqslant \sum A_{ij}(y) \xi_i \xi_j \leqslant \alpha_2 |\xi|^2. \end{cases} \tag{3.5}$$

Theorem 3.1 *Let $e = (1, 0, \dots, 0) \in \mathbb{R}^N$ and $\varepsilon > 0$. Let $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$ satisfy the form described in the previous page. Under the assumptions* (3.3), (3.4), *and* (3.5), *consider the reaction–diffusion equation*

$$\begin{cases} u_t(t,x,y) = \varepsilon \nabla \cdot (A(y) \nabla u)(t,x,y) + f(x,y,u), \text{ for } (t,x,y) \in \mathbb{R} \times \Omega, \\ \nu \cdot A \nabla u = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \times \partial\omega. \end{cases} \tag{3.6}$$

Assume, furthermore, that A and f satisfy one of the following two alternatives:

$$\begin{cases} \exists\, \alpha > 0,\ \forall y \in \omega,\ A(y)e = \alpha e, \\ f'_u(x,y,0) = \zeta(y), \text{ for all } (x,y) \in \overline{\Omega}, \end{cases} \tag{3.7}$$

or

$$\begin{cases} f'_u(x,y,0) = \zeta \text{ is constant}, \\ \forall\, y \in \omega,\ A(y)e = \alpha(y)e, \text{ where} \\ y \mapsto \alpha(y) \text{ is a positive, } (L_1, \dots, L_d)\text{-periodic function over } \overline{\omega}. \end{cases} \tag{3.8}$$

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Then,

$$\lim_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} = 2\sqrt{\max_{\overline{\omega}} \zeta}\sqrt{\max_{\overline{\omega}} eAe}. \tag{3.9}$$

Before going further, we mention that the family of domains for which Theorem 3.1 holds is wide. An infinite cylinder $\mathbb{R}\times B_{\mathbb{R}^{N-1}}(y_0,R)$ (where $R>0$, and $B_{\mathbb{R}^{N-1}}(y_0,R)$ is the Euclidian ball of center y_0 and radius R) is an archetype of such domains. In these cylinders, $\omega = B_{\mathbb{R}^{N-1}}(y_0,R)$, l is any positive real number, and $d=0$. The whole space $\mathbb{R}^N$ is another archetype of the domain Ω where $d=N-1$, $\omega=\mathbb{R}^{N-1}$ and $\{l, L_1,\dots,L_d\}$ is any family of positive real numbers.

Remark 3.2 In Theorem 3.1, the domain $\Omega=\mathbb{R}\times\omega$ is invariant in the direction of $e=(1,0,\dots,0)$ which is parallel to Ae (in both cases (3.7) and (3.8)). Also, the assumption that the entries of A do not depend on x, yields that $\nabla.(Ae)\equiv 0$ over Ω. On the other hand, it is easy to find a diffusion matrix A and a non-linearity f which satisfy, together, the assumptions of Theorem 3.1 while one of $eAe(y)$ and $\zeta(y)$ is *not constant*. Referring to part (c) of Theorem 2.5, one obtains

$$\forall\varepsilon>0,\quad 0<\frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \lneqq 2\sqrt{\max_{y\in\overline{\omega}} \zeta(y)}\sqrt{\max_{y\in\overline{\omega}} eAe(y)}.$$

However, Theorem 3.1 implies that

$$\lim_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} = 2\sqrt{\max_{y\in\overline{\omega}} \zeta(y)}\sqrt{\max_{y\in\overline{\omega}} eAe(y)}.$$

On the other hand, if $\Omega=\mathbb{R}\times\omega$ as in Theorem 3.1, $A=Id$ and $f=f(u)$, Theorem 2.5 yields that $c^*_{\Omega,\varepsilon Id,0,f}(e) = 2\sqrt{\varepsilon}\sqrt{f'(0)}$, for all $\varepsilon>0$. □

In the same context, one can also find the limit when the diffusion factor goes to zero, but in the presence of an advection field in the form of shear flows.

Theorem 3.3 *Assume that $e=(1,0,\dots,0)\in\mathbb{R}^N$, the domain $\Omega=\mathbb{R}\times\omega$ has the same form as in Theorem 3.1, and the coefficients f and A satisfy* (3.3 *and* 3.4) *and* (3.5)*, respectively. Assume, furthermore, that for all $y\in\overline{\omega}$, there exists $\alpha(y)$ positive so that $A(y)e=\alpha(y)e$ in $\overline{\omega}$. Consider, in addition, an advective shear flow $q=(q_1(y),0,\dots,0)$ ($y\in\overline{\omega}$) which is $(L_1,\dots,L_d)$-periodic with respect to y. Assume that ε is a positive parameter and consider the parametric reaction–advection–diffusion problem*

$$\begin{cases} u_t = \varepsilon\nabla\cdot(A(y)\nabla u) + q_1(y)\,\partial_x u(t,x,y) + f(x,y,u), & t\in\mathbb{R},\ (x,y)\in\Omega,\\ \nu\cdot A\,\nabla u(t,x,y)=0, & t\in\mathbb{R},\quad (x,y)\in\partial\Omega, \end{cases} \tag{3.10}$$

where $q\not\equiv 0$ over $\mathbb{R}\times\overline{\omega}$ and q has a zero average. Then,

$$\lim_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,q,f}(e) = \max_{y\in\overline{\omega}}(-q_1(y)) = \max_{\overline{\omega}}(-q.e). \tag{3.11}$$

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The situation in this result is more general than that considered in part (b) of Corollary 4.5 in [4]. In details, the coefficients A and f can be both non-constant. Meanwhile, in the result of [4], the coefficients considered were assumed to satisfy the alternative (3.7).

After having the exact value of $\lim_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,0,f}(e)/\sqrt{\varepsilon}$, we move now to investigate the limit of the minimal speed of propagation, considered as a function of the period of the coefficients of the reaction–diffusion equation set in the whole space $\mathbb{R}^N$, when the periodicity parameter tends to $+\infty$. By making some change in variables, we will find a link between this problem and Theorem 3.1.

Theorem 3.4 *Let $e = (1, 0, \dots, 0) \in \mathbb{R}^N$. An element $z \in \mathbb{R}^N$ is represented as $z = (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$. Assume that $f = f(x, y, u)$ and $A = A(y)$ satisfy* (3.3), (3.4) *and* (3.5) *with $\omega = \mathbb{R}^{N-1}$, $d = N - 1$ and $l = L_1 = \cdots = L_{N-1} = 1$. (That is, the domain and the coefficients of the equation are $(1, 1, \dots, 1)$ periodic with respect to y.) Assume furthermore, that A and f satisfy either* (3.7) *or* (3.8). *For each $L > 0$, and $(x, y) \in \mathbb{R}^N$, let $A_L(y) = A\left(\frac{y}{L}\right)$ and $f_L(x, y, u) = f\left(\frac{x}{L}, \frac{y}{L}, u\right)$. Consider the reaction–diffusion problem*

$$\begin{aligned} u_t(t, x, y) &= \nabla \cdot (A_L \nabla u)(t, x, y) + f_L(x, y, u), \ (t, x, y) \in \mathbb{R} \times \mathbb{R}^N \\ &= \nabla \cdot \left(A\left(\frac{y}{L}\right) \nabla u \right)(t, x, y) + f\left(\frac{x}{L}, \frac{y}{L}, u\right), \ (t, x, yx) \in \mathbb{R} \times \mathbb{R}^N, \end{aligned} \tag{3.12}$$

whose coefficients are $(L, \dots, L)$-periodic with respect to $(x, y) \in \mathbb{R}^N$. Then,

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N, A_L, 0, f_L}(e) = 2\sqrt{\max_{y\in\mathbb{R}^{N-1}} \zeta(y)}\sqrt{\max_{y\in\mathbb{R}^{N-1}} e.Ae(y)}. \tag{3.13}$$

The above theorem gives the limit of the minimal speed of propagation in the direction of $e = (1, 0, \dots, 0)$ as the periodicity parameter $L \to +\infty$. The domain is the whole space $\mathbb{R}^N$ which is $(L, \dots, L)$-periodic whatever the positive number L be. However, one can find

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N, A_L, Lq_L, f_L}(e)$$

whenever q is a shear flow advection. Namely, in the same manner that Theorem 3.1 implies Theorem 3.4, one can prove that Theorem 3.3 implies Theorem 3.5.

Theorem 3.5 *Let $e = (1, 0, \dots, 0) \in \mathbb{R}^N$. Assume that $f = f(x, y, u)$ and $A = A(y)$ satisfy* (3.3), (3.4) *and* (3.5) *with $\omega = \mathbb{R}^{N-1}$, $d = N - 1$ and $l = L_1 = \cdots = L_{N-1} = 1$. (That is, the domain and the coefficients of the equation are $(1, 1, \dots, 1)$-periodic with respect to y in $\mathbb{R}^{N-1}$.) Assume, furthermore, that for all $y \in \mathbb{R}^{N-1}$, there exists $\alpha(y)$ positive so that $A(y)e = \alpha(y)e$ in $\mathbb{R}^{N-1}$. Let $q = (q_1(y), 0, \dots, 0)$ for all $y \in \mathbb{R}^{N-1}$ such that $q_1 \not\equiv 0$ over $\mathbb{R}^{N-1}$, q is $(1, \dots, 1)$-periodic with respect to y and q_1 has a zero average. Then,*

$$\lim_{L\to+\infty} c^*_{\mathbb{R}^N, A_L, Lq_L, f_L}(e) = \max_{y\in\mathbb{R}^{N-1}} (-q_1(y)) = \max_{\mathbb{R}^{N-1}} (-q.e). \tag{3.14}$$

In the proof of Theorem 3.3 (which implies Theorem 3.5), the assumption that the advection q is in the form of shear flows plays an important role in reducing the elliptic equation involved by the variational formula (7.13) below. Namely, since

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$q = (q_1(y), 0, \ldots, 0)$ and since $e = (1, 0, \ldots, 0)$, then the terms $q(x, y) \cdot \nabla_{x,y}\psi$ and $q(x, y) \cdot e$ (in the general elliptic equation) become equal to $q_1(y)\partial_x\psi$ and $q_1(y)$, respectively. As a consequence, and due to the uniqueness of the principal eigenfunction ψ up to multiplication by a constant, we are able to choose ψ independent of x, and hence, obtain a symmetric elliptic operator (without drift) whose principal eigenvalue was given by the variational formula (7.15) below (see Section 7 for more details).

Remark 3.6 After the above explanations, we find that the techniques used to prove Theorem 3.3 which implies 3.5, will no longer work in the presence of a general periodic advection field satisfying (2.3).

Concerning the influence of advection, we mention that the limit of $c^*_{\Omega,A,Bq,f}(e)/B$ as $B \to +\infty$ (in the general periodic setting) is not yet given explicitly as a function of the direction e and the coefficients A, q and f. For more details one can see Theorem 4.1 in [4]. However, the problem of front propagation in an infinite cylinder with an underlying shear flow was widely studied in [1, 8]. In the case of strong advection, assume that $\Omega = \mathbb{R} \times \omega$, where ω is a bounded smooth subset of $\mathbb{R}^{N-1}$, $q = (q_1(y), 0, \ldots, 0)$, $y \in \omega$, and $f = f(u)$ is a (KPP) non-linearity. It was proved, in [16], that

$$\lim_{B \to +\infty} \frac{c^*_{\Omega,A,Bq,f}(e)}{B} = \gamma, \tag{3.15}$$

where

$$\gamma = \sup_{\psi \in D} \int_\omega q_1(y)\, \psi^2\, dy,$$

$$D = \left\{ \psi \in H^1(w), \quad \int_\omega |\nabla\psi|^2\, dy \leqslant f'(0) \text{ and } \int_\omega \psi^2\, dy = 1 \right\}.$$

4 Minimal speed within large diffusion factors or within small period coefficients

After having the limit of $c^*_{\Omega,\varepsilon A,0,f}(e)/\sqrt{\varepsilon}$ as $\varepsilon \to 0^+$, and after knowing that this limit depends on $\max_{y \in \overline{w}} \zeta(y)$ and $\max_{y \in \overline{w}} eAe(y)$, we investigate now the limit of $c^*_{\Omega,MA,q,f}(e)/\sqrt{M}$ as the diffusion factor M tends to $+\infty$, and we try to answer this question in a situation which is more general than that we considered in the previous section (in the case where the diffusion factor was going to 0^+). That is in the presence of an advection field and in a domain Ω which satisfies (2.1) and which may take more forms other than those of Section 3. We will find that in the case of large diffusion, the limit will depend on

$$\fint_C \zeta(x, y)\, dx\, dy := \frac{1}{|C|} \int_C \zeta(x, y)\, dx\, dy \quad \text{and} \quad \fint_C \tilde{e}A\tilde{e}(x, y)\, dx\, dy := \frac{1}{|C|} \int_C \tilde{e}A\tilde{e}(x, y)\, dx\, dy,$$

where C denotes the periodicity cell of the domain Ω.

Theorem 4.1 *Under the assumptions* (2.1) *for* Ω, (2.3) *for the advection* q, (2.4) *and* (2.5) *for the non-linearity* $f = f(x, y, u)$, *let* e *be any unit direction of* $\mathbb{R}^d$. *Assume that the diffusion*

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matrix $A = A(x,y)$ *satisfies* (2.2) *together with* $\nabla \cdot A\tilde{e} \equiv 0$ *over* Ω, *and* $\nu \cdot A\tilde{e} = 0$ *over* $\partial\Omega$. *For each* $M > 0$ *and* $0 \leqslant \gamma \leqslant 1/2$, *consider the following reaction–advection–diffusion equation*

$$\begin{cases} u_t = M\nabla \cdot (A(x,y)\nabla u) + M^{\gamma} q(x,y) \cdot \nabla u + f(x,y,u), \ t \in \mathbb{R}, \ (x,y) \in \Omega, \\ \nu \cdot A \nabla u(t,x,y) = 0, \ t \in \mathbb{R}, \ (x,y) \in \partial\Omega. \end{cases}$$

Then

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,MA,M^{\gamma} q,f}(e)}{\sqrt{M}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\sqrt{\fint_C \zeta(x,y)\,dx\,dy},$$

where C *is the periodicity cell of* Ω.

Remark 4.2

- The setting in Theorem 4.1 is more general than that in Theorem 3.1, where $\Omega = \mathbb{R}\times\omega$, $\tilde{e} = (1,0,\ldots,0)$ and $A\tilde{e} = \alpha(y)\tilde{e}$. Under the assumptions of Theorem 3.1, the domain Ω is invariant in the direction of $A\tilde{e}$, which is that of $\tilde{e}$. Consequently, if ν denotes the outward normal on $\partial\Omega = \mathbb{R}\times\partial\omega$, one gets $\nu \cdot A\tilde{e} = \alpha(y)\,\nu\cdot\tilde{e} = 0$ over $\partial\Omega$, while $\nabla\cdot(A\tilde{e}) = \partial/\partial x\alpha(y) = 0$ over Ω. Moreover, in Theorem 3.1, we have only reaction and diffusion terms. That is $q \equiv 0$. Therefore, considering the setting of Theorem 3.1, and taking βA as a parametric diffusion matrix, one consequently knows the limits of $c^*_{\Omega,\beta A,0,f}(e)/\sqrt{\beta}$ as $\beta \to 0^+$ (Theorem 3.1) and as $\beta \to +\infty$ (Theorem 4.1).
- The other observation in Theorem 4.1 is that the limit does not depend on the advection field q. This may play an important role in drawing counter examples to answer many different questions. For example, the variation of the minimal speed of propagation with respect to the diffusion factor and with respect to diffusion matrices which are symmetric positive definite.
- Another important feature, in Theorem 4.1, is that the order of M in the denominator of the ratio $c^*_{\Omega,MA,M^{\gamma} q,f}(e)/\sqrt{M}$ is equal to $1/2$. It is independent of γ. Consequently, *the case where the advection is null* and there is only a reaction–diffusion equation follows, in particular, from the previous theorem. That is

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,MA,0,f}(e)}{\sqrt{M}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\sqrt{\fint_C \zeta(x,y)\,dx\,dy}.$$

- The previous point leads us to conclude that the presence of an advection with a factor M^{γ}, where $0 \leqslant \gamma \leqslant 1/2$, will have no more effect on the ratio $c^*_{\Omega,MA,M^{\gamma} q,f}(e)/\sqrt{M}$ as soon as the diffusion factor M gets very large.

As far as the limit of the minimal speed of propagation within small periodic coefficients in the reaction–diffusion equation is concerned, the following theorem, which mainly depends on Theorem 4.1, treats this problem:

Theorem 4.3 *Let* $\Omega = \mathbb{R}^N$. *Assume that* $A = A(x,y)$, $q = q(x,y)$ *and* $f = f(x,y,u)$ *are* $(1,\ldots,1)$*-periodic with respect to* $(x,y) \in \mathbb{R}^N$, *and that they satisfy* (2.2), (2.3), (2.4)

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and (2.5) *with* $L_1 = \cdots = L_N = 1$. *Let* e *be any unit direction of* $\mathbb{R}^N$, *such that* $\nabla \cdot A\tilde{e} \equiv 0$ *over* $\mathbb{R}^N$. *For each* $L > 0$, *let* $A_L(x,y) = A(x/L, y/L)$, $q_L(x,y) = q(x/L, y/L)$ *and* $f_L(x,y,u) = f(x/L, y/L, u)$, *where* $(x,y) \in \mathbb{R}^N$. *Consider the problem*

$$\begin{aligned} u_t(t,x,y) &= \nabla \cdot (A_L \nabla u)(t,x,y) + q_L \cdot \nabla u(t,x,y) + f_L(x,y,u),\ (t,x,y) \in \mathbb{R} \times \mathbb{R}^N, \\ &= \nabla \cdot \left(A\left(\frac{x}{L}, \frac{y}{L}\right) \nabla u \right)(t,x,y) + q\left(\frac{x}{L}, \frac{y}{L}\right) \cdot \nabla u(t,x,y) + f\left(\frac{x}{L}, \frac{y}{L}, u\right), \end{aligned} \tag{4.1}$$

whose coefficients are $(L, \ldots, L)$*-periodic with respect to* $(x,y) \in \mathbb{R}^N$. *Then,*

$$\lim_{L \to 0^+} c^*_{\mathbb{R}^N, A_L, q_L, f_L}(e) = 2 \sqrt{\fint_C \tilde{e} A \tilde{e}(x,y)\, dx\, dy} \sqrt{\fint_C \zeta(x,y)\, dx\, dy},$$

where, in this setting, $C = [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^N$.

The above result gives the limit in any space dimension. It depends on the assumption $\nabla \cdot (A\tilde{e}) \equiv 0$ in $\mathbb{R}^N$. However, if one takes $N = 1$, and denotes the diffusion coefficient by $a = a(x)$, $x \in \mathbb{R}$, then the previous result holds under the assumptions that a satisfies (2.2) and $da/dx \equiv 0$ in $\mathbb{R}$. In other words, it holds when a is a positive constant. Thus, it is interesting to mention that, in the one-dimensional case, the above limit was given in [13] and [17] within a general diffusion coefficient (which may not be constant over $\mathbb{R}$). In details, assume that $f = f(x,u) = (\zeta(x) - u)u$ is a 1-periodic (KPP) non-linearity satisfying (2.4) with (2.5), and $\mathbb{R} \ni x \mapsto a(x)$ is a 1-periodic function which satisfies $0 < \alpha_1 \leqslant a(x) \leqslant \alpha_2$, for all $x \in \mathbb{R}$, where α_1 and α_2 are two positive constants. For each $L > 0$, consider the reaction–diffusion equation

$$\partial_t u(t,x) = \frac{\partial}{\partial x}\left(a\left(\frac{x}{L}\right) \frac{\partial u}{\partial x} \right)(t,x) + \left[\zeta\left(\frac{x}{L}\right) - u(t,x) \right] u(t,x) \quad \text{for} \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{4.2}$$

It was derived in [13] and, formally, in [17] that

$$\lim_{L \to 0^+} c^*_{\mathbb{R}, a_L, 0, f_L}(e) = 2 \sqrt{< a >_H \cdot \int_0^1 \zeta(x)}, \tag{4.3}$$

where $< a >_H$ denotes the harmonic mean of the map $x \mapsto a(x)$ over $[0,1]$.

5 Minimal speed within small or large reaction coefficients

In this section, the parameter of the reaction–advection–diffusion problem is the coefficient B multiplied by the non-linearity f. In fact, it follows from Theorem 2.6 in [3] (recalled via Theorem 2.5 in the present paper) that the map $B \mapsto c^*_{\Omega,A,q,Bf}(e)/\sqrt{B}$ remains, with the assumption $\nu . A\tilde{e} = 0$ on $\partial\Omega$, bounded by two positive constants as B gets very large. Therefore, it is interesting to find the limit of $c^*_{\Omega,A,q,Bf}(e)/\sqrt{B}$ as $B \to +\infty$ even in some particular situations. Moreover, it is important to find the limit of the same quantity as $B \to 0^+$. We start with the case where $B \to +\infty$ and then we move to that where $B \to 0^+$.

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Theorem 5.1 *Let $e = (1, 0, \dots, 0) \in \mathbb{R}^N$ and $B > 0$. Assume that $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$, A and f satisfy the same assumptions of Theorem 3.1. That is, f and A satisfy* (3.3), (3.4) *and* (3.5), *and one of the two alternatives* (3.7) *and* (3.8). *Consider the reaction–diffusion equation*

$$\begin{cases} u_t(t,x,y) = \nabla \cdot (A(y)\nabla u)(t,x,y) + B f(x,y,u), \text{ for } (t,x,y) \in \mathbb{R} \times \Omega, \\ \nu \cdot A\nabla u = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R} \times \partial\omega. \end{cases} \tag{5.1}$$

Then,

$$\lim_{B \to +\infty} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} = 2\sqrt{\max_{y \in \overline{\omega}} \zeta(y)} \sqrt{\max_{y \in \overline{\omega}} eAe(y)}. \tag{5.2}$$

We mention that one can find the coefficients A and f, and the domain Ω of the problem (5.1) satisfying all the assumptions of Theorem 5.1, which are the same of Theorem 3.1, including one of the alternatives (3.7) and (3.8) while one of ζ and eAe is *not constant*. Owing to Theorem 1.10 in [3], it follows that

$$\forall B > 0, \quad c^*_{\Omega,A,0,Bf}(e) \lneqq 2\sqrt{B} \sqrt{\max_{y \in \overline{\omega}} \zeta(y)} \sqrt{\max_{y \in \overline{\omega}} eAe(y)},$$

which is equivalent to saying that

$$\frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} \lneqq 2\sqrt{\max_{y \in \overline{\omega}} \zeta(y)} \sqrt{\max_{y \in \overline{\omega}} eAe(y)}.$$

Therefore, there are *heterogeneous* settings in which the result found in Theorem 5.1 does not follow trivially.

We move now to study the limit when the reaction factor B tends to 0^+. However, the situation will be more general than that in Theorem 5.1 because it will consider reaction–advection–diffusion equations rather than considering reaction–diffusion equations only.

Theorem 5.2 *Under the assumptions* (2.1) *for Ω,* (2.3) *for the advection q,* (2.4) *and* (2.5) *for the non-linearity $f = f(x,y,u)$, let e be any unit direction of $\mathbb{R}^d$. Assume that the diffusion matrix $A = A(x,y)$ satisfies* (2.2) *together with $\nabla \cdot A\tilde{e} \equiv 0$ over Ω, and $\nu \cdot A\tilde{e} = 0$ over $\partial\Omega$. For each $B > 0$ and $\gamma \geqslant 1/2$, consider the following reaction–advection–diffusion equation*

$$\begin{cases} u_t = \nabla \cdot (A(x,y)\nabla u) + B^\gamma q(x,y) \cdot \nabla u + B f(x,y,u), \quad t \in \mathbb{R}, \ (x,y) \in \Omega, \\ \nu \cdot A \nabla u(t,x,y) = 0, \quad t \in \mathbb{R}, \quad (x,y) \in \partial\Omega. \end{cases}$$

Then

$$\lim_{B \to 0^+} \frac{c^*_{\Omega,A,B^\gamma q,Bf}(e)}{\sqrt{B}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy} \sqrt{\fint_C \zeta(x,y)\,dx\,dy},$$

where C is the periodicity cell of Ω.

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Having the above result one can mark a sample of notes:

The order of B in the denominator of the ratio $c^*_{\Omega,A,B^\gamma q,Bf}(e)/\sqrt{B}$ is independent of γ (it is equal to $1/2$). Thus, whenever the advection is null, one gets

$$\lim_{B\to 0^+} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\,\sqrt{\fint_C \zeta(x,y)\,dx\,dy}.$$

Therefore, one concludes that the presence of an advection with a factor B^γ, where $\gamma \geqslant 1/2$, will have no more effect on the limit of the ratio $c^*_{\Omega,A,B^\gamma q,Bf}(e)/\sqrt{B}$ as the reaction factor B gets very small.

On the other hand, it is easy to check that the assumptions in Theorem 5.2 are more general than those in Theorem 5.1. Consequently, once we are in the more strict setting, which is that of Theorem 5.1, we are able to know both limits of $c^*_{\Omega,A,0,Bf}(e)/\sqrt{B}$ as $B \to +\infty$ and as $B \to 0^+$.

6 Variations of the minimal speed with respect to diffusion and reaction factors and with respect to periodicity parameters

After having studied the limits and the asymptotic behaviours of the functions $\varepsilon \mapsto c^*_{\Omega,\varepsilon A,0,f}(e)/\sqrt{\varepsilon}$, $M \mapsto c^*_{\Omega,MA,M^\gamma q,f}(e)/\sqrt{M}$ (for very large M and for $0 \leqslant \gamma \leqslant 1/2$), $B \mapsto c^*_{\Omega,A,B^\gamma q,Bf}(e)/\sqrt{B}$ ($\gamma \geqslant 1/2$) and $L \mapsto c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$, where L is a periodicity parameter, we move now to investigate the variations of these functions with respect to the diffusion and reaction factors and with respect the periodicity parameter L. The present section will be devoted to discuss and answer these questions.

We sketch first the form of the domain. $\Omega \subseteq \mathbb{R}^N$ is assumed to be in the form $\mathbb{R} \times \omega$ which was taken in Section 3. As a review, $\Omega = \mathbb{R} \times \omega \subseteq \mathbb{R}^N$, where $\omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-d-1}$ ($d \geqslant 0$). If $d = 0$, the subset ω is a bounded open subset of $\mathbb{R}^{N-1}$. While, in the case where $1 \leqslant d \leqslant N-1$, ω is a $(L_1,\dots,L_d)$-periodic open domain of $\mathbb{R}^{N-1}$ which satisfies (2.1); and hence, Ω is a $(l,L_1,\dots,L_d)$-periodic subset of $\mathbb{R}^N$ that satisfies (2.1) with $l > 0$. An element of $\Omega = \mathbb{R} \times \omega$ will be represented as $z = (x,y)$ where $y \in \omega \subseteq \mathbb{R}^d \times \mathbb{R}^{N-1-d}$. With a domain of such form, we have:

Theorem 6.1 *Let* $e = (1,0,\dots,0) \in \mathbb{R}^N$. *Assume that* Ω *has the form* $\mathbb{R} \times \omega$ *which is described above, and that the diffusion matrix* $A = A(y)$ *satisfies* (3.5) *together with the assumption*

$$A(x,y)e = A(y)e = \alpha(y)e, \text{ for all } (x,y) \in \mathbb{R} \times \overline{\omega}, \tag{6.1}$$

where $y \mapsto \alpha(y)$ *is a positive* $(L_1,\dots,L_d)$*-periodic function defined over* $\overline{\omega}$. *The non-linearity* f *is assumed to satisfy* (3.3) *and* (3.4). *Moreover, one assumes that, at least, one of* $\tilde{e}\cdot A\tilde{e}$ *and* ζ *is not constant. Besides, the advection field* q *(when it exists) is in the form* $q(x,y) = (q_1(y),0,\dots,0)$ *where* q_1 *has a zero average over* C, *the periodicity cell of* ω. *For each* $\beta > 0$ *consider the reaction–advection–diffusion problem*

$$\begin{cases} u_t = \beta\,\nabla\cdot(A(y)\nabla u) + \sqrt{\beta}\,q_1(y)\,\partial_x u + f(x,y,u), \ t \in \mathbb{R}, \ (x,y) \in \mathbb{R}\times\omega, \\ \nu\cdot A\,\nabla u(t,x,y) = 0, \quad t \in \mathbb{R}, \ (x,y) \in \partial\Omega. \end{cases}$$

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Then the map $\beta \mapsto c^*_{\Omega,\beta A,\sqrt{\beta}\, q,f}(e)/\sqrt{\beta}$ *is decreasing in* $\beta > 0$, *and by Theorem 4.1, one has*

$$\lim_{\beta\to+\infty} \frac{c^*_{\Omega,\beta A,\sqrt{\beta}\, q,f}(e)}{\sqrt{\beta}} = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(y)\,dy}\,\sqrt{\fint_C \zeta(y)\,dy},$$

where C *is the periodicity cell of* ω.

Remark 6.2 In the same setting of Theorem 6.1 but with no advection, that is $q_1 \equiv 0$, we still have $\beta \mapsto c^*_{\Omega,\beta A,0,f}(e)/\sqrt{\beta}$ as a decreasing map in $\beta > 0$. Moreover, if one of the alternatives (3.7) and (3.8) holds and there is no advection, Theorem 3.1 yields that

$$\lim_{\beta\to 0^+} \frac{c^*_{\Omega,\beta A,0,f}(e)}{\sqrt{\beta}} = 2\sqrt{\max_{y\in\overline{\omega}} \tilde{e}A\tilde{e}(y)}\,\sqrt{\max_{y\in\overline{\omega}} \zeta(y)}.$$

The preceding result yields another one concerned in the variation of the minimal speeds with respect to the periodicity parameter L. In the following, the domain will be the whole space $\mathbb{R}^N$. We choose the diffusion matrix $A(x,y) = A(y)$, the shear flow q and reaction term f to be $(1,\dots,1)$-periodic and to satisfy some restrictions. For each $L > 0$, we assign the diffusion matrix $A_L(x,y) = A(x/L, y/L)$, the advection field $q_L(x,y) = q(x/L, y/L)$ and the non-linearity $f_L = f(x/L, y/L, u)$ and we are going to study the variation, with respect to the periodicity parameter L, of the minimal speed $c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$, which corresponds to the reaction–advection–diffusion equation within the $(L,\dots,L)$-periodic coefficients A_L, q_L and f_L:

Theorem 6.3 *Let* $e = (1,0,\dots,0) \in \mathbb{R}^N$. *An element* $z \in \mathbb{R}^N$ *is represented as* $z = (x,y) \in \mathbb{R}\times\mathbb{R}^{N-1}$. *Assume that* $A(x,y) = A(y)$ *(for all* $(x,y) \in \mathbb{R}^N$*) and* $f(x,y,u)$ *satisfy* (3.3), (3.4) *and* (3.5) *with* $\omega = \mathbb{R}^{N-1}$, $d = N-1$ *and* $l = L_1 = \cdots = L_{N-1} = 1$. *Assume furthermore, that for all* $y \in \mathbb{R}^{N-1}$, $A(x,y)e = A(y)e = \alpha(y)e$, *where* $y \mapsto \alpha(y)$ *is a positive* $(1,\dots,1)$*-periodic function defined over* $\mathbb{R}^{N-1}$ *and that, at least, one of* $\tilde{e}\cdot A\tilde{e}$ *and* ζ *is not constant. Let* q *be an advection field satisfying* (2.3) *and having the form* $q(x,y) = (q_1(y),0,\dots,0)$ *for each* $(x,y) \in \mathbb{R}^N$. *Consider the reaction–advection–diffusion problem,*

$$\begin{gathered}\forall\,(t,x,y) \in \mathbb{R}\times\mathbb{R}^N,\\ u_t(t,x,y) = \nabla\cdot(A_L(y)\nabla u)(t,x,y) + (q_1)_L(y)\partial_x u(t,x,y) + f_L(x,y,u),\end{gathered} \tag{6.2}$$

whose coefficients are $(L,\dots,L)$*-periodic with respect to* $(x,y) \in \mathbb{R}^N$.

Then, the map $L \mapsto c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$ *is increasing in* $L > 0$.

Remark 6.4 The assumptions of Theorem 6.3 cannot be fulfilled whenever $N = 1$. However, assuming that $N = 1$ and that the function

$$\frac{\zeta}{<\zeta>_A} + \frac{<a>_H}{a}$$

is not identically equal to 2 (where $a(x)$ is the diffusion factor, $<a>_H$ and $<\zeta>_A$ are, respectively, the harmonic mean of $x \mapsto a(x)$ and arithmetic mean of $x \mapsto \zeta(x)$ over $[0,1]$),

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it was proved, in [13], that $L \mapsto c^*_{\mathbb{R}^N, a_L, q_L, f_L}(e)$ is increasing in L when L is close to 0. In particular, if a is constant and ζ is not constant, or if μ is constant and a is not constant, then $L \mapsto c^*_{\mathbb{R}^N, a_L, q_L, f_L}(e)$ is increasing when L is close to 0.

Concerning now the variation with respect to the reaction factor B, we have the following:

Theorem 6.5 *Assume that $\Omega = \mathbb{R} \times \omega$ and the coefficients A, q and f satisfy the same assumptions of Theorem 6.1. Let $e = (1, 0, \ldots, 0)$ and for each $B > 0$, consider the reaction–advection–diffusion problem*

$$\begin{cases} u_t = \nabla \cdot (A(y)\nabla u) + \sqrt{B}\, q_1(y)\, \partial_x u + Bf(x, y, u), & t \in \mathbb{R},\ (x, y) \in \mathbb{R} \times \omega, \\ \nu \cdot A\, \nabla u(t, x, y) = 0, & t \in \mathbb{R},\ (x, y) \in \partial\Omega. \end{cases}$$

Then, the map $B \mapsto \frac{c^*_{\Omega, A, \sqrt{B} q, Bf}(e)}{\sqrt{B}}$ is increasing in $B > 0$.

As a first note, we mention that Theorem 6.5 holds also in the case where there is no advection. On the other hand, Berestycki, Hamel and Nadirashvili [3] proved that the map $B \mapsto c^*_{\Omega, A, q, Bf}(e)$ is increasing in $B > 0$ under the assumptions (2.1), (2.2), (2.3), (2.4) and (2.5) which are less strict than the assumptions considered in our present theorem. However, the present theorem is concerned in the variation of the map $B \mapsto c^*_{\Omega, A, \sqrt{B} q, Bf}(e)/\sqrt{B}$ rather than that of $B \mapsto c^*_{\Omega, A, q, Bf}(e)$.

Remark 6.6 Owing to the same justifications given after Theorem 3.5, one concludes the importance of taking, in Section 6, an advection in the form of shear flows. To study the variations of the minimal speeds as in Theorems 6.1, 6.3 and 6.5, but in a more general framework (general advection fields, general diffusion, etc.…), formula 2.7 remains an important tool. However, we will no longer have variational formulations as (7.62) below. These problems remain open in the general periodic framework.

7 Proofs of the announced results

In this section, we are going to demonstrate the theorems announced in Sections 3, 4, 5 and 6. We will proceed in four subsections, each devoted to proving the results announced in a corresponding section.

7.1 Proofs of Theorems 3.1, 3.3 and 3.4

Proof of Theorem 3.1. Under the assumptions of Theorem 3.1, we can apply the variational formula (2.7) of the minimal speed. Consequently,

$$c^*_{\Omega, \varepsilon A, 0, f}(e) = \min_{\lambda > 0} \frac{k_{\Omega, e,\ \varepsilon A,\ 0, \zeta}(\lambda)}{\lambda}, \tag{7.1}$$

where $k_{\Omega, e, \varepsilon A, 0, \zeta}(\lambda)$ is the first eigenvalue (for each $\lambda,\ \varepsilon > 0$) of the eigenvalue problem

$$\begin{cases} L_{\Omega, e,\ \varepsilon A,\ 0, \zeta, \lambda}\, \psi = k_{\Omega, e,\ \varepsilon A,\ 0, \zeta}(\lambda)\, \psi(x, y) \text{ over } \mathbb{R} \times \omega; \\ \nu \cdot A \nabla \psi = 0 \quad \text{on } \mathbb{R} \times \partial\omega, \end{cases} \tag{7.2}$$

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and

$$L_{\Omega,e,\,\varepsilon A,\,0,\zeta,\lambda}\psi(x,y) = \varepsilon\nabla\cdot(A(y)\nabla\psi(x,y)) - 2\,\varepsilon\,\lambda Ae\cdot\nabla\psi(x,y) + [\varepsilon\,\lambda^2 eA(y)e - \lambda\,\varepsilon\nabla\cdot(A(y)e) + \zeta(y)]\psi(x,y),$$

for all $(x,y)\in\mathbb{R}\times\omega$.

Initially, the boundary condition in (7.2) is $\nu\cdot A\nabla\psi = \lambda\,\nu\cdot Ae$ on $\partial\Omega = \mathbb{R}\times\partial\omega$; where $\nu(x,y)$ is the unit outward normal at $(x,y)\in\partial\Omega$. However, $\Omega = \mathbb{R}\times\omega$ is invariant in the direction of e which is that of Ae in both alternatives (3.7) and (3.8). Consequently, $\nu\cdot Ae\equiv 0$ on $\partial\Omega$.

We recall that for all $\lambda>0$, and for all $\varepsilon>0$, we have $k_{\Omega,e,\,\varepsilon A,\,0,\zeta}(\lambda)>0$. Also, the first eigenfunction of (7.2) is positive over $\overline{\Omega} = \mathbb{R}\times\overline{\omega}$, and it is unique up to multiplication by a non-zero constant.

In our present setting, whether in (3.7) or (3.8) and due to the assumption (3.4), one concludes that the coefficients in $L_{\Omega,e,\,\varepsilon A,\,0,\zeta,\lambda}$ are independent of x. Moreover, in both alternatives (3.7) and (3.8), the direction of Ae is the same of $e=(1,0,\dots,0)$. On the other hand, since $\Omega=\mathbb{R}\times\omega$, then for each $(x,y)\in\partial\Omega$, we have $\nu(x,y)=(0;\nu_\omega(y))$, where $\nu_\omega(y)$ is the outward unit normal on $\partial\omega$ at y. Consequently, the first eigenfunction of (7.2) is independent of x and *the eigenvalue problem* (7.2) *is reduced to*

$$\begin{cases} L_{\Omega,e,\,\varepsilon A,\,0,\,\zeta,\,\lambda}\phi : = \varepsilon\nabla\cdot(A(y)\nabla\phi(y)) + [\varepsilon\,\lambda^2 eA(y)e+\zeta(y)]\phi(y) \\ \qquad\qquad = k_{\Omega,e,\,\varepsilon A,\,0,\,\zeta}(\lambda)\,\phi \quad \text{over } \omega; \\ \nu(x,y)\cdot A(y)\nabla\phi(y) = (0;\nu_\omega(y))\cdot A(y)\nabla\phi(y) = 0 \quad \text{on } \mathbb{R}\times\partial\omega, \end{cases} \tag{7.3}$$

where $\phi=\phi(y)$ is positive over $\overline{\omega}$, L-periodic (since the domain ω and the coefficients of $L_{\Omega,e,\,\varepsilon A,0,\zeta,\lambda}$ are L-periodic), unique up to multiplication by a constant, and belongs to $C^2(\overline{\omega})$.

In the case where $d\geqslant 1$, let $C\subseteq\mathbb{R}^{N-1}$ denote the periodicity cell of ω. Otherwise, $d=0$ and one takes $C=\omega$. In both cases, C is bounded. Multiplying the first line of (7.3) by ϕ, and integrating by parts over C, one gets

$$-k_{\Omega,e,\,\varepsilon A,\,0,\,\zeta}(\lambda) = \frac{\varepsilon\displaystyle\int_C \nabla\phi\cdot A(y)\nabla\phi\,dy - \int_C [\varepsilon\lambda^2 eA(y)e + \zeta(y)]\,\phi^2(y)\,dy}{\displaystyle\int_C \phi^2(y)\,dy}. \tag{7.4}$$

One also notes that, in this present setting, the operator $L_{\Omega,e,\,\varepsilon A,0,\zeta,\lambda}$ is self-adjoint and its coefficients are $(L_1,\dots,L_d)$-periodic with respect $(y_1,\dots,y_d)$. Consequently, $-k_{\Omega,e,\,\varepsilon A,\,0,\zeta}(\lambda)$ has the following variational characterisation:

$$-k_{\Omega,e,\,\varepsilon A,\,0,\,\zeta}(\lambda) = \min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\varepsilon\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi\,dy - \int_C [\varepsilon\lambda^2 eA(y)e + \zeta(y)]\,\varphi^2(y)\,dy}{\displaystyle\int_C \varphi^2(y)\,dy}. \tag{7.5}$$

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In what follows, we will assume that (3.7) is the alternative that holds. That is, $eAe = \alpha$ is constant. The proof can be imitated easily whenever we assume that (3.8) holds.

The function $y \mapsto \zeta(y)$ is continuous and $(L_1, \ldots, L_d)$-periodic over $\overline{\omega}$, whose periodicity cell C is a bounded subset of $\mathbb{R}^{N-1}$ (whether $d = 0$ or $d \geqslant 1$). Let $y_0 \in \overline{C} \subseteq \overline{\omega}$ such that $\max_{y \in \overline{w}} \zeta(y) = \zeta(y_0)$ (trivially, this also holds when ζ is constant). Consequently, we have

$$\forall\, \varphi \in H^1(C) \setminus \{0\},\ \frac{\varepsilon \displaystyle\int_C \nabla\varphi \cdot A\nabla\varphi - \int_C (\varepsilon\alpha\lambda^2 + \zeta(y))\varphi^2}{\displaystyle\int_C \varphi^2(y)\, dy} \geqslant -[\varepsilon\alpha\lambda^2 + \zeta(y_0)].$$

This yields that

$$\forall\, \varepsilon > 0,\ \forall\, \lambda > 0,\ -k_{\Omega,e,\ \varepsilon A,\ 0,\zeta}(\lambda) \geqslant -[\varepsilon\alpha\lambda^2 + \zeta(y_0)]. \tag{7.6}$$

Consequently,

$$\forall\, \varepsilon > 0,\ \forall\, \lambda > 0,\ \frac{k_{\Omega,e,\ \varepsilon A,\ 0,\zeta}(\lambda)}{\lambda} \leqslant \lambda\,\alpha\varepsilon + \frac{\zeta(y_0)}{\lambda}. \tag{7.7}$$

However, the function $\lambda \mapsto \lambda\alpha\varepsilon + \zeta(y_0)/\lambda$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(\varepsilon) = \sqrt{\zeta(y_0)/\alpha\varepsilon}$. This minimum is equal to $2\sqrt{\zeta(y_0)} \times \sqrt{\alpha\,\varepsilon}$. From (7.7), we conclude that

$$\frac{k_{\Omega,e,\ \varepsilon A,\ 0,\zeta}(\lambda(\varepsilon))}{\lambda(\varepsilon)} \leqslant 2\sqrt{\alpha\varepsilon}\sqrt{\zeta(y_0)}.$$

Finally, (2.7) implies that $c^*_{\Omega,\varepsilon A,0,f}(e) = \min_{\lambda>0} k_{\Omega,e,\varepsilon A,0,\zeta}(\lambda)/\lambda \leqslant 2\sqrt{\alpha\,\varepsilon}\sqrt{\zeta(y_0)}$, or equivalently

$$\forall \varepsilon > 0,\ \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \leqslant 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{7.8}$$

We pass now to prove the other sense of the inequality for $\liminf_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,0,f}(e)/\sqrt{\varepsilon}$. We will consider formula (7.5), and then organize a suitable function ψ which leads us to a lower bound of $\liminf_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,0,f}(e)/\sqrt{\varepsilon}$.

We have $\zeta(y_0) > 0$. Let δ be such that $0 < \delta < \zeta(y_0)$. Thus $0 < \zeta(y_0) - \delta < \max_{\overline{\omega}} \zeta(y)$. The continuity of ζ, over $\overline{C} \subseteq \overline{\omega}$, yields that there exists an open and bounded set $U \subset \overline{C}$ such that

$$\forall\, y \in \overline{U},\ \zeta(y_0) - \delta \leqslant \zeta(y). \tag{7.9}$$

Designate by ψ, a function in $\mathscr{D}(C)$ (a $C^\infty(C)$ function whose support is compact), with $\operatorname{supp}\psi \subseteq \overline{U}$, and $\int_U \psi^2 = 1$. One will have,

$$\begin{aligned}
&\forall \lambda > 0,\ \forall\, \varepsilon > 0,\\
&-k_{\Omega,e,\ \varepsilon A,\ 0,\zeta}(\lambda) \leqslant \varepsilon \int_U \nabla\psi \cdot A(y)\nabla\psi\, dy - \int_U [\varepsilon\lambda^2 eA(y)e + \zeta(y)]\,\psi^2(y)\, dy\\
&\qquad\qquad \leqslant \varepsilon \int_U \nabla\psi \cdot A(y)\nabla\psi\, dy - [\varepsilon\lambda^2\alpha + \zeta(y_0) - \delta] \int_U \psi^2(y)\, dy\\
&\qquad\qquad \leqslant \varepsilon \int_U \alpha_2|\nabla\psi|^2 - [\varepsilon\lambda^2\alpha + \zeta(y_0) - \delta],\ \text{by (3.5)},
\end{aligned}$$

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or equivalently

$$\frac{k_{\Omega,e,\,\varepsilon A,\,0,\zeta}(\lambda)}{\lambda} \geqslant \lambda\alpha\varepsilon + \frac{1}{\lambda}\,\beta(\varepsilon), \tag{7.10}$$

where $\beta(\varepsilon) = \zeta(y_0) - \delta - \varepsilon \int_U \alpha_2 |\nabla\psi|^2$. Choosing $0 < \varepsilon < \zeta(y_0) - \delta/\alpha_2 \int_U |\nabla\psi|^2$ (this is possible), we get $\beta(\varepsilon) > 0$.

The map $\lambda \mapsto \lambda\alpha\varepsilon + 1/\lambda\ \beta(\varepsilon)$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(\varepsilon) = \sqrt{\beta(\varepsilon)/\varepsilon\alpha}$. This minimum is equal to $2\sqrt{\varepsilon\,\alpha}\sqrt{\beta(\varepsilon)}$.

Now, referring to formula (7.10), one gets

$$\text{For } \varepsilon \text{ small enough,} \quad \frac{k_{\Omega,e,\,\varepsilon A,\,0,\zeta}(\lambda)}{\lambda} \geqslant 2\sqrt{\varepsilon\alpha}\,\sqrt{\beta(\varepsilon)} \quad \text{for all } \lambda > 0.$$

Together with (2.7), we conclude that

$$\text{for } \varepsilon \text{ small enough,} \quad \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \geqslant 2\sqrt{\beta(\varepsilon)}\sqrt{\alpha}. \tag{7.11}$$

Consequently,

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} &\geqslant \liminf_{\varepsilon\to 0^+} 2\sqrt{\beta(\varepsilon)}\sqrt{\alpha} \\ &= 2\sqrt{\zeta(y_0) - \delta}\sqrt{\alpha} \quad \text{(since } \psi \text{ is independent of } \varepsilon),\end{aligned}$$

and this holds for all $0 < \delta < \zeta(y_0)$. Therefore, one can conclude that

$$\liminf_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} \geqslant 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{7.12}$$

Finally, the inequalities (7.8) and (7.12) imply that $\lim_{\varepsilon\to 0^+} \frac{c^*_{\Omega,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}}$ exists, and it is equal to

$$2\sqrt{\alpha}\sqrt{\zeta(y_0)} = 2\sqrt{\max_{\overline{\omega}} eA(y)e}\sqrt{\max_{\overline{\omega}} \zeta(y)}.$$

We note that the same ideas of this proof can be easily applied in the case where the assumption (3.8) holds. In (3.8), we have ζ is constant; however, eAe is not in general. Meanwhile the converse is true in the case (3.7). The little difference is that, in the case of (3.8), we choose the subset U (of the proof done above) around the point y_0 where eAe attains its maximum and then we continue by the same way used above. □

Proof of Theorem 3.3 We have

$$c^*_{\Omega,\varepsilon A,0,f}(e) = \min_{\lambda>0} \frac{k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)}{\lambda}, \tag{7.13}$$

where (due to the facts that q is a shear flow, $e = (1, 0, \ldots, 0)$ and e is an eigenvector of the matrix $A(y)$ for all $y \in \overline{\omega}$) $k_{\Omega,e,\varepsilon A,q,\zeta}(\lambda)$ is the principal eigenvalue of the problem

$$\begin{cases} L_{\Omega,e,\varepsilon A,q,\zeta,\lambda}\psi(x,y) = k_{\Omega,e,\varepsilon A,q,\zeta}(\lambda)\,\psi(x,y) & \text{over } \mathbb{R}\times\omega; \\ v\cdot A\nabla\psi = 0 & \text{on } \mathbb{R}\times\partial\omega, \end{cases}$$

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with

$$L_{\Omega,e,\varepsilon A,\,q,\zeta,\lambda}\,\psi = \varepsilon\nabla\cdot(A(y)\nabla\psi) - 2\varepsilon\lambda\,\alpha(y)\,\partial_x\psi + q_1(y)\partial_x\psi \\ + [\varepsilon\,\lambda^2 eA(y)e - \lambda q_1(y) + \zeta(y)]\psi \text{ over } \mathbb{R}\times\omega. \tag{7.14}$$

The uniqueness of the principal eigenfunction ψ up to multiplication by a constant, yields that one can choose ψ independent of x. Hence, the elliptic operator $L_{\Omega,e,\varepsilon A,\,q,\zeta,\lambda}$ can be reduced to the symmetric operator

$$L_{\Omega,e,\varepsilon A,\,q,\zeta,\lambda}\,\psi = \varepsilon\nabla\cdot(A(y)\nabla\psi) + [\varepsilon\,\lambda^2 eA(y)e - \lambda q_1(y) + \zeta(y)]\psi.$$

Consequently,

$$\forall\lambda>0,\ \forall\varepsilon>0,\quad -k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda) \\ = \min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\varepsilon\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi dy + \lambda\int_C q_1(y)\varphi^2 - \int_C [\lambda^2\varepsilon eA(y)e + \zeta(y)]\,\varphi^2(y)\,dy}{\displaystyle\int_C \varphi^2(y)\,dy}. \tag{7.15}$$

Formula (7.15) yields that

$$\forall\lambda>0,\ \forall\varepsilon>0,\ -k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda) \geqslant -\lambda\max_{y\in\overline{\omega}}(-q_1(y)) - \lambda^2\varepsilon\max_{y\in\overline{\omega}} eA(y)e - \max_{y\in\overline{\omega}}\zeta(y),$$

or equivalently

$$\forall\lambda>0,\ \forall\varepsilon>0,\ \frac{k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)}{\lambda} \leqslant \max_{y\in\overline{\omega}}(-q_1(y)) + \lambda\varepsilon\max_{y\in\overline{\omega}} eA(y)e + \frac{\max\limits_{y\in\overline{\omega}}\zeta(y)}{\lambda}.$$

Putting $\lambda=\lambda(\varepsilon)=\sqrt{\frac{\max_{y\in\overline{\omega}}\zeta(y)}{\varepsilon\max_{y\in\overline{\omega}e\cdot A(y)e}}}>0$ into the last inequality yields that

$$\min_{\lambda>0}\frac{k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)}{\lambda} \leqslant \max_{y\in\overline{\omega}}(-q_1(y)) + 2\sqrt{\varepsilon}\sqrt{\max_{y\in\overline{\omega}} e\cdot A(y)e}\sqrt{\max_{y\in\overline{\omega}}\zeta(y)},$$

and hence,

$$\limsup_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,q,f}(e) \leqslant \max_{y\in\overline{\omega}}(-q_1(y)). \tag{7.16}$$

Now, we take $y_0\in C$ (C is the periodicity cell of ω) such that $\max_{y\in\overline{\omega}}(-q_1(y)) = -q_1(y_0)>0$ (since q is periodic with respect to y, $q_1\not\equiv 0$ and q_1 has a zero average) and we take $\delta>0$ such $-q_1(y_0)-\delta>0$. It follows, from the continuity of q_1, that there exists an open subset $U\subset C$ such that $y_0\in U$ and

$$\forall y\in\overline{U},\ -q_1(y)\geqslant\max_{y\in\overline{\omega}}(-q_1(y))-\delta.$$

Let ψ be a function in $\mathscr{D}(C)$ with $\operatorname{supp}\psi\subseteq\overline{U}$, and $\int_U\psi^2=1$. Referring to (7.15), it follows that

$$\forall\lambda>0,\forall\varepsilon>0,\quad \frac{k_{\Omega,e,\,\varepsilon A,\,q,\zeta}(\lambda)}{\lambda} \geqslant -q_1(y_0)-\delta+\lambda\varepsilon\min_{y\in\overline{\omega}} e\cdot Ae + \frac{1}{\lambda}\,\beta(\varepsilon), \tag{7.17}$$

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where $\beta(\varepsilon) = \min_{y\in\overline{\omega}} \zeta(y) - \varepsilon \int_U \alpha_2 |\nabla\psi|^2 > 0$ for a small enough $\varepsilon > 0$ ($\alpha_2 > 0$ is the constant appearing in (3.5)).

It follows from (7.17) that

$$\forall \lambda > 0, \forall \varepsilon > 0, \quad \frac{k_{\Omega,e,\ \varepsilon A,\ q,\zeta}(\lambda)}{\lambda} \geqslant -q_1(y_0) - \delta + 2\sqrt{\varepsilon}\sqrt{\min_{y\in\overline{\omega}} e\cdot Ae}\sqrt{\beta(\varepsilon)}.$$

Together with (7.13), and since $\delta > 0$ is arbitrary, one gets

$$\liminf_{\varepsilon\to 0^+} c^*_{\Omega,\varepsilon A,q,f}(e) \geqslant -q_1(y_0) = \max_{y\in\overline{\omega}}(-q_1(y)). \tag{7.18}$$

Finally, (7.16) and (7.18) complete the proof of Theorem 3.3. □

Proof of Theorem 3.4 Consider the change of variables

$$v(t,x,y) = u(t,Lx,Ly), \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}.$$

The function u satisfies (3.12) if and only if v satisfies

$$v_t(t,x,y) = \frac{1}{L^2}\,\nabla\cdot(A(y)\nabla v)(t,x,y) + f(x,y,v) \text{ over } \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{N-1}. \tag{7.19}$$

Consequently,

$$\forall\, L > 0,\ c^*_{\mathbb{R}^N, A_L, 0, f_L}(e) = L\, c^*_{\mathbb{R}^N, \frac{1}{L^2}A, 0, f}(e) \tag{7.20}$$

Taking $\varepsilon = 1/L^2$, and applying Theorem 3.1 to problem (7.19), one then has

$$\lim_{L\to+\infty} \frac{c^*_{\mathbb{R}^N,\frac{1}{L^2}A,0,f}(e)}{\sqrt{\frac{1}{L^2}}} = \lim_{\varepsilon\to 0^+} \frac{c^*_{\mathbb{R}^N,\varepsilon A,0,f}(e)}{\sqrt{\varepsilon}} = 2\sqrt{\max_{y\in\mathbb{R}^{N-1}} \zeta(y)}\sqrt{\max_{y\in\mathbb{R}^{N-1}} eA(y)e}. \tag{7.21}$$

Finally, (7.20) together with (7.21) complete the proof of Theorem 3.4. □

7.2 Proofs of Theorems 4.1 and 4.3

Proof of Theorem 4.1 The proof will be divided into three steps:

Step 1. According to Theorem 2.4, and since $\nu\cdot A\tilde{e} = 0$ on $\partial\Omega$, the minimal speeds $c^*_{\Omega,MA,M^\gamma q,f}(e)$ are given by:

$$\forall M > 0,\ c^*_{\Omega,MA,\,M^\gamma q,f}(e) = \min_{\lambda>0} \frac{k_{\Omega,e,\ MA,\ M^\gamma q,\zeta}(\lambda)}{\lambda},$$

where $k_{\Omega,e,MA,M^\gamma q,\zeta}(\lambda)$ and $\psi^{\lambda,M}$ denote the unique eigenvalue and the positive L-periodic eigenfunction of the problem

$$\begin{aligned} &M\nabla\cdot(A\nabla\psi^{\lambda,M}) - 2M\lambda\tilde{e}\cdot A\nabla\psi^{\lambda,M} + M^\gamma q\cdot\nabla\psi^{\lambda,M} + [\lambda^2 M\,\tilde{e}A\tilde{e} - \lambda M^\gamma q\cdot\tilde{e} + \zeta]\psi^{\lambda,M} \\ &\quad = k_{\Omega,e,\ MA,\ M^\gamma q,\zeta}(\lambda)\psi^{\lambda,M} \text{ in } \Omega, \end{aligned}$$

with $\nu\cdot A\nabla\psi^{\lambda,M} = 0$ on $\partial\Omega$.

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For each $\lambda > 0$ and $M > 0$, let $\lambda' = \lambda\sqrt{M}$, and let $k_{\Omega,e,MA,M^\gamma q,\zeta}(\lambda) = \mu(\lambda', M)$. Consequently,

$$\forall M > 0, \quad \frac{c^*_{\Omega,MA,M^\gamma q,f^{(e)}}}{\sqrt{M}} = \min_{\lambda'>0} \frac{\mu(\lambda', M)}{\lambda'}, \tag{7.22}$$

where $\mu(\lambda', M)$ and $\psi^{\lambda',M}$ are the first eigenvalue and the unique, positive L-periodic (with respect to x) eigenfunction of

$$\begin{aligned} &M\nabla\cdot(A\nabla\psi^{\lambda',M}) - 2\lambda'\sqrt{M}\tilde{e}\cdot A\nabla\psi^{\lambda',M} + M^\gamma q\cdot\nabla\psi^{\lambda',M} \\ &\quad + \left[\lambda'^2\tilde{e}A\tilde{e} - \frac{\lambda'}{M^{\frac{1}{2}-\gamma}}\, q\cdot\tilde{e} + \zeta\right]\psi^{\lambda',M} \;=\; \mu(\lambda', M)\psi^{\lambda',M} \text{ in } \Omega, \end{aligned} \tag{7.23}$$

with $\nu\cdot A\nabla\psi^{\lambda',M} = 0$ on $\partial\Omega$.

Owing to the uniqueness, up to multiplication by positive constants, of the first eigenfunction of (7.23), one may assume that

$$\forall\,\lambda' > 0,\ \forall\, M > 0,\ \|\psi^{\lambda',M}\|_{L^2(C)} = 1. \tag{7.24}$$

Moreover, for each $M > 0$, $\min_{\lambda'>0} \frac{\mu(\lambda',M)}{\lambda'}$ is attained at $\lambda'_M > 0$. Thus,

$$\forall M > 0, \quad \frac{c^*_{\Omega,MA,M^\gamma q,f^{(e)}}}{\sqrt{M}} = \min_{\lambda'>0} \frac{\mu(\lambda', M)}{\lambda'} = \frac{\mu(\lambda'_M, M)}{\lambda'_M}. \tag{7.25}$$

The above characterization of $c^*_{\Omega,MA,M^\gamma q,f}(e)/\sqrt{M}$ will be used in the next steps in order to prove that $\liminf_{M\to+\infty} c^*_{\Omega,MA,M^\gamma q,f}(e)/\sqrt{M}$ (resp. $\limsup_{M\to+\infty} c^*_{\Omega,MA,M^\gamma q,f}(e)/\sqrt{M}$) is greater than (resp. less than) $2\sqrt{⨍_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\,\sqrt{⨍_C \zeta(x,y)\,dx\,dy}$; and hence, complete the proof.

Step 2. Fix $\lambda' > 0$ and $M > 0$. We divide (7.23) by $\psi^{\lambda',M}$ then, using the facts $\nabla.A\tilde{e} \equiv 0$ in Ω and $\nu\cdot A\tilde{e} = 0$ on $\partial\Omega$, we integrate by parts over the periodicity cell C. It follows from (2.3) and the L-periodicity of A, ζ and $\psi^{\lambda',M}$ that

$$\int_C \frac{\nabla\psi^{\lambda',M}\cdot A\nabla\psi^{\lambda',M}}{\left(\psi^{\lambda',M}\right)^2} + \lambda'^2\int_C \tilde{e}A\tilde{e} + \int_C \zeta = \mu(\lambda', M)|C|, \tag{7.26}$$

where $|C|$ denotes the Lebesgue measure of C. Let

$$m_0 = ⨍_C \tilde{e}A\tilde{e} = \frac{1}{|C|}\int_C \tilde{e}A(x,y)\tilde{e}\,dx\,dy \quad \text{and} \quad m = ⨍_C \zeta(x,y)\,dx\,dy.$$

One concludes that

$$\forall\lambda' > 0,\ \forall M > 0, \quad \mu(\lambda', M) \geqslant \lambda'^2 ⨍_C \tilde{e}A\tilde{e} + ⨍_C \zeta = \lambda'^2 m_0 + m,$$

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whence

$$\forall \lambda' > 0,\ \forall M > 0, \quad \frac{\mu(\lambda', M)}{\lambda'} \geqslant \lambda' m_0 + \frac{m}{\lambda'}. \tag{7.27}$$

The right side of (7.27) attains its minimum over $\mathbb{R}^+$ at $\lambda_0' = \sqrt{\frac{m}{m_0}}$. This minimum is equal to $2\sqrt{m_0 m}$.

Consequently, for any $M > 0$, $\frac{c^*_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}} = \min_{\lambda'>0} \frac{\mu(\lambda',M)}{\lambda'} \geqslant 2\sqrt{m_0 m}$. This yields that

$$\liminf_{M\to+\infty} \frac{c^*_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}} \geqslant 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\sqrt{\fint_C \zeta(x,y)\,dx\,dy}. \tag{7.28}$$

Step 3. Fix $\lambda' > 0$ and $M > 0$. Multiply (7.23) by $\psi^{\lambda',M}$ and integrate by parts over C. Owing to the L-periodicity of Ω, A, ζ and $\psi^{\lambda',M}$, and due to the facts that $\int_C (\psi^{\lambda',M})^2 = 1$, $\nabla \cdot A\tilde{e} \equiv 0$ in Ω, and that $\nu \cdot A\tilde{e} = 0$ on $\partial\Omega$, together with (2.3), one gets

$$\begin{aligned} -M\int_C \nabla\psi^{\lambda',M} \cdot A\nabla\psi^{\lambda',M} + \lambda'^2 \int_C \tilde{e}A\tilde{e}\,(\psi^{\lambda',M})^2 + \int_C \zeta\,(\psi^{\lambda',M})^2 \\ - \frac{\lambda'}{M^{\frac{1}{2}-\gamma}} \int_C q\cdot\tilde{e}\,(\psi^{\lambda',M})^2 \ = \ \mu(\lambda', M), \end{aligned} \tag{7.29}$$

whence

$$\forall \lambda' > 0,\ \forall M > 0,\ 0 < \mu(\lambda', M) \leqslant \lambda'^2\alpha + \beta + \frac{\lambda'}{M^{\frac{1}{2}-\gamma}} ||\,(q\cdot\tilde{e})^-\,||_\infty,$$

where $\alpha = \max_{(x,y)\in\overline{\Omega}} \tilde{e}A\tilde{e}(x,y)$ and $\beta = \max_{(x,y)\in\overline{\Omega}} \zeta(x,y)$. Together with (7.27), one gets

$$\forall \lambda' > 0,\ \forall M > 0,\ 0 < \lambda'^2 m_0 + m \leqslant \mu(\lambda', M) \leqslant \lambda'^2\alpha + \beta + \frac{\lambda'}{M^{\frac{1}{2}-\gamma}} ||\,(q\cdot\tilde{e})^-\,||_\infty. \tag{7.30}$$

If $\gamma = \frac{1}{2}$, then $\frac{\lambda'}{M^{\frac{1}{2}-\gamma}} ||\,(q\cdot\tilde{e})^-\,||_\infty = \lambda' ||\,(q\cdot\tilde{e})^-\,||_\infty$. On the other hand, if $0 \leqslant \gamma < \frac{1}{2}$, then

$$\frac{\lambda'}{M^{\frac{1}{2}-\gamma}} ||\,(q\cdot\tilde{e})^-\,||_\infty \to 0 \quad \text{as} \quad M \to +\infty.$$

Consequently, the right side of (7.30) is bounded above by a positive constant B which does not depend on M and γ. This yields that

$$\forall \lambda' > 0, \quad 0 < \limsup_{M\to+\infty} \mu(\lambda', M) < +\infty.$$

On the other hand, it follows from (2.2) and (7.29) that $\forall \lambda' > 0, \forall M > 0$,

$$\begin{aligned} 0 &\leqslant \alpha_1 \int_C |\nabla\psi^{\lambda',M}|^2 \leqslant \int_C \nabla\psi^{\lambda',M} \cdot A\nabla\psi^{\lambda',M} \\ &\leqslant \frac{1}{M}\left[-\mu(\lambda', M) + \lambda'^2 \int_C \tilde{e}A\tilde{e}\,(\psi^{\lambda',M})^2 + \int_C \zeta\,(\psi^{\lambda',M})^2 - \frac{\lambda'}{M^{\frac{1}{2}-\gamma}} \int_C q\cdot\tilde{e}\,(\psi^{\lambda',M})^2\right] \\ &< \frac{B}{M}. \end{aligned}$$

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Meanwhile, $\lim_{M\to+\infty}\frac{B}{M}=0$, one then gets

$$\begin{cases}\forall\lambda^{'}>0, & \lim\limits_{M\to+\infty}\int_C|\nabla\psi^{\lambda^{'},M}|^2=0,\\ \forall\lambda^{'}>0,\ \forall M>0, & \int_C(\psi^{\lambda^{'},M})^2=1.\end{cases}\tag{7.31}$$

Fix $\lambda^{'}>0$, and let $(M_n)_n$ be a sequence converging to $+\infty$ as $n\to+\infty$ and such that $\mu(\lambda^{'},M_n)\to l^{\lambda^{'},(M_n)}$ as $n\to+\infty$. It follows, from (7.31), that $||\psi^{\lambda^{'},M_n}||_{H^1(C)}\to 1$ as $n\to+\infty$. Thus, the sequence $(\psi^{\lambda^{'},M_n})_n$ is bounded in $H^1(C)$. Therefore, there exists a function $\psi^{\lambda^{'},\infty}\in H^1(C)$ such that, up to extraction of some subsequence, the functions $(\psi^{\lambda^{'},M_n})_n$ converge in $L^2(C)$ strong, $H^1(C)$ weak and almost everywhere in C, to the function $\psi^{\lambda^{'},\infty}$. Consequently, and owing to (7.31), $\psi^{\lambda^{'},\infty}$ satisfies

$$\int_C\left(\psi^{\lambda^{'},\infty}\right)^2=1,\ \text{and}\tag{7.32}$$

$$\left(\int_C|\nabla\psi^{\lambda^{'},\infty}|^2\right)^{\frac{1}{2}}\leqslant\liminf_{M_n\to+\infty}\left(\int_C|\nabla\psi^{\lambda^{'},M_n}|^2\right)^{\frac{1}{2}}=0.\tag{7.33}$$

From (7.33), it follows that for all $\lambda^{'}>0$, the function $\psi^{\lambda^{'},\infty}$ is almost everywhere constant over C. On the other hand, the elliptic regularity applied on equation (7.23) for $M=M_n$, implies that $\forall\lambda^{'}>0$, the function $\psi^{\lambda^{'},\infty}$ is continuous over $\overline{C}$. Consequently, referring to (7.32), one gets

$$\forall\lambda^{'}>0,\quad \psi^{\lambda^{'},\infty}=\frac{1}{\sqrt{|C|}}\ \text{over}\ \overline{C}.\tag{7.34}$$

Consider now equation (7.23). Fix $\lambda^{'}$, take $M=M_n$, and integrate by parts over C. It follows, from (2.2), (2.3) and the assumptions $\nabla.A\tilde{e}\equiv 0$ over Ω with $\nu.A\tilde{e}=0$ on $\partial\Omega$, that $\int_C M_n\nabla\cdot(A\nabla\psi^{\lambda^{'},M_n})=0$, $\int_C -2\lambda'\sqrt{M_n}\tilde{e}\cdot A\nabla\psi^{\lambda^{'},M_n}=0$, and $\int_C q\cdot\nabla\psi^{\lambda^{'},M_n}=0$. Hence,

$$-\frac{\lambda^{'}}{M_n^{\frac{1}{2}-\gamma}}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n}+\lambda^{'2}\int_C\tilde{e}\cdot A\tilde{e}\,\psi^{\lambda^{'},M_n}+\int_C\zeta\,\psi^{\lambda^{'},M_n}=\mu(\lambda^{'},M_n)\int_C\psi^{\lambda^{'},M_n}.\tag{7.35}$$

Meanwhile, the functions $\psi^{\lambda^{'},M_n}$ converge to the constant function $\psi^{\lambda^{'},\infty}$ in $L^2(C)$ strong; and hence, in $L^1(C)$ strong (C is bounded, so $L^2(C)$ is embedded in $L^1(C)$). Let $M_n\to+\infty$ in (7.35):

In case $\gamma=1/2$, one has

$$\frac{\lambda^{'}}{M_n^{\frac{1}{2}-\gamma}}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n}=\lambda^{'}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n}\to\lambda^{'}\psi^{\lambda^{'},\infty}\int_C q\cdot\tilde{e}=0,$$

as $n\to+\infty$ (from (2.3)). Also, in the case $0\leqslant\gamma<1/2$, one trivially has

$$\frac{\lambda^{'}}{M_n^{\frac{1}{2}-\gamma}}\int_C q\cdot\tilde{e}\,\psi^{\lambda^{'},M_n}\to 0\ \text{ as }\ n\to+\infty.$$

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Moreover, $\tilde{e}A\tilde{e}$ and ζ are in $L^\infty(C)$. Thus, as $M_n \to +\infty$ in (7.35), we get

$$\lambda'^2 \psi^{\lambda',\infty} \int_C \tilde{e}A\tilde{e} + \psi^{\lambda',\infty} \int_C \zeta = l^{\lambda',(M_n)}\, \psi^{\lambda',\infty} |C|.$$

One concludes that

$$\forall \lambda' > 0, \quad \frac{l^{\lambda',(M_n)}}{\lambda'} = \lambda' \fint_C \tilde{e}A\tilde{e} + \frac{\fint_C \zeta}{\lambda'} = \lambda' m_0 + \frac{m}{\lambda'}. \tag{7.36}$$

Whence for $\lambda' = \lambda'_0 = \sqrt{\frac{m}{m_0}}$, one gets $\frac{l^{\lambda'_0,(M_n)}}{\lambda'_0} = 2\sqrt{m_0 m}$.

On the other hand, for all M_n,

$$\frac{c^*_{\Omega,M_nA,M_n^\gamma q,f}(e)}{\sqrt{M_n}} = \inf_{\lambda'>0} \frac{\mu(\lambda', M_n)}{\lambda'} \leqslant \frac{\mu(\lambda'_0, M_n)}{\lambda'_0}. \tag{7.37}$$

Passing $M_n \to +\infty$, one gets $\limsup_{M_n\to+\infty} \frac{c^*_{\Omega,M_nA,M_n^\gamma q,f}(e)}{\sqrt{M_n}} \leqslant \frac{l^{\lambda'_0,(M_n)}}{\lambda'_0} = 2\sqrt{m_0 m}$, and this holds for all sequences $\{M_n\}_n$ converging to $+\infty$. Thus,

$$\limsup_{M\to+\infty} \frac{c^*_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}} \leqslant 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\ \sqrt{\fint_C \zeta(x,y)\,dx\,dy}. \tag{7.38}$$

Having (7.28) together with (7.38), the proof of Theorem 4.1 is complete. □

Proof of Theorem 4.3 We will consider the change of variables similar to that made in the proof of Theorem 3.4:

$$v(t,x,y) = u(t,Lx,Ly), \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}^N.$$

After the same calculations done there, one gets that u satisfies (4.1) if and only if v satisfies

$$v_t(t,x,y) = \frac{1}{L^2}\nabla\cdot(A(x,y)\nabla v)(t,x,y) + \frac{1}{L}\, q\cdot\nabla v(t,x,y) + f(x,y,v) \text{ over } \mathbb{R}\times\mathbb{R}^N. \tag{7.39}$$

Consequently,

$$\forall\, L>0,\ c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e) = L\, c^*_{\mathbb{R}^N,\frac{1}{L^2}A,\frac{1}{L}q,f}(e). \tag{7.40}$$

On the other hand, the coefficients and the domain of problem (7.39) satisfy all the assumptions of Theorem 4.1. Taking $M = 1/L^2$ and $\gamma = 1/2$, then (7.39) can be rewritten as

$$v_t(t,x,y) = M\,\nabla\cdot(A(x,y)\nabla v)(t,x,y) + M^{\frac{1}{2}}\, q\cdot\nabla v(t,x,y) + f(x,y,v) \text{ over } \mathbb{R}\times\mathbb{R}^N.$$

In this situation, the periodicity cell of the whole space $\mathbb{R}^N$ is $C = [0,1]\times\cdots\times[0,1]$.

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It follows, from Theorem 4.1, that

$$\begin{aligned}\lim_{L\to 0^+} \frac{c^*_{\mathbb{R}^N,\frac{1}{L^2}A,\frac{1}{L}q,f}(e)}{\sqrt{\frac{1}{L^2}}} &= \lim_{M\to+\infty} \frac{c^*_{\mathbb{R}^N,M\,A,M^{\frac{1}{2}}q,f}(e)}{\sqrt{M}} \\ &= 2\sqrt{\iint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\sqrt{\iint_C \zeta(x,y)\,dx\,dy}.\end{aligned} \tag{7.41}$$

Having (7.40) together with (7.41), the proof of Theorem 4.3 is complete. □

7.3 Proofs of Theorems 5.1 and 5.2

Proof of Theorem 5.1 The main ideas of this proof are similar to those in the demonstration of Theorem 3.1. Applying the variational formula (2.7) of the minimal speed, one gets

$$c^*_{\Omega,A,0,Bf}(e) = \min_{\lambda>0} \frac{k_{\Omega,e,A,0,B\zeta}(\lambda)}{\lambda}, \tag{7.42}$$

where $k_{\Omega,e,A,0,B\zeta}(\lambda)$ is the first eigenvalue (for each λ, $B>0$) of the eigenvalue problem:

$$\begin{cases} L_{\Omega,e,A,0,B\zeta,\lambda}\,\psi(x,y) = k_{\Omega,e,A,0,B\zeta}(\lambda)\,\psi(x,y) \text{ over } \mathbb{R}\times\omega; \\ \nu\cdot A\nabla\psi = 0 \quad \text{on } \mathbb{R}\times\partial\omega, \end{cases} \tag{7.43}$$

and

$$\begin{aligned} L_{\Omega,e,A,0,B\zeta,\lambda}\psi(x,y) &= \nabla\cdot(A(y)\nabla\psi(x,y)) - 2\lambda Ae\cdot\nabla\psi(x,y) \\ &\quad + [\lambda^2 eA(y)e - \lambda\nabla\cdot(A(y)e) + B\zeta(y)]\psi(x,y), \end{aligned}$$

for each $(x,y)\in\mathbb{R}\times\omega$.

We recall that for all $\lambda>0$, and for all $B>0$, we have $k_{\Omega,e,A,0,B\zeta}(\lambda)>0$. Also, the first eigenfunction of (7.43) is positive over $\overline{\Omega}=\mathbb{R}\times\overline{\omega}$, and it is unique up to multiplication by a non-zero constant.

Moreover, whether in (3.7) or (3.8) and due to (3.4), one concludes that the coefficients in $L_{\Omega,e,A,0,B\zeta,\lambda}$ are independent of x. Hence, the first eigenfunction of (7.43) is independent of x and the eigenvalue problem (7.43) is reduced to

$$\begin{cases} L_{\Omega,e,A,0,B\zeta,\lambda}\phi := \nabla\cdot(A(y)\nabla\phi(y)) + [\lambda^2 eA(y)e + B\zeta(y)]\phi(y) \\ \qquad\qquad = k_{\Omega,e,A,0,B\zeta}(\lambda)\,\phi \quad \text{over } \omega; \\ \nu(x,y)\cdot A(y)\nabla\phi(y) = (0;\nu_\omega(y))\cdot A(y)\nabla\phi(y) = 0 \quad \text{on } \mathbb{R}\times\partial\omega, \end{cases} \tag{7.44}$$

where $\phi=\phi(y)$ is positive over $\overline{\omega}$, L-periodic (since the domain ω and the coefficients of $L_{\Omega,e,A,0,B\zeta,\lambda}$ are L-periodic), unique up to multiplication by a constant, and belongs to $C^2(\overline{\omega})$.

In the case where $d\geqslant 1$, let $C\subseteq\mathbb{R}^{N-1}$ denote the periodicity cell of ω. Otherwise, $d=0$ and one takes $C=\omega$. In both cases, C is bounded. Multiplying the first line of

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(7.44) by ϕ, and integrating by parts over C, one gets

$$-k_{\Omega,e,A,0,B\zeta}(\lambda) = \frac{\displaystyle\int_C \nabla\phi \cdot A(y)\nabla\phi\, dy - \int_C [\lambda^2 eA(y)e + B\,\zeta(y)]\,\phi^2(y)\, dy}{\displaystyle\int_C \phi^2(y)\, dy}. \tag{7.45}$$

One also notes that, in this present setting, the operator $L_{\Omega,e,A,0,B\zeta,\lambda}$ is self-adjoint and its coefficients are $(L_1,\dots,L_d)$-periodic with respect $(y_1,\dots,y_d)$. Consequently, $-k_{\Omega,e,A,0,B\zeta}(\lambda)$ has the following variational characterisation:

$$-k_{\Omega,e,A,0,B\zeta}(\lambda) = \min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\displaystyle\int_C \nabla\varphi \cdot A(y)\nabla\varphi\, dy - \int_C [\lambda^2 eA(y)e + B\,\zeta(y)]\,\varphi^2(y)\, dy}{\displaystyle\int_C \varphi^2(y)\, dy}. \tag{7.46}$$

In what follows, we will assume that (3.7) is the alternative that holds. That is, $eAe = \alpha$ is constant. The proof can be imitated easily whenever we assume that (3.8) holds.

The function $y \mapsto \zeta(y)$ is continuous and $(L_1,\dots,L_d)$-periodic over $\overline{\omega}$, whose periodicity cell C is a bounded subset of $\mathbb{R}^{N-1}$ (whether $d = 0$ or $d \geqslant 1$). Let $y_0 \in \overline{C} \subseteq \overline{\omega}$ such that $\max_{y\in\overline{\omega}} \zeta(y) = \zeta(y_0)$ (trivially, this also holds when ζ is constant). Consequently, we have

$$\forall\ \varphi \in H^1(C)\setminus\{0\},\ \frac{\displaystyle\int_C \nabla\varphi \cdot A\nabla\varphi - \int_C (\alpha\lambda^2 + B\,\zeta(y))\varphi^2}{\displaystyle\int_C \varphi^2(y)\, dy} \geqslant -[\alpha\lambda^2 + B\,\zeta(y_0)].$$

This yields that

$$\forall\, B > 0,\ \forall\,\lambda > 0, -k_{\Omega,e,A,0,B\,\zeta}(\lambda) \geqslant -[\alpha\lambda^2 + B\,\zeta(y_0)]. \tag{7.47}$$

Consequently,

$$\forall\, B > 0,\ \forall\,\lambda > 0,\ \frac{k_{\Omega,e,A,0,B\zeta}(\lambda)}{\lambda} \leqslant \lambda\,\alpha + \frac{B\,\zeta(y_0)}{\lambda}. \tag{7.48}$$

However, the function $\lambda \mapsto \lambda\alpha + (B\,\zeta(y_0)/\lambda)$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(B) = \sqrt{\frac{B\,\zeta(y_0)}{\alpha}}$. This minimum is equal to $2\sqrt{B\zeta(y_0)} \times \sqrt{\alpha}$.

From (7.48), we conclude that $\frac{k_{\Omega,e,A,0,B\zeta}(\lambda(B))}{\lambda(B)} \leqslant 2\sqrt{B\,\alpha}\sqrt{\zeta(y_0)}$.

Finally, (2.7) implies that

$$c^*_{\Omega,A,0,Bf}(e) = \min_{\lambda>0} \frac{k_{\Omega,e,A,0,B\zeta}(\lambda)}{\lambda} \leqslant 2\sqrt{B\,\alpha}\sqrt{\zeta(y_0)},$$

or equivalently

$$\forall\, B > 0,\ \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} \leqslant 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{7.49}$$

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We pass now to prove the other sense of the inequality for $\liminf_{B\to+\infty} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}$. We will consider formula (7.5), and then organize a suitable function ψ which leads us to a lower bound of $\liminf_{B\to+\infty} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}$.

We have $\zeta(y_0) > 0$. Let δ be such that $0 < \delta < \zeta(y_0)$. Thus $0 < \zeta(y_0) - \delta < \max_{\overline{\omega}} \zeta(y)$. The continuity of ζ, over $\overline{C} \subseteq \overline{\omega}$, yields that there exists an open and bounded set $U \subset \overline{C}$ such that

$$\zeta(y_0) - \delta \leqslant \zeta(y), \ \forall\, y \in \overline{U}. \tag{7.50}$$

Designate by ψ, a function in $\mathscr{D}(C)$ (a $C^\infty(C)$ function whose support is compact), with $\operatorname{supp}\psi \subseteq \overline{U}$, and $\int_U \psi^2 = 1$. One will have,

$$\begin{aligned}
&\forall \lambda > 0, \forall\, B > 0,\\
&-k_{\Omega,e,A,0,B}\,\zeta(\lambda) \leqslant \int_U \nabla\psi \cdot A(y)\nabla\psi\, dy - \int_U [\lambda^2 eA(y)e + B\,\zeta(y)]\,\psi^2(y)\, dy\\
&\qquad\qquad \leqslant \int_U \nabla\psi \cdot A(y)\nabla\psi\, dy - [\lambda^2\alpha + B\,(\zeta(y_0) - \delta)] \text{ (by (7.50))}\\
&\qquad\qquad \leqslant \int_U \alpha_2|\nabla\psi|^2 - [\lambda^2\alpha + B\,(\zeta(y_0) - \delta)] \text{ by (3.5)},
\end{aligned}$$

or equivalently

$$\frac{k_{\Omega,e,A,0,B}\,\zeta(\lambda)}{\lambda} \geqslant \lambda\alpha + \frac{B}{\lambda}\,\rho(B), \tag{7.51}$$

where $\rho(B) = \zeta(y_0) - \delta - \frac{1}{B}\int_U \alpha_2|\nabla\psi|^2$. Choosing B large enough, we get $\rho(B) > 0$ (this is possible since $\zeta(y_0) - \delta > 0$ and also $\int_U \alpha_2|\nabla\psi|^2 > 0$). The map $\lambda \mapsto \lambda\alpha + \frac{B}{\lambda}\,\rho(B)$ attains its minimum, over $\mathbb{R}^+$, at $\lambda(\varepsilon) = \sqrt{\frac{B\,\rho(B)}{\alpha}}$. This minimum is equal to $2\sqrt{B\,\alpha}\,\sqrt{\rho(B)}$.

Now, referring to formula (7.51), one gets,

$$\text{for } B \text{ large enough, } \frac{k_{\Omega,e,A,0,B}\zeta(\lambda)}{\lambda} \geqslant 2\sqrt{B\,\alpha}\,\sqrt{\rho(B)} \quad \text{for all } \lambda > 0.$$

Together with (2.7), we conclude that

$$\text{for } B \text{ large enough, } \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} \geqslant 2\sqrt{\rho(B)}\sqrt{\alpha}. \tag{7.52}$$

Consequently,

$$\begin{aligned}
\liminf_{B\to+\infty} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} &\geqslant \liminf_{B\to+\infty} 2\sqrt{\rho(B)}\sqrt{\alpha}\\
&= 2\sqrt{\zeta(y_0) - \delta}\sqrt{\alpha} \quad \text{(since } \psi \text{ is independent of } B),
\end{aligned}$$

and this holds for all $0 < \delta < \zeta(y_0)$. Therefore, one can conclude that

$$\liminf_{B\to+\infty} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}} \geqslant 2\sqrt{\alpha}\sqrt{\zeta(y_0)}. \tag{7.53}$$

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Finally, the inequalities (7.49) and (7.53) imply that $\lim_{B\to+\infty} \frac{c^*_{\Omega,A,0,Bf}(e)}{\sqrt{B}}$ exists, and it is equal to $2\sqrt{\alpha}\sqrt{\zeta(y_0)} = 2\sqrt{\max_{\overline{\omega}} eA(y)e}\sqrt{\max_{\overline{\omega}} \zeta(y)}$.

The above proof was done while assuming that the alternative (3.7) holds. The same ideas of this proof can be easily applied in the case where alternative (3.8) holds. In (3.8), we have ζ as constant; however, eAe is not in general. Meanwhile the converse is true in the case (3.7). The little difference is that, in the case of (3.8), we choose the subset U (of the proof done above) around the point y_0 where eAe attains its maximum and then we continue by the same way used above. □

Proof of Theorem 5.2 According to Theorem 2.4, and since $\nu \cdot A\tilde{e} = 0$ on $\partial\Omega$, the minimal speeds $c^*_{\Omega,A,B^\gamma q,Bf}(e)$ are given by

$$\forall B > 0,\ c^*_{\Omega,A,B^\gamma q,Bf}(e) = \min_{\lambda>0} \frac{k_{\Omega,e,A,B^\gamma q,B\zeta}(\lambda)}{\lambda},$$

where $k_{\Omega,e,A,B^\gamma q,B\zeta}(\lambda)$ and $\psi^{\lambda,B}$ denote the unique eigenvalue and the positive L-periodic eigenfunction of the problem

$$\nabla\cdot(A\nabla\psi^{\lambda,B}) - 2\lambda\tilde{e}\cdot A\nabla\psi^{\lambda,B} + B^\gamma q\cdot\nabla\psi^{\lambda,B} + \left[\lambda^2\,\tilde{e}A\tilde{e} - \lambda B^\gamma q\cdot\tilde{e} + B\,\zeta\right]\psi^{\lambda,B}$$
$$= k_{\Omega,e,A,B^\gamma q,B\zeta}(\lambda)\,\psi^{\lambda,B} \text{ in } \Omega, \text{ with } \nu\cdot A\nabla\psi = \nu\cdot A\nabla\psi^{\lambda,B} = 0 \text{ on } \partial\Omega.$$

For each $\lambda > 0$ and $B > 0$, let $\lambda' = \lambda/\sqrt{B}$, and let $k_{\Omega,e,A,B^\gamma q,B\zeta}(\lambda) = \mu(\lambda',B)$. Consequently,

$$\forall B > 0,\ \frac{c^*_{\Omega,A,B^\gamma q,Bf}(e)}{\sqrt{B}} = \min_{\lambda'>0} \frac{\mu(\lambda',B)}{\lambda' B}, \tag{7.54}$$

where $\mu(\lambda',B)$ and $\psi^{\lambda',B}$ are the first eigenvalue and the unique, positive L-periodic (with respect to x) eigenfunction of

$$\nabla\cdot(A\nabla\psi^{\lambda',B}) - 2\lambda'\sqrt{B}\tilde{e}\cdot A\nabla\psi^{\lambda',B} + B^\gamma q\cdot\nabla\psi^{\lambda',B}$$
$$+ [\lambda'^2 B\,\tilde{e}A\tilde{e} - \lambda' B^{\gamma+\frac{1}{2}}\,q\cdot\tilde{e} + B\zeta]\psi^{\lambda',B} = \mu(\lambda',B)\psi^{\lambda',B} \text{ in } \Omega, \tag{7.55}$$

with $\nu\cdot A\nabla\psi^{\lambda',B} = 0$ on $\partial\Omega$.

Owing to the uniqueness, up to multiplication by positive constants, of the first eigenfunction of (7.55), one may assume that

$$\forall\,\lambda' > 0,\ \forall\,B > 0,\ \|\psi^{\lambda',B}\|_{L^2(C)} = 1. \tag{7.56}$$

Moreover, for each $B > 0$, $\min_{\lambda'>0} \frac{\mu(\lambda',B)}{\lambda' B}$ is attained at $\lambda'_B > 0$. Thus,

$$\forall B > 0,\ \frac{c^*_{\Omega,A,B^\gamma q,Bf}(e)}{\sqrt{B}} = \min_{\lambda'>0} \frac{\mu(\lambda',B)}{\lambda' B} = \frac{\mu(\lambda'_B,B)}{B\,\lambda'_B}. \tag{7.57}$$

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Having the above characterisation, one can now imitate the steps 2 and 3 in the proof of Theorem 4.1 to prove that

$$\liminf_{B\to 0^+} c^*_{\Omega,A,\,B^\gamma q,\,Bf}(e)/\sqrt{B}$$

(resp. $\limsup_{B\to 0^+} c^*_{\Omega,A,B^\gamma q,\,Bf}(e)/\sqrt{B}$) is greater than (resp. less than)

$$2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\,\sqrt{\fint_C \zeta(x,y)\,dx\,dy};$$

and hence, complete the proof of Theorem 5.2. □

7.4 Proofs of Theorems 6.1, 6.3 and 6.5

Proof of Theorem 6.1 Referring to Theorem 2.4, it follows that for each $\beta > 0$, we have

$$\frac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}} = \min_{\lambda>0} \frac{k_{\Omega,e,\,\beta A,\,\sqrt{\beta}\,q,\zeta}(\lambda)}{\lambda\sqrt{\beta}},$$

where $k_{\Omega,e,\,\beta A,\,\sqrt{\beta}\,q,\zeta}(\lambda)$ is the first eigenvalue of the problem

$$\begin{cases} L_{\Omega,e,\,\beta A,\,\sqrt{\beta}\,q,\zeta,\lambda}\psi(x,y) = k_{\Omega,e,\,\beta A,\,\sqrt{\beta}\,q,\zeta}(\lambda)\,\psi(x,y) \text{ over } \mathbb{R}\times\omega; \\ \nu.A\nabla\psi = 0 \text{ on } \mathbb{R}\times\partial\omega, \end{cases} \tag{7.58}$$

where

$$L_{\Omega,e,\beta A,\sqrt{\beta}\,q,\zeta,\lambda}\,\psi = \beta\nabla\cdot(A(y)\nabla\psi) - 2\beta\lambda\,\alpha(y)\,\partial_x\psi + \sqrt{\beta}\,q_1(y)\partial_x\psi$$
$$+ [\beta\,\lambda^2 eA(y)e - \lambda\sqrt{\beta}\,q_1(y) + \zeta(y)]\psi \text{ over } \mathbb{R}\times\omega.$$

The boundary condition follows so from the facts that $\Omega = \mathbb{R}\times\omega$, $e = (1,0,\ldots,0)$ and that $A(y)e = \alpha(y)e$ over ω. These yield that $\nu\cdot Ae = 0$ over $\partial\Omega$ and $\nabla\cdot Ae = 0$. Moreover, for each $(x,y)\in\partial\Omega$, we have $\nu(x,y) = (0;\nu_\omega(y))$, where $\nu_\omega(y)$ is the outward unit normal on $\partial\omega$ at y.

On the other hand, the function ψ is positive, $(L_1,\ldots,L_d)$-periodic with respect to y, and unique up to multiplication by non-zero constants. Meanwhile, the coefficients A, q and ζ are independent of x. Thus the eigenfunction ψ will be independent of x and our eigenvalue problem is reduced to

$$\begin{cases} \beta\nabla\cdot(A(y)\nabla\psi(y)) + \left[\beta\,\lambda^2 eA(y)e - \lambda\sqrt{\beta}q_1(y) + \zeta(y)\right]\psi(y) \\ \quad = k_{\Omega,e,\,\beta A,\,\sqrt{\beta}q,\zeta}(\lambda)\psi(y) \text{ for all } y\in\omega, \\ \nu(x,y)\cdot A(y)\nabla\psi(y) = (0;\nu_\omega(y))\cdot A(y)\nabla\psi(y) = 0 \text{ on } \mathbb{R}\times\partial\omega. \end{cases} \tag{7.59}$$

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For each $\lambda > 0$ and $\beta > 0$, let $\lambda' = \lambda\sqrt{\beta}$, and let $k_{\Omega,e,\beta A,\sqrt{\beta}\,q,\zeta}(\lambda) = \mu(\lambda', \beta)$. Since for each $\beta > 0$, $\min_{\lambda>0} \frac{k_{\Omega,e,\beta A,\sqrt{\beta}\,q,\zeta}(\lambda)}{\lambda}$ is attained at $\lambda(\beta)$, it follows that

$$\forall\, \beta > 0, \quad \frac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}} = \min_{\lambda'>0} \frac{\mu(\lambda', \beta)}{\lambda'}, \tag{7.60}$$

where $\mu(\lambda', \beta)$ is the first eigenvalue of the problem

$$\begin{cases} L^{\beta}_{\lambda'}\psi = \beta\nabla\cdot(A(y)\nabla\psi) + [\lambda'^2 eA(y)e - \lambda' q_1(y) + \zeta(y)]\psi = \mu(\lambda', \beta)\psi \text{ in } \omega, \\ v\cdot A\nabla\psi = 0 \text{ on } \partial\omega. \end{cases} \tag{7.61}$$

The elliptic operator $L^{\beta}_{\lambda'}$ in (7.61) is self-adjoint. Consequently, the first eigenvalue $\mu(\lambda', \beta)$ has the following characterisation[2]:

$$\begin{aligned} &\forall \lambda' > 0, \forall \beta > 0, \quad -\mu(\lambda', \beta) \\ &= \min_{\varphi\in H^1(C)\setminus\{0\}} \frac{\beta\displaystyle\int_C \nabla\varphi\cdot A(y)\nabla\varphi dy + \lambda'\int_C q_1(y)\varphi^2 - \int_C [\lambda'^2 eA(y)e + \zeta(y)]\varphi^2(y)\,dy}{\displaystyle\int_C \varphi^2(y)\,dy} \\ &= \min_{\varphi\in H^1(C)\setminus\{0\}} R(\lambda', \beta, \varphi). \end{aligned} \tag{7.62}$$

For each λ' and $\beta > 0$, $\varphi \mapsto R(\lambda', \beta, \varphi)$ attains its minimum over $H^1(C)\setminus\{0\}$ at $\psi^{\lambda',\beta}$, the eigenfunction of the problem (7.61). On the other hand, $\beta \mapsto R(\lambda', \beta, \varphi)$ is increasing as an affine function in β. Consequently, fixing $\lambda' > 0$ and taking $\beta > \beta' > 0$ we get

$$\begin{aligned} -\mu(\lambda', \beta) = R(\lambda', \beta, \psi^{\lambda',\beta}) &> R(\lambda', \beta', \psi^{\lambda',\beta}) \\ &\geqslant \min_{\varphi\in H^1(C)\setminus\{0\}} R(\lambda', \beta', \varphi) = -\mu(\lambda', \beta'). \end{aligned} \tag{7.63}$$

In other words, for all $\lambda' > 0$, the function $\beta \mapsto \mu(\lambda', \beta)$ is decreasing. Concerning now the function $\beta \mapsto c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)/\sqrt{\beta}$, one takes randomly $\beta > \beta' > 0$, hence

$$\begin{aligned} \frac{c^*_{\Omega,\beta' A,\sqrt{\beta'}\,q,f}(e)}{\sqrt{\beta'}} = \frac{\mu(\lambda'(\beta'), \beta')}{\lambda'(\beta')} &> \frac{\mu(\lambda'(\beta'), \beta)}{\lambda'(\beta')} \\ &\geqslant \min_{\lambda'>0} \frac{\mu(\lambda', \beta)}{\lambda'} = \frac{c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)}{\sqrt{\beta}}, \end{aligned}$$

which means that the function $\beta \mapsto c^*_{\Omega,\beta A,\sqrt{\beta}\,q,f}(e)/\sqrt{\beta}$ is decreasing.

[2] To have an idea, multiply (7.61) by the positive, $(L_1,\ldots,L_d)$-periodic function ψ and integrate by parts over the periodicity cell C of the the domain ω.

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Finally, when $\beta \to +\infty$, one can easily check that the hypothesis of Theorem 4.1 are satisfied; hence, one has the limit at $+\infty$, and that completes the proof of Theorem 6.1. □

Proof of Theorem 6.3 Consider the change of variables $v(t,x,y) = u(t,Lx,Ly)$, for any $(t,x,y) \in \mathbb{R} \times \mathbb{R}^N$. One consequently has,

$$\forall\, L > 0,\ c^*_{\mathbb{R}^N, A_L, q_L, f_L}(e) = L\, c^*_{\mathbb{R}^N, \frac{1}{L^2}A, \frac{1}{L}q, f}(e). \tag{7.64}$$

Taking $\beta = 1/L^2$, then

$$v_t(t,x,y) = \beta\, \nabla \cdot (A(y)\nabla v)(t,x,y) + \sqrt{\beta}\, q_1(y)\, \partial_x v(t,x,y) + f(x,y,v) \text{ over } \mathbb{R} \times \mathbb{R}^N.$$

Owing to Theorem 6.1, the function $\beta \mapsto c^*_{\mathbb{R}^N, \beta A, \sqrt{\beta} q, f}(e)/\sqrt{\beta}$ is decreasing in $\beta > 0$. Besides, $L \mapsto 1/L^2$ is decreasing in $L > 0$. Together with (7.64), one obtains that the function $L \mapsto c^*_{\mathbb{R}^N, A_L, q_L, f_L}(e)$ is increasing in $L > 0$ which completes the proof of Theorem 6.3. □

Proof of Theorem 6.5 Referring to Theorem 2.4, it follows that for each $B > 0$, we have:

$$\frac{c^*_{\Omega, A, \sqrt{B}\, q, Bf}(e)}{\sqrt{B}} = \min_{\lambda > 0} \frac{k_{\Omega, e, A, \sqrt{B}\, q, B\zeta}(\lambda)}{\lambda \sqrt{B}}.$$

Owing to the same justifications explained in the proof of Theorem 6.1, $k_{\Omega, e, A, \sqrt{B} q, B\zeta}(\lambda)$ is the first eigenvalue of the problem

$$\begin{cases} \nabla \cdot (A(y)\nabla\psi(y)) + [\lambda^2 e \cdot Ae - \lambda\sqrt{B} q_1(y) + B\zeta(y)]\psi(y) = k_{\Omega, e, A, \sqrt{B} q, B\zeta}(\lambda)\psi \text{ in } \omega, \\ \qquad v(x,y) \cdot A(y)\nabla\psi(y) = (0; v_\omega(y)) \cdot A(y)\nabla\psi(y) = 0 \ \text{ on } \ \mathbb{R} \times \partial\omega. \end{cases} \tag{7.65}$$

For each $\lambda > 0$ and $B > 0$, let $\lambda' = \lambda/\sqrt{B}$ and $k_{\Omega, e, A, \sqrt{B}\, q, B\zeta}(\lambda) = \mu(\lambda', B)$. The first eigenvalue $\mu(\lambda', B)$ has the following characterisation:

$$\begin{aligned} &\forall \lambda' > 0,\ \forall B > 0,\ -\frac{\mu(\lambda', B)}{\lambda' B} \\ &= \min_{\substack{\|\varphi\|_{L^2(C)} = 1 \\ \varphi \in H^1(C) \setminus \{0\};}} \frac{\displaystyle\int_C \nabla\varphi \cdot A(y)\nabla\varphi\, dy}{\lambda' B} + \int_C q_1 \varphi^2 - \lambda' \int_C eAe\varphi^2 - \frac{\displaystyle\int_C \zeta(y)\varphi^2(y)\, dy}{\lambda'} \\ &= \min_{\substack{\|\varphi\|_{L^2(C)} = 1 \\ \varphi \in H^1(C) \setminus \{0\}}} R(\lambda', B, \varphi). \end{aligned} \tag{7.66}$$

On the other hand, $B \mapsto R(\lambda', B, \varphi)$ is decreasing in $B > 0$. Consequently, fixing $\lambda' > 0$ and taking $0 < B < B'$,

$$-\frac{\mu(\lambda', B)}{\lambda' B} = R(\lambda', B, \psi^{\lambda', B}) > R(\lambda', B', \psi^{\lambda', B}) \geqslant \min_{\substack{\|\varphi\|_{L^2(C)} = 1 \\ \varphi \in H^1(C) \setminus \{0\};}} R(\lambda', B', \varphi) = -\frac{\mu(\lambda', B')}{\lambda' B'}.$$

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In other words, for all $\lambda' > 0$, the function $B \mapsto \mu(\lambda', B)/\lambda' B$ is increasing in $B > 0$. Now, we take randomly $0 < B < B'$. Thus,

$$\begin{aligned}\frac{c^*_{\Omega,A,\sqrt{B'}q,B'f}(e)}{\sqrt{B'}} &= \min_{\lambda'>0} \frac{\mu(\lambda', B')}{\lambda' B'} = \frac{\mu(\lambda'_{B'}, B')}{\lambda'_{B'} \times B'} \\ &> \frac{\mu(\lambda'_{B'}, B)}{\lambda'_{B'} \times B} \geqslant \min_{\lambda'>0} \frac{\mu(\lambda', B)}{\lambda' B} = \frac{c^*_{\Omega,A,\sqrt{B}q,Bf}(e)}{\sqrt{B}},\end{aligned}$$

which means that $B \mapsto c^*_{\Omega,A,\sqrt{B}q,Bf}(e)/\sqrt{B}$ is increasing in $B > 0$. □

8 Applications to homogenisation problems

The reaction–advection–diffusion problem set in a heterogeneous periodic domain Ω satisfying (2.1) generates a homogenization problem:

Let $e \in \mathbb{R}^d$ be a vector of unit norm. Assume that Ω, A, q and f are $(L_1, \ldots, L_d)$-periodic and that they satisfy (2.1), (2.2), (2.3), (2.4) and (2.5).

For each $\varepsilon > 0$, let $\Omega^\varepsilon = \varepsilon\Omega$ and consider the following re-scales:

$$\forall (x,y) \in \Omega^\varepsilon, \quad A_\varepsilon(x,y) = A\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), q_\varepsilon(x,y) = q\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \text{ and } f_\varepsilon(x,y) = f\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

The coefficients A_ε, q_ε and f_ε together with the domain Ω^ε are $(\varepsilon L_1, \ldots, \varepsilon L_d)$-periodic, and they satisfy properties similar to those of A, q, f and Ω.

Consider the parametric reaction–advection–diffusion problem

$$(P_\varepsilon) \begin{cases} u^\varepsilon_t(t,x,y) = \nabla \cdot (A_\varepsilon \nabla u^\varepsilon)(t,x,y) + q_\varepsilon \cdot \nabla u^\varepsilon + f_\varepsilon(x,y,u^\varepsilon), \ t \in \mathbb{R}, \ (x,y) \in \Omega^\varepsilon, \\ \nu^\varepsilon \cdot A_\varepsilon \nabla u^\varepsilon(t,x,y) = 0, \quad t \in \mathbb{R}, \ (x,y) \in \partial\Omega^\varepsilon, \end{cases}$$

where $\nu^\varepsilon(x,y)$ denotes the outward unit normal on $\partial\Omega^\varepsilon$ at the point (x,y).

Owing to the results found by Berestycki and Hamel in Section 6 of [2], and since the coefficients A_ε, f_ε and q_ε together with the domain Ω^ε satisfy all the necessary assumptions, it follows that the problem (P_ε) admits a minimal speed of propagation $c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e) > 0$ such that (P_ε) has a solution u^ε in the form of a pulsating front within a speed c if and only if $c \geqslant c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e) > 0$.

In this section, we investigate the limit of the parametric minimal speeds $c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e)$ (whose parameter is ε) of the problems $(P_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0^+$. In other words, we search the limit of these minimal speeds as the periodicity cell $C^\varepsilon = \varepsilon C$ becomes a very small size. On the other hand, we study although not the most general setting, the variation of the map $\varepsilon \mapsto c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e)$ in $\varepsilon > 0$.

Theorem 8.1 *Let $e \in \mathbb{R}^d$ be a unit vector, and let $\Omega \subseteq \mathbb{R}^N$ be a domain which is L-periodic and satisfying* (2.1). *Assume that $A = A(x,y)$, $q = q(x,y)$ and $f = f(x,y,u)$ are L-periodic and that they satisfy* (2.2), (2.3), (2.4) *and* (2.5) *together with the assumptions $\nabla.A\tilde{e} \equiv 0$ on $\overline{\Omega}$ and $\nu.A\tilde{e} = 0$ on $\partial\Omega$. For each $\varepsilon > 0$, consider the problem*

$$\begin{cases} u^\varepsilon_t(t,x,y) = \nabla \cdot (A_\varepsilon \nabla u^\varepsilon)(t,x,y) + q_\varepsilon \cdot \nabla u^\varepsilon + f_\varepsilon(x,y,u^\varepsilon), \ t \in \mathbb{R}, \ (x,y) \in \Omega^\varepsilon, \\ \nu^\varepsilon \cdot A_\varepsilon \nabla u^\varepsilon(t,x,y) = 0, \quad t \in \mathbb{R}, \ (x,y) \in \partial\Omega^\varepsilon, \end{cases} \tag{8.1}$$

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where A_ε, f_ε and q_ε are the coefficients defined in the beginning of this section. Then, the minimal speed $c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e)$ of pulsating travelling fronts propagating in the direction of e and solving (8.1) *satisfies*

$$\lim_{\varepsilon\to 0^+} c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e) = 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\,\sqrt{\fint_C \zeta(x,y)\,dx\,dy}, \tag{8.2}$$

where C is the periodicity cell of Ω and $\tilde{e} = (e, 0, \ldots, 0) \in \mathbb{R}^N$.

Proof As a first notice, we mention that the domain Ω^ε is the image of Ω by a dilation whose center is the origin $O(0,\ldots,0)$ and whose scale factor is equal to ε. Consequently,

$$\text{for each } \varepsilon > 0,\ (\varepsilon x, \varepsilon y) \in \Omega^\varepsilon \quad \text{if and only if} \quad (x,y) \in \Omega, \text{ and}$$

$$(\varepsilon x, \varepsilon y) \in \partial\Omega^\varepsilon \quad \text{if and only if} \quad (x,y) \in \partial\Omega.$$

Moreover,

$$\forall \varepsilon > 0,\ \forall (x,y) \in \partial\Omega,\ v^\varepsilon(\varepsilon x, \varepsilon y) = v(x,y).$$

Consider now, for each $\varepsilon > 0$, the following change of variables:

$$v^\varepsilon(t,x,y) = u^\varepsilon(t, \varepsilon x, \varepsilon y)\,; \quad (t,x,y) \in \mathbb{R}\times\Omega.$$

One gets

$$\begin{aligned}
&\forall (t,x,y) \in \mathbb{R}\times\Omega,\ v^\varepsilon_t(t,x,y) = u^\varepsilon_t(t,\varepsilon x,\varepsilon y),\\
&\quad \nabla_{x,y}\cdot(A(x,y)\nabla v^\varepsilon)(t,x,y) = \nabla_{x,y}\cdot(A_\varepsilon \nabla u^\varepsilon)(t,\varepsilon x,\varepsilon y) = \varepsilon^2\,\nabla\cdot(A_\varepsilon\nabla u^\varepsilon)(t,\varepsilon x,\varepsilon y),
\end{aligned}$$

and

$$\begin{aligned}
v_\varepsilon(\varepsilon x,\varepsilon y)\cdot[A_\varepsilon\nabla u^\varepsilon](t,\varepsilon x,\varepsilon y) &= v(x,y)\cdot A\left(\frac{\varepsilon x}{\varepsilon},\frac{\varepsilon y}{\varepsilon}\right)\nabla u^\varepsilon(t,\varepsilon x,\varepsilon y)\\
&= \frac{1}{\varepsilon}\, v(x,y)\cdot A(x,y)\nabla v^\varepsilon(t,x,y) \text{ on } \mathbb{R}\times\partial\Omega.
\end{aligned} \tag{8.3}$$

The boundary condition in (8.1) yields that $v_\varepsilon(\varepsilon x,\varepsilon y)\cdot[A_\varepsilon\nabla u^\varepsilon](t,\varepsilon x,\varepsilon y) = 0$, for all $(t,x,y) \in \mathbb{R}\times\partial\Omega$ (which is equivalent to say, for all $(t,\varepsilon x,\varepsilon y) \in \mathbb{R}\times\partial\Omega^\varepsilon$). It follows from (8.3) that

$$\forall (t,x,y) \in \mathbb{R}\times\partial\Omega, \quad v\cdot A\nabla v^\varepsilon(t,x,y) = 0.$$

One can now conclude that for each $\varepsilon > 0$, u^ε satisfies (8.1) if and only if v^ε satisfies

$$\begin{cases} v^\varepsilon_t(t,x,y) = \dfrac{1}{\varepsilon^2}\nabla\cdot(A\nabla v^\varepsilon)(t,x,y) + \dfrac{1}{\varepsilon}\, q\cdot\nabla v^\varepsilon + f(x,y,v^\varepsilon),\ t\in\mathbb{R},\ (x,y)\in\Omega,\\ v\cdot A\,\nabla v^\varepsilon(t,x,y) = 0,\ t\in\mathbb{R},\ (x,y)\in\partial\Omega. \end{cases} \tag{8.4}$$

Having the assumptions (2.1), (2.2), (2.3), (2.4) and (2.5) on Ω, A, q and f, one gets that problem (8.4) admits, for each $\varepsilon > 0$, a minimal speed of propagation denoted by $c^*_{\Omega,(\frac{1}{\varepsilon})^2 A,\frac{1}{\varepsilon}q,f}(e)$.

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Moreover, due to the change of variables between u^ε and v^ε, it follows that for each $\varepsilon > 0$, u^ε is a pulsating travelling front propagating in the direction of e within a speed c and solving (8.1) if and only if v^ε is a pulsating travelling front propagating in the direction of e within a speed $\frac{c}{\varepsilon}$ and solving (8.4). This yields that

$$\forall \varepsilon > 0, \quad c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e) = \varepsilon\, c^*_{\Omega, (\frac{1}{\varepsilon})^2 A, \frac{1}{\varepsilon} q, f}(e) = c^*_{\Omega, MA, \sqrt{M} q, f}(e)/\sqrt{M}, \tag{8.5}$$

where $M = (1/\varepsilon)^2$.

As $\varepsilon \to 0^+$, the variable $M \to +\infty$. Applying Theorem 4.1, with $\gamma = \frac{1}{2}$, one gets that

$$\lim_{M \to +\infty} \frac{c^*_{\Omega, MA, \sqrt{M} q, f}(e)}{\sqrt{M}} = 2\sqrt{\fint_C \tilde{e} A \tilde{e}(x,y)\, dx\, dy} \sqrt{\fint_C \zeta(x,y)\, dx\, dy}.$$

Therefore, $\lim_{\varepsilon \to 0^+} c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e) = 2\sqrt{\fint_C \tilde{e} A \tilde{e}(x,y)\, dx\, dy} \sqrt{\fint_C \zeta(x,y)\, dx\, dy}$, and the proof of Theorem 8.1 is complete. □

Remark 8.2 It is worth noticing that, in formula 8.2, the homogenized speed depends on the averages of the diffusion and reaction coefficients, but it does not depend on the advection.

We move now to study the variation of the map $\varepsilon \mapsto c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e)$ with respect to $\varepsilon > 0$. In other words, we want to check the monotonicity behaviour of the parametric minimal speed of propagation, whose parameter $\varepsilon > 0$, as the periodicity cell of the domain of propagation shrinks or enlarges within a ratio ε. In this study, we will consider the same situation of Theorem 6.1 and also the same notations introduced in the beginning of Section 8.

Theorem 8.3 *Let $e = (1, 0, \ldots, 0)$. Assume that Ω has the form $\mathbb{R} \times \omega$ where ω may or may not be bounded (precisely described in Section 3) and that the diffusion matrix $A = A(y)$ satisfies (3.5) together with the assumption that e is an eigenvector of $A(y)$ for all $y \in \overline{\omega}$, that is*

$$A(x,y)e = A(y)e = \alpha(y)e, \text{ for all } (x,y) \in \mathbb{R} \times \overline{\omega}; \tag{8.6}$$

where $y \mapsto \alpha(y)$ is a positive $(L_1, \ldots, L_d)$-periodic function defined over $\overline{\omega}$. The non-linearity f is assumed to satisfy (3.3) and (3.4). Assume further more that the advection field q (when it exists) is in the form $q(x,y) = (q_1(y), 0, \ldots, 0)$ where q_1 has a zero average over C, the periodicity cell of ω. For $\varepsilon > 0$ consider the reaction–advection–diffusion problem

$$\begin{cases} \forall\, t \in \mathbb{R},\ \forall\, (x,y) \in \Omega^\varepsilon = \mathbb{R} \times \varepsilon\, \omega, \\ u^\varepsilon_t(t,x,y) = \nabla \cdot (A_\varepsilon \nabla u^\varepsilon)(t,x,y) \ + \ q_\varepsilon \cdot \nabla u^\varepsilon \ + \ f_\varepsilon(x,y,u^\varepsilon); \\ v^\varepsilon \cdot A_\varepsilon\, \nabla u^\varepsilon(t,x,y) = 0,\ t \in \mathbb{R},\ (x,y) \in \partial\Omega^\varepsilon. \end{cases} \tag{8.7}$$

Then, the map $\varepsilon \mapsto c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e)$ is increasing in $\varepsilon > 0$.

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Proof of Theorem 8.3 For each $\varepsilon > 0$, we consider the change of variables

$$v^\varepsilon(t,x,y) = u^\varepsilon(t,\varepsilon x,\varepsilon y)\,; \quad (t,x,y) \in \mathbb{R}\times\Omega.$$

Owing to the justifications shown in the proof of Theorem 8.1, one consequently obtains

$$\forall \varepsilon > 0, \quad c^*_{\Omega^\varepsilon,\, A_\varepsilon,\, q_\varepsilon, f_\varepsilon}(e) \quad = \varepsilon\, c^*_{\Omega,\left(\frac{1}{\varepsilon}\right)^2 A,\, \frac{1}{\varepsilon} q, f}(e) = c^*_{\Omega,\beta A,\, \sqrt{\beta}\, q, f}(e)/\sqrt{\beta}, \tag{8.8}$$

where $\beta(\varepsilon) = (1/\varepsilon)^2$.

Applying Theorem 6.1, it follows that the map $\eta_1 : \beta \mapsto c^*_{\Omega,\beta A,\,\sqrt{\beta}\,q,f}(e)/\sqrt{\beta}$ is decreasing in $\beta > 0$. On the other hand, the map $\eta_2 : \varepsilon \mapsto \beta(\varepsilon)$ is also decreasing in $\varepsilon > 0$. Therefore, $\varepsilon \mapsto c^*_{\Omega^\varepsilon, A_\varepsilon, q_\varepsilon, f_\varepsilon}(e)$, which is the composition $\eta_1 \circ \eta_2$, is increasing in $\varepsilon > 0$ and this completes our proof. □

Other homogenization results, concerning reaction–advection–diffusion problems, were given in the case of a combustion-type non-linearity $f = f(u)$ satisfying

$$\begin{cases} \exists\, \theta \in (0,1),\ f(s) = 0 \text{ for all } s \in [0,\theta]\,,\ f(s) > 0 \text{ for all } s \in (\theta,1),\ f(1) = 0, \\ \exists \rho \in (0, 1-\theta), \quad f \text{ is non-increasing on } [1-\rho, 1]\,. \end{cases} \tag{8.9}$$

Consider the equation

$$u^\varepsilon_t(t,x) = \nabla\cdot(A(\varepsilon^{-1}x)\nabla u^\varepsilon) + \varepsilon^{-1} q(\varepsilon^{-1}x)\cdot\nabla u^\varepsilon + f(u^\varepsilon) \quad \text{in } \mathbb{R}^N, \tag{8.10}$$

where the non-linearity f satisfies (8.9), and the drift and diffusion coefficients q and A satisfy the general assumptions (2.2) and (2.3), with periodicity 1 in all variables $x_1,\ldots,x_N$. Fix a unit vector e of $\mathbb{R}^N$. From Berestycki and Hamel [2], it follows that for each $\varepsilon > 0$, problem (8.10) admits a unique pulsating front $(c_\varepsilon, u^\varepsilon)$ such that

$$u^\varepsilon(t,x) = \phi^\varepsilon(x\cdot e + c_\varepsilon t,\, x)$$

where $\phi^\varepsilon(s,x)$ is $(\varepsilon,\ldots,\varepsilon)$-periodic in x that satisfies $\phi^\varepsilon(-\infty,.) = 0$ and $\phi^\varepsilon(+\infty,.) = 1$. The functions u^ε are actually unique up to shifts in time, and one can assume that $\max_{\mathbb{R}^N}\phi^\varepsilon(0,.) = \theta$.

Concerning problem (8.10), Heinze [15] proved that

$$\text{as } \ \varepsilon \to 0^+,\ c_\varepsilon \to c_0 > 0 \text{ and } u^\varepsilon(t,x) \to u_0(x\cdot e + c_0 t) \text{ weakly in } H^1_{\mathrm{loc}},$$

where (c_0, u_0) is the unique solution of the one-dimensional homogenised equation

$$\begin{cases} a^* u_0'' - c_0 u_0' + f(u_0) = 0 \text{ in } \mathbb{R}, \\ u_0(-\infty) = 0 < u_0 < u_0(+\infty) = 1 \text{ in } \mathbb{R},\ u_0(0) = \theta \end{cases} \tag{8.11}$$

and a^* is a positive constant determined in [15].

In Theorem 1 of [10], the homogenization limit was combined with the singular high activation limit for the reaction (one can also see [11] in this context) while the diffusion matrix was taken $A = Id_{\mathbb{R}^N}$. More precisely, the non-linearity had the form $f_\varepsilon(u) = \frac{1}{\varepsilon}\beta(\frac{u}{\varepsilon})$

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with $\beta(s)$ a Lipschitz function satisfying

$$\beta(s) > 0 \text{ in } (0,1) \text{ and } \beta(s) = 0 \text{ otherwise.}$$

These non-linearities approach a Dirac mass at $u = 1$.

9 Open problems

In all the results of this paper, we deal with non-linearities of the "KPP" type. In the periodic framework of this paper, pulsating travelling fronts exist also with other types of non-linearities (see Theorems 2.13 and 2.14 in [2]). Namely, they exist when $f = f(x, y, u)$ is of the "combustion"-type satisfying

$$\begin{cases} f \text{ is globally Lipschitz-continuous in } \overline{\Omega} \times \mathbb{R}, \\ \forall\, (x,y) \in \overline{\Omega},\ \forall\, s \in (-\infty, 0] \cup [1, +\infty),\ f(s,x,y) = 0, \\ \exists\, \rho \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall\, 1-\rho \leqslant s \leqslant s' \leqslant 1,\ f(x,y,s) \geqslant f(x,y,s'), \end{cases} \tag{9.1}$$

and

$$\begin{cases} f \text{ is } L\text{-periodic with respect to } x, \\ \exists\, \theta \in (0,1),\ \forall (x,y) \in \overline{\Omega},\ \forall\, s \in [0,\theta],\ f(x,y,s) = 0, \\ \forall\, s \in (\theta,1),\ \exists\, (x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0, \end{cases} \tag{9.2}$$

or when $f = f(x, y, u)$ is of the "ZFK" (for Zeldovich–Frank–Kamenetskii)-type satisfying (9.1) and

$$\begin{cases} f \text{ is } L\text{-periodic with respect to } x, \\ \exists\, \delta > 0, \text{ the restriction of } f \text{ to } \overline{\Omega} \times [0,1] \text{ is of class } C^{1,\delta}, \\ \forall\, s \in (0,1),\ \exists\, (x,y) \in \overline{\Omega} \text{ such that } f(x,y,s) > 0. \end{cases} \tag{9.3}$$

In particular, the "KPP" non-linearities are of the "ZFK" type.

Recently, El Smaily [12] gave min–max and max–min formulae for the speeds of propagation of problem (2.6) taken with a "ZFK" or a "combustion" non-linearity. These formulae, together with the results of this paper, can give important estimates for the parametric minimal speeds of the problem (2.6) when f is a "ZFK" non-linearity which is not of the "KPP" type. Indeed, if f is a "ZFK" non-linearity, one can find a "KPP" function $h = h(x, y, u)$ such that

$$\forall (x,y,u) \in \overline{\Omega} \times \mathbb{R},\ f(x,y,u) \leqslant h(x,y,u).$$

Referring to formula (1.17) in El Smaily [12], one can conclude that

$$\forall M > 0, \forall B > 0,\ \forall \gamma \in \mathbb{R},\quad c^*_{\Omega,MA,\, M^\gamma q, Bf}(e) \leqslant c^*_{\Omega,MA,\, M^\gamma q, Bh}(e).$$

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Moreover, if f is a "ZFK" non-linearity satisfying the additional assumption

$$\forall (x,y) \in \overline{\Omega},\ f'_u(x,y,0) > 0, \tag{9.4}$$

then one can find a "KPP" function $g = g(x,y,u)$ such that $g \leqslant f$ in $\overline{\Omega} \times \mathbb{R}$, and thus

$$\begin{gathered} \forall M > 0, \forall B > 0,\ \forall \gamma \in \mathbb{R}, \\ c^*_{\Omega,MA,M^\gamma q,Bg}(e) \leqslant c^*_{\Omega,MA,M^\gamma q,Bf}(e) \leqslant c^*_{\Omega,MA,M^\gamma q,Bh}(e). \end{gathered} \tag{9.5}$$

As a consequence, under the assumptions that $0 \leqslant \gamma \leqslant 1/2$, $\nu \cdot A\tilde{e} = 0$ on $\partial\Omega$, and $\nabla \cdot A\tilde{e} \equiv 0$ in Ω, Theorem 4.1 implies that

$$\limsup_{M\to+\infty} \frac{c^*_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}} \leqslant 2\sqrt{\iint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\ \sqrt{\iint_C g'_u(x,y,0)\,dx\,dy}, \tag{9.6}$$

and

$$\liminf_{M\to+\infty} \frac{c^*_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}} = 2\sqrt{\iint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\ \sqrt{\iint_C h'_u(x,y,0)\,dx\,dy}\ > 0. \tag{9.7}$$

If f is a "combustion" non-linearity, then problem (2.6) admits a solution (c,u) where $c = c_{\Omega,A,q,f}(e) > 0$ is unique and $u = u(t,x,y)$ is increasing in t and it is unique up to a translation in t. Taking g as a "KPP" non-linearity such that $g \geqslant f$ in $\overline{\Omega} \times \mathbb{R}$ and using Theorem 4.1, it follows that

$$\begin{gathered} \limsup_{M\to+\infty} \frac{c_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}} \leqslant 2\sqrt{\iint_C \tilde{e}A\tilde{e}(x,y)dx\,dy}\ \sqrt{\iint_C g'_u(x,y,0)\,dx\,dy} \\ \text{together with} \quad \liminf_{M\to+\infty} \frac{c_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}} \geqslant 0. \end{gathered} \tag{9.8}$$

Similarly, one can get several estimates concerning the case of a small diffusion factors, small (resp. large) reaction factors, or small (resp. large) periodicity parameters.

The above motivation gives several upper and lower estimates for the parametric speeds of propagation. However, the exact limits are not known. This leads us to ask about the asymptotics of the minimal speeds of propagation with respect to diffusion, reaction and periodicity factors in the "ZFK" case and about the asymptotics of the unique parametric speed of propagation in the "combustion" case. These studies should help, as it was done in Section 8, in solving some homogenisation problems in the "ZFK" case.

Besides, Theorem 8.1 gives the limit of $c^*_{\Omega^\varepsilon,A_\varepsilon,q_\varepsilon,f_\varepsilon}(e)$ as $\varepsilon \to 0^+$. However, finding the homogenised equation of (8.1) in the "KPP" remains an open problem.

10 Conclusions

As we mentioned in the beginning of this paper, our first aim was to give a complete and rigorous analysis of the *minimal speed of propagation* of pulsating travelling fronts solving *parametric heterogeneous* reaction–advection–diffusion equations in a periodic framework. In the paper of Berestycki, Hamel and Nadirashvili [3], several upper and lower estimates for the parametric minimal speed of propagation were given (see Theorems 2.6 and 2.10

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in [3]). However, the exact asymptotic behaviours of the minimal speed with respect to diffusion and reaction factors and with respect to the periodicity parameter L were not given there. In this paper, we determined the exact asymptotes of the minimal speed in the "KPP" periodic framework. In Sections 3, 4 and 5, we proved that (under some assumptions on A, q, f and Ω) the asymptotes of the parametric minimal speed are either

$$2\sqrt{\max_{\overline{\omega}} \zeta}\sqrt{\max_{\overline{\omega}} eAe} \quad \text{or} \quad 2\sqrt{\fint_C \tilde{e}A\tilde{e}(x,y)\,dx\,dy}\sqrt{\fint_C \zeta(x,y)\,dx\,dy}.$$

(see Theorems 3.1, 3.4, 4.1, 4.3, 5.1 and 5.2 above). Moreover, we found in Section 3 that the presence of an advection field, in the general form or in the form of shear flows, changes the asymptotic behaviour of the minimal speed within a small diffusion (see Theorem 3.3 and Remark 3.6). Conversely, we proved in Section 4 that the presence of a general advection field $M^\gamma q$ (where q satisfies (2.3)) has no effect on $\lim_{M\to+\infty} \frac{c^*_{\Omega,MA,M^\gamma q,f}(e)}{\sqrt{M}}$ whenever $0 \leqslant \gamma \leqslant 1/2$ (see Theorem 4.1). Furthermore, we studied, in a particular periodic framework, the variations of the maps $\beta \mapsto \frac{c^*_{\Omega,\beta A,\sqrt{\beta}q,f}(e)}{\sqrt{\beta}}$ and $L \mapsto c^*_{\mathbb{R}^N,A_L,q_L,f_L}(e)$ and $B \mapsto \frac{c^*_{\Omega,A,\sqrt{B}q,Bf}(e)}{\sqrt{B}}$ with respect to the positive variables β, L and B, respectively. Roughly speaking, we found that the first and the third maps have opposite senses of variations (see Theorems 6.1 and 6.5). On the other hand, Theorems 6.3 and 8.3 yield that the minimal speed increases when the medium undergoes a dilation whose scale factor is greater than 1.

The second aim was to find the homogenized "KPP" minimal speed. We achieved this goal in Section 8 (Theorem 8.1) under the assumptions of free divergence on $A(x,y)\tilde{e}$ and invariance of the domain in the direction $A(x,y)\tilde{e}$. This was an application to the results obtained in Section 4. The found homogenized speed should play an important role in finding the homogenized reaction–advection–diffusion equation in the "KPP" case. In a forthcoming paper [13], we find also the homogenised speed in the one-dimensional case but in a more general setting (in fact, the assumption of divergence free is equivalent to the assumption that the diffusion term $x \mapsto a(x)$ is constant over $\mathbb{R}$ in the case $N = 1$).

All the mathematical results obtained in this paper can be applied to study some *spreading* phenomena. Referring to the results of Weinberger [30], one can conclude that the *spreading speed* is equal to the "KPP" *minimal speed of propagation* in the periodic framework under some assumptions on the initial data $u_0 := u_0(x,y) = u(0,x,y)$ which is defined on a periodic domain Ω of $\mathbb{R}^N$. In such a setting, all our results can be applied to give rigorous answers on the asymptotic behaviour of the *parametric spreading speed* with respect to diffusion and reaction factors and with respect to the periodicity parameter.

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