



# Infinitesimal Hilbertianity of Weighted Riemannian Manifolds

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*Abstract.* The main result of this paper is the following: any *weighted* Riemannian manifold  $(M, g, \mu)$ , i.e., a Riemannian manifold  $(M, g)$  endowed with a generic non-negative Radon measure  $\mu$ , is *infinitesimally Hilbertian*, which means that its associated Sobolev space  $W^{1,2}(M, g, \mu)$  is a Hilbert space.

We actually prove a stronger result: the abstract tangent module (à la Gigli) associated with any weighted reversible Finsler manifold  $(M, F, \mu)$  can be isometrically embedded into the space of all measurable sections of the tangent bundle of  $M$  that are 2-integrable with respect to  $\mu$ .

By following the same approach, we also prove that all weighted (sub-Riemannian) Carnot groups are infinitesimally Hilbertian.

## 1 Introduction

*General Overview* In the rapidly expanding theory of geometric analysis over metric measure spaces  $(X, d, m)$  a key role is played by the notion of Sobolev space  $W^{1,2}(X, d, m)$  that has been proposed in [11] (see also [5, 26]). In general, the space  $W^{1,2}(X, d, m)$  has a Banach space structure, but is not necessarily a Hilbert space. Those metric measure spaces  $(X, d, m)$  whose associated Sobolev space  $W^{1,2}(X, d, m)$  is Hilbert, are said to be *infinitesimally Hilbertian* [16]. This choice of terminology is due to the fact that such requirement captures, in a sense, the property of being a *Hilbert-like* space at arbitrarily small scales.

Infinitesimally Hilbertian spaces are particularly relevant in several situations. For instance, in the framework of synthetic lower Ricci curvature bounds (in the sense of Lott–Villani [24] and Sturm [27, 28]) known as the *CD condition*, the infinitesimal Hilbertianity assumption has been used to single out the Riemannian structures among the Finslerian ones, thus bringing forth the well-established notion of *RCD space* [7, 8, 16]. We refer to the surveys [2, 29, 30] for a detailed account of the vast literature concerning the CD/RCD conditions.

The main purpose of the present paper is to prove that any geodesically complete Riemannian manifold  $(M, g)$  is *universally infinitesimally Hilbertian*, meaning that  $(M, d_g, \mu)$  is infinitesimally Hilbertian for any Radon measure  $\mu \geq 0$  on  $M$ , where  $d_g$  stands for the distance on  $M$  induced by the Riemannian metric  $g$ . This will be achieved as an immediate consequence of the following result: given a geodesically complete, reversible Finsler manifold  $(M, F)$  and a non-negative Radon measure  $\mu$

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on  $M$ , it holds that the *abstract* tangent module  $L^2_\mu(TM)$  associated with  $(M, F, \mu)$  in the sense of Gigli [18] can be isometrically embedded into the *concrete* space of all  $L^2(\mu)$ -sections of the tangent bundle  $TM$  of  $M$ . We will also describe how to obtain the corresponding results in the setting of weighted Carnot groups.

*Motivation and Related Works* Our interest in universally infinitesimally Hilbertian metric spaces is mainly motivated by the study of metric-valued Sobolev maps, as we will describe. Given a metric measure space  $(X, d_X, m)$  and a complete separable metric space  $(Y, d_Y)$ , one of the possible ways to define the space  $S^2(X; Y)$  of *weakly differentiable* maps from  $X$  to  $Y$  is via post-composition [21]. As shown in [20, Theorem 3.3], any Sobolev map  $u \in S^2(X; Y)$  can be naturally associated with an  $L^0(m)$ -linear and continuous operator  $du: L^0_m(TX) \rightarrow (u^*L^0_\mu(T^*Y))^*$ , where the finite Borel measure  $\mu$  is defined as  $\mu := u_*(|Du|^2m)$ ; the map  $du$  is called *differential*. (See [20, §2] for a brief summary of the terminology used above.) We underline that the measure  $\mu$  is not given *a priori*, but rather depends on the map  $u$  itself in a non-trivial manner. This implies that the target module of  $du$  might possess a very complicated structure. One of the reasons why we focus on universally infinitesimally Hilbertian spaces  $(Y, d_Y)$  is that the cotangent module  $L^0_\mu(T^*Y)$  is a Hilbert module regardless of the chosen measure  $\mu$ . In particular, the target space  $(u^*L^0_\mu(T^*Y))^*$  of the differential  $du$  is a Hilbert module as well and can be canonically identified with  $u^*L^0_\mu(TY)$ . This allows for more refined calculus tools and nicer functional-analytic properties; see [18] for the related discussion. Even more importantly, to show that the abstract tangent module  $L^0_\mu(TY)$  isometrically embeds into some geometric space of sections would provide a more concrete representation of the differential operator  $du$ .

The results contained in this paper were proved in [19] for the particular case in which the Finsler manifold  $(M, F)$  under consideration is the Euclidean space  $\mathbb{R}^n$  equipped with any norm  $\|\cdot\|$ . In fact, the structure of our proofs follows along the path traced by [19]. We also mention it was proved that locally  $CAT(\kappa)$  spaces are universally infinitesimally Hilbertian [13]. We recall that these are geodesic metric spaces whose sectional curvature is (locally) bounded from above by  $\kappa \in \mathbb{R}$  in the sense of Alexandrov. The motivation behind such a result is that if used in conjunction with the notion of differential operator for metric-valued Sobolev maps discussed above, it could be helpful in order to study the regularity properties of harmonic maps from finite-dimensional RCD spaces to  $CAT(0)$  spaces.

*Outline of the Work* In Section 2 we briefly recall the basics of Sobolev calculus on metric measure spaces and the language of  $L^2$ -normed  $L^\infty$ -modules proposed by Gigli [18].

Section 3 is entirely devoted to Finsler geometry. After a short introduction to a few basic concepts, we will be concerned with the approximation of Lipschitz functions by  $C^1$ -functions. Our new contribution in this regard, namely Theorem 3.6, constitutes a more local version of similar results that were proved in [9, 15, 22].

The core of the paper is Section 4. In Proposition 4.2 we exploit the above-mentioned approximation result to bridge the gap between the abstract Sobolev space associated with a weighted Finsler manifold  $(M, F, \mu)$  and the true differentials of functions in  $C^1_c(M)$ . This represents the key passage to build a quotient projection map from the space  $\Gamma_2(T^*M; \mu)$  of all  $L^2(\mu)$ -sections of  $T^*M$  to  $L^2_\mu(T^*M)$  (Lemma 4.5,

Proposition 4.6). We thus obtain by duality an isometric embedding of  $L^2_\mu(TM)$  into the space  $\Gamma_2(TM; \mu)$  of all  $L^2(\mu)$ -sections of  $TM$  (Theorem 4.7). As a direct corollary, any weighted Riemannian manifold is infinitesimally Hilbertian (Theorem 4.11).

In Section 5 we provide an alternative proof of Theorem 4.11, which does not rely upon Theorem 4.7. This approach combines the analogue of Theorem 4.11 for the Euclidean space proved in [19] with a localisation argument. Nonetheless, we preferred to follow the first approach in order to place the emphasis on Theorem 4.7, because of its independent interest.

Finally, in Section 6 we recall the basic notions in the theory of Carnot groups, we prove a smoothing result for Lipschitz functions on a sub-Finsler Carnot group  $\mathbb{G}$  equipped with the induced Carnot–Carathéodory distance  $d_{CC}$  (Theorem 6.2), and we build an isometric embedding of the tangent module  $L^2_\mu(T\mathbb{G})$  associated with a weighted sub-Finsler Carnot group  $(\mathbb{G}, d_{CC}, \mu)$  into the space  $\Gamma_2(H\mathbb{G}; \mu)$  of all  $L^2(\mu)$ -sections of the horizontal bundle of  $\mathbb{G}$  (Theorem 6.3). We can thus conclude that any weighted sub-Riemannian Carnot group is infinitesimally Hilbertian.

## 2 Preliminaries on Metric Measure Spaces

### 2.1 Notation on Metric Spaces

Consider a metric space  $(X, d)$ . Given any  $x \in X$  and  $r > 0$ , we denote by  $B_r^X(x)$  the open ball in  $(X, d)$  with center  $x$  and radius  $r$ . More generally, we denote by  $B_r^X(E)$  the  $r$ -neighbourhood of any set  $E \subseteq X$ . We shall sometimes work with metric spaces having the property that the closure of any ball is compact: such spaces are said to be *proper*.

We shall use the notation  $LIP(X)$  to indicate the family of all real-valued Lipschitz functions defined on  $X$ , while  $LIP_c(X)$  will be the set of all functions in  $LIP(X)$  having compact support. Given any  $f \in LIP(X)$ , let us introduce the following quantities.

(i) *Global Lipschitz constant.* Let  $E \subseteq X$  be a given non-empty set. Then we denote by  $Lip(f; E)$  the Lipschitz constant of  $f|_E$ , i.e., we set  $Lip(f; E) := 0$  if  $E$  is a singleton, and

$$Lip(f; E) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \mid x, y \in E, x \neq y \right\}$$

otherwise. For the sake of brevity, we shall write  $Lip(f)$  instead of  $Lip(f; X)$ .

(ii) *Local Lipschitz constant.* We define the function  $lip(f): X \rightarrow [0, +\infty)$  as

$$lip(f)(x) := \overline{\lim}_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}$$

for every accumulation point  $x \in X$  and  $lip(f)(x) := 0$  for every isolated point  $x \in X$ .

(iii) *Asymptotic Lipschitz constant.* We define the function  $lip_a(f): X \rightarrow [0, +\infty)$  as  $lip_a(f)(x) := \inf_{r>0} Lip(f; B_r^X(x))$  for every accumulation point  $x \in X$  and  $lip_a(f)(x) := 0$  for every isolated point  $x \in X$ .

It can be readily checked that  $lip(f) \leq lip_a(f) \leq Lip(f)$  is satisfied in  $X$ .

## 2.2 Sobolev Calculus on Metric Measure Spaces

For our purposes, by a *metric measure space* we mean a triple  $(X, d, m)$ , where

$$(2.1) \quad \begin{aligned} &(X, d) \text{ is a complete and separable metric space and} \\ &m \neq 0 \text{ is a non-negative Radon measure on } (X, d). \end{aligned}$$

In order to introduce the notion of Sobolev space  $W^{1,2}(X, d, m)$  proposed by L. Ambrosio, N. Gigli, and G. Savaré [5], we need to fix some notation. We say that a continuous curve  $\gamma: [0, 1] \rightarrow X$  is *absolutely continuous* provided there exists  $f \in L^1(0, 1)$  such that  $d(\gamma_t, \gamma_s) \leq \int_s^t f(r) dr$  holds for every  $t, s \in [0, 1]$  with  $s < t$ . The minimal 1-integrable function (in the almost everywhere sense) that can be chosen as  $f$  is called the *metric speed* of  $\gamma$  and is denoted by  $|\dot{\gamma}|$ . It holds that  $|\dot{\gamma}_t| = \lim_{h \rightarrow 0} d(\gamma_{t+h}, \gamma_t)/|h|$  for almost every  $t \in (0, 1)$  [4, Theorem 1.1.2].

A *test plan* on  $X$  is any Borel probability measure  $\pi$  on  $C([0, 1], X)$  with the following properties.

- There exists a constant  $C > 0$  such that  $(e_t)_* \pi \leq Cm$  holds for every  $t \in [0, 1]$ , where the evaluation map  $e_t: C([0, 1], X) \rightarrow X$  is given by  $e_t(\gamma) := \gamma_t$  and  $(e_t)_* \pi$  stands for the pushforward measure of  $\pi$  under  $e_t$ .
- It holds that  $\iint_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) < +\infty$ , with the convention that  $\int_0^1 |\dot{\gamma}_t|^2 dt := +\infty$  when the curve  $\gamma$  is not absolutely continuous.

In particular, any test plan is concentrated on the family of all absolutely continuous curves on  $X$ .

**Definition 2.1** ([5]) We define the *Sobolev space*  $W^{1,2}(X, d, m)$  as the set of all functions  $f \in L^2(m)$  with the following property: there exists  $G \in L^2(m)$  such that  $\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) \leq \iint_0^1 G(\gamma_t) |\dot{\gamma}_t| dt d\pi(\gamma)$  for every test plan  $\pi$  on  $X$ . Any such function  $G$  is said to be a *weak upper gradient* of  $f$ . The minimal weak upper gradient of the function  $f$ , intended in the  $m$ -almost everywhere sense, is denoted by  $|Df|$ .

The Sobolev space  $W^{1,2}(X, d, m)$  is a Banach space if endowed with the norm  $\|f\|_{W^{1,2}(X, d, m)} := (\|f\|_{L^2(m)}^2 + \| |Df| \|^2_{L^2(m)})^{1/2}$  for every  $f \in W^{1,2}(X, d, m)$ , but in general it is not a Hilbert space. For this reason, the following definition is meaningful.

**Definition 2.2** (Infinitesimal Hilbertianity) We say that the metric measure space  $(X, d, m)$  is *infinitesimally Hilbertian* provided its associated Sobolev space  $W^{1,2}(X, d, m)$  is a Hilbert space.

An important property of minimal weak upper gradients is their lower semicontinuity.

**Proposition 2.3** ([5]) Let  $(f_n)_{n \in \mathbb{N}} \subseteq W^{1,2}(X, d, m)$  satisfy  $f_n \rightarrow f$  in  $L^2(m)$  for some  $f \in L^2(m)$ . Suppose also that  $|Df_n| \rightarrow G$  weakly in  $L^2(m)$  for some  $G \in L^2(m)$ . Then  $f \in W^{1,2}(X, d, m)$ , and the inequality  $|Df| \leq G$  holds  $m$ -almost everywhere in  $X$ .

We point out that  $W^{1,2}(X, d, m)$  contains all Lipschitz functions on  $X$  having compact support. More precisely, given any function  $f \in \text{LIP}_c(X)$  it holds that

$$(2.2) \quad |Df| \leq \text{lip}(f) \quad \text{in the } m\text{-almost everywhere sense.}$$

On proper spaces, Lipschitz functions with compact support are *dense in energy* in  $W^{1,2}(X, d, m)$ .

**Theorem 2.4** (Ambrosio–Gigli–Savaré [6]) *Suppose  $(X, d, m)$  is a proper metric measure space. Fix any Sobolev function  $f \in W^{1,2}(X, d, m)$ . Then there exists a sequence  $(f_n)_{n \in \mathbb{N}} \subseteq \text{LIP}_c(X)$  such that  $f_n \rightarrow f$  and  $\text{lip}_a(f_n) \rightarrow |Df|$  in  $L^2(m)$  as  $n \rightarrow \infty$ .*

### 2.3 Abstract Tangent and Cotangent Modules

Consider a metric measure space  $(X, d, m)$ . We assume that the reader is familiar with the language of  $L^2(m)$ -normed  $L^\infty(m)$ -modules [17, 18].

We just recall that there is a unique couple  $(L^2_m(T^*X), d)$ , where  $L^2_m(T^*X)$  is an  $L^2(m)$ -normed  $L^\infty(m)$ -module called the *cotangent module* and  $d: W^{1,2}(X, d, m) \rightarrow L^2_m(T^*X)$  is a linear operator called the *differential*, such that the following two conditions are satisfied.

- It holds that  $|df| = |Df|$  in the  $m$ -almost everywhere sense for every  $f \in W^{1,2}(X, d, m)$ .
- The set  $\{df : f \in W^{1,2}(X, d, m)\}$  generates  $L^2_m(T^*X)$  in the sense of modules.

The module dual of  $L^2_m(T^*X)$  is called the *tangent module* and is denoted by  $L^2_m(TX)$ .

A fundamental property of the differential, which follows from Proposition 2.3, is that it is a closed operator [18, Theorem 2.2.9].

**Proposition 2.5** (Closure of  $d$ ) *Let  $(f_n)_{n \in \mathbb{N}} \subseteq W^{1,2}(X, d, m)$  be a sequence satisfying*

$$f_n \rightharpoonup f, \text{ weakly in } L^2(m), \quad df_n \rightharpoonup \omega, \text{ weakly in } L^2_m(T^*X),$$

*for some  $f \in L^2(m)$  and  $\omega \in L^2_m(T^*X)$ . Then  $f \in W^{1,2}(X, d, m)$  and  $df = \omega$ .*

The following result is taken from [18, Proposition 2.2.10].

**Proposition 2.6** (Reflexivity of the Sobolev space) *The following conditions are equivalent.*

- (i) *The Sobolev space  $W^{1,2}(X, d, m)$  is reflexive.*
- (ii) *Given any bounded sequence  $(f_n)_{n \in \mathbb{N}} \subseteq W^{1,2}(X, d, m)$ , there exist  $f \in W^{1,2}(X, d, m)$  and a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  such that  $(f_{n_k}, df_{n_k}) \rightharpoonup (f, df)$  weakly in  $L^2(m) \times L^2_m(T^*X)$ .*

*In particular, if  $L^2_m(T^*X)$  is reflexive, then  $W^{1,2}(X, d, m)$  is reflexive.*

Finally, we point out that

$$(2.3) \quad W^{1,2}(X, d, m) \text{ is a Hilbert space} \iff L^2_m(TX) \text{ is a Hilbert module,}$$

as proved in [18, Proposition 2.3.17].

### 3 Some Properties of Finsler Manifolds

#### 3.1 Definition and Basic Results

For our purposes, by a *manifold* we shall always mean a connected differentiable manifold of class  $C^\infty$ . Given a manifold  $M$  and a point  $x \in M$ , we denote by  $T_x M$  the tangent space of  $M$  at  $x$  and by  $\exp_x$  the exponential map at  $x$ . We make use of the notation  $TM = \bigsqcup_{x \in M} T_x M$  to indicate the tangent bundle of  $M$ . Moreover, we denote by  $T_x^* M$  and  $T^* M$  the cotangent space of  $M$  at  $x$  and the cotangent bundle of  $M$ , respectively. We now briefly report the definition of a Finsler structure over a manifold; see [10] for a thorough account about this topic.

Let  $V$  be a given finite-dimensional vector space over  $\mathbb{R}$ . Then a *Minkowski norm* on  $V$  is a functional  $F: V \rightarrow [0, +\infty)$  having the following properties.

- (i) *Positive definiteness.* Given any  $v \in V$ , we have that  $F(v) = 0$  if and only if  $v = 0$ .
- (ii) *Triangle inequality.* It holds that  $F(v + w) \leq F(v) + F(w)$  for every  $v, w \in V$ .
- (iii) *Positive homogeneity.* We have that  $F(\lambda v) = \lambda F(v)$  for every  $v \in V$  and  $\lambda \geq 0$ .
- (iv) *Regularity.* The function  $F$  is continuous on  $V$  and of class  $C^\infty$  on  $V \setminus \{0\}$ .
- (v) *Strong convexity.* Given any  $v \in V \setminus \{0\}$ , it holds that the quadratic form

$$(3.1) \quad V \ni w \longmapsto \frac{1}{2} d^2(F^2)_v[w, w]$$

is positive definite. (The expression in (3.1) stands for the second differential of  $F^2$  at  $v$ .)

In particular, any Minkowski norm is an asymmetric norm.

**Definition 3.1** (Finsler manifold) A *Finsler manifold* is a couple  $(M, F)$ , where  $M$  is a given manifold and  $F: TM \rightarrow [0, +\infty)$  is a continuous function satisfying the following properties.

- (i) The function  $F$  is of class  $C^\infty$  on  $TM \setminus \{0\}$ .
  - (ii) The functional  $F(x, \cdot): T_x M \rightarrow [0, +\infty)$  is a Minkowski norm for every  $x \in M$ .
- Moreover, we say that  $(M, F)$  is *reversible* provided each function  $F(x, \cdot)$  is symmetric, i.e.,

$$(3.2) \quad F(x, -v) = F(x, v) \quad \text{for every } x \in M \text{ and } v \in T_x M.$$

Condition (3.2) is equivalent to requiring that each  $F(x, \cdot)$  is a (symmetric) norm on  $T_x M$ .

We point out that any Riemannian manifold is a special case of a reversible Finsler manifold. (This is an abuse of notation. More precisely, if  $(M, g)$  is a Riemannian manifold, then  $(M, F)$  is a reversible Finsler manifold, where we set  $F(x, v) := g_x(v, v)^{1/2}$  for every  $x \in M$  and  $v \in T_x M$ .)

**Definition 3.2** (Finsler distance) Let  $(M, F)$  be a reversible Finsler manifold. Given any piecewise  $C^1$ -curve  $\gamma: [0, 1] \rightarrow M$ , we define its *Finsler length* as  $\ell_F(\gamma) := \int_0^1 F(\gamma_t, \dot{\gamma}_t) dt$ . Then we define the *Finsler distance*  $d_F(x, y)$  between two points

$x, y \in M$  as

$$(3.3) \quad d_F(x, y) := \inf \{ \ell_F(\gamma) \mid \gamma: [0, 1] \rightarrow M \text{ piecewise } C^1 \text{ with } \gamma_0 = x \text{ and } \gamma_1 = y \}.$$

A *Finsler geodesic* is a  $C^1$ -curve on  $M$  that is locally a stationary point of the length functional.

**Remark 3.3** When  $(M, F)$  is a (not reversible) Finsler manifold, one has that formula (3.3) defines a quasi-distance on  $M$ , rather than a distance in the usual sense. Our main approximation result, namely Theorem 3.6, still holds true even in the case of general Finsler manifolds (this can be achieved with minor modifications of the arguments, as we shall see). Nevertheless, we prefer to focus our attention on the reversible case, the reason being that the language of Sobolev calculus and (co)tangent modules is so far available just for metric structures.

For a proof of the ensuing result in the Finsler case, see [10, Theorem 6.6.1].

**Theorem 3.4** (Hopf–Rinow) *Let  $(M, F)$  be a reversible Finsler manifold. Then the following four conditions are equivalent.*

- (i) *The Finsler manifold  $(M, F)$  is geodesically complete, i.e., any constant speed geodesic can be extended to a geodesic defined on the whole real line.*
- (ii) *The metric space  $(M, d_F)$  is complete.*
- (iii) *Given any  $x \in M$ , it holds that the exponential map  $\exp_x$  is defined on the whole  $T_x M$ .*
- (iv) *The metric space  $(M, d_F)$  is proper.*

### 3.2 Smooth Approximation of Lipschitz Functions

In the sequel, we shall need the following result concerning the biLipschitz behaviour of the exponential map on sufficiently small balls.

**Theorem 3.5** (Deng–Hou [12]) *Let  $(M, F)$  be a reversible Finsler manifold. Fix a point  $x \in M$  and some constant  $\varepsilon > 0$ . Then there exists a radius  $r > 0$  such that the exponential map  $\exp_x: B_r^{T_x M}(0) \rightarrow B_r^M(x)$  is a  $(1 + \varepsilon)$ -biLipschitz  $C^1$ -diffeomorphism.*

We now present a new result about regularisation of Lipschitz functions on a reversible Finsler manifold  $(M, F)$ . Roughly speaking, it states that any Lipschitz function  $f: M \rightarrow \mathbb{R}$  can be uniformly approximated by functions of class  $C^1$  whose Lipschitz constant is locally controlled by that of  $f$ . This represents a local variant of the approximation theorem proved in [15].

**Theorem 3.6** *Let  $(M, F)$  be a reversible Finsler manifold. Fix a Lipschitz function  $f \in \text{LIP}(M)$  and some constants  $\delta, \varepsilon, \lambda > 0$ . Then there exists a function  $g \in C^1(M)$  with  $\text{spt}(g) \subseteq B_\delta^M(\text{spt}(f))$  such that*

$$(3.4) \quad |g(x) - f(x)| \leq \varepsilon \quad \text{and} \quad \text{lip}_a(g)(x) \leq \text{Lip}(f; B_\delta^M(x)) + \lambda$$

for every  $x \in M$ .

**Proof** We divide the proof into several steps.

*Step 1:* Set-up. Fix any  $r > 0$  such that  $r \leq \delta/2$  and

$$(3.5) \quad (2r + r^2) \operatorname{Lip}(f) + r \leq \lambda.$$

Theorem 3.5 grants that for any  $x \in M$  we can pick a radius  $r_x \in (0, r)$  such that the exponential map  $\exp_x : B_{r_x}^{T_x M}(0) \rightarrow B_{r_x}^M(x)$  is a  $(1 + r)$ -biLipschitz  $C^1$ -diffeomorphism. Hence we can choose a sequence  $(x_i)_{i \in \mathbb{N}} \subseteq M$  such that the family  $(B_i)_{i \in \mathbb{N}}$ , where we set  $B_i := B_{r_{x_i}}^M(x_i)$  for all  $i \in \mathbb{N}$ , is a locally finite open covering of  $M$ . Given any  $i \in \mathbb{N}$ , we fix a linear isomorphism  $I_i : \mathbb{R}^n \rightarrow T_{x_i} M$ , where  $n := \dim(M)$ . Let us define the norm  $\|\cdot\|_i$  on  $\mathbb{R}^n$  as

$$\|v\|_i := F(x_i, I_i(v)) \quad \text{for every } v \in \mathbb{R}^n.$$

Since any two norms on  $\mathbb{R}^n$  are equivalent, there exists  $C_i \geq 1$  such that

$$(3.6) \quad \frac{1}{C_i} \|v\|_i \leq |v| \leq C_i \|v\|_i \quad \text{for every } v \in \mathbb{R}^n.$$

We define the chart  $\varphi_i : B_i \rightarrow \mathbb{R}^n$  as

$$\varphi_i(x) := (\exp_{x_i} \circ I_i)^{-1}(x) \quad \text{for every } x \in B_i.$$

Therefore,  $\varphi_i$  is a  $(1 + r)$ -biLipschitz  $C^1$ -diffeomorphism from  $(B_i, d_F)$  to  $(\varphi_i(B_i), \|\cdot\|_i)$ . Moreover, let us fix a smooth partition of unity  $(\psi_i)_{i \in \mathbb{N}}$  subordinated to the covering  $(B_i)_{i \in \mathbb{N}}$ .

- The functions  $\psi_i$  belong to  $C_c^\infty(M)$ .
- $0 \leq \psi_i \leq 1$  and  $\operatorname{spt}(\psi_i) \subseteq B_i$  for every  $i \in \mathbb{N}$ .
- $\sum_{i \in \mathbb{N}} \psi_i(x) = 1$  holds for every  $x \in M$ .

Finally, for any  $i \in \mathbb{N}$ , we call  $\mathcal{A}_i := \{j \in \mathbb{N} : B_j \cap B_i \neq \emptyset\}$ , we denote by  $n_i \in \mathbb{N}$  the cardinality of the set  $\mathcal{A}_i$ , and we define  $m_i := \max\{n_j : j \in \mathcal{A}_i\} \in \mathbb{N}$ . Then it is immediate to check that

$$(3.7) \quad n_i \leq m_j \quad \text{for every } i \in \mathbb{N} \text{ and } j \in \mathcal{A}_i.$$

*Step 2:* Construction of  $g$ . First fix a family  $(\rho_k)_{k \in \mathbb{N}}$  of smooth mollifiers on  $\mathbb{R}^n$ .

- The functions  $\rho_k$  are symmetric and belong to  $C_c^\infty(\mathbb{R}^n)$ .
- $\rho_k \geq 0$  and  $\operatorname{spt}(\rho_k) \subseteq B_{1/k}^{\mathbb{R}^n}(0)$  for every  $k \in \mathbb{N}$ .
- $\int_{\mathbb{R}^n} \rho_k(v) \, dv = 1$  holds for every  $k \in \mathbb{N}$ .

For any  $i \in \mathbb{N}$  we can choose a McShane extension  $f_i : (\mathbb{R}^n, \|\cdot\|_i) \rightarrow \mathbb{R}$  of  $f \circ \varphi_i^{-1} : \varphi_i(B_i) \rightarrow \mathbb{R}$ , namely  $f_i$  is a Lipschitz function with  $\operatorname{Lip}(f_i) \leq (1 + r) \operatorname{Lip}(f; B_i)$  that coincides with  $f \circ \varphi_i^{-1}$  on the set  $\varphi_i(B_i)$ . Now we define  $f_i^k : \mathbb{R}^n \rightarrow \mathbb{R}$  for any  $i, k \in \mathbb{N}$  as

$$f_i^k(v) := (f_i * \rho_k)(v) = \int_{\mathbb{R}^n} f_i(v + w) \rho_k(w) \, dw \quad \text{for every } v \in \mathbb{R}^n.$$

It is well known that each function  $f_i^k$  is of class  $C^\infty$ . Pick a sequence  $(k_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$  such that for every  $i \in \mathbb{N}$ ,



$$(3.8) \quad \frac{(1+r) \operatorname{Lip}(f; B_i) C_i}{k_i} \leq \varepsilon, \\ \frac{\operatorname{Lip}(\psi_i) (1+r) \operatorname{Lip}(f; B_i) C_i}{k_i} \leq \frac{r}{m_i}.$$

Then we define  $g_i := f_i^{k_i}$  for all  $i \in \mathbb{N}$  and

$$(3.9) \quad g(x) := \sum_{i \in \mathbb{N}} \psi_i(x) (g_i \circ \varphi_i)(x) \quad \text{for every } x \in M.$$

Clearly  $g$  belongs to the space  $C^1(M)$ .

Step 3: Properties of  $g$ . Given  $i \in \mathbb{N}$  and  $v \in \mathbb{R}^n$ , it holds that

$$(3.10) \quad |g_i(v) - f_i(v)| = \left| \int_{\mathbb{R}^n} f_i(v+w) \rho_{k_i}(w) dw - \int_{\mathbb{R}^n} f_i(v) \rho_{k_i}(w) dw \right| \\ \leq \int_{\mathbb{R}^n} |f_i(v+w) - f_i(v)| \rho_{k_i}(w) dw \\ \leq \operatorname{Lip}(f_i) \int_{B_{1/k_i}^{\mathbb{R}^n}(0)} \|w\| \rho_{k_i}(w) dw \\ \stackrel{(3.6)}{\leq} \frac{(1+r) \operatorname{Lip}(f; B_i) C_i}{k_i} \int_{\mathbb{R}^n} \rho_{k_i}(w) dw \\ = \frac{(1+r) \operatorname{Lip}(f; B_i) C_i}{k_i} \stackrel{(3.8)}{\leq} \varepsilon.$$

Accordingly, one has that

$$|g(x) - f(x)| \stackrel{(3.9)}{\leq} \sum_{i \in \mathbb{N}} \psi_i(x) |g_i \circ \varphi_i - f|(x) \leq \sum_{i \in \mathbb{N}} \psi_i(x) |g_i - f \circ \varphi_i^{-1}|(\varphi_i(x)) \\ \stackrel{(3.10)}{\leq} \varepsilon \sum_{i \in \mathbb{N}} \psi_i(x) = \varepsilon,$$

which proves the first line of (3.4). Moreover, calling  $S$  the set of all  $i \in \mathbb{N}$  such that the center of the ball  $B_i$  does not lie in  $B_r^M(\operatorname{spt}(f))$ , we have for any  $i \in S$  that  $f|_{B_i} \equiv 0 \Rightarrow f_i \equiv 0 \Rightarrow g_i \equiv 0$ , whence accordingly  $g = \sum_{i \in \mathbb{N} \setminus S} \psi_i g_i \circ \varphi_i$ . This shows that

$$\operatorname{spt}(g) \subseteq \bigcup_{i \in \mathbb{N} \setminus S} B_i \subseteq B_{2r}^M(\operatorname{spt}(f)) \subseteq B_\delta^M(\operatorname{spt}(f)).$$

Step 4: Properties of  $\operatorname{lip}_a(g)$ . Given  $i \in \mathbb{N}$  and  $v, w \in \mathbb{R}^n$ , it holds that

$$(3.11) \quad |g_i(v) - g_i(w)| \leq \int_{\mathbb{R}^n} |f_i(v+u) - f_i(w+u)| \rho_{k_i}(u) du \\ \leq (1+r) \operatorname{Lip}(f; B_i) \|v - w\|_i.$$

Now fix  $x \in M$  and denote  $\mathcal{J}_x := \{i \in \mathbb{N} : x \in B_i\}$ . Pick any  $i \in \mathcal{J}_x$  and notice that  $\mathcal{J}_x \subseteq \mathcal{A}_i$ . Since the set  $\mathcal{J}_x$  is finite, we can choose a radius  $s_x > 0$  satisfying  $B_{s_x}^M(x) \subseteq B_j$

for all  $j \in \mathbb{J}_x$ . Hence for every  $y, z \in B_{s_x}^M(x)$  one has that

$$\begin{aligned}
 |g(y) - g(z)| &\leq \underbrace{\left| \sum_{j \in \mathbb{J}_x} [\psi_j(y) - \psi_j(z)] (g_j \circ \varphi_j - f)(y) \right|}_{=:(A)} \\
 &\quad + \left| \sum_{j \in \mathbb{J}_x} \psi_j(z) [(g_j \circ \varphi_j)(y) - (g_j \circ \varphi_j)(z)] \right| \\
 &\leq \underbrace{\sum_{j \in \mathbb{J}_x} |\psi_j(y) - \psi_j(z)| |(g_j \circ \varphi_j - f)(y)|}_{=:(A)} \\
 &\quad + \underbrace{\sum_{j \in \mathbb{J}_x} \psi_j(z) |(g_j \circ \varphi_j)(y) - (g_j \circ \varphi_j)(z)|}_{=:(B)}.
 \end{aligned}$$

We estimate the quantities (A) and (B) separately. First, observe that

$$\begin{aligned}
 (A) &\stackrel{(3.10)}{\leq} d_F(y, z) \sum_{j \in \mathbb{J}_x} \text{Lip}(\psi_j) \frac{(1+r) \text{Lip}(f; B_j) C_j}{k_j} \\
 &\leq d_F(y, z) \sum_{j \in \mathcal{A}_i} \frac{\text{Lip}(\psi_j) (1+r) \text{Lip}(f; B_j) C_j}{k_j} \\
 &\stackrel{(3.8)}{\leq} d_F(y, z) \sum_{j \in \mathcal{A}_i} \frac{r}{m_j} \stackrel{(3.7)}{\leq} r d_F(y, z).
 \end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
 (B) &\stackrel{(3.11)}{\leq} (1+r) \sum_{j \in \mathbb{J}_x} \psi_j(z) \text{Lip}(f; B_j) \|\varphi_j(y) - \varphi_j(z)\|_j \\
 &\leq (1+r)^2 d_F(y, z) \sum_{j \in \mathbb{J}_x} \psi_j(z) \text{Lip}(f; B_j) \\
 &\leq (1+r)^2 d_F(y, z) \text{Lip}(f; B_{2r}^M(x)) \sum_{j \in \mathbb{J}_x} \psi_j(z) \\
 &\leq [\text{Lip}(f; B_\delta^M(x)) + (2r+r^2) \text{Lip}(f)] d_F(y, z).
 \end{aligned}$$

Therefore, we finally conclude that, for any  $y, z \in B_{s_x}^M(x)$ , it holds that

$$\begin{aligned}
 |g(y) - g(z)| &\leq [\text{Lip}(f; B_\delta^M(x)) + (2r+r^2) \text{Lip}(f) + r] d_F(y, z) \\
 &\stackrel{(3.5)}{\leq} [\text{Lip}(f; B_\delta^M(x)) + \lambda] d_F(y, z).
 \end{aligned}$$

This shows that  $\text{lip}_a(g)(x) \leq \text{Lip}(g; B_{s_x}^M(x)) \leq \text{Lip}(f; B_\delta^M(x)) + \lambda$  for every  $x \in M$ , thus proving the second line in (3.4). Hence the statement is achieved. ■

**Remark 3.7** On general Finsler manifolds the exponential map is only of class  $C^1$ . Moreover, as proved by Akbar-Zadeh [1], the exponential map is of class  $C^2$  if and only if it is smooth. The family of those Finsler manifolds having this property (that are said to be of *Berwald type*) contains all Riemannian manifolds. We observe that if

$(M, F)$  is of Berwald type, then the approximating function  $g$  in Theorem 3.6 can be chosen to be smooth (by the same proof).

### 4 Main Result

Let us consider a geodesically complete, reversible Finsler manifold  $(M, F)$  and a non-negative Radon measure  $\mu$  on the metric space  $(M, d_F)$ , which will remain fixed for the whole section.

**Remark 4.1** Observe that  $(M, d_F, \mu)$  is a metric measure space, in the sense of (2.1). Indeed, the metric space  $(M, d_F)$  is complete (by Theorem 3.4) and separable (as the manifold  $M$  is second-countable by definition).

#### 4.1 Density in Energy of $C^1$ Functions

Let  $f \in C_c^1(M)$  be given. Then we denote by  $\underline{d}f$  its differential, which is a continuous section of the cotangent bundle  $T^*M$ . For brevity, let us set

$$|\underline{d}f|(x) := F^*(x, \underline{d}f(x)) \quad \text{for every } x \in M,$$

where  $F^*(x, \cdot)$  stands for the dual norm of  $F(x, \cdot)$ . Observe that the function  $f$  can be viewed as an element of the Sobolev space  $W^{1,2}(M, d_F, \mu)$  and that

$$(4.1) \quad |\underline{d}f| \leq |\underline{d}f| \quad \text{in the } \mu\text{-almost everywhere sense,}$$

as a consequence of (2.2) and the fact that  $\text{lip}(f) = |\underline{d}f|$ .

**Proposition 4.2** (Density in energy of  $C^1$  functions) *Let  $f \in W^{1,2}(M, d_F, \mu)$  be given. Then there exists a sequence  $(f_k)_{k \in \mathbb{N}} \subseteq C_c^1(M)$  such that  $f_k \rightarrow f$  and  $|\underline{d}f_k| \rightarrow |\underline{d}f|$  in  $L^2(\mu)$  as  $k \rightarrow \infty$ .*

**Proof** First of all, we know from Theorem 2.4 that there exists a sequence  $(g_k)_{k \in \mathbb{N}} \subseteq \text{LIP}_c(M)$  such that  $g_k \rightarrow f$  and  $\text{lip}_a(g_k) \rightarrow |\underline{d}f|$  in  $L^2(\mu)$ . (Recall that  $(M, d_F)$  is proper by Theorem 3.4.) Now fix  $k \in \mathbb{N}$  and observe that Theorem 3.6 provides us with a sequence  $(g_k^i)_{i \in \mathbb{N}} \subseteq C_c^1(M)$  with

$$(4.2) \quad \begin{aligned} |g_k^i(x) - g_k(x)| &\leq \frac{1}{i}, \quad \text{spt}(g_k^i) \subseteq B_{1/i}^M(\text{spt}(g_k)), \\ \text{lip}_a(g_k^i)(x) &\leq \text{Lip}(g_k; B_{1/i}^M(x)) + \frac{1}{i} \end{aligned}$$

for every  $i \in \mathbb{N}$  and  $x \in M$ . Notice that the first two lines in (4.2) yield

$$\lim_i \|g_k^i - g_k\|_{L^2(\mu)} \leq \lim_i \mu(B_{1/i}^M(\text{spt}(g_k)))^{1/2}/i = 0,$$

while the third one grants that  $\overline{\lim}_i |\underline{d}g_k^i|(x) = \overline{\lim}_i \text{lip}_a(g_k^i)(x) \leq \text{lip}_a(g_k)(x)$  for every  $x \in M$ . Since we also have that

$$|\underline{d}g_k^i| \leq \chi_{B_{1/i}^M(\text{spt}(g_k))}(\text{Lip}(g_k) + 1) \in L^2(\mu) \quad \text{for all } i \in \mathbb{N},$$

it follows from the reverse Fatou lemma that  $\overline{\lim}_i \|\underline{d}g_k^i\|_{L^2(\mu)} \leq \|\text{lip}_a(g_k)\|_{L^2(\mu)}$ . Therefore, a diagonal argument gives us a sequence  $(i_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$  such that the functions  $f_k := g_k^{i_k}$  satisfy  $f_k \rightarrow f$  in  $L^2(\mu)$  and

$$(4.3) \quad \overline{\lim}_{k \rightarrow \infty} \|\underline{d}f_k\|_{L^2(\mu)} \leq \|\underline{d}f\|_{L^2(\mu)}.$$

In particular, both the sequences  $(|\underline{d}f_k|)_k$  and  $(|df_k|)_k$  are bounded in  $L^2(\mu)$  by (4.3) and (4.1). Thus (up to subsequence) it holds that  $|df_k| \rightharpoonup h$  and  $|\underline{d}f_k| \rightharpoonup h'$  weakly for some  $h, h' \in L^2(\mu)$ . Then Proposition 2.3 grants that  $|df| \leq h \leq h'$  holds  $\mu$ -almost everywhere in  $M$ . Given that in

$$\begin{aligned} \|\underline{d}f\|_{L^2(\mu)} &\leq \|h'\|_{L^2(\mu)} \leq \overline{\lim}_{k \rightarrow \infty} \|\underline{d}f_k\|_{L^2(\mu)} \\ &\leq \overline{\lim}_{k \rightarrow \infty} \|\underline{d}f_k\|_{L^2(\mu)} \stackrel{(4.3)}{\leq} \|\underline{d}f\|_{L^2(\mu)}, \end{aligned}$$

all inequalities are actually equalities, it holds that  $\|h'\|_{L^2(\mu)} = \|\underline{d}f\|_{L^2(\mu)} = \lim_k \|\underline{d}f_k\|_{L^2(\mu)}$ . Hence, we conclude that  $h' = |df|$  in the  $\mu$ -almost everywhere sense and accordingly  $|\underline{d}f_k| \rightarrow |df|$  in  $L^2(\mu)$ . ■

### 4.2 Concrete Tangent and Cotangent Modules

We define the *concrete* tangent/cotangent modules associated with  $(M, d_F, \mu)$  as

$$\begin{aligned} \Gamma_2(TM; \mu) &:= \text{space of all } L^2(\mu)\text{-sections of } TM, \\ \Gamma_2(T^*M; \mu) &:= \text{space of all } L^2(\mu)\text{-sections of } T^*M. \end{aligned}$$

The space  $\Gamma_2(TM; \mu)$  has a natural structure of  $L^2(\mu)$ -normed  $L^\infty(\mu)$ -module if endowed with the usual vector space structure and the following pointwise operations

$$(fv)(x) := f(x)v(x) \in T_xM \quad \text{and} \quad |v|(x) := F(x, v(x))$$

for  $\mu$ -almost every  $x \in M$  and for any  $v \in \Gamma_2(TM; \mu)$  and  $f \in L^\infty(\mu)$ . Similarly,  $\Gamma_2(T^*M; \mu)$  is an  $L^2(\mu)$ -normed  $L^\infty(\mu)$ -module.

Standard verifications show that the modules  $\Gamma_2(TM; \mu)$  and  $\Gamma_2(T^*M; \mu)$  have local dimension equal to  $n := \dim(M)$ , whence they are separable [25, Remark 5]. Furthermore, it holds that

$$\begin{aligned} \Gamma_2(T^*M; \mu) &\text{ is the module dual of } \Gamma_2(TM; \mu), \\ \Gamma_2(TM; \mu) &\text{ is the module dual of } \Gamma_2(T^*M; \mu). \end{aligned}$$

In particular, they are both reflexive as Banach spaces [18, Corollary 1.2.18]. It can also be readily proved that

$$(4.4) \quad \{ \underline{d}f : f \in C_c^1(M) \} \text{ generates } \Gamma_2(T^*M; \mu) \text{ in the sense of modules,}$$

where each element  $\underline{d}f$  can be viewed as an element of  $\Gamma_2(T^*M; \mu)$  as it is a continuous section of the cotangent bundle  $T^*M$  and its associated pointwise norm  $|\underline{d}f|$  has compact support.

**Remark 4.3** We emphasise that in (4.4) it is essential to consider  $C^1$ -functions (as opposed to Lipschitz functions). The reason is that  $C^1$ -functions are everywhere differentiable, thus, in particular,  $\mu$ -almost everywhere differentiable (independently of the chosen measure  $\mu$ ), while a Lipschitz function might be not differentiable at any point of a set of positive  $\mu$ -measure.

**Lemma 4.4** *If  $(M, F)$  is a Riemannian manifold, then  $\Gamma_2(TM; \mu)$  is a Hilbert module. Conversely, if  $\Gamma_2(TM; \mu)$  is a Hilbert module and  $\text{spt}(\mu) = M$ , then  $(M, F)$  is a Riemannian manifold.*

**Proof** First suppose that  $(M, F)$  is a Riemannian manifold, i.e., that each norm  $F(x, \cdot)$  satisfies the parallelogram identity. Then for any  $v, w \in \Gamma_2(TM; \mu)$ , it holds that for  $\mu$ -almost every  $x \in M$ ,

$$\begin{aligned} |v + w|^2(x) + |v - w|^2(x) &= F(x, (v + w)(x))^2 + F(x, (v - w)(x))^2 \\ &= 2F(x, v(x))^2 + 2F(x, w(x))^2 \\ &= 2|v|^2(x) + 2|w|^2(x), \end{aligned}$$

thus showing that  $\Gamma_2(TM; \mu)$  is a Hilbert module.

Now suppose that the concrete tangent module  $\Gamma_2(TM; \mu)$  is a Hilbert module and  $\text{spt}(\mu) = M$ . Let  $U$  be the domain of some chart on  $M$ . Then one can easily build a sequence  $(v_i)_{i \in \mathbb{N}}$  of continuous vector fields on  $U$  such that

$$(4.5) \quad (v_i(x))_{i \in \mathbb{N}} \text{ is dense in } T_x M \quad \text{for every } x \in U.$$

Hence, for  $\mu$ -almost every  $x \in U$ , we have that the identity

$$\begin{aligned} F(x, (v_i + v_j)(x))^2 + F(x, (v_i - v_j)(x))^2 &= |v_i + v_j|^2(x) + |v_i - v_j|^2(x) \\ &= 2|v_i|^2(x) + 2|v_j|^2(x) \\ &= 2F(x, v_i(x))^2 + 2F(x, v_j(x))^2 \end{aligned}$$

holds for every  $i, j \in \mathbb{N}$ . Since the function  $F: TM \rightarrow [0, +\infty)$  is continuous and any set of full  $\mu$ -measure is dense in  $M$ , we deduce from property (4.5) that the norm  $F(x, \cdot)$  satisfies the parallelogram identity for every point  $x \in U$ . By arbitrariness of  $U$ , we thus conclude that  $(M, F)$  is a Riemannian manifold. ■

### 4.3 The Isometric Embedding $L^2_\mu(TM) \hookrightarrow \Gamma_2(TM; \mu)$

The aim of this concluding subsection is to investigate the relation between the abstract (co)tangent module and the concrete one. The argument goes as follows: the natural projection map  $P: \Gamma_2(T^*M; \mu) \rightarrow L^2_\mu(T^*M)$  (Lemma 4.5) is a quotient map (Proposition 4.6), whence its adjoint operator  $\iota: L^2_\mu(TM) \rightarrow \Gamma_2(TM; \mu)$  is an isometric embedding (Theorem 4.7). As a consequence, the Sobolev space  $W^{1,2}(M, d_F, \mu)$  is a Hilbert space as soon as  $(M, F)$  is a Riemannian manifold (Theorem 4.11). Such results are essentially taken from [19], where the Euclidean case has been treated; in any case, we provide their full proof here for completeness.

In light of inequality (4.1), there is a natural projection operator  $P$  from the concrete cotangent module  $\Gamma_2(T^*M; \mu)$  to the abstract cotangent module  $L^2_\mu(T^*M)$ . The characterisation of such an operator is the subject of the following result.

**Lemma 4.5** (The projection  $P$ ) *There exists a unique  $L^\infty(\mu)$ -linear and continuous operator  $P: \Gamma_2(T^*M; \mu) \rightarrow L^2_\mu(T^*M)$  such that  $P(\underline{d}f) = df$  for every  $f \in C^1_c(M)$ . Moreover, it holds that*

$$(4.6) \quad |P(\underline{\omega})| \leq |\underline{\omega}| \text{ } \mu\text{-almost everywhere, for every } \underline{\omega} \in \Gamma_2(T^*M; \mu).$$

**Proof** We denote by  $\mathcal{V}$  the vector space of all elements of  $\Gamma_2(T^*M; \mu)$  that can be written in the form  $\sum_{i=1}^k \chi_{E_i} \underline{d}f_i$ , where  $(E_i)_{i=1}^k$  is a Borel partition of  $M$  and  $(f_i)_{i=1}^k \subseteq C^1_c(M)$ . Recall that  $\mathcal{V}$  is dense in  $\Gamma_2(T^*M; \mu)$  by (4.4). Since  $P$  is required to be  $L^\infty(\mu)$ -linear and to satisfy  $P(\underline{d}f) = df$  for all  $f \in C^1_c(M)$ , we are forced to set

$$(4.7) \quad P(\underline{\omega}) := \sum_{i=1}^k \chi_{E_i} df_i, \text{ for every } \underline{\omega} = \sum_{i=1}^k \chi_{E_i} \underline{d}f_i \in \mathcal{V}.$$

The well posedness of such a definition stems from the validity of the  $\mu$ -almost everywhere inequality

$$(4.8) \quad \left| \sum_{i=1}^k \chi_{E_i} \underline{d}f_i \right| = \sum_{i=1}^k \chi_{E_i} |\underline{d}f_i| \stackrel{(4.1)}{\leq} \sum_{i=1}^k \chi_{E_i} |\underline{d}f_i| = \left| \sum_{i=1}^k \chi_{E_i} \underline{d}f_i \right|,$$

which also ensures that the map  $P: \mathcal{V} \rightarrow L^2_\mu(T^*M)$  is linear continuous and accordingly can be uniquely extended to a linear continuous operator  $P: \Gamma_2(T^*M; \mu) \rightarrow L^2_\mu(T^*M)$ . Another consequence of (4.8) is that  $P$  satisfies the inequality (4.6). Finally, by suitably approximating any element of the space  $L^\infty(\mu)$  with a sequence of simple functions, we deduce from (4.7) that the map  $P$  is  $L^\infty(\mu)$ -linear. This completes the proof of the statement. ■

Given any  $\omega \in L^2_\mu(T^*M)$ , we infer from (4.6) that  $|\omega| \leq |\underline{\omega}|$  holds  $\mu$ -almost everywhere for any  $\underline{\omega} \in \Gamma_2(T^*M; \mu)$  such that  $P(\underline{\omega}) = \omega$ , so that the estimate

$$(4.9) \quad |\omega| \leq \operatorname{ess\,inf}_{\underline{\omega} \in P^{-1}(\omega)} |\underline{\omega}| \text{ holds } \mu\text{-almost everywhere in } M.$$

The next result shows that the inequality in (4.9) is actually an equality, thus proving that the operator  $P$  is a *quotient map*. The proof relies upon Proposition 4.2.

**Proposition 4.6** ( $P$  is a quotient map) *The operator  $P$  satisfies the following property.*

$$(4.10) \quad \text{For any } \omega \in L^2_\mu(T^*M), \text{ there exists } \underline{\omega} \in P^{-1}(\omega) \text{ such that } |\omega| = |\underline{\omega}| \text{ in the } \mu\text{-almost everywhere sense.}$$

*In particular, it holds that the map  $P$  is surjective.*

**Proof** We divide the proof into three steps.

*Step 1:* (4.10) for  $\omega = df$ . Let  $f \in W^{1,2}(M, d_F, \mu)$  be fixed. By Proposition 4.2, we can pick a sequence  $(f_k)_{k \in \mathbb{N}} \subseteq C^1_c(M)$  such that  $f_k \rightarrow f$  and  $|\underline{d}f_k| \rightarrow |df|$  in  $L^2(\mu)$ . In particular, it holds that  $(\underline{d}f_k)_{k \in \mathbb{N}}$  is bounded in  $\Gamma_2(T^*M; \mu)$ . Since  $\Gamma_2(T^*M; \mu)$  is reflexive, we have (up to subsequence) that  $\underline{d}f_k \rightharpoonup \underline{\omega}$  weakly in  $\Gamma_2(T^*M; \mu)$  for some  $\underline{\omega} \in \Gamma_2(T^*M; \mu)$ . Since the map  $P$  is linear and continuous, it holds that

$df_k = P(\underline{df}_k) \rightharpoonup P(\underline{\omega})$  weakly in  $L^2_\mu(T^*M)$ . Then Proposition 2.5 grants that  $P(\underline{\omega}) = df$ . Moreover, the  $\mu$ -almost everywhere inequality  $|df| = |P(\underline{\omega})| \leq |\underline{\omega}|$  follows from (4.6). Hence, from  $\|\underline{\omega}\|_{L^2(\mu)} \leq \underline{\lim}_k \|\underline{df}_k\|_{L^2(\mu)} = \|df\|_{L^2(\mu)}$  we deduce that  $|df| = |\underline{\omega}|$  is satisfied in the  $\mu$ -almost everywhere sense. This proves the claim (4.10) for all  $\omega = df$  with  $f \in W^{1,2}(M, d_F, \mu)$ .

*Step 2:* (4.10) for  $\omega$  simple. Let  $\omega \in L^2_\mu(T^*M)$  be of the form  $\omega = \sum_{i=1}^k \chi_{E_i} df_i$ , where  $(E_i)_{i=1}^k$  is a Borel partition of  $M$  and  $(f_i)_{i=1}^k \subseteq W^{1,2}(M, d_F, \mu)$ . From Step 1 we know that there exist elements  $\underline{\omega}_1, \dots, \underline{\omega}_k \in \Gamma_2(T^*M; \mu)$  such that  $P(\underline{\omega}_i) = df_i$  and  $|df_i| = |\underline{\omega}_i|$   $\mu$ -almost everywhere for all  $i = 1, \dots, k$ . Now let  $\underline{\omega} := \sum_{i=1}^k \chi_{E_i} \underline{\omega}_i \in \Gamma_2(T^*M; \mu)$ . Then the  $L^\infty(\mu)$ -linearity of  $P$  ensures that  $P(\underline{\omega}) = \omega$ , which together with the  $\mu$ -almost everywhere identity

$$|\omega| = \left| \sum_{i=1}^k \chi_{E_i} df_i \right| = \sum_{i=1}^k \chi_{E_i} |df_i| = \sum_{i=1}^k \chi_{E_i} |\underline{\omega}_i| = \left| \sum_{i=1}^k \chi_{E_i} \underline{\omega}_i \right| = |\underline{\omega}|$$

grant that the claim (4.10) holds whenever  $\omega$  is a simple 1-form.

*Step 3:* (4.10) for general  $\omega$ . Fix  $\omega \in L^2_\mu(T^*M)$ . Since simple 1-forms are dense in  $L^2_\mu(T^*M)$ , we can choose a sequence  $(\omega_k)_{k \in \mathbb{N}} \subseteq L^2_\mu(T^*M)$  of simple 1-forms converging to  $\omega$ . Given  $k \in \mathbb{N}$ , there exists an element  $\underline{\omega}_k \in \Gamma_2(T^*M; \mu)$  such that  $P(\underline{\omega}_k) = \omega_k$  and  $|\omega_k| = |\underline{\omega}_k|$   $\mu$ -almost everywhere by Step 2. In particular, the sequence  $(\omega_k)_{k \in \mathbb{N}}$  is bounded in the reflexive space  $\Gamma_2(T^*M; \mu)$ , whence there exists  $\underline{\omega} \in \Gamma_2(T^*M; \mu)$  such that (up to subsequence) we have  $\underline{\omega}_k \rightharpoonup \underline{\omega}$ . Since  $P$  is linear and continuous, we deduce that  $\omega_k = P(\underline{\omega}_k) \rightharpoonup P(\underline{\omega})$ . On the other hand, it holds that  $\omega_k \rightarrow \omega$  by assumption, thus necessarily  $P(\underline{\omega}) = \omega$ . Finally, we have  $|\omega| = |P(\underline{\omega})| \leq |\underline{\omega}|$   $\mu$ -almost everywhere by (4.6) and

$$\|\underline{\omega}\|_{L^2(\mu)} \leq \underline{\lim}_{k \rightarrow \infty} \|\underline{\omega}_k\|_{L^2(\mu)} = \underline{\lim}_{k \rightarrow \infty} \|\omega_k\|_{L^2(\mu)} = \|\omega\|_{L^2(\mu)},$$

so that  $|\omega| = |\underline{\omega}|$  in the  $\mu$ -almost everywhere sense. This shows the claim (4.10) for any  $\omega \in L^2_\mu(T^*M)$ . ■

Our main result is the following: the adjoint operator  $\iota$  of the map  $P$  is an isometric embedding of the abstract tangent module  $L^2_\mu(TM)$  into the concrete tangent module  $\Gamma_2(TM; \mu)$ . This is achieved by duality in the ensuing theorem, as a consequence of the fact that  $P$  is a quotient map.

**Theorem 4.7** (The isometric embedding  $\iota$ ) *Let  $(M, F)$  be a geodesically complete, reversible Finsler manifold and  $\mu$  a non-negative Radon measure on  $(M, d_F)$ . Let us denote by  $\iota: L^2_\mu(TM) \rightarrow \Gamma_2(TM; \mu)$  the adjoint map of  $P: \Gamma_2(T^*M; \mu) \rightarrow L^2_\mu(T^*M)$ , i.e., the unique  $L^\infty(\mu)$ -linear and continuous operator satisfying*

$$(4.11) \quad \underline{\omega}(\iota(v)) = P(\underline{\omega})(v), \text{ } \mu\text{-almost everywhere, for every } v \in L^2_\mu(TM) \text{ and } \underline{\omega} \in \Gamma_2(T^*M; \mu).$$

*Then it holds that*

$$(4.12) \quad |\iota(v)| = |v|, \text{ } \mu\text{-almost everywhere, for every } v \in L^2_\mu(TM).$$

*In particular, the operator  $\iota$  is an isometric embedding and  $L^2_\mu(TM)$  is a finitely-generated module.*

**Proof** First the  $\mu$ -almost everywhere inequality  $|P(\underline{\omega})(v)| \leq |\underline{\omega}||v|$ , which is granted by (4.6), shows that the element  $\iota(v)$  in (4.11) is well defined and that the map  $\iota$  is  $L^\infty(\mu)$ -linear continuous. The same inequality also implies that  $|\iota(v)| \leq |v|$  holds  $\mu$ -almost everywhere for any fixed  $v \in L^2_\mu(TM)$ . On the other hand, pick any  $\omega \in L^2_\mu(T^*M)$  such that  $|\omega| \leq 1$   $\mu$ -almost everywhere in  $M$ . Proposition 4.6 provides us with some element  $\underline{\omega} \in \Gamma_2(T^*M; \mu)$  satisfying  $P(\underline{\omega}) = \omega$  and  $|\underline{\omega}| = |\omega|$  in the  $\mu$ -almost everywhere sense. Therefore,

$$\begin{aligned} \omega(v) &= P(\underline{\omega})(v) \leq \operatorname{ess\,sup}_{|\underline{\omega}'| \leq 1} P(\underline{\omega}')(v) \stackrel{(4.11)}{=} \operatorname{ess\,sup}_{|\underline{\omega}'| \leq 1} \underline{\omega}'(\iota(v)) \\ &= |\iota(v)| \mu\text{-almost everywhere in } M, \end{aligned}$$

whence accordingly we conclude that

$$|v| = \operatorname{ess\,sup}_{|\omega| \leq 1} \omega(v) \leq |\iota(v)| \mu\text{-almost everywhere in } M.$$

This proves that the identity (4.12) is satisfied. The last statement now directly follows from the fact that the module  $\Gamma_2(TM; \mu)$  has local dimension equal to  $n$ . ■

**Remark 4.8** In general, the isometric embedding  $\iota: L^2_\mu(TM) \rightarrow \Gamma_2(TM; \mu)$  provided by Theorem 4.7 might not be an isomorphism. More precisely, we have that

(4.13)  $\iota$  is an isomorphism if and only if  $|\underline{df}| = \operatorname{lip}(f)$  holds  $\mu$ -almost everywhere for every  $f \in C^1_c(M)$ .

Indeed, it can be readily checked that  $\iota$  is surjective if and only if  $P$  is injective, which is in turn equivalent to saying that  $P$  preserves the pointwise norm. Moreover, it is sufficient to check the latter condition on the elements of  $\{\underline{df} : f \in C^1_c(M)\}$ , as these are generators of  $\Gamma_2(T^*M; \mu)$ . Finally, we recall that, for any function  $f \in C^1_c(M)$ , we have that  $P(\underline{df}) = df$  and  $|\underline{df}| = \operatorname{lip}(f)$ . All in all, we can conclude that the property (4.13) is verified.

We now provide an example in which  $\iota$  fails to be an isomorphism, even if the measure  $\mu$  has full support. Choose any sequence  $(a_k)_{k \in \mathbb{N}}$  of positive real numbers such that  $\sum_{k=0}^\infty a_k < +\infty$ , enumerate the rational numbers as  $(q_k)_{k \in \mathbb{N}}$ , and define the finite Borel measure  $\mu$  on  $\mathbb{R}$  as  $\mu := \sum_{k=0}^\infty a_k \delta_{q_k}$ , where  $\delta_{q_k}$  is the Dirac delta at  $q_k$ . Therefore,  $W^{1,2}(\mathbb{R}, d_{\text{Eucl}}, \mu) = L^2(\mu)$  and all its elements have null minimal weak upper gradient [5, Remark 4.12]. Thus, in particular,  $|\underline{df}| = 0$  holds for every  $f \in C^1_c(M)$ . On the other hand, there clearly exist functions  $f \in C^1_c(M)$  such that  $\operatorname{lip}(f) > 0$  on some Borel set having positive  $\mu$ -measure. Thanks to (4.13) we conclude that the map  $\iota$  is not an isomorphism.

**Corollary 4.9** *Let  $(M, F)$  be a geodesically complete, reversible Finsler manifold. Let  $\mu$  be a non-negative Radon measure on  $(M, d_F)$ . Then the Sobolev space  $W^{1,2}(M, d_F, \mu)$  is reflexive.*



**Proof** Theorem 4.7 says that  $L^2_\mu(TM)$  is finitely generated, whence it is reflexive [18, Theorem 1.4.7]. This implies that  $W^{1,2}(M, d_F, \mu)$  is reflexive by Proposition 2.6. ■

**Remark 4.10** We point out that Corollary 4.9 can be alternatively deduced as a consequence of a result by Ambrosio, Colombo, and Di Marino [3, Corollary 7.5], as we shall sketch.

Fix  $\bar{x} \in M$ . We call  $M_k$  the closed ball of radius  $k \in \mathbb{N}$  centered at  $\bar{x}$  and we set  $\mu_k := \mu|_{M_k}$ . By properness of  $(M, d_F)$  and the Bishop–Gromov inequality we know that each metric space  $(M_k, d_F)$  is doubling; accordingly,  $W^{1,2}(M_k, d_F, \mu_k)$  is reflexive for every  $k \in \mathbb{N}$  by [3, Corollary 7.5]. Now pick a bounded sequence  $(f_i)_{i \in \mathbb{N}}$  in  $W^{1,2}(M, d_F, \mu)$ . A diagonalisation argument together with Proposition 2.6 and [16, Proposition 2.6] grant the existence of a function  $f \in W^{1,2}(M, d_F, \mu)$  and of a (not relabeled) subsequence of  $(f_i)_{i \in \mathbb{N}}$  such that  $(f_i, df_i) \rightharpoonup (f, df)$  in the weak topology of  $L^2(\mu) \times L^2_\mu(T^*M)$ . This yields the reflexivity of  $W^{1,2}(M, d_F, \mu)$  by Proposition 2.6.

We conclude by focusing on the special case of Riemannian manifolds. By combining Theorem 4.7 with Lemma 4.4 we can immediately obtain the following result. (A word on notation: given a Riemannian manifold  $(M, g)$ , we denote by  $d_g$  the distance induced by the metric  $g$ .)

**Theorem 4.11** (Weighted Riemannian manifolds are infinitesimally Hilbertian) *Let  $(M, g)$  be a geodesically complete Riemannian manifold. Fix any non-negative Radon measure  $\mu$  on  $(M, d_g)$ . Then the metric measure space  $(M, d_g, \mu)$  is infinitesimally Hilbertian.*

**Proof** Let us define  $F(x, \nu) := g_x(\nu, \nu)^{1/2}$  for every  $x \in M$  and  $\nu \in T_xM$ , so that  $(M, F)$  is a reversible Finsler manifold (and  $d_F = d_g$ ). Consider the embedding  $\iota: L^2_\mu(TM) \rightarrow \Gamma_2(TM; \mu)$  provided by Theorem 4.7. Since  $\iota$  preserves the pointwise norm and  $\Gamma_2(TM; \mu)$  is a Hilbert module (by Lemma 4.4), we deduce that  $L^2_\mu(TM)$  is a Hilbert module as well. This grants that the Sobolev space  $W^{1,2}(M, d_F, \mu)$  is a Hilbert space by (2.3), thus proving the statement. ■

## 5 Alternative Proof of Theorem 4.11

Here we provide an alternative proof of Theorem 4.11. Instead of deducing it as a corollary of Theorem 4.7, we rather make use of the following fact that has been achieved in [19].

- (5.1) Let  $(V, \langle \cdot, \cdot \rangle)$  be a finite-dimensional scalar product space. Let  $\nu \geq 0$  be any Radon measure on  $(V, \langle \cdot, \cdot \rangle)$ . Then  $(V, \langle \cdot, \cdot \rangle, \nu)$  is infinitesimally Hilbertian.

(Actually, the result is proved for  $V = \mathbb{R}^d$  equipped with the Euclidean distance, but, as observed in [19, Remark 2.11], the very same proof works for any finite-dimensional scalar product space.)

Let  $f, g \in W^{1,2}(M, d_g, \mu)$  be fixed. In order to prove the claim, it is enough to show that

$$(5.2) \quad |D(f + g)|^2 + |D(f - g)|^2 = 2|Df|^2 + 2|Dg|^2$$

holds  $\mu$ -almost everywhere on  $M$ . Fix any  $\varepsilon > 0$ . By using Theorem 3.5 and the Lindelöf property of  $(M, d_g)$ , we can find two sequences  $(x_i)_{i \in \mathbb{N}} \subseteq M$  and  $(r_i)_{i \in \mathbb{N}} \subseteq (0, +\infty)$  satisfying the following properties.

- Calling  $V_i$  the closed ball in  $(M, d_g)$  having radius  $r_i$  and center  $x_i$ , it holds that  $(V_i)_{i \in \mathbb{N}}$  is a cover of  $M$ .
- Calling  $W_i$  the closed ball in  $(T_{x_i}M, g_{x_i})$  having radius  $r_i$  and center 0, it holds that each exponential map  $\exp_{x_i}$  is  $(1 + \varepsilon)$ -biLipschitz between  $W_i$  and  $V_i$ .

For any  $i \in \mathbb{N}$ , let us denote by  $\varphi_i: V_i \rightarrow W_i$  the inverse map of  $\exp_{x_i}|_{W_i}$ . Define  $\mu_i := \mu|_{V_i}$  and  $\nu_i := (\varphi_i)_* \mu_i$ . Then  $\varphi_i$  is a map of bounded deformation from  $(V_i, d_g, \mu_i)$  to  $(W_i, g_{x_i}, \nu_i)$ , with inverse of bounded deformation (see [18, Definition 2.4.1] for the notion of a map of bounded deformation). Therefore [18, (2.4.1)] ensures that for every  $h \in W^{1,2}(V_i, d_g, \mu_i)$  one has that  $h \circ \varphi_i^{-1} \in W^{1,2}(W_i, g_{x_i}, \nu_i)$  and that

$$(5.3) \quad \frac{|Dh| \circ \varphi_i^{-1}}{1 + \varepsilon} \leq |D(h \circ \varphi_i^{-1})| \leq (1 + \varepsilon) |Dh| \circ \varphi_i^{-1}$$

holds  $\nu_i$ -almost everywhere on  $T_{x_i}M$ . Furthermore, we know from [16, Proposition 2.6] that, for any  $h \in W^{1,2}(M, d_g, \mu)$  and  $i \in \mathbb{N}$ , one has that  $\chi_{V_i}h \in W^{1,2}(V_i, d_g, \mu_i)$  and that

$$(5.4) \quad |D(\chi_{V_i}h)| = |Dh|$$

holds  $\mu_i$ -almost everywhere on  $V_i$ . Now let us set  $f_i := (\chi_{V_i}f) \circ \varphi_i^{-1}$  and  $g_i := (\chi_{V_i}g) \circ \varphi_i^{-1}$  for every  $i \in \mathbb{N}$ . We have that the Sobolev space  $W^{1,2}(T_{x_i}M, g_{x_i}, \nu_i) \simeq W^{1,2}(W_i, g_{x_i}, \nu_i)$  is a Hilbert space by (5.1), whence accordingly

$$(5.5) \quad |D(f_i + g_i)|^2 + |D(f_i - g_i)|^2 = 2|Df_i|^2 + 2|Dg_i|^2$$

holds  $\nu_i$ -almost everywhere on  $T_{x_i}M$ . By combining (5.3), (5.4), and (5.5), we conclude that

$$(5.6) \quad \frac{2|Df|^2 + 2|Dg|^2}{(1 + \varepsilon)^4} \leq |D(f + g)|^2 + |D(f - g)|^2 \leq (1 + \varepsilon)^4 (2|Df|^2 + 2|Dg|^2)$$

holds  $\mu_i$ -almost everywhere for every  $i \in \mathbb{N}$ . This implies that (5.6) is satisfied  $\mu$ -almost everywhere on  $M$ , so by letting  $\varepsilon \searrow 0$ , we finally obtain (5.2), as required.

**Remark 5.1** It seems to us that Theorem 4.7 also could be deduced from the results in [19] via a suitable localisation argument, but with a much more involved proof. For this reason, we chose the presentation seen in Section 4.

## 6 Infinitesimal Hilbertianity of Weighted Carnot Groups

In this conclusive section we prove that all (sub-Riemannian) Carnot groups are infinitesimally Hilbertian when equipped with any non-negative Radon measure.

Since the techniques we shall use closely follow along the lines of Section 4, we will omit some details. More specifically, we first replace Theorem 3.6 with an approximation result tailored for Carnot groups (see Theorem 6.2), and then, to conclude, we employ the same functional-analytic machinery as in Section 4.3.

### 6.1 Preliminaries on Carnot Groups

We recall the basics in the theory of Carnot groups.

A *sub-Finsler Carnot group* of rank  $k \geq 0$  and step  $s \geq 1$  is a connected, simply connected Lie group  $\mathbb{G}$ , whose associated Lie algebra  $\mathfrak{g}$  admits a stratification  $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s$ , such that we have the following.

- $\mathfrak{g}_1, \dots, \mathfrak{g}_s$  are linear subspaces of  $\mathfrak{g}$  satisfying  $\mathfrak{g}_s \neq \{0\}$ ,  $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$  for all  $i = 1, \dots, s - 1$ , and  $[\mathfrak{g}_1, \mathfrak{g}_s] = \{0\}$ , where by  $[\cdot, \cdot]$  we denote the Lie bracket of smooth vector fields on  $\mathbb{G}$ .
- The degree-one stratum  $\mathfrak{g}_1$  has dimension  $k$  and is equipped with a norm  $\|\cdot\|$ .

When  $\|\cdot\|$  is induced by a scalar product, we say that  $\mathbb{G}$  is a *sub-Riemannian Carnot group*.

The homogeneous dimension of  $\mathbb{G}$  is defined as  $d := \sum_{i=1}^s i \dim(\mathfrak{g}_i)$ , while  $\text{Vol}_{\mathbb{G}}$  stands for the bi-invariant Haar measure on  $\mathbb{G}$  (which is unique up to a positive multiplicative constant). We call  $\{\delta_\lambda\}_{\lambda>0}$  the one-parameter group of dilations of  $\mathbb{G}$ , namely  $\delta_\lambda$  is the unique Lie group automorphism of  $\mathbb{G}$  such that  $(\delta_\lambda)_*(v) = \lambda^i v$  for every  $i = 1, \dots, s$  and  $v \in \mathfrak{g}_i$ , where by  $(\delta_\lambda)_*$  we denote the Lie algebra automorphism of  $\mathfrak{g}$  associated with  $\delta_\lambda$ . Given any point  $x \in \mathbb{G}$ , we define the *horizontal fiber*  $H_x\mathbb{G} \leq T_x\mathbb{G}$  as  $H_x\mathbb{G} := d_e L_x(\mathfrak{g}_1)$ , where  $e$  is the identity of  $\mathbb{G}$ , the space  $T_e\mathbb{G}$  is identified with  $\mathfrak{g}$ , and  $L_x: \mathbb{G} \rightarrow \mathbb{G}$  is the left-translation map  $L_x(y) := x \cdot y$ . We endow  $H_x\mathbb{G}$  with the norm  $\|v\|_x := \|d_x L_{x^{-1}}(v)\|$ . The *horizontal bundle* of  $\mathbb{G}$  is defined as  $H\mathbb{G} := \bigsqcup_{x \in \mathbb{G}} H_x\mathbb{G}$ .

A piecewise  $C^1$ -curve  $\gamma: [0, T] \rightarrow \mathbb{G}$  is said to be *horizontal* provided it holds that  $\dot{\gamma}_t \in H_{\gamma_t}\mathbb{G}$  for almost every  $t \in [0, T]$ . Its *horizontal length* is given by  $\ell_H(\gamma) := \int_0^T \|\dot{\gamma}_t\|_{\gamma_t} dt$ . The *Carnot–Carathéodory distance* between two points  $x, y \in \mathbb{G}$  is defined as  $d_{CC}(x, y) := \inf \ell_H(\gamma)$ , where the infimum is taken among all horizontal curves  $\gamma$  joining  $x$  to  $y$ . The resulting metric space  $(\mathbb{G}, d_{CC})$  is complete and separable. Observe that  $d_{CC}$  is left-invariant, *i.e.*, it holds that  $d_{CC}(z \cdot x, z \cdot y) = d_{CC}(x, y)$  for all  $x, y, z \in \mathbb{G}$ . Moreover,  $d_{CC}$  is one-homogeneous with respect to the dilations  $\delta_\lambda$ , which means that  $d_{CC}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{CC}(x, y)$  holds for every  $\lambda > 0$  and  $x, y \in \mathbb{G}$ . (See [23] for a complete presentation of the basic theory of sub-Finsler Carnot groups.)

### 6.2 The Embedding Theorem on Weighted Carnot Groups

Let  $\mathbb{G}$  be a sub-Finsler Carnot group. Given any  $f \in C_c^1(\mathbb{G})$  and  $x \in \mathbb{G}$ , we define the element  $\underline{d}_H f(x) \in H_x^*\mathbb{G} := (H_x\mathbb{G})^*$  as the restriction of the differential of  $f$  at  $x$  to  $H_x\mathbb{G}$ . This way we obtain a section  $\underline{d}_H f$  of the Banach bundle  $H^*\mathbb{G} := \bigsqcup_{x \in \mathbb{G}} H_x^*\mathbb{G}$ , where each fiber  $H_x^*\mathbb{G}$  is equipped with the dual norm  $\|\cdot\|_x^*$ .

**Lemma 6.1** *Let  $\mathbb{G}$  be a sub-Finsler Carnot group. Fix any  $f \in C_c^1(\mathbb{G})$ . Then  $f \in \text{LIP}(\mathbb{G})$  and  $\text{lip}(f)(x) = \text{lip}_a(f)(x) = \|\underline{d}_H f(x)\|_x^*$  for every  $x \in \mathbb{G}$ .*

**Proof** Fix  $x \in \mathbb{G}$  and  $\varepsilon > 0$ . By continuity of the function  $x \mapsto \|\underline{d}_H f(x)\|_x^*$ , there exists  $r > 0$  such that  $\|\underline{d}_H f(y)\|_y^* \leq \|\underline{d}_H f(x)\|_x^* + \varepsilon$  for every  $y \in B_{2r}^{\mathbb{G}}(x)$ . Pick  $y, z \in B_r^{\mathbb{G}}(x)$ . Calling  $\mathcal{C}_{yz}$  the set of all horizontal curves  $\gamma$  joining  $y$  to  $z$  with  $\ell_H(\gamma) < 2r$ , it holds  $d_{\text{CC}}(y, z) = \inf\{\ell_H(\gamma) : \gamma \in \mathcal{C}_{yz}\}$ . Each curve  $\gamma \in \mathcal{C}_{yz}$  lies entirely in  $B_{2r}^{\mathbb{G}}(x)$  and the function  $f \circ \gamma$  is absolutely continuous, thus we have that

$$\begin{aligned} |f(y) - f(z)| &\leq \int_0^T \left| \frac{d}{dt} f(\gamma_t) \right| dt \leq \int_0^T \|\underline{d}_H f(\gamma_t)\|_{\gamma_t}^* \|\dot{\gamma}_t\|_{\gamma_t} dt \\ &\leq (\|\underline{d}_H f(x)\|_x^* + \varepsilon) \ell_H(\gamma). \end{aligned}$$

By first taking the infimum over  $\gamma \in \mathcal{C}_{yz}$  and then letting  $\varepsilon \searrow 0$ , we deduce that

$$(6.1) \quad |f(y) - f(z)| \leq \|\underline{d}_H f(x)\|_x^* d_{\text{CC}}(y, z) \quad \text{for every } y, z \in B_{2r}^{\mathbb{G}}(x).$$

Calling  $C$  the maximum of  $x \mapsto \|\underline{d}_H f(x)\|_x^*$  on  $\mathbb{G}$ , we infer from (6.1) that  $f$  is locally  $C$ -Lipschitz (thus  $\text{Lip}(f) \leq C$ , as  $(\mathbb{G}, d_{\text{CC}})$  is a length space) and that  $\text{lip}_a(f)(x) \leq \|\underline{d}_H f(x)\|_x^*$  for all  $x \in \mathbb{G}$ .

To conclude the proof, it remains to show that  $\|\underline{d}_H f(x)\|_x^* \leq \text{lip}(f)(x)$  for every  $x \in \mathbb{G}$ . To this end, fix  $x \in \mathbb{G}$  such that  $\|\underline{d}_H f(x)\|_x^* > 0$ . Choose any unit-speed horizontal curve  $\gamma: [0, 1] \rightarrow \mathbb{G}$  such that  $\gamma_0 = x$  and  $\underline{d}_H f(x)[\dot{\gamma}_0] = \|\underline{d}_H f(x)\|_x^*$ . Therefore, we finally conclude that

$$\begin{aligned} \|\underline{d}_H f(x)\|_x^* &= \underline{d}_H f(x)[\dot{\gamma}_0] = \lim_{t \searrow 0} \frac{f(\gamma_t) - f(\gamma_0)}{t} \\ &\leq \text{lip}(f)(x) \overline{\lim}_{t \searrow 0} \frac{d_{\text{CC}}(\gamma_t, \gamma_0)}{t} \leq \text{lip}(f)(x), \end{aligned}$$

where the last inequality follows from the fact that  $d_{\text{CC}}(\gamma_t, \gamma_0) \leq \ell_H(\gamma|_{[0,t]}) = t$  for all  $t > 0$ . ■

We present an approximation result on Carnot groups that is the analogue of Theorem 3.6.

**Theorem 6.2** *Let  $\mathbb{G}$  be a sub-Finsler Carnot group. Let  $f \in \text{LIP}_c(\mathbb{G})$  and  $\varepsilon > 0$  be fixed. Then there exists a function  $g \in C_c^\infty(\mathbb{G})$  such that*

$$(6.2) \quad \begin{aligned} \text{spt}(g) &\subseteq B_\varepsilon^{\mathbb{G}}(\text{spt}(f)), \\ |f(x) - g(x)| &\leq \varepsilon \quad \text{for every } x \in \mathbb{G}, \\ \text{lip}_a(g)(x) &\leq \text{Lip}(f; B_\varepsilon^{\mathbb{G}}(x)) \quad \text{for every } x \in \mathbb{G}. \end{aligned}$$

**Proof** Fix a symmetric kernel of mollification  $\rho$  on  $\mathbb{G}$ , i.e., a function  $\rho \in C_c^\infty(\mathbb{G})$  with  $0 \leq \rho \leq 1$  such that  $\int \rho \, d\text{Vol}_{\mathbb{G}} = 1$ ,  $\text{spt}(\rho) \subseteq B_1^{\mathbb{G}}(e)$ , and  $\rho(x^{-1}) = \rho(x)$  for every  $x \in \mathbb{G}$ . Given any  $\lambda > 0$ , we set  $\rho_\lambda(x) := \lambda^{-d} \rho(\delta_{1/\lambda}(x))$  for every  $x \in \mathbb{G}$ . It can be readily checked that  $\int \rho_\lambda \, d\text{Vol}_{\mathbb{G}} = 1$  and that  $\text{spt}(\rho_\lambda) \subseteq B_\lambda^{\mathbb{G}}(e)$  for every  $\lambda > 0$ . Now let us define  $(\rho_\lambda * f)(x) := \int \rho_\lambda(w) f(w^{-1} \cdot x) \, d\text{Vol}_{\mathbb{G}}(w)$  for every  $x \in \mathbb{G}$ . It is well known that  $\rho_\lambda * f \in C_c^\infty(\mathbb{G})$  and  $\text{spt}(\rho_\lambda * f) \subseteq B_\lambda^{\mathbb{G}}(\text{spt}(f))$  [14].

Given that the map  $\mathbb{G} \times \mathbb{G} \ni (w, x) \mapsto w^{-1} \cdot x \in \mathbb{G}$  is continuous and the set  $\text{spt}(f)$  is bounded, there exists  $\lambda_0 \in (0, \varepsilon/2)$  such that  $d_{CC}(w^{-1} \cdot x, x) \leq \varepsilon / \max\{\text{Lip}(f), 2\}$  holds for every  $w \in B_{\lambda_0}^{\mathbb{G}}(e)$  and  $x \in B_{3\varepsilon/2}^{\mathbb{G}}(\text{spt}(f))$ . Let  $g := \rho_{\lambda_0} * f$ . First,  $\text{spt}(g) \subseteq B_{\varepsilon/2}^{\mathbb{G}}(\text{spt}(f))$  by construction, thus proving the first line of (6.2). Moreover, for every point  $x \in B_{\varepsilon/2}^{\mathbb{G}}(\text{spt}(f))$  we have that

$$\begin{aligned} |f(x) - g(x)| &\leq \int_{B_{\lambda_0}^{\mathbb{G}}(e)} \rho_{\lambda_0}(w) |f(w^{-1} \cdot x) - f(x)| \, d\text{Vol}_{\mathbb{G}}(w) \\ &\leq \text{Lip}(f) \int_{B_{\lambda_0}^{\mathbb{G}}(e)} \rho_{\lambda_0}(w) \, d_{CC}(w^{-1} \cdot x, x) \, d\text{Vol}_{\mathbb{G}}(w) \leq \varepsilon, \end{aligned}$$

while  $|f(x) - g(x)| = 0$  for every  $x \in \mathbb{G} \setminus B_{\varepsilon/2}^{\mathbb{G}}(\text{spt}(f))$ , which gives the second line of (6.2). Finally, to get the third line, it is clearly enough to prove that  $\text{Lip}(g; B_{\lambda_0}^{\mathbb{G}}(x)) \leq \text{Lip}(f; B_{\varepsilon}^{\mathbb{G}}(x))$  for all  $x \in \mathbb{G}$ . Such a property is satisfied when  $x \notin B_{\varepsilon}^{\mathbb{G}}(\text{spt}(f))$ , since in this case  $g = 0$  on  $B_{\lambda_0}^{\mathbb{G}}(x)$ . Then let us suppose that  $x \in B_{\varepsilon}^{\mathbb{G}}(\text{spt}(f))$ . Given any  $y, z \in B_{\lambda_0}^{\mathbb{G}}(x) \subseteq B_{3\varepsilon/2}^{\mathbb{G}}(\text{spt}(f))$ , it holds that

$$\begin{aligned} |g(y) - g(z)| &\leq \int_{B_{\lambda_0}^{\mathbb{G}}(e)} \rho_{\lambda_0}(w) |f(w^{-1} \cdot y) - f(w^{-1} \cdot z)| \, d\text{Vol}_{\mathbb{G}}(w) \\ &\leq \text{Lip}(f; B_{\varepsilon}^{\mathbb{G}}(x)) \int_{B_{\lambda_0}^{\mathbb{G}}(e)} \rho_{\lambda_0}(w) \, d_{CC}(w^{-1} \cdot y, w^{-1} \cdot z) \, d\text{Vol}_{\mathbb{G}}(w) \\ &= \text{Lip}(f; B_{\varepsilon}^{\mathbb{G}}(x)) \, d_{CC}(y, z), \end{aligned}$$

whence the sought inequality  $\text{Lip}(g; B_{\lambda_0}^{\mathbb{G}}(x)) \leq \text{Lip}(f; B_{\varepsilon}^{\mathbb{G}}(x))$  follows. Hence, (6.2) is proved. ■

Fix a sub-Finsler Carnot group  $\mathbb{G}$  of rank  $k$  and a non-negative Radon measure  $\mu$  on  $(\mathbb{G}, d_{CC})$ . We call  $\Gamma_2(H\mathbb{G}; \mu)$  and  $\Gamma_2(H^*\mathbb{G}; \mu)$  the spaces of  $L^2(\mu)$ -sections of  $H\mathbb{G}$  and  $H^*\mathbb{G}$ , respectively. These spaces have a natural structure of  $L^2(\mu)$ -normed  $L^\infty(\mu)$ -module with respect to the usual pointwise operations. It can be readily checked that  $\Gamma_2(H^*\mathbb{G}; \mu)$  has local dimension equal to  $k$ , is generated by  $\{\underline{d}_H f : f \in C_c^\infty(\mathbb{G})\}$ , and its module dual is  $\Gamma_2(H\mathbb{G}; \mu)$ . In particular,  $\Gamma_2(H^*\mathbb{G}; \mu)$  is reflexive (as a Banach space) and is the module dual of  $\Gamma_2(H\mathbb{G}; \mu)$ .

By arguing as in Section 4, one can prove the following statements.

- Given any  $f \in W^{1,2}(\mathbb{G}, d_{CC}, \mu)$ , there exists a sequence  $(f_n)_n \subseteq C_c^\infty(\mathbb{G})$  such that  $f_n \rightarrow f$  and  $|\underline{d}_H f_n| \rightarrow |df|$  in  $L^2(\mu)$ , where we set  $|\underline{d}_H f_n|(x) := \|\underline{d}_H f_n(x)\|_x^*$  for  $\mu$ -almost every  $x \in \mathbb{G}$ . This can be proved as in Proposition 4.2 (but replacing Theorem 3.6 with Theorem 6.2).
- It holds that  $|df| \leq |\underline{d}_H f|$  in the  $\mu$ -almost everywhere sense for all  $f \in C_c^\infty(\mathbb{G})$ , by Lemma 6.1 and (2.2).
- It makes sense (Lemma 4.5) to define the projection map  $P: \Gamma_2(H^*\mathbb{G}; \mu) \rightarrow L_\mu^2(T^*\mathbb{G})$  as the unique module morphism satisfying  $P(\underline{d}_H f) = df$  for every  $f \in C_c^\infty(\mathbb{G})$ .
- By mimicking the proof of Proposition 4.6, it is possible to show that, for any  $\omega \in L_\mu^2(T^*\mathbb{G})$ , there exists  $\underline{\omega} \in P^{-1}(\omega)$  such that  $|\omega| = |\underline{\omega}|$  holds  $\mu$ -almost everywhere in  $\mathbb{G}$ .

Finally, arguing as in Theorem 4.7, Corollary 4.9, and Theorem 4.11, we conclude by pointing out that we can obtain the following embedding result.

**Theorem 6.3** *Let  $\mathbb{G}$  be a sub-Finsler Carnot group. Let  $\mu \geq 0$  be a Radon measure on  $(\mathbb{G}, d_{CC})$ . Consider the adjoint map  $\iota$  of  $P$ , i.e., the unique module morphism  $\iota: L^2_\mu(T\mathbb{G}) \rightarrow \Gamma_2(H\mathbb{G}; \mu)$  satisfying  $\omega(\iota(v)) = P(\omega)(v)$  in the  $\mu$ -almost everywhere sense for every  $v \in L^2_\mu(T\mathbb{G})$  and  $\omega \in \Gamma_2(H^*\mathbb{G}; \mu)$ . Then  $|\iota(v)| = |v|$ ,  $\mu$ -almost everywhere for every  $v \in L^2_\mu(T\mathbb{G})$ . In particular, the Sobolev space  $W^{1,2}(\mathbb{G}, d_{CC}, \mu)$  is reflexive. Moreover, if  $\mathbb{G}$  is a sub-Riemannian Carnot group, then the metric measure space  $(\mathbb{G}, d_{CC}, \mu)$  is infinitesimally Hilbertian.*

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