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n-T-COTORSION-FREE MODULES

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Abstract. In order to better unify the tilting theory and the Auslander–Reiten theory, Xi introduced a general transpose called the relative transpose. Originating from this, we introduce and study the cotranspose of modules with respect to a left A-module T called n-T-cotorsion-free modules. Also, we give many properties and characteristics of n-T-cotorsion-free modules under the help of semi-Wakamatsu-tilting modules ${}_AT$.

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1. Introduction and preliminaries. In the history of the representation theory of Artin algebra, the Auslander–Reiten theory plays an intensely crucial role. In particular, the transpose is a powerful tool in this theory. The generalization of the transpose has been studied by a multitude of authors. For instance, let C be a semidualizing R-bimodule; a transpose $\mathrm{Tr}_C\, M$ of an R-module M with respect to C was introduced in [6]. Later, Geng [5] used $\mathrm{Tr}_C\, M$ to develop further the generalized Gorenstein dimension with respect to C in the setting of two-sided Noetherian rings. Especially, she generalized the Auslander–Bridger formula to the generalized Gorenstein dimension case. The dual of the transpose was studied in [7] and the relative transpose of an R-module was considered in [10].

Auslander and Bridger introduced n-torsion-free modules and obtained an approximation theory for finitely generated modules when n-syzygy modules and n-torsion-free modules coincide in [2]. Tang and Huang [7] introduced and demonstrated the cotranspose of modules with respect to a semidualizing module C. Moreover, they introduced n-C-cotorsion-free modules and manifested that n-C-cotorsion-free modules have many dual properties of n-torsion-free modules.

Based on [10], we introduce the notion of n-T-cotorsion-free modules in this paper. It turns out that a host of paramount results on the n-C-cotorsion modules is still true in this paper.

First, we recall the definition of transposes in [1] and introduce the relative cotransposes that are dual to the relative transposes in [10]:

Let $P^1 \to P_0 \to M \to 0$ be a minimal projective presentation of M. Applying the functor $\mathrm{Hom}_A(-, A)$, we obtain an exact sequence of right A-modules:

$$0 \longrightarrow \mathrm{Hom}_A(M, A) \longrightarrow \mathrm{Hom}_A(P^0, A) \xrightarrow{f} \mathrm{Hom}_A(P^1, A) \longrightarrow C \longrightarrow 0.$$

We denote the cokernel of f by $\mathrm{Tr}M$ and call it the transpose of M, i.e., $C = \mathrm{Tr}M$.

Let M be a left A-module in $\mathrm{Copre}({}_AT)$, that is, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1 \qquad (*).$$

Applying $\mathrm{Hom}({}_AT, -)$ to $(*)$, we call $c\Sigma_T(M) := \mathrm{coker} f^1_*$ the cotranspose of M with respect to T, or T-cotranspose of M.

We mainly prove the following conclusions:

THEOREM 1.1. *If M lies in* $\mathrm{Copre}({}_AT)$*, then there exists an exact sequence*

$$0 \longrightarrow \mathrm{Tor}_2^B(T, c\Sigma_T(M)) \longrightarrow T \otimes_B (T, M) \xrightarrow{\theta_M} M \longrightarrow \mathrm{Tor}_1^B(T, c\Sigma_T(M)) \longrightarrow 0,$$

where θ_M is the natural homomorphism, given by $t \otimes f \mapsto f(t)$ for any $t \in T, f \in M_*$.

THEOREM 1.2. *If ${}_AT$ is semi-Wakamatsu-tilting, M has an* $\mathrm{add}T$*-coresolution* ($\sharp$) *and* $n \geq 1$. *Then, the following statements are equivalent:*

(1) $\mathrm{co}\Omega^n_T(M)$ *is n-T-cotorsion free and*
(2) there exists an exact sequence $0 \to M \to X \to Y \to 0$ such that X is right n-T-orthogonal and $\mathrm{add}T$*-id* $(Y) \leq n - 1$.

THEOREM 1.3. *Assume that ${}_AT$ is semi-Wakamatsu-tilting, M has an* $\mathrm{add}T$*-coresolution* (§) *and* $n \geq 1$. *Then* $\mathrm{co}\Omega^i_T(M)$ *is i-T-cotorsion free for all $1 \leq i \leq n$ if and only if T-cograde* $\mathrm{Ext}^i(T, M) \geq i - 1$ *for all $1 \leq i \leq n$.*

Let A be an Artin R-algebra, that is, R is a commutative Artin ring and A is an R-algebra which is finitely generated as an R-module. The category of finitely generated left A-modules will be denoted by A-mod. Throughout this paper, all modules are invariably finitely generated.

Let $\mathcal{X}$ be a subcategory of A-mod and M be a left A-module. A homomorphism f: $X \to M$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation (or $\mathcal{X}$-precover) of M if the induced morphism $\mathrm{Hom}(X', f)$ is surjective for all $X' \in \mathcal{X}$. Dually, a homomorphism f: $M \to X$ with $X \in \mathcal{X}$ is called a left $\mathcal{X}$-approximation (or $\mathcal{X}$-preenvelope) of M if the induced morphism $\mathrm{Hom}(f, X')$ is surjective for all $X' \in \mathcal{X}$. For further details, see [3, 4]. An $\mathcal{X}$-resolution of M is an exact sequence:

$$\cdots \longrightarrow X^n \longrightarrow X^{n-1} \longrightarrow \cdots \longrightarrow X^1 \longrightarrow X^0 \longrightarrow M \longrightarrow 0,$$

with $X^i \in \mathcal{X}$ for all $i \geq 0$. In addition, if the exact sequence is $\mathrm{Hom}(\mathcal{X}, -)$-exact, then the exact sequence is called a proper $\mathcal{X}$-resolution of M. Dually, we can define the notion of $\mathcal{X}$-coresolution and proper $\mathcal{X}$-coresolution. We say that M has $\mathcal{X}$-projective dimension $\leq m$, denoted by $\mathcal{X}$-pd $(M) \leq m$, if there is an $\mathcal{X}$-resolution of M of the form $0 \to X_m \to \cdots \to X^1 \to X^0 \to M \to 0$. Let T be a module in A-mod. We denote by B the endomorphism algebra of T, thus T is a A–B bimodule in the natural manner.

Throughout this paper, we shall fix such a triple (A, T, B) and $\mathrm{add}({}_AT)$ stands for the additive category generated by T. We denote the following full subcategories of A-mod:

$\mathrm{Cogen}({}_AT) = \{M \in A\text{-mod} \mid$ there is an injective morphism from M to T^n, $n \in N\}$.

$\mathrm{Copre}({}_AT) = \{M \in A\text{-mod} \mid$ there is an exact sequence $0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1$ with $T^i \in \mathrm{add}T$ for $i = 0,1\}$.

$\mathrm{Coapp}({}_AT) = \{M \in A\text{-mod} \mid$ there is an exact sequence $0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1$ such that $\mathrm{coker}(f^0) \in \mathrm{Cogen}(T)$ and f^0 is an $\mathrm{add}T$-preenvelope of $M\}$.

Dually, we can define the subcategories $\mathrm{Gen}(T)$ whose objects are the A-modules M which are generated by ${}_AT$, and the subcategories $\mathrm{Pre}(T)$ whose objects are those modules

M which posses an exact sequence of form $T^1 \xrightarrow{f^1} T^0 \xrightarrow{f^0} M \longrightarrow 0$. The notion of App($T$) can similarly to define.

For simplicity, we shall denote the functor Hom$({}_AT, -)$ by $(-)_*$. Especially, $0 \to L \to M \to N \to 0$ is called Hom$(T, -)$-exact exact sequence if $0 \to L_* \to M_* \to N_* \to 0$ is an exact sequence.

This paper is organized as follows: in Section 3, we introduce the cotranspose of modules with respect to a left A-module T called n-T-cotorsion-free modules and give a characterization of these modules (Theorem 2.9). In particular, the proof of Theorem 1.2 (i.e., Theorem 2.12 in this section) is presented. In Section 3, we give the definition of T-cograde and prove Theorem 1.3 (i.e., Theorem 3.3 in this section).

2. *n-T*-cotorsion-free modules. In this section, we introduce the definition of n-T-cotorsion-free modules and give a characterization on n-T-cotorsion-free modules (Theorem 2.9). Also, we show that n-T-torsion-free modules have a close relationship (Theorem 2.12) with right n-T-orthogonal modules.

The following lemmas are useful in the course of our discussion.

LEMMA 2.1 ([10], Lemma 2.1(3)). *If $M \in$ Gen(T), then the evaluation map θ_M : $T \otimes_B (T, M) \longrightarrow M$ is surjective. If $M \in$ App(T), then θ_M is bijective. Conversely, if θ_M is bijective, then $M \in$ App(T). In particular, if $M \in$ add$T$, then θ_M is bijective.*

LEMMA 2.2. *If $T^i \in$ add$T$, then $\mathrm{Tor}_n^B(T_B, \mathrm{Hom}(T, T^i)) = 0$, $n \geq 1$.*

Proof. It follows from [8, Chapter 3 and Theorem 4]. □

THEOREM 2.3. *If M lies in Copre$({}_AT)$, then there exists an exact sequence:*

$$0 \longrightarrow \mathrm{Tor}_2^B(T, c\Sigma_T(M)) \longrightarrow T \otimes_B (T, M) \xrightarrow{\theta_M} M \longrightarrow \mathrm{Tor}_1^B(T, c\Sigma_T(M)) \longrightarrow 0,$$

where θ_M is the natural homomorphism, given by $t \otimes f \mapsto f(t)$ for any $t \in T, f \in M_*$.

Proof. Essentially, it is key to obtain the kernel and cokernel of θ_M. By applying the functor $(-)_*$ to the sequence $(*)$, we have an exact sequence in mod-B:

$$0 \longrightarrow M_* \xrightarrow{f_*^0} T_*^0 \xrightarrow{f_*^1} T_*^1 \longrightarrow c\Sigma_T(M) \longrightarrow 0. \qquad (\natural)$$

Let $f^1 = i\pi$ (where $\pi : T^0 \to \mathrm{Im}\ f^0$ and $i : \mathrm{Im}\ f^0 \to T^1$) and $f_*^1 = i'\pi'$ (where $\pi' : T_*^0 \to \mathrm{Im}\ f_*^0$ and $i' : \mathrm{Im}\ f_*^0 \to T_*^1$) be the natural decompositions of f^0 and f_*^0, respectively. Since $\theta_{T_*^0}$ is an isomorphism and $\mathrm{Tor}_1^B(T_B, T_*^0) = 0$ by Lemmas 2.1 and 2.2, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^B(T, \mathrm{Im}\ f_*^0) & \longrightarrow & T \otimes_B M_* & \longrightarrow & T \otimes_B T_*^0 & \xrightarrow{1_T \otimes \pi'} & T \otimes_B \mathrm{Im}\ f_*^0 & \longrightarrow & 0\ , \\ & & & & \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_{T_*^0}} & & \downarrow{\scriptstyle h} & & \\ & & 0 & \longrightarrow & M & \longrightarrow & T^0 & \xrightarrow{\pi} & \mathrm{Im}\ f^0 & \longrightarrow & 0 \end{array}$$

where h is an induced homomorphism. It follows that $\pi \cdot \theta_{T_*^0} = h \cdot (1_T \otimes \pi')$. Hence, by the snake lemma, we know that $\ker\theta_M \cong \mathrm{Tor}_1^B(T, \mathrm{Im} f_*^0)$ and $\mathrm{coker}\theta_M \cong \ker h$. Moreover, applying the functor $T \otimes_B -$ to the exact sequence:

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$$0 \longrightarrow \mathrm{Im}f_*^0 \xrightarrow{i'} T_*^1 \longrightarrow c\Sigma_T(M) \longrightarrow 0,$$

and noting that $\mathrm{Tor}_1^B(T_B, T_*^1) = 0$ by Lemma 2.2, one can get the following exact sequence:

$$0 \longrightarrow \mathrm{Tor}_1^B(T, c\Sigma_T(M)) \longrightarrow T \otimes_B \mathrm{Im}f_*^0 \xrightarrow{1_T \otimes i'} T \otimes_B T_*^1 \longrightarrow T \otimes_B c\Sigma_T(M) \longrightarrow 0,$$

and the isomorphism:

$$\mathrm{Tor}_1^B(T, \mathrm{Im}\, f_*^0) \cong \mathrm{Tor}_2^B(T, c\Sigma_T(M)).$$

Notice that there are facts: $f^0 \cdot \theta_{T^0} = \theta_{T^1} \cdot (1_T \otimes f_*^0)$, $f_*^0 = i'\pi'$, and $1_T \otimes f_*^0 = 1_T \otimes i'\pi' = (1_T \otimes i')(1_T \otimes \pi')$. Then, we have $i \cdot h \cdot (1_T \otimes \pi') = i \cdot \pi \cdot \theta_{T^0} = f^0 \cdot \theta_{T^0} = \theta_{T^1} \cdot (1_T \otimes f_*^0) = \theta_{T^1} \cdot (1_T \otimes i')(1_T \otimes \pi')$. There is a commutative diagram:

$$\begin{array}{ccccc}
T \otimes_B T_*^0 & \xrightarrow{1_T \otimes \pi'} & T \otimes_B \mathrm{Im}f_*^0 & \xrightarrow{1_T \otimes i'} & T \otimes_B T_*^1 \\
\downarrow{\scriptstyle \theta_{T^0}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \theta_{T^1}} \\
T^0 & \xrightarrow{\pi} & f_*^0 & \xrightarrow{i} & T^1 \\
\| & & \downarrow{\scriptstyle i} & & \| \\
T^0 & \xrightarrow{f^0} & T^1 & = & T^1
\end{array}$$

Also, note that i is monic and θ_{T^1} is an isomorphism, so $\mathrm{coker}\theta_M \cong \ker h \cong \ker(1_T \otimes i') \cong \mathrm{Tor}_1^B(T, c\Sigma_T(M))$. Consequently, the desired exact sequence is obtained. □

We introduce the following definition of n-T-cotorsion-free modules by the above result.

DEFINITION 2.4. Let M be a finitely generated left A-module in $\mathrm{Copre}({}_AT)$. Then M is called n-T-cotorsion free if $\mathrm{Tor}_i^B(T, c\Sigma_T(M)) = 0$ for all $1 \le i \le n$. If $\mathrm{Tor}_i^B(T, c\Sigma_T(M)) = 0$ for all $i \ge 1$, then M is called ∞-T-cotorsion free.

REMARK 2.5.

(1) *If M is in $\mathrm{add}({}_AT)$, then M is ∞-T-cotorsion free. This is an exceedingly useful fact in remaining discussion.*
(2) *If M is n-T-cotorsion free, then M is m-T-cotorsion free for any $m \le n$.*

The following result will be used frequently in this paper.

COROLLARY 2.6. *Let M be a finitely generated left A-module in* $\mathrm{Copre}({}_AT)$. *Then, we have*

(1) *M is 1-T-cotorsion free if and only if θ_M is epimorphism.*
(2) *M is 2-T-cotorsion free if and only if θ_M is isomorphism if and only if $M \in \mathrm{App}(T)$.*
(3) *For all $n \ge 3$, M is n-T-cotorsion free if and only if θ_M is isomorphism and $\mathrm{Tor}_1^B(T, M_*) = 0$ for any $1 \le i \le n-2$.*

Proof. We just prove the result (3).

($\Rightarrow$) Assume that M is n-T-cotorsion free, then θ_M is an isomorphism by Theorem 2.3. Applying $T \otimes_B -$ to the exact sequence ($\natural$), we can deduce that $\mathrm{Tor}_i^B(T, c\Sigma_T(M)) \cong \mathrm{Tor}_{i-2}^B(T, M_*)$ by dimension shifting for any $i \ge 3$. Then by the definition of n-T-cotorsion

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free, we have $\mathrm{Tor}_i^B(T, c\Sigma_T(M)) = 0$, $1 \le i \le n$. Therefore, $\mathrm{Tor}_{i-2}^B(T, M_*) = 0$ for all $1 \le i \le n-2$.

($\Leftarrow$) By the assumptions and the above discussion, one can imply that $\mathrm{Tor}_i^B(T, c\Sigma_T(M)) \cong \mathrm{Tor}_{i-2}^B(T, M_*) = 0$, $3 \le i \le n$. But we have already obtained $\mathrm{Tor}_{1,2}^B(T, c\Sigma_T(M)) = 0$ by Theorem 2.3. Accordingly, M is n-T-cotorsion free, as desired. □

PROPOSITION 2.7. *Let L be n-T-cotorsion free. If there is a* $\mathrm{Hom}(T, -)$*-exact exact sequence* $0 \to L \to M \to N \to 0$, *then M is n-T-cotorsion free if and only if so is N.*

Proof. By the assumption, we can obtain a new exact sequence $0 \to L_* \to M_* \to N_* \to 0$ in mod-B. Then, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & T\otimes_B L_* & \longrightarrow & T\otimes_B M_* & \longrightarrow & T\otimes_B N_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_L} & & \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_N} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

and the following exact sequence:

$$\mathrm{Tor}_i^B(T, L_*) \to \mathrm{Tor}_i^B(T, M_*) \to \mathrm{Tor}_i^B(T, N_*) \to \mathrm{Tor}_{i-1}^B(T, L_*),\ i \ge 2.$$

Thus the assertion follows easily from the snake lemma and Corollary 2.6. □

LEMMA 2.8. *Let M be in* $\mathrm{Copre}({}_AT)$. *Then the following conclusions hold:*

(1) M is 1-T-cotorsion free if and only if M admits a surjective add$({}_AT)$-precover.
(2) M is 2-T-cotorsion free if and only if there is a $\mathrm{Hom}_A({}_AT, -)$*-exact exact sequence* $0 \to M \to T^0 \to T^1$, *where T^0 and T^1 are in* $\mathrm{add}({}_AT)$.

Proof. (1) ($\Rightarrow$) Assume that M is 1-T-cotorsion free. Hence, θ_M is a surjection by Theorem 2.3. Note that there is an exact sequence $B^{(X)} \to M_* \to 0$, where $X = \mathrm{Hom}(B, M_*)$. By applying the functor $T \otimes_B -$, we can get an epimorphism $T^{(X)} \to T \otimes_B M_* \to 0$, which induces an epimorphism $T^{(X)} \to M \to 0$ because θ_M is epic. Accordingly, we get the desired $\mathrm{add}({}_AT)$-precover.

($\Leftarrow$) Suppose that M admits a surjective $\mathrm{add}({}_AT)$-precover $K^0 \to M \to 0$. We consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc} T\otimes_B K^0_* & \longrightarrow & T\otimes_B M_* & \longrightarrow & 0 \\ \downarrow{\scriptstyle \theta_{K^0}} & & \downarrow{\scriptstyle \theta_M} & & \\ K^0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Because θ_{K^0} is an isomorphism by Lemma 2.1, one can imply that θ_M is epic. That is, M is 1-T-cotorsion free.

(2) ($\Rightarrow$) Assume that M is 2-T-cotorsion free, by the above argument, there exists an $\mathrm{Hom}_A(T, -)$-exact exact sequence $0 \to N \to K^0 \to M \to 0$. Now it is enough to prove that N is 1-T-cotorsion free. We consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & T\otimes_B N_* & \longrightarrow & T\otimes_B K^0_* & \longrightarrow & T\otimes_B M_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_N} & & \downarrow{\scriptstyle \theta_{K^0}} & & \downarrow{\scriptstyle \theta_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & K^0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Because both θ_{K^0} and θ_M are isomorphism by Lemma 2.1 and Theorem 2.3, θ_N is epimorphism by the snake lemma. Then N is 1-T-cotorsion free, i.e., there exists an exact sequence

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$K^1 \to N \to 0$, where $K^1 \in \operatorname{add}T$. Thus, we get the spliced sequence $K^1 \to K^0 \to M \to 0$, as desired.

($\Leftarrow$) Put $W = \ker(T^0 \to M)$. Then W is 1-T-cotorsion free by the proof of the result (1), i.e., θ_W is epic. Based on the above commutative diagram, it implies that θ_M is an isomorphism. Therefore, M is 2-T-cotorsion free. □

THEOREM 2.9. *Let M be in* $\operatorname{Copre}({}_AT)$ *and $n \geq 1$. Then M is n-T-cotorsion free if and only if there exists a* $\operatorname{Hom}_A({}_AT, -)$*-exact exact sequence:*

$$T^{n-1} \longrightarrow \cdots \longrightarrow T^1 \longrightarrow T^0 \longrightarrow M \longrightarrow 0,$$

where T^i is in add$({}_AT)$ for any $0 \leq i \leq n-1$.

Proof. We proceed by induction on n. By Lemma 2.8, the case $n \leq 2$ is clear. Suppose that $n \geq 3$ and M is n-T-cotorsion free. Then, θ_M is an isomorphism and $\operatorname{Tor}_i^B(T, M_*) = 0$ for any $1 \leq i \leq n-2$ by Corollary 2.6. Moreover, by induction hypothesis, there exists an exact sequence $0 \to N \to K^0 \to M \to 0$ in mod-A with $K^0 \in \operatorname{add}T$ such that $0 \to N_* \to K^0_* \to M_* \to 0$ is still exact with K^0_* projective. Hence, $\operatorname{Tor}_i^B(T, N_*) \cong \operatorname{Tor}_{i+1}^B(T, M_*) = 0$ for $1 \leq i \leq n-3$. Moreover, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T \otimes_B N_* & \longrightarrow & T \otimes_B K^0_* & \longrightarrow & T \otimes_B M_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_N} & & \downarrow{\scriptstyle \theta_{K^0}} & & \downarrow{\scriptstyle \theta_M} & & \\
0 & \longrightarrow & N & \longrightarrow & K^0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Because θ_{K^0} is an isomorphism by Lemma 2.1, θ_N is also an isomorphism. Accordingly, we get N is (n-1)-T-cotorsion free by Corollary 2.6 and the desired sequence follows from the induction hypothesis.

Conversely, suppose that there exists a $\operatorname{Hom}_A({}_AT, -)$-exact exact sequence:

$$T^{n-1} \longrightarrow \cdots \longrightarrow T^1 \longrightarrow T^0 \longrightarrow M \longrightarrow 0,$$

where T^i is in add$({}_AT)$ for $0 \leq i \leq n-1$. Set $N = \operatorname{Im}(T^1 \to T^0)$. Then $0 \to N_* \to T^0_* \to M_* \to 0$ is exact with T^0_* projective. By the induction hypothesis, N is (n-1)-T-cotorsion free, θ_N is an isomorphism and $\operatorname{Tor}_i^B(T, N_*) = 0$ for $1 \leq i \leq n-3$ by Corollary 2.6. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & T \otimes_B N_* & \longrightarrow & T \otimes_B K^0_* & \longrightarrow & T \otimes_B M_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_N} & & \downarrow{\scriptstyle \theta_{K^0}} & & \downarrow{\scriptstyle \theta_M} & & \\
0 & \longrightarrow & N & \longrightarrow & K^0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

By the same technology, we get $\operatorname{Tor}_i^B(T, M_*) = 0$ and $\operatorname{Tor}_i^B(T, N_*) \cong \operatorname{Tor}_{i+1}^B(T, M_*) = 0$ for all $1 \leq i \leq n-3$, i.e., $\operatorname{Tor}_i^B(T, M_*) = 0$ for $1 \leq i \leq n-2$. Consequently, M is n-T-cotorsion free by Corollary 2.6. □

There is an immediate consequence of Theorem 2.9:

COROLLARY 2.10. *Let M be in* $\operatorname{Copre}({}_AT)$*. The following statements are equivalent:*

(1) M is 1*-T-cotorsion free;*

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(2) there is an exact sequence $0 \to N \to K^0 \to M \to 0$ *with* $K^0 \in \mathrm{add}T$ *and* $\mathrm{Ext}^1_A(T, N) = 0$*; and*

(3) there exists an epimorphism add$T$*-precover of* M.

We say a module ${}_AT$ is self-orthogonal if $\mathrm{Ext}^i(T, T) = 0$ for any $i \geq 1$. Recall that an A-module T is Wakamatsu-tilting [9] provided that

(1) $\mathrm{End}_B T \cong A$, where $B := \mathrm{End}_A T$ and

(2) $\mathrm{Ext}^i_A(T, T) = 0 = \mathrm{Ext}^i_B(T, T) = 0$ for all $i > 0$. In order to give more characteristics on n-T-cotorsion-free modules, we give the following definition:

DEFINITION 2.11. A module ${}_AT$ is called semi-Wakamatsu-tilting if $B := \mathrm{End}_A T$ and ${}_AT$ is self-orthogonal.

If ${}_AT$ is semi-Wakamatsu-tilting, Corollary 2.10 suggests that there are some relationships between n-T-cotorsion-free modules and the functor $\mathrm{Ext}^i(T, -)$. Assume that M has an add$T$-coresolution, i.e., there is an exact sequence:

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1 \longrightarrow \cdots \xrightarrow{f^i} T^i \longrightarrow \cdots \qquad (\sharp)$$

with $T^i \in \mathrm{add}T$ for all $i \geq 0$. $\mathrm{co}\Omega^i_T(M) = \mathrm{Im} f^i$ is called an nth T-cosyzygy of M for any $i \geq 0$. In particular, put $\mathrm{co}\Omega^0_T(M) = M$. We denote that add$T$-id (M) := inf $\{n \mid$ there exists an add$T$-coresolution of $M : 0 \to M \to T^0 \to T^1 \to \cdots \to T^n \to 0$ in mod-$A\}$. In the following part of this section, we always assume that M has an add$T$-coresolution. A module M is called right n-T-orthogonal if $\mathrm{Ext}^i_A(T, M) = 0$ for all $1 \leq i \leq n$ and right T-orthogonal if $\mathrm{Ext}^i_A(T, M) = 0$ for all $i \geq 1$.

THEOREM 2.12. *If* ${}_AT$ *is semi-Wakamatsu-tilting,* M *has an* add$T$*-coresolution* ($\sharp$) *and* $n \geq 1$. *Then the following statements are equivalent:*

(1) $\mathrm{co}\Omega^n_T(M)$ *is* n*-*T*-cotorsion free and*

(2) there exists an exact sequence $0 \to M \to X \to Y \to 0$ *such that* X *is right* n*-*T*-orthogonal and* add$T$*-id* $(Y) \leq n - 1$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathrm{co}\Omega^n_T(M)$ is n-T-cotorsion free. By Theorem 2.9, there is an exact sequence $0 \to N^0 \to K^0 \to \mathrm{co}\Omega^n_T(M) \to 0$ with $K^0 \in \mathrm{add}T$, N^0 $(n-1)$-T-cotorsion free, and $\mathrm{Ext}^1(T, N^0) = 0$. Consider the pullback of $K^0 \to \mathrm{co}\Omega^n_T(M)$ and $T^n \to \mathrm{co}\Omega^n_T(M)$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{co}\Omega^{n-1}_T(M) & = & \mathrm{co}\Omega^{n-1}_T(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^0 & \longrightarrow & X^0 & \longrightarrow & T^{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^0 & \longrightarrow & K^0 & \longrightarrow & \mathrm{co}\Omega^n_T(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

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When $n = 1$, it follows from the middle row in the above diagram that $\mathrm{Ext}^1_A(T, X^0) = 0$, since $\mathrm{Ext}^1(T, N^0) = 0 = \mathrm{Ext}^1(T, T^{n-1})$. Hence, the middle column in the above diagram is the desired exact sequence.

Now, consider the case $n \geq 2$. Note that the second row in the above diagram is $\mathrm{Hom}(T, -)$-exact since $\mathrm{Ext}^1_A(T, N^0) = 0$. Combining with that T^{n-1} is $(n-1)$-T-cotorsion free by Lemma 2.1, we get X^0 is $(n-1)$-T-cotorsion free by Proposition 2.7. By Theorem 2.9, there is an exact sequence $0 \to Z^0 \to U^0 \to X^0 \to 0$, where $U^0 \in \mathrm{add}T$, Z^0 is $(n-2)$-T-cotorsion free, and $\mathrm{Ext}^1_A(T, Z^0) = 0$. One can consider the following pullback of $U^0 \to X^0$ and $\mathrm{co}\Omega_T^{n-1}(M) \to X^0$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Z^0 & = & Z^0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Y^0 & \to & U^0 & \to & K^0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathrm{co}\Omega_T^{n-1}(M) & \to & X^0 & \to & K^0 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It follows that $\mathrm{add}T$-id $(Y^0) \leq 1$ and $\mathrm{Ext}^{1,2}(T, Z^0) = 0$. Notice that we obtain an exact sequence $0 \to Z^0 \to Y^0 \to \mathrm{co}\Omega_T^{n-1}(M) \to 0$. Combining with the exact sequence $0 \to \mathrm{co}\Omega_T^{n-2}(M) \to T^{n-2} \to \mathrm{co}\Omega_T^{n-1}(M) \to 0$, we can also have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{co}\Omega_T^{n-2}(M) & = & \mathrm{co}\Omega_T^{n-2}(M) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Z^0 & \to & X^1 & \to & T^{n-2} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & Z^0 & \to & Y^0 & \to & \mathrm{co}\Omega_T^{n-1}(M) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It follows from the middle row in the above diagram that $\mathrm{Ext}^{1,2}(T, X^1) = 0$. Therefore, the middle column in the above diagram is the desired exact sequence if $n = 2$.

Now, assume $n \geq 3$. Since Z^0 is $(n-2)$-T-cotorsion free and $\mathrm{Ext}^1(T, Z^0) = 0$, X^1 is $(n-2)$-T-cotorsion free by Proposition 2.7. We have an exact sequence $0 \to Z^1 \to U^1 \to X^1 \to 0$, with $U^1 \in \mathrm{add}T$, Z^1 being $(n-3)$-T-cotorsion free, and

$\mathrm{Ext}^1(T, Z^1) = 0$ by Theorem 2.9. Repeating the above discussion, and so on, we eventually obtain the desired exact sequence.

(2) ⇒ (1) Since add$T$-id $(Y) \leq n-1$, we have the following exact sequence:

$$0 \longrightarrow Y \xrightarrow{g^0} L^0 \xrightarrow{g^1} L^1 \longrightarrow \cdots \xrightarrow{g^{n-1}} L^{n-1} \longrightarrow 0,$$

with $L^i \in \mathrm{add}T$ for all $0 \leq i \leq n-1$. Set $\mathrm{Im}g^i = Y^i$ for all $0 \leq i \leq n-1$. It is clear that $\mathrm{Ext}^i(Y^j, T) = 0$ for all $i \geq 1$ and $0 \leq j \leq n-1$ because ${}_AT$ is semi-Wakamatsu-tilting.

Denote $\mathrm{co}\Omega^0_T(M) := M$, $X^0 := X$, $Y^0 := Y$. Then, we have an exact sequence

$$0 \to \mathrm{co}\Omega^0_T(M) \to X^0 \to Y^0 \to 0 \qquad (*^0)$$

by the assumption. Since M has an add$T$-coresolution (♯), there is an exact sequence:

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1 \longrightarrow \cdots \xrightarrow{f^i} T^i \longrightarrow \cdots,$$

with $T^i \in \mathrm{add}T$ for all $i \geq 0$ and $\mathrm{co}\Omega^i_T(M) = \mathrm{Im}f^i$.

First, the exact sequence $(*^0)$ and the morphism f^0 induce a pushout:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{co}\Omega^0_T(M) & \longrightarrow & T^0 & \longrightarrow & \mathrm{co}\Omega^1_T(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X^0 & \longrightarrow & H^0 & \longrightarrow & \mathrm{co}\Omega^1_T(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y^0 & = & Y^0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathrm{Ext}^1(Y^0, T^0) = 0$, we have $H^0 \cong Y^0 \bigoplus T^0$ from the second column in the above diagram. Therefore, we can obtain a short exact sequence $0 \to H^0 \to L^0 \bigoplus T^0 \to Y^1 \to 0$ that is induced by the exact sequence $0 \to Y^0 \to L^0 \to Y^1 \to 0$. Then, we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X^0 & \longrightarrow & H^0 & \longrightarrow & \mathrm{co}\Omega^1_T(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X^0 & \longrightarrow & L^0 \bigoplus T^0 & \longrightarrow & X^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y^1 & = & Y^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

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In particular, we obtain an exact sequence:

$$0 \to X^0 \to L^0 \bigoplus T^0 \to X^1 \to 0 \qquad (\sharp^0).$$

Note that we also have an exact sequence:

$$0 \to \mathrm{co}\Omega_T^1(M) \to X^1 \to Y^1 \to 0 \qquad (*^1).$$

Now, By repeating the same process for $(*^0)$ to the exact sequence $(*^1)$, and so on, we obtain some exact sequences $0 \to X^i \to L^i \bigoplus T^i \to X^{i+1} \to 0$ with $0 \leq i \leq n-1$. By dimension shifting, we have $\mathrm{Ext}^{1\leq j\leq n-i}(T, X^i) = 0$ for all $1 \leq i \leq n-1$. Then, there is an exact sequence $0 \to X_*^i \to (L^i \oplus T^i)_* \to X_*^{i+1} \to 0$.

Next, we will prove that X^i is i-T-cotorsion free.

When $i = 1$, we consider the following natural commutative diagram:

$$\begin{array}{ccccc} T\otimes_B (L^0\oplus T^0)_* & \longrightarrow & T\otimes_B X_*^1 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \theta_{(L^0\oplus T^0)_*}} & & \downarrow{\scriptstyle \theta_{X^1}} & & \\ L^0\oplus T^0 & \longrightarrow & X^1 & \longrightarrow & 0 \end{array}$$

It follows from the snake lemma that θ_{X^1} is surjective. That is, X^1 is 1-T-cotorsion free.

When $i = 2$, we consider the following natural commutative diagram:

$$\begin{array}{ccccccccc} & & T\otimes_B X_*^1 & \longrightarrow & T\otimes_B (L^1\oplus T^1)_* & \longrightarrow & T\otimes_B X_*^2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{X^1}} & & \downarrow{\scriptstyle \theta_{L^1\oplus T^1}} & & \downarrow{\scriptstyle \theta_{X^2}} & & \\ 0 & \longrightarrow & X^1 & \longrightarrow & L^1\oplus T^1 & \longrightarrow & X^2 & \longrightarrow & 0 \end{array}$$

It follows from the snake lemma that θ_{X^2} is an isomorphism. That is, X_2 is 2-T-cotorsion free.

For the case $i = 3$, we consider the following natural commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^B(T, X_*^3) & \longrightarrow & T\otimes_B X_*^2 & \longrightarrow & T\otimes_B (L^2\oplus T^2)_* & \longrightarrow & T\otimes_B X_*^3 & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \theta_{X^2}} & & \downarrow{\scriptstyle \theta_{L^2\oplus T^2}} & & \downarrow{\scriptstyle \theta_{X^3}} & & \\ & & 0 & \longrightarrow & X^2 & \longrightarrow & L^2\oplus T^2 & \longrightarrow & X^3 & \longrightarrow & 0 \end{array}$$

It follows from the above diagram that θ_{X^3} is an isomorphism and $\mathrm{Tor}_1^B(T, X_*^3) = 0$. Thus X_3 is 3-T-cotorsion free by Corollary 2.6. Iterating the argument above, we can finally get that X^n is n-T-cotorsion free. Repeating a similar argument, it is clear to see that $\mathrm{co}\Omega_T^n(M) \cong X^n$. Thus $\mathrm{co}\Omega_T^n(M)$ is n-T-cotorsion free. □

PROPOSITION 2.13. *Suppose that* ${}_AT$ *is semi-Wakamatsu-tilting and* M *has an* add$T$*-coresolution. If* $\mathrm{co}\Omega_T^n(M)$ *is* ∞*-*T*-cotorsion free for some* $n \geq 1$*, then there exists*

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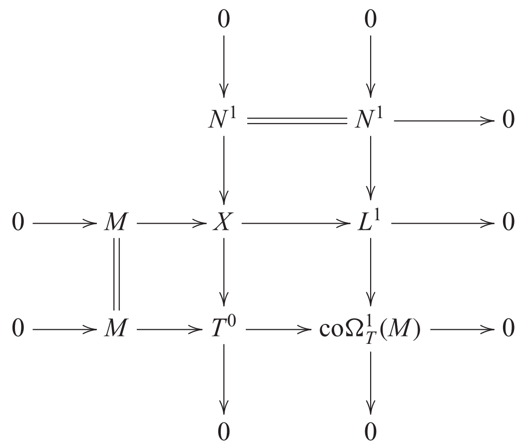
an exact sequence $0 \to M \to X \to Y \to 0$ *such that* $X \in \infty$-T-*torsion free and* add$T$-*id* $(Y) \leq n-1$.

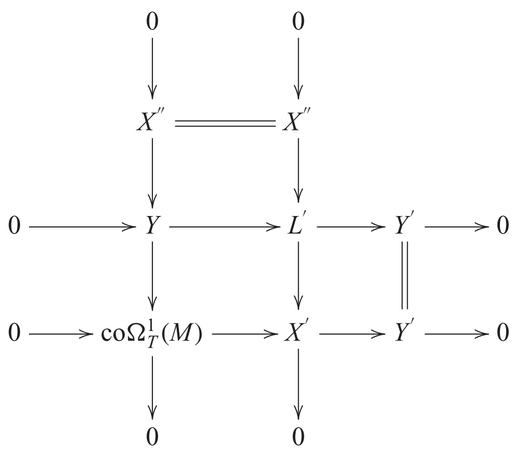
Proof. We will prove the result by induction on n.

When $n = 1$, since $\mathrm{co}\Omega^1_T(M)$ is ∞-T-cotorsion free, there exists an exact sequence $0 \to N^1 \to L^1 \to \mathrm{co}\Omega^1_T(M) \to 0$, where $L^1 \in \mathrm{add}T$, N^1 is ∞-T-cotorsion free and $\mathrm{Ext}^1(T, N^1) = 0$ by Theorem 2.9. We can consider the following pullback diagram:



It is easy to show that the middle row in the above diagram is just the desired exact sequence.

Now suppose that $n \geq 2$. By the hypothesis, we obtain an exact sequence $0 \to \mathrm{co}\Omega^1_T(M) \to X' \to Y' \to 0$ with X' ∞-T-cotorsion free and add$T$-id $(Y') \leq n-2$. Also, there is an exact sequence $0 \to X'' \to L' \to X' \to 0$ with X'' ∞-T-cotorsion free and $L' \in \mathrm{add}T$ and $\mathrm{Ext}^1(T, X'') = 0$ by Theorem 2.9. Hence, we have the following pullback diagram:



It follows from the middle row in the above diagram that add$T$-id $(Y) \leq n-1$. Moreover, we consider the following pullback diagram:

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$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X'' & = & X'' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & T^0 & \longrightarrow & \mathrm{co}\Omega^1_T(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{Ext}^1(T, N'') = 0$, we imply that the second column in this diagram is $\mathrm{Hom}(T, -)$-exact. From Proposition 2.7, we get that X is ∞-T-cotorsion free. Thus the middle row in the above diagram is just desired. □

3. T-cograde and T-cotorsion-freeness. In this section, M is in A-mod, we give the definition of T-cograde M and show that $\mathrm{co}\Omega^i_T(M)$ is i-T-cotorsion free if and only if T-cograde $\mathrm{Ext}^i(T, M) \geq i - 1$, for any $1 \leq i \leq n$.

Assume that M has an add$T$-coresolution,

$$0 \longrightarrow M \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^{n-1} \longrightarrow T^n \longrightarrow 0 \qquad (\S)$$

with $T^i \in \mathrm{add}T$ for all $i \geq 0$. Applying $\mathrm{Hom}(T, -)$ to the exact sequence:

$$0 \longrightarrow \mathrm{co}\Omega^{n-1}_T(M) \xrightarrow{\lambda^{n-1}} T^{n-1} \xrightarrow{\rho^n} \mathrm{co}\Omega^n_T(M) \longrightarrow 0,$$

we can obtain the following exact sequence:

$$0 \longrightarrow (\mathrm{co}\Omega^{n-1}_T(M))_* \xrightarrow{\lambda^{n-1}_*} T^{n-1}_* \xrightarrow{\rho^n_*} (\mathrm{co}\Omega^n_T(M))_* \longrightarrow \mathrm{Ext}^1(T, \mathrm{co}\Omega^{n-1}_T(M)) \longrightarrow 0.$$

It is easy to show that $\mathrm{Ext}^1(T, \mathrm{co}\Omega^{n-1}_T(M)) \cong \mathrm{Ext}^n(T, M)$. Set $Q = \mathrm{Im}\rho^n_*$. We get two new exact sequences:

$$0 \longrightarrow (\mathrm{co}\Omega^{n-1}_T(M))_* \xrightarrow{\lambda^{n-1}_*} T^{n-1}_* \xrightarrow{\beta} Q \longrightarrow 0 \qquad (3.1)$$

and

$$0 \longrightarrow Q \xrightarrow{\alpha} (\mathrm{co}\Omega^n_T(M))_* \longrightarrow \mathrm{Ext}^n(T, M) \longrightarrow 0. \qquad (3.2)$$

Applying the functor $\mathrm{Hom}(T, -)$ to the exact sequence (3.1), we have the following commutative diagram:

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$$\begin{array}{ccccccccc} & & T\otimes_B(\mathrm{co}\Omega_T^{n-1}(M))_* & \xrightarrow{1\otimes\lambda_*^{n-1}} & T\otimes_B T_*^{n-1} & \xrightarrow{1\otimes\beta} & T\otimes_B Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{\mathrm{co}\Omega_T^{n-1}(M)}} & & \downarrow{\scriptstyle \theta_{T^{n-1}}} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & \mathrm{co}\Omega_T^{n-1}(M) & \xrightarrow{\lambda^{n-1}} & T^{n-1} & \xrightarrow{\rho^n} & \mathrm{co}\Omega_T^{n}(M) & \longrightarrow & 0 \end{array} \tag{3.3}$$

Similarly, applying the functor $\mathrm{Hom}(T, -)$ to the exact sequence (3.2), we have the following diagram with exact row:

$$\begin{array}{ccccccc} T\otimes_B Q & \xrightarrow{1\otimes\alpha} & T\otimes_B(\mathrm{co}\Omega_T^{n}(M))_* & \longrightarrow & T\otimes_B \mathrm{Ext}^n(T,M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \theta_{\mathrm{co}\Omega_T^{n}(M)}} & & & & \\ \mathrm{co}\Omega_T^{n}(M) & = & \mathrm{co}\Omega_T^{n}(M) & & & & \end{array} \tag{3.4}$$

From the right square in diagram (3.3), it is easy to verify that the square in diagram (3.4) is commutative.

LEMMA 3.1. *Suppose that ${}_AT$ is semi-Wakamatsu-tilting and M has an* $\mathrm{add}T$-*coresolution* (§)*, then the following conclusions hold:*

(1) $\mathrm{co}\Omega_T^1(M)$ *is* 1-T-*cotorsion free.*
(2) For any $n \geq 2$, $\ker(\theta_{co\Omega_T^n(M)}) \cong T\otimes_B \mathrm{Ext}^n(T,M)$.

Proof.

(1) It is trivial.
(2) If $n \geq 2$, then the morphism $\theta_{\mathrm{co}\Omega_T^{n-1}(M)}$ is an epimorphism by (1). Hence, the morphism g in diagram (3.3) is an isomorphism since the morphism $\theta_{T^{n-1}}$ is an isomorphism. It follows from diagram (3.4) and the snake lemma that $\ker(\theta_{\mathrm{co}\Omega_T^n(M)}) \cong T\otimes_B \mathrm{Ext}^n(T,M)$. □

DEFINITION 3.2. Let N be in A-mod. The T-cograde of N with respect to T, denoted by T-cograde N, is defined to be the integer $n = \inf\{i \mid \mathrm{Tor}^i(T,N) \neq 0\}$ and ∞ if such integer does not exist.

THEOREM 3.3. *Assume that ${}_AT$ is semi-Wakamatsu-tilting, M has an* $\mathrm{add}T$-*coresolution* (§) *and* $n \geq 1$. *Then* $\mathrm{co}\Omega_T^i(M)$ *is i-T-cotorsion free for all* $1 \leq i \leq n$ *if and only if T-cograde* $\mathrm{Ext}^i(T,M) \geq i-1$ *for all* $1 \leq i \leq n$.

Proof. We will prove the result by induction on n.

For the case $n = 1$, the conclusion follows from Lemma 3.1(1).

Suppose that $n = 2$. Then, $\mathrm{co}\Omega_T^2(M)$ is 2-T-cotorsion free if and only if the morphism $\theta_{\mathrm{co}\Omega_T^2(M)}$ is an isomorphism. By Lemma 3.1(1), the morphism $\theta_{\mathrm{co}\Omega_T^2(M)}$ is surjective. So $\mathrm{co}\Omega_T^2(M)$ is 2-T-cotorsion free if and only if the morphism $\theta_{\mathrm{co}\Omega_T^2(M)}$ is monic. It follows from Lemma 3.1(2) that

$$\ker(\theta_{\mathrm{co}\Omega_T^n(M)}) \cong T\otimes_B \mathrm{Ext}^2(T,M).$$

Hence, $\mathrm{co}\Omega_T^2(M)$ is 2-T-cotorsion free if and only if $T\otimes_B \mathrm{Ext}^2(T,M) = 0$, i.e., T-cograde $\mathrm{Ext}^2(T,M) \geq 1$.

Now we assume that $n \geq 3$.

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($\Rightarrow$) Assume that $\mathrm{co}\Omega^i_T(M)$ is i-T-cotorsion free for all $1 \le i \le n$, we only need to prove that T-cograde $\mathrm{Ext}^n(T, M) \ge n-1$. By Lemma 3.1(2), we have $0 = \ker(\theta_{\mathrm{co}\Omega^n_T(M)}) \cong T \otimes_B \mathrm{Ext}^n(T, M)$. Applying the functor $\mathrm{Hom}(T, -)$ to the exact sequence (3.2), we get the following new exact sequence:

$$\mathrm{Tor}^B_1(T, \mathrm{co}\Omega^n_T(M)_*) \longrightarrow \mathrm{Tor}^B_1(T, \mathrm{Ext}^n(T, M)) \longrightarrow$$

$$\longrightarrow T \otimes_B Q \xrightarrow{1\otimes\alpha} T \otimes_B (\mathrm{co}\Omega^n_T(M))_* \longrightarrow T \otimes_B \mathrm{Ext}^n(T, M) \longrightarrow 0.$$

By induction hypothesis, we know that the morphism $\theta_{co\Omega^{n-1}_T(M)}$ is an isomorphism. Therefore, the morphism g in diagram (3.3) is also an isomorphism. Thus, the morphism $1 \otimes \alpha$ in diagram (3.4) is monic. Since $\mathrm{Tor}^B_1(T, (\mathrm{co}\Omega^n_T M)_*) = 0$ by Corollary 2.6, we have $\mathrm{Tor}^B_1(T, \mathrm{Ext}^n(T, M)) = 0$ from the exact sequence above. Hence, T-cograde $\mathrm{Ext}^n(T, M)$ ≥ 2. Combining with the exact sequences (3.1) and (3.2), we have

$$0 = \mathrm{Tor}^B_i(T, (\mathrm{co}\Omega^{n-1}_T M)_*) \cong \mathrm{Tor}^B_{i+1}(T, Q),$$

for all $1 \le i \le n-3$ by the assumption and Corollary 2.6.

By dimension shifting, we obtain that $\mathrm{Tor}^B_i(T, Q) \cong \mathrm{Tor}^B_{i+1}(T, \mathrm{Ext}^n(T, M))$ for $1 \le i \le n-3$. But $\mathrm{Tor}^B_{n-2}(T, Q) \ncong \mathrm{Tor}^B_{n-1}(T, \mathrm{Ext}^n(T, M))$. Therefore,

$$\mathrm{Tor}^B_j(T, \mathrm{Ext}^n(T, M)) = 0,$$

for any $3 \le j \le n-2$.

For the case $j = 2$, by the assumption, we have that the morphism $\theta_{\mathrm{co}\Omega^{n-1}_T(M)}$ is an isomorphism. It follows that the morphism $1 \otimes \lambda^{n-1}_*$ in diagram (3.3) is injective, and $\mathrm{Tor}^B_1(T, Q) = 0$. Therefore, $0 = \mathrm{Tor}^B_1(T, Q) \cong \mathrm{Tor}^B_2(T, \mathrm{Ext}^n(T, M))$. Consequently, $\mathrm{Tor}^B_k(T, \mathrm{Ext}^n(T, M)) = 0$ for all $0 \le k \le n-2$. That is, T-cograde $\mathrm{Ext}^n(T, M) \ge n-1$.

($\Leftarrow$) Assume that the assertion holds for the case $n-1$. That is, if T-cograde $\mathrm{Ext}^i(T, M) \ge i-1$, then $\mathrm{co}\Omega^i_T(M)$ is i-T-cotorsion free, $1 \le i \le n-1$. Suppose that T-cograde $\mathrm{Ext}^i(T, M) \ge i-1$ for all $1 \le i \le n$, it suffices to show that $\mathrm{co}\Omega^n_T(M)$ is n-T-cotorsion free by the induction hypothesis. Note that the morphism $\theta_{\mathrm{co}\Omega^{n-1}_T(M)}$ is an isomorphism by Corollary 2.6. It follows that the morphism g is an isomorphism and $\mathrm{Tor}^B_1(T, Q) = 0$ in diagram (3.3). Because T-cograde $\mathrm{Ext}^n(T, M) \ge n-1$, the morphism $1 \otimes \alpha$ is an isomorphism in diagram (3.4). Thus, the morphism $\theta_{\mathrm{co}\Omega^n_T(M)}$ is an isomorphism by the snake lemma.

Next, we only need to prove that $\mathrm{Tor}^B_i(T, (\mathrm{co}\Omega^{n-1}_T M)_*) = 0$ for all $1 \le i \le n-2$ by Corollary 2.6. From the exact sequence (3.1), we have that

$$\mathrm{Tor}^B_{i+1}(T, Q) \cong \mathrm{Tor}^B_i(T, (\mathrm{co}\Omega^{n-1}_T M)_*) = 0,$$

for all $1 \le i \le n-3$ by the assumption and Corollary 2.6. Since T-cograde $\mathrm{Ext}^i(T, M) \ge n-1$, we have that $\mathrm{Tor}^B_j(T, \mathrm{Ext}^n(T, M)) = 0$ for any $1 \le j \le n-2$, and that $\mathrm{Tor}^B_j(T, (\mathrm{co}\Omega^n_T M)_*) \cong \mathrm{Tor}^B_j(T, Q)$ for $1 \le j \le n-3$ from the exact sequence (3.2). Consequently,

$$\mathrm{Tor}^B_j(T, (\mathrm{co}\Omega^n_T M)_*) = 0$$

for $2 \le j \le n-3$. It follows from the assumption and Corollary 2.6 that $\mathrm{Tor}^B_{n-3}(T, (\mathrm{co}\Omega^n_T M)_*) = 0$. Therefore, we have that

$$\mathrm{Tor}^B_{n-3}(T, (co\Omega^n_T M)_*) \cong \mathrm{Tor}^B_{n-2}(T, Q) = 0$$

from the exact sequence (3.1). Moreover,

$$\mathrm{Tor}_{n-2}^{B}(T, (\mathrm{co}\Omega_T^n M)_*) = 0$$

from the exact sequence (3.2), since $\mathrm{Tor}_{n-2}^{B}(T, \mathrm{Ext}^n(T, M)) = 0$. Moreover, in former portion, we have proved that $\mathrm{Tor}_1^B(T, Q) = 0$, hence we also have that

$$\mathrm{Tor}_1^B(T, (\mathrm{co}\Omega_T^n M)_*) \cong \mathrm{Tor}_1^B(T, Q) = 0$$

from the exact sequence (3.2). Thus, $\mathrm{Tor}_i^B(T, (\mathrm{co}\Omega_T^n M)_*) = 0$ for all $1 \le i \le n-2$. □

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