Canad. Math. Bull. Vol. **64** (2), 2021, pp. 452–473
http://dx.doi.org/10.4153/S0008439520000569

Finite descent obstruction for Hilbert modular varieties

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Abstract. Let S be a finite set of primes. We prove that a form of finite Galois descent obstruction is the only obstruction to the existence of $\mathbb{Z}_S$-points on integral models of Hilbert modular varieties, extending a result of D. Helm and F. Voloch about modular curves. Let L be a totally real field. Under (a special case of) the absolute Hodge conjecture and a weak Serre's conjecture for mod ℓ representations of the absolute Galois group of L, we prove that the same holds also for the $\mathcal{O}_{L,S}$-points.

1 Introduction

A leading problem in arithmetic geometry is to determine whether an equation with coefficients in a number field F has any solutions. Since there can be no algorithm determining whether a given Diophantine equation is soluble in the integers $\mathbb{Z}$, one usually tries to understand the problem under strong constraints of the geometry of the variety defined by such equation or by assuming the existence of many *local* solutions. In the case of curves, for example, Skorobogatov [38] asked whether the Brauer-Manin obstruction is the only obstruction to the existence of rational points. The question, or variations thereof, attracted the attention of Bruin, Harari, Helm, Poonen, Stoll, and Voloch among others. In particular, Helm and Voloch [26] studied a form of the finite Galois descent obstruction for the integral points of modular curves. The goal of our paper is to present a class of arbitrarily large dimensional varieties that can be treated similarly to curves. More precisely, we give sufficient conditions for the existence of $\mathcal{O}_{L,S}$-integral points on (twists of) Hilbert modular varieties associated with K, where both L and K are totally real fields.

1.1 What is a Point of a Shimura Variety?

A point of a Shimura variety attached to a Shimura datum (G, X) corresponds to a Hodge structure (once a faithful linear representation of the group G is fixed). Of course, not every Hodge structure can arise in this way. Even when the Shimura variety parametrises motives, there is no description of the Hodge structures coming

Received by the editors June 3, 2020; revised July 14, 2020.
Published online on Cambridge Core July 22, 2020.
This work was supported by the Engineering and Physical Sciences Research Council [EP/ L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), and University College London.
AMS subject classification: **11F41, 11G35, 14G35, 11F80.**
Keywords: Hilbert modular varieties, descent obstruction, Galois representations, Serre's conjecture, Abelian varieties of GL_2-type, totally real fields.

from geometry, nor a conjecture predicting this. However, Shimura varieties have canonical models over number fields. Hence, we can associate with an algebraic point a Galois representation, and, conjecturally at least, we can predict which ℓ-adic Galois representations come from geometry. This is the content of the Fontaine–Mazur conjecture [19]. Our study of rational and integral points of Hilbert modular varieties begins with an attempt to understand when a suitable system of Galois representations comes from an abelian variety with $\mathcal{O}_K$-multiplication; see Section 2.1 for a more precise formulation of the question.

Our strategy arises from predictions of the Langlands' programme, which link the worlds of

$$\text{Automorphic forms} \longleftrightarrow \text{Motives} \longleftrightarrow \text{Galois representations.}$$

We refer the reader to [12] for an introduction to this circle of ideas. More precisely, from a system of Galois representations that "looks like" the one coming from an abelian variety with $\mathcal{O}_K$-multiplication, we want to produce, via Serre's modularity conjecture, a Hilbert modular form over L with Fourier coefficients in K. Eichler–Shimura theory attaches to this modular form an abelian variety over L with $\mathcal{O}_K$-multiplication, which will correspond to an L-point on the Hilbert modular variety for K. If $L = \mathbb{Q}$, Serre's conjecture is known to hold true by the work of Khare and Wintenberger [27], and the Eichler–Shimura theory has been worked out by Shimura [37]. If $L \neq \mathbb{Q}$, to make such a strategy work, we need to assume Serre's conjecture for the totally real field L and also (a special case of) the absolute Hodge conjecture, where the latter is required by Blasius in [6] in order to attach abelian varieties to Hilbert modular forms. In the next section, we present in more detail the main results of the paper.

1.2 Main Results

Let L, K be totally real extensions of $\mathbb{Q}$ and set

$$n_L := [L : \mathbb{Q}] \quad \text{and} \quad n_K := [K : \mathbb{Q}].$$

We denote by w a place of L and by v a place of K. In what follows, one should think of L as the *field of definition* and of K as the *Hecke field*. We denote by $\mathcal{O}_L$ and $\mathcal{O}_K$ the rings of integers of L and K, by L_w (resp. K_v) the completion of L at w (resp. of K at v) and by $\mathcal{O}_{L_w}$ (resp. $\mathcal{O}_{K_v}$) the ring of integers of L_w (resp. K_v). Finally, G_L denotes the absolute Galois group of L.

Let S be a finite set of places of L (including all archimedean places), and consider a system of Galois representations

$$(\mathcal{S}) \qquad \rho_v : G_L \longrightarrow \mathrm{GL}_2(K_v)$$

for every finite place v of K, such that:

- $\mathcal{S}.1$ $\{\rho_v\}_v$ is a weakly compatible system of Galois representations (see Definition 3.1);
- $\mathcal{S}.2$ $\det(\rho_v) = \chi_\ell$, where χ_ℓ is the ℓ-adic cyclotomic character and $v \mid \ell$;
- $\mathcal{S}.3$ the residual representation $\bar{\rho}_v$ is finite flat at $w \mid \ell$, for all $v \mid \ell$ such that $\bar{\rho}_v$ is irreducible and ℓ is not divisible by any prime in S;

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$\mathcal{S}.4$ $\bar{\rho}_v$ is absolutely irreducible for all but finitely many v;
$\mathcal{S}.5$ the field generated by the trace of $\rho_v(\mathrm{Frob}_w)$ for every w is K.

To make it clear which case is conjectural and which is not, we separate the statement of our first theorem into two cases depending on whether $n_L = 1$ or $n_L > 1$.

Theorem 1.A *If $L = \mathbb{Q}$, there exists an n_K-dimensional abelian variety $A/\mathbb{Q}$ with $\mathcal{O}_K$-multiplication, such that, for every v, the v-adic Tate module of A, denoted by $T_v A$, is isomorphic to ρ_v as representation of $G_{\mathbb{Q}}$.*

Theorem 1.B *Assume $n_L > 1$. Under the validity of the absolute Hodge–conjecture (more precisely Conjecture 1.1) and a suitable generalisation of Serre's conjecture (Conjecture 1.2), there exists an n_K-dimensional abelian variety A/L with $\mathcal{O}_K$-multiplication, such that, for every v, $T_v A$ is isomorphic to ρ_v as representation of G_L.*

We apply the above to study the finite descent obstruction, as explored in [38, 39, 25], of Hilbert modular varieties. A recap is given in Section 4.1. More precisely, denote by Y_K the Hilbert modular variety associated with K. Let $\mathfrak{N}$ be an ideal in $\mathcal{O}_K$ and denote by $Y_K(\mathfrak{N})$ the moduli space of n_K-dimensional abelian varieties, principally $\mathcal{O}_K$-polarized and with $\mathfrak{N}$-level structure (see Section 2.2.1 for a precise definition). As a corollary of the above theorems, we prove that the finite Galois descent obstruction (as defined in Section 4) is the only obstruction to the existence of S-integral points on integral models of twists of Hilbert modular varieties, denoted by $\mathcal{Y}_K(\mathfrak{N})$, over the ring of S-integers of a totally real field L, generalising [26, Theorem 3]. Assume that S contains the places of bad reduction of $Y_K(\mathfrak{N})$. The set $\mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$ is defined in Section 4.1. We prove the following theorem.

Theorem 2 *If $n_L > 1$, assume that the conjectures of Theorem 1.B hold. Let $\mathcal{Y}_\rho$ be the S-integral model of a twist of $\mathcal{Y}_K(\mathfrak{N})$, corresponding to a representation $\rho : G_L \to GL_2(\mathcal{O}_K/\mathfrak{N})$. If $\mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$ is non-empty, then $\mathcal{Y}_\rho(\mathcal{O}_{L,S})$ is non-empty.*

In the work of Helm–Voloch, $\mathcal{Y}$ is the integral model of an affine curve. In the case of curves, there are also other tools to establish (variants of) such results, without invoking Serre's conjecture. Indeed, as noticed after the proof of [26, Theorem 3], Stoll [39, Corollary 8.8] proved a similar result, under some extra assumptions, knowing that a factor of the Jacobian of such modular curves has finite Mordell–Weil and Tate–Shafarevich groups. The goal of this paper is to push Helm–Voloch's strategy to a particular class of varieties of arbitrarily large dimension and whose associated Albanese variety is trivial (see Theorem 2.2), thereby showing that the method could also be applied to study L-points.

Another reason for studying rational points of Hilbert modular varieties is the following. By [13, Theorem 1], every smooth projective geometrically connected curve $C/\overline{\mathbb{Q}}$ of genus at least two admits a non-constant $\overline{\mathbb{Q}}$-morphism to either a Hilbert or a Quaternionic modular variety for K, where K is a totally real number field depending on C and the choice of a Belyi function $\beta : C \to \mathbb{P}^1$. See [13, Remark 2] for a detailed description of the ambiguities of such a construction. Inspired by [32, Theorem 5.2],

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where the role of the *Belyi embedding* is played by the Kodaira–Parshin construction [30], we have the following corollary of Theorem 2. For simplicity, we consider rational points of projective curves, even if our main theorems are about integral points.

Corollary *Let $C/\mathbb{Q}$ be a smooth projective curve of genus $g \geq 2$. Assume the following:*

(i) $C(\mathbb{A}_{\mathbb{Q}})^{f\text{-}cov} \neq \emptyset$;

(ii) *there exist two totally real number fields L, K and a non-constant L-morphism*

$$f : C_L := C \times_{\mathbb{Q}} L \longrightarrow Y_K(\mathfrak{N}),$$

where $Y_K(\mathfrak{N})$ denotes, as above, the Hilbert modular variety for K of some level $\mathfrak{N}$;

(iii) *if $L \neq \mathbb{Q}$, the conjectures of Theorem 1.B hold true.*

Then $C(\mathbb{Q}) \neq \emptyset$.

Stoll [39, Conjecture 9.1] conjectured that every smooth projective curve is *very good*. That is, the closure of $C(\mathbb{Q})$ in the adelic points of C is equal to $C(\mathbb{A}_{\mathbb{Q}})^{\text{f-ab}}$. For the definition of $C(\mathbb{A}_{\mathbb{Q}})^{\text{f-ab}}$, we refer the reader to [39, Definition 6.1]. For the moment, we just need to know that

$$C(\mathbb{Q}) \subset C(\mathbb{A}_{\mathbb{Q}})^{\text{f-cov}} \subset C(\mathbb{A}_{\mathbb{Q}})^{\text{f-ab}} \subset C(\mathbb{A}_{\mathbb{Q}}).$$

Hence, Stoll's conjecture predicts the following implication:

$$C(\mathbb{A}_{\mathbb{Q}})^{\text{f-cov}} \neq \emptyset \Longrightarrow C(\mathbb{Q}) \neq \emptyset,$$

which we are able to prove, as a consequence of Theorem 2, for curves satisfying (i), (ii), and (iii) from the above corollary.

Proof Fix a point $(P_w) \in C(\mathbb{A}_{\mathbb{Q}})^{\text{f-cov}}$, which is not empty by (i). Let X/L be the image of C in $Y := Y_K(\mathfrak{N})$ under the map f of assumption (ii). Notice that, since $C(\mathbb{A}_{\mathbb{Q}})^{\text{f-cov}} \neq \emptyset$, $C(\mathbb{A}_L)^{\text{f-cov}} \neq \emptyset$, and therefore, by [39, Proposition 5.9.], $X(\mathbb{A}_L)^{\text{f-cov}}$ and $Y(\mathbb{A}_L)^{\text{f-cov}}$ are not empty. The untwisted version of Theorem 2, for a suitable choice of a finite set S of places of L, implies that $Y(L) \neq \emptyset$ (this is the only step where (iii) is needed). The proof of Theorem 2 (*cf.* Section 4.3) actually guarantees the existence of a point in $Y(L)$ inducing the fixed $f(P_w) \in Y(\mathbb{A}_{L,S})^{\text{f-cov}}$. Since $Y(L)$ injects into $Y(\mathbb{A}_{L,S})^{\text{f-cov}}$, there exists a unique $Q \in Y(L)$ inducing $f(P_w) \in Y(\mathbb{A}_L)^{\text{f-cov}}$. Moreover, by construction, $f(P_w)$ lies in $X(\mathbb{A}_L)^{\text{f-cov}}$, and $X(L) = Y(L) \cap X(\mathbb{A}_L)^{\text{f-cov}}$. Eventually, we conclude that Q lies in $X(L)$.

Let Z be the finite $\mathbb{Q}$-subscheme of C given by the $G_{\mathbb{Q}}$-orbit of the pull-back of Q along the surjective map

$$C_L \longrightarrow X.$$

By construction, $(P_w) \in C(\mathbb{A}_{\mathbb{Q}})^{\text{f-cov}} \subset C(\mathbb{A}_{\mathbb{Q}})^{\text{f-ab}}$ lies in $Z(\mathbb{A}_{\mathbb{Q}})$. In particular, [39, Theorem 8.2] implies that $Z(\mathbb{Q}) \neq \emptyset$. That is, C has at least one $\mathbb{Q}$-rational point, concluding the proof of the corollary. ∎

1.3 Conjectures

We briefly state the conjectures appearing in Theorem 1.B.

1.3.1 Absolute Hodge Conjecture

We only give a brief overview for the purpose of understanding the conjecture. For more details, we refer the reader to Deligne–Milne's paper [16, Section 6], where Deligne's category of *absolute motives* is described. Let $X, Y/\mathbb{C}$ be smooth projective varieties. A morphism of Hodge structures between their Betti cohomology groups corresponds to a Hodge class in the cohomology of $X \times Y$:

$$\operatorname{Hom}\big(H^*_{\mathrm{Betti}}(X,\mathbb{Q}), H^*_{\mathrm{Betti}}(Y,\mathbb{Q})\big) \cong H^{2*}_{\mathrm{Betti}}(X \times Y, \mathbb{Q}).$$

We say that a Hodge class $\alpha \in H^{2i}_{\mathrm{Betti}}(X \times Y, \mathbb{Q})$, or a morphism of Hodge structures between their H^i's, is absolute Hodge if, for every automorphism σ of $\mathbb{C}$, the class $\alpha^\sigma \in H^{2i}(X^\sigma \times Y^\sigma, \mathbb{C})$ is again a Hodge class. With this definition, we can split the classical Hodge conjecture into two parts:

$$\text{Hodge classes} = \text{Absolute Hodge classes} = \text{Algebraic cycles}.$$

For the purpose of this paper, the following conjecture is enough.

Conjecture 1.1 *If $X, Y/\mathbb{C}$ are smooth projective complex varieties such that, for some i, we have an isomorphism of Hodge structures*

$$H^i_{\mathrm{Betti}}(X,\mathbb{Q}) \cong H^i_{\mathrm{Betti}}(Y,\mathbb{Q}),$$

then there exists an absolute Hodge class inducing this isomorphism.

More precisely, Conjecture 1.1 will be applied when X is a Picard modular variety and Y is an abelian variety.

1.3.2 Serre's Weak Conjecture Over Totally Real Fields

We now explain the version of Serre's conjecture we need to assume to obtain the main theorems when $L \neq \mathbb{Q}$ (see, for example, [10, Conjecture 1.1], where it is referred to as a folklore generalisation of Serre's conjecture). For more details, we refer the reader to the introduction of [10] and references therein. Given a prime ℓ, we denote by $\mathbb{F}_\ell$ a finite field with ℓ elements and by $\overline{\mathbb{F}}_\ell$ a fixed algebraic closure of $\mathbb{F}_\ell$.

Conjecture 1.2 *Let ℓ be any odd prime and $\overline{\rho} : G_L \to GL_2(\overline{\mathbb{F}}_\ell)$ be an irreducible and totally odd[1] Galois representation. Then there exists some Hilbert modular eigenform f for L such that $\overline{\rho}$ is isomorphic to the reduction mod λ of $\rho_{f,\lambda}$, where $\rho_{f,\lambda}$ is the λ-adic Galois representation attached to f and λ is a prime of the Hecke field of f dividing ℓ.*

Remark 1.1 This is usually referred to as *weak* Serre's conjecture, because there is no explicit recipe to compute the weight $k(\overline{\rho})$ and the level $N(\overline{\rho})$. It has been proven [23,

[1] Here, totally odd means that $\det(\overline{\rho}(c)) = -1$ for all n_L complex conjugations c.

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22] that the *refined* version follows from the weak version under some assumptions. We state the results we need in Theorem 3.4.

When $L = \mathbb{Q}$, this was proved by Khare and Wintenberger [27]. When $L \neq \mathbb{Q}$, Conjecture 1.2 is known when the coefficient field is $\mathbb{F}_3$ (Langlands–Tunnell [29, 42]), but we really need to assume that the conjecture holds for all (but finitely many) ℓ's. Indeed, our strategy follows the lines of the proof of modularity theorems assuming Serre's conjecture: starting from a system of Galois representations, we produce a Hilbert modular form whose Fourier coefficients are equal to the traces of Frobenii modulo infinitely many primes, and hence are equal as elements of $\mathcal{O}_K$.

Finally, a potential version of the above conjecture was proved in [41, Theorem 1.6]. There, Taylor proves a potential modularity result: if $\bar{\rho} : G_L \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ is a totally odd irreducible representation with determinant equal to the cyclotomic character, then there exists L'/L a finite totally real Galois extension such that all the primes of L above ℓ split in L', and there exists f a Hilbert modular form for L' such that $\bar{\rho}_{f,\lambda'}$ is isomorphic to $\bar{\rho}$ restricted to $G_{L'}$.

1.4 Related Work

We compare our results with [32, Theorems 3.1 and 3.7] (later also extended to the moduli space of K3 surfaces by the first author [3, Theorem 1.3] and Klevdal [28, Theorem 1.1], where a finite extension of the base field is, however, required). In the approach of Patrikis, Voloch, and Zarhin, there are no restrictions on the base field, whereas here it is crucial for L to be a totally real field. We believe that it is easier to make the results of this paper unconditional. We notice here that the absolute Hodge conjecture is not enough for such papers. In [32, 3, 28], the Hodge conjecture is not only needed to descend complex abelian varieties over number fields. Finally, the version of Serre's conjecture we are assuming here is always about GL_2-coefficients, and so is certainly easier than the full Fontaine–Mazur conjecture [19].

Thanks to the recent breakthroughs on potential modularity over CM fields [1], it should be possible to extract from the main result of [1, Section 7.1] the following. Let $\{\rho_v\}_v$ be a compatible system as in Section 1.2, and let $n_L > 1$. Under the validity of Conjecture 1.1, there exist a totally real extension L'/L and n_K-dimensional abelian variety A/L' with $\mathcal{O}_K$-multiplication, such that, for every v, $T_v A$ is isomorphic to ρ_v as representations of $G_{L'}$.

Since Conjecture 1.1 can be avoided in many interesting cases, as recalled in Remark 2.7, a potential but unconditional version of Theorem 2 can therefore be obtained. Unfortunately, the extension L'/L depends on the system of Galois representations $\{\rho_v\}_v$, and it is not easy to control a priori its degree over L.

1.5 Outline of the Paper

In Section 2, we collect all the results we need about Hilbert modular forms (especially how Eichler–Shimura works in this setting). In Section 3, which is the heart of the paper, we prove Theorems 1.A and 1.B. We then explain how these results are related to the finite descent obstruction for Hilbert modular varieties in Section 4, eventually proving Theorem 2.

2 Recap on Hilbert Modular Varieties and Modular Forms

We recall some general facts about Shimura varieties. The reader interested only in Hilbert modular varieties can skip Section 2.1, which is not fundamental for the main results. We then focus on Hilbert modular varieties and Hilbert modular forms, explaining how they give rise to certain principally polarised abelian varieties.

2.1 A Question on Rational Points on Shimura Varieties

Let $\mathbb{S}$ denote the real torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$, and let $\mathbb{A}_{\mathbb{Q}}^f$ be the finite adeles of $\mathbb{Q}$. A Shimura datum is a pair (G, X) where G is a reductive $\mathbb{Q}$-algebraic group and X a $G(\mathbb{R})$-orbit in the set of morphisms of $\mathbb{R}$-algebraic groups $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, satisfying the Shimura–Deligne axioms ([15, Conditions 2.1.1(1-3)]); furthermore, in what follows, we also assume that G is the generic Mumford–Tate group on X. The Shimura–Deligne axioms imply that the connected components of X are hermitian symmetric domains and that faithful representations of G induce variations of polarisable $\mathbb{Q}$-Hodge structures on X. Let $\widetilde{K}$ be a compact open subgroup of $G(\mathbb{A}_{\mathbb{Q}}^f)$ and set

$$\mathrm{Sh}_{\widetilde{K}}(G, X) := G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_{\mathbb{Q}}^f)/\widetilde{K}\big).$$

Let X^+ be a connected component of X and let $G(\mathbb{Q})^+$ be the stabiliser of X^+ in $G(\mathbb{Q})$. The above double coset set is a disjoint union of quotients of X^+ by the arithmetic groups $\Gamma_g := G(\mathbb{Q})^+ \cap gKg^{-1}$, where g runs through a set of representatives for the finite double coset set $G(\mathbb{Q})^+\backslash G(\mathbb{A}_{\mathbb{Q}}^f)/K$. Baily and Borel [2] proved that $\mathrm{Sh}_{\widetilde{K}}(G, X)$ has a unique structure of a quasi-projective complex algebraic variety. Thanks to the work of Borovoi, Deligne, Milne, and Milne–Shih, among others, the $\mathbb{C}$-scheme

$$\mathrm{Sh}(G, X) = G(\mathbb{Q})\backslash\Big(X \times G\big(\mathbb{A}_{\mathbb{Q}}^f\big)\Big),$$

together with its $G(\mathbb{A}_{\mathbb{Q}}^f)$-action, can be naturally defined over a number field $E := E(G, X) \subset \mathbb{C}$ called the *reflex field* of (G, X). That is, there exists an E-scheme $\mathrm{Sh}(G, X)_E$ with an action of $G(\mathbb{A}_{\mathbb{Q}}^f)$ whose base change to $\mathbb{C}$ gives $\mathrm{Sh}(G, X)$ with its $G(\mathbb{A}_{\mathbb{Q}}^f)$-action.

Let F be a finite extension of E such that there exists a point $x \in \mathrm{Sh}_{\widetilde{K}}(G, X)_E(F)$. With such a point, we can naturally associate a continuous group homomorphism

$$\rho_x : G_F := \mathrm{Gal}(\overline{F}/F) \longrightarrow \widetilde{K} \subset G(\mathbb{A}_{\mathbb{Q}}^f).$$

This paper is motivated by the following question.

Question 2.1 *Let F be a field as above and let $\rho : G_F \to \widetilde{K} \subset G(\mathbb{A}_{\mathbb{Q}}^f)$ be a Galois representation. What are necessary and sufficient conditions such that there exists $x \in Sh_{\widetilde{K}}(G, X)_E(F)$ and $\rho = \rho_x$?*

When the Shimura variety has a natural interpretation as a moduli space of motives, the above question is naturally related to the Fontaine–Mazur conjecture [19]. Indeed, both aim to predict when a Galois representation comes from the

ℓ-adic (or adelic in our case) realisation of a motive. Even when they are not motivical (see [4, Remark 1.6] for a discussion about this), representations arising in this way enjoy nice properties. For an example of geometric flavour, we refer the reader to [4, Theorem 1.3].

2.2 Hilbert Modular Varieties

Let $F/\mathbb{Q}$ be a totally real extension of degree n_F and fix $\{\sigma_i\}_{i=1}^{n_F}$ the set of real embeddings of F into $\mathbb{C}$. We let G be the $\mathbb{Q}$-algebraic group obtained as the Weil restriction of GL_2 from F to $\mathbb{Q}$ and let X be n_F copies of $\mathcal{H}^{\pm} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) \neq 0\}$, on which $G(\mathbb{Q}) = \mathrm{GL}_2(F)$ acts on the i-th component via σ_i. That is,

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau_1, \ldots, \tau_{n_F})\right)_i = \frac{\sigma_i(a)\tau_i + \sigma_i(b)}{\sigma_i(c)\tau_i + \sigma_i(d)}.$$

In this case, the reflex field of (G, X) is $\mathbb{Q}$, and the set of geometrically connected components of the Shimura variety $S := \mathrm{Sh}_{G(\widehat{\mathbb{Z}})}(G, X)$ is $\mathrm{Pic}(\mathcal{O}_F)^+$, where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$ (different choices of level structure will appear later).

Remark 2.1 To obtain a Shimura variety from the above construction, it is fundamental that F is totally real. Indeed, if F is a number field, and G is an algebraic F-group, then the real points of $\mathrm{Res}_{F/\mathbb{Q}}G$ have a structure of Hermitian symmetric space if and only if F is a totally real field and the symmetric space associated with each real embedding of F is Hermitian.

It is interesting to notice here a first difference between modular curves (*i.e.*, when $F = \mathbb{Q}$) and higher dimensional Hilbert modular varieties. We recall the following folklore result (see [6, Section 2.3.2.]) to see how it follows from Matsushima's formula [8, Theorem VII.5.2].

Theorem 2.2 *Let (G, X) be a Shimura datum as above and let $\widetilde{K}$ be a neat[2] subgroup of $G(\mathbb{A}_{\mathbb{Q}}^f)$. Consider $S_{\widetilde{K}}$, the Shimura variety associated with (G, X) and $\widetilde{K}$. Unless $n_F = 1$, the first group of Betti cohomology of $S_{\widetilde{K}}$ with rational coefficients is trivial. In particular, there are no non-constant maps from $S_{\widetilde{K}}$ to an abelian variety.*

To have a better interpretation as moduli space, we actually consider the subgroup G^* of G given by its elements whose determinant is in $\mathbb{Q}$. More precisely, we let G^* be the pull-back of

$$\det : G \longrightarrow \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\mathbb{G}_m$$

[2] A neat subgroup of $G(\mathbb{A}_{\mathbb{Q}}^f)$ is an open compact subgroup such that every element of $\widetilde{K} \cap G(\mathbb{Q})$ is neat. Recall that an element g of $G(\mathbb{Q})$ is called *neat* if the subgroup of $\overline{\mathbb{Q}}^{\times}$ generated by the eigenvalues of g in some faithful representation V of G is free (that is, there are no nontrivial elements of finite order). This is independent of V, as all faithful representations W are obtained from V via sums, tensor products, duals, and subquotients; hence, the group in question is the set of eigenvalues that occur in W.

to $\mathbb{G}_m/\mathbb{Q}$. The Shimura variety $Y_F := \mathrm{Sh}_{G^*(\widehat{\mathbb{Z}})}(G^*, X^*)(\mathbb{C})$ is connected and comes with a finite map to $S/\mathbb{C}$. It is a quasi projective n_F-dimensional $\mathbb{Q}$-scheme.

In the next section, we present the moduli problem solved by Y_F. It will be also clear from such moduli interpretation that the reflex field of Y_F is the field of rational numbers.

2.2.1 Hilbert Modular Varieties as Moduli Spaces

As explained, for example, in [17, Section 3], the Shimura variety Y_F represents the (coarse) moduli space for triplets (A, α, λ) where:

- A is a complex abelian variety of dimension n_F;
- $\alpha : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$ is a morphism of rings;
- $\lambda : A \to A^*$ is a principal $\mathcal{O}_F$-polarisation.

By A^*, we denoted the $\mathcal{O}_F$-dual abelian variety of A, *i.e*,. it is defined as $\mathrm{Ext}^1(A, \mathcal{O}_F \otimes \mathbb{G}_m)$. Otherwise, one can obtain such abelian variety considering $A^\vee$ (the dual of A, in the standard sense) and tensoring it over $\mathcal{O}_F$ with the different ideal of the extension $F/\mathbb{Q}$. By principal $\mathcal{O}_F$-polarisation we mean an isomorphism $\lambda : A \to A^*$, such that the induced map $A \to A^\vee$ is a polarisation.

Furthermore, the Shimura variety of level

$$U_0(\mathfrak{N}) := \left\{\gamma \in G(\widehat{\mathbb{Z}}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod \mathfrak{N}\right\},$$

where $\mathfrak{N}$ is an integral ideal of $\mathcal{O}_F$, parametrises triplets as above, equipped with a $\mathfrak{N}$-level structure as follows. We fix an isomorphism of $\mathcal{O}_F$-modules

$$\left(\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F\right)^2 \to A[\mathfrak{N}]$$

making the following diagram commutative:

$$\begin{array}{ccc} \left((\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F)^2\right)^2 & \longrightarrow & A[\mathfrak{N}]^2 \\ \downarrow{\scriptstyle \psi_{\mathfrak{N}}} & & \downarrow{\scriptstyle e_{\lambda,\mathfrak{N}}} \\ \mathcal{O}_F \otimes \mathbb{Z}/N\mathbb{Z} & \longrightarrow & \mathcal{O}_F \otimes \mu_N, \end{array}$$

where $(N) = \mathbb{Z} \cap \mathfrak{N}$, $\psi_{\mathfrak{N}}$ is the pairing given by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $e_{\lambda,\mathfrak{N}}$ is the perfect Weil pairing on $A[\mathfrak{N}]$ induced by the $\mathcal{O}_F$-polarisation λ, and $\mathcal{O}_F \otimes \mathbb{Z}/N\mathbb{Z} \to \mathcal{O}_F \otimes \mu_N$ is an arbitrarily chosen isomorphism. When a level structure is needed, we always assume that $N > 3$ in order to have a *fine* moduli space. A rational point of $Y_F(\mathfrak{N}) := \mathrm{Sh}_{U_0(\mathfrak{N}) \cap G^*}(G^*, X)$ then represents a triple, as above, together with such level structure. In Section 4.2, we will see a similar description for $\mathcal{O}_{L,S}$-points of an *integral* model of $Y_F(\mathfrak{N})$.

2.3 Eichler–Shimura Theory

We discuss Eichler–Shimura theory for classical and Hilbert modular forms, reviewing results that attach opportune abelian varieties to modular forms.

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2.3.1 Classical Modular Eigenform

The following is [37, Theorem 7.14, p. 183 and Theorem 7.24, p. 194].

Theorem 2.3 (Shimura) *Let f be a holomorphic newform of weight 2 with rational Fourier coefficients* $(a_n(f))_n$. *There exists an elliptic curve* $E/\mathbb{Q}$ *such that, for all primes p at which E has good reduction, one has*

$$a_p(f) = 1 - N_p(E) + p,$$

where $N_p(E)$ *denotes the number of points of the reduction mod p of E, over the field with p-elements. In other words, up to a finite number of Euler factors,* $L(s, E/\mathbb{Q})$ *and* $L(s, f)$ *coincide.*

More generally, let $K(f)$ *be the subfield of* $\mathbb{C}$ *generated over* $\mathbb{Q}$ *by* $(a_n(f))_n$ *for all n. Then there exists an abelian variety* $A/\mathbb{Q}$ *and an isomorphism* $K(f) \cong End^0(A)$ *with the following properties:*

- $\dim(A) = [K(f) : \mathbb{Q}]$;
- *up to a finite number of Euler factors at primes at which A has good reduction,* $L(s, A/\mathbb{Q}, K(f))$ *coincides with* $L(s, f)$*;*

where the L-function $L(s, A/\mathbb{Q}, K(f))$ *is defined by the product of the following local factors where v is a prime of* $K(f)$ *not dividing* ℓ

$$\det\left(1 - \ell^{-s}\,\mathrm{Frob}_\ell\,|T_v(A)\right).$$

Shimura considers the Jacobian of the modular curve of level N and takes the quotient by the kernel of the homomorphism giving the Hecke action on *f*. What happens if we want to produce an abelian variety with such properties, defined over our totally real field *F*, when $\mathbb{Q} \subsetneq F$? Here is where Hilbert modular forms come into play. In the next section, we recall what we need from such theory, and explain Blasius' generalisation of Theorem 2.3 and why the absolute Hodge conjecture is needed.

2.3.2 Hilbert Modular Forms for *F*

Let $\mathcal{H}_F$ denote n_F copies of the upper half plane $\mathcal{H}^+$. We consider subgroups $\Gamma \subset \mathrm{GL}_2(\mathcal{O}_F)$ of the form $U_0(\mathfrak{N}) \cap G(\mathbb{Q})^+$. Moreover, for $\lambda \in F$ and $\underline{r} = (r_1, \ldots, r_{n_F}) \in \mathbb{Z}^{n_F}$, we write $\lambda^{\underline{r}} = \lambda_1^{r_1} \cdots \lambda_{n_F}^{r_{n_F}}$, where $\lambda_i = \sigma_i(\lambda)$. Similarly, if $\underline{\tau} = (\tau_1, \ldots, \tau_{n_F}) \in \mathcal{H}_F$, we write $\underline{\tau}^\lambda = \tau_1^{r_1} \cdots \tau_{n_F}^{r_{n_F}}$.

Definition 2.1 A Hilbert modular form of level $\mathfrak{N}$ and weight $(\underline{r}, w) \in \mathbb{Z}^{n_F} \times \mathbb{Z}$, with $r_i \equiv w \mod 2$ (and trivial nebentype character) is a holomorphic function $f : \mathcal{H}_F \to \mathbb{C}$ such that

$$f(\gamma \cdot \underline{\tau}) = (\det\gamma)^{-\underline{r}/2}(c\underline{\tau} + d)^{\underline{r}} f(\underline{\tau}),$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{N}) \cap G(\mathbb{Q})^+$ and for every $\underline{\tau} = (\tau_1, \ldots, \tau_{n_F}) \in \mathcal{H}_F$.

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Since Hilbert modular forms are holomorphic functions on $\mathcal{H}_F$ invariant under the lattice $\mathcal{O}_F$, they admit a q-expansion over $\mathcal{O}_F^\vee = \mathfrak{d}^{-1}$, where $q = e^{2\pi i \sum \tau_i}$; see [24, Definition 3.1] for more details.

2.3.3 Hecke Operators and Hilbert Eigenforms

On the space of Hilbert modular forms of level

$$U_0(\mathfrak{N}) \cap G(\mathbb{Q})^+,$$

one has Hecke operators $T(\mathfrak{n})$ for every integral ideal of $\mathcal{O}_F$ coprime with $\mathfrak{N}$. The definition is analogous to the one for classical modular forms. For example, if $\mathfrak{p}$ does not divide $\mathfrak{N}$ and x is a totally positive generator of $\mathfrak{p}$, one defines

$$(T(\mathfrak{p})f)(\underline{\tau}) := \mathrm{Nm}(\mathfrak{p}) f(x \cdot \underline{\tau}) + \frac{1}{\mathrm{Nm}(\mathfrak{p})} \sum_{a \in \mathcal{O}_F/\mathfrak{p}} f(\gamma_a \cdot \underline{\tau}),$$

where $\gamma_a := \left(\begin{smallmatrix} 1 & a \\ 0 & x \end{smallmatrix}\right)$. We recall the following definition.

Definition 2.2 A cuspidal Hilbert modular form (*i.e.*, such that the 0-th Fourier coefficient $a_0(f)$ vanishes) is an eigenform if it is an eigenvector for every Hecke operator $T(\mathfrak{n})$.

As in the case of classical modular forms, if f is an eigenform, normalised so that $a_1(f) = 1$, then the eigenvalues of the Hecke operators are the Fourier coefficients, *i.e.*, $T(\mathfrak{n})f = a_{\mathfrak{n}}(f) \cdot f$; moreover, they are algebraic integers lying in the number field $K(f) := \mathbb{Q}((a_{\mathfrak{n}}(f))_{\mathfrak{n}})$, as shown in [36, §2].

2.3.4 Eichler–Shimura for Hilbert Modular Forms

Blasius and Rogawski [7], Carayol [11], and Taylor [40] proved that to any Hilbert eigenform, one can attach representations of G_F, similarly to the classical case. More precisely, one has the following result.

Theorem 2.4 *If f is a Hilbert eigenform for F of weight $(\underline{r}, t)$, level $\mathfrak{N}$, and trivial nebentype character and $K(f)$ is the number field generated by its eigenvalues, then for every finite place λ of $K(f)$, there is an irreducible 2-dimensional Galois representation*

$$\rho_{f,\lambda} : G_F \longrightarrow GL_2(K(f)_\lambda)$$

such that for every prime $w \nmid \mathfrak{N}\,\mathrm{Nm}_{K(f)/\mathbb{Q}}(\lambda)$ in F, $\rho_{f,\lambda}$ is unramified at w and

$$\det(1 - X\rho_{f,\lambda}(\mathrm{Frob}_w)) = X^2 - a_w(f)X + \mathrm{Nm}_{F/\mathbb{Q}}^{t-1}(w).$$

Assume that f is of weight $(2, \ldots, 2)$. As in the classical case, we would like to have such Galois representations to be attached to opportune abelian varieties. The existence of abelian varieties associated with f was first considered by Oda in [31], and Blasius gave a conjectural solution to such a problem.

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Theorem 2.5 (Blasius) *Let f be a Hilbert eigenform for F of parallel weight 2. Denote by $K(f)$ the number field generated by the $a_w(f)$ for all w. Assume Conjecture 1.1. There exists a $[K(f):\mathbb{Q}]$-dimensional abelian variety A_f/F with $\mathcal{O}_{K(f)}$-multiplication such that for all but finitely many of the finite places w of F at which A_f has good reduction, we have*

$$L\big(s, A_f, K(f)\big) = L(f, s),$$

where the L-function $L(s, A_f, K(f))$ is defined by the product of local factors

$$\det\big(1 - \mathrm{Nm}(w)^{-s}\,\mathrm{Frob}_w\,|T_v(A)^{I_w}\big),$$

where v is a prime of K, w is a prime of F, and w and v lay above distinct rational primes.

Proof If $K(f) = \mathbb{Q}$, this is precisely [6, Theorem 1, p. 3]. As noticed by Blasius [6, 1.10], the proof easily adapts to the general case (where the necessary changes are hinted at in the remarks in [6, Sections 5.4., 5.7., and 7.6.]). ∎

Remark 2.6 The proof is completely different from the one of Shimura, since, as noticed in Theorem 2.2, we cannot obtain a non-trivial abelian variety as quotient of the Albanese variety of a Hilbert modular variety. Blasius instead considers the symmetric square of the automorphic representation of $\mathrm{GL}_{2,F}$ associated with the Hilbert eigenform; it is an automorphic representation of $\mathrm{GL}_{3,F,}$ and its base change to a quadratic imaginary field appears in the middle degree cohomology of a Picard modular variety. He then considers the associated motive and shows that its Betti realisation is the symmetric square of a polarised Hodge structure of type $(1,0),(0,1)$. This gives a complex abelian variety A and Conjecture 1.1 (applied to the product of the Picard modular variety and A) allows us to conclude that A is defined over a number field containing F. He then finds the desired abelian variety inside the restriction of scalars of A over F.

Remark 2.7 Theorem 2.5 is known to hold unconditionally in many interesting cases (for example, when n_F is odd, by the work of Hida). For more details, we refer the reader to [6, Theorem 3] and references therein. The proof of such unconditional cases actually follows Shimura's proof of Theorem 2.3, rather than the strategy described in the previous remark.

2.4 A Remark on Polarisations

To use Theorems 2.3 and 2.5 to produce F-points of a Hilbert modular variety, we need, of course, the abelian varieties produced to be principally polarised (up to isogeny would actually be enough for our applications, if the isogeny is defined over the base field F). An abelian variety over an algebraically closed base field always admits an isogeny to a principally polarised abelian variety. But since the same does not hold over number fields, some considerations are needed. The first observation is that every weight one Hodge structure of dimension 2 with an action by $\mathcal{O}_{K(f)}$ is automatically $\mathcal{O}_{K(f)}$-polarised, as explained for example in [17, Appendix B].

As noticed in [6, Remark 5.7.], Blasius first finds a principally polarised abelian variety A over a finite extension L'/F. Actually, we can assume that A has a principal $\mathcal{O}_{K(f)}$-polarisation λ. As explained above, the proof then considers the Weil restriction of A to F, which is again principally $\mathcal{O}_{K(f)}$-polarised. It is not hard to see that the construction of [6, section 7] behaves well with respect to the $\mathcal{O}_{K(f)}$-polarisation, and, therefore, the proof actually produces an $\mathcal{O}_{K(f)}$-polarised abelian variety over F.

3 Producing Abelian Varieties via Serre's Conjecture

In this section, we prove Theorems 1.A and 1.B. As in Section 1.2, consider two totally real fields L and K. We work with a compatible system of Galois representations of G_L with values in $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{GL}_2)(\mathbb{A}^f_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{A}^f_K)$, where $\mathbb{A}^f_K$ denotes the finite adeles of K that "looks like" an algebraic point of the Hilbert modular variety for K. We then produce a Hilbert modular form for L of weight $(2, \dots, 2)$, and of opportune explicit conductor. Eventually, we obtain an abelian variety over L that allows us to produce an L-rational point on the Hilbert modular variety for K.

3.1 Weakly Compatible Systems

The definition of weakly compatible families presented is due to Serre, who called them *strictly compatible* in [35, p. I-11]. It follows from the Weil conjectures that the ℓ-adic Tate modules of abelian varieties form a weakly compatible system of Galois representations.

Definition 3.1 (Weakly compatible system) A system $\{\rho_v : G_L \to \mathrm{GL}_2(K_v)\}_v$ is *weakly compatible* if there exists a finite set of places S of L such that the following hold.

(i) For all places w of L, ρ_v is unramified outside the set S_v. Here, we denote by S_v the union of S and all the primes of L dividing ℓ where ℓ is the residue characteristic of K_v.

(ii) For all $w \notin S_v$, denoting by Frob_w a Frobenius element at w, the characteristic polynomial of $\rho_v(\mathrm{Frob}_w)$ has K-rational coefficients, and it is independent of v.

Recall that ρ_v is said to be unramified at a place w of L if the image of the inertia at w is trivial. If ρ_v is attached to the v-adic cohomology of a smooth proper variety defined over a number field, the smooth and proper base change theorems (see, for example, [14, I, Theorems 5.3.2 and 4.1.1]) imply that ρ_v is unramified at every place $w \notin S_v$ such that X has good reduction at w.

3.2 Key Proposition

Fix a finite set of places S of L, containing all the archimedean places. To prove Theorem 1.A and Theorem 1.B, we need the following proposition.

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Proposition 3.1 *Assume Conjecture 1.2 and let $\{\rho_v\}_v$ be a system of representations satisfying conditions ($\mathcal{S}$.1)– ($\mathcal{S}$.4). For every $w \notin S$, let $a_w \in \mathcal{O}_K$ be the trace of $\rho_v(\mathrm{Frob}_w)$. Then there exists f a normalised Hilbert eigenform for L with Fourier coefficients in $\mathcal{O}_K$, such that for every $w \notin S$, $a_w(f) = a_w$. Moreover, f is of weight $(2, \dots, 2)$ and conductor divisible only by primes in S.*

Remark 3.2 If we start with an abelian variety A/L with $\mathcal{O}_K$-multiplication, we can produce a system

$$\rho_v : G_L \longrightarrow \mathrm{GL}(T_v(A)),$$

which satisfies the four conditions of ($\mathcal{S}$) for S the union of infinite places and the set of places of bad reduction; see [33, §3]. Proposition 3.1 hence implies that A is modular; *i.e.*, there exists a Hilbert modular form for the totally real field L such that

$$L(A/L, s) = L(f, s)$$

up to a finite number of Euler factors. Unconditionally, it has been proven that elliptic curves over real quadratic fields are modular (see [21]), and, more generally, the work of Taylor and Kisin implies that elliptic curves over L become modular (in this sense) after a totally real extension L'/L. See [9, Theorem 1.16] and reference therein.

In the proof of Proposition 3.1, we use Conjecture 1.2 and the following result due to Serre (for the proof, see [34, 4.9.4]).

Proposition 3.3 (Serre) *Let q be a power of ℓ. Let $r : G_E \to GL_2(\mathbb{F}_q)$ be a continuous homomorphism, where $q = \ell^t$ and E is a local field of residue characteristic $p \neq \ell$ and discrete valuation v_E. Let $e_E := v_E(p)$ and $c \geq 0$ be an integer such that the image via r of the wild inertia of E has cardinality p^c. We denote by $n(r, E)$ the exponent of the conductor of r. We have*

$$n(r, E) \leq 2\Big(1 + e_E \cdot c + \tfrac{e_E}{p-1}\Big).$$

We also need to compute the weight and the conductor of the modular forms produced by Conjecture 1.2. As anticipated in Remark 1.1, this is a known result under some assumptions. The weight part stated in the following theorem is a special case of the work [23]; the conductor part follows from automorphy lifting methods or can be seen as a consequence of the main theorem of [22].

Theorem 3.4 [23, 22] *Let $\ell > 5$ and $\bar{\rho} : G_L \to GL_2(\overline{\mathbb{F}}_\ell)$ be an irreducible totally odd representation such that its determinant is the cyclotomic character and it is finite flat at all places $w \mid \ell$. Assume, furthermore, that $\bar{\rho}$ satisfies the Taylor–Wiles assumption; namely,*

$$\text{(TW)} \qquad \bar{\rho}_{|G_{L(\zeta_\ell)}} \text{ is irreducible,}$$

where ζ_ℓ is a primitive ℓ-th root of unity. Then if $\bar{\rho}$ is modular, there exists a Hilbert modular form of parallel weight 2 and conductor equal to the Artin conductor of $\bar{\rho}$ giving rise to $\bar{\rho}$.

We are now ready to prove Proposition 3.1.

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Proof Our goal is to apply Serre's conjecture to $\bar{\rho}_v$, the reduction modulo v of the representation ρ_v, for infinitely many $v \notin S_K$, where S_K is the following finite set of primes of K:

$$S_K = \{v : v \mid \ell \text{ and } w \mid \ell \text{ for some } w \in S \text{ or } \ell \text{is ramified in } L\}.$$

Let v be such a prime, and let ℓ be its residue characteristic.

We first compute the conductor $N(\bar{\rho}_v)$. Since $\bar{\rho}_v$ is unramified outside S_v, the conductor is divisible only by primes in S. We then apply Proposition 3.3 to $E = L_w$ and $r = \bar{\rho}_v$. The image of $\bar{\rho}_v$ s contained in $\mathrm{GL}_2(\mathbb{F}_{\ell^t})$, where $t \le [K : \mathbb{Q}] = n_K$. The cardinality of this group is $\ell^t(\ell^{2t} - 1)(\ell^t - 1)$. Let W_w denote the wild inertia subgroup of G_{L_w}. If ℓ satisfies the following congruences:

$$(\star) \qquad \ell^{n_K} \not\equiv \begin{cases} \pm 1 \mod p & \text{if } p \ne 2,3 \\ \pm 1 \mod 8 & \text{if } p = 2, \\ \pm 1, 4, 7 \mod 9 & \text{if } p = 3, \end{cases}$$

then the same congruences hold for ℓ^t, and hence $\bar{\rho}_v(W_w)$ is trivial if $p \ne 2, 3$ and is at most p^5 if $p = 2$ and at most p if $p = 3$. Hence for $v \notin S$ laying above ℓ satisfying the above conditions, using that $e_E \le [L : \mathbb{Q}] = n_L$, the inequality of Proposition 3.3 implies that

$$n_K(\bar{\rho}_v, L_w) \le \begin{cases} 2(1 + n_L) & \text{if } p \ne 2, 3, \\ 2(1 + 6n_L) & \text{if } p = 2, \\ 2(1 + 2n_L) & \text{if } p = 3. \end{cases}$$

Writing $\mathfrak{p}_w$ for the prime ideal of L corresponding to w, we hence find that the conductor of $\bar{\rho}_v$ divides

$$\mathfrak{C} := \prod_{\substack{w \in S, \\ w \nmid 2,3}} \mathfrak{p}_w^{2+2n_L} \cdot \prod_{\substack{w \in S, \\ w \mid 2}} \mathfrak{p}_w^{2+12n_L} \cdot \prod_{\substack{w \in S, \\ w \mid 3}} \mathfrak{p}_w^{2+4n_L}.$$

Finally, notice that ρ_v is odd thanks to the condition on the determinant, and, moreover, [5, Proposition 5.3.2] implies that there exists a density one set of primes such that $(\bar{\rho}_v)_{|G_{L(\zeta_\ell)}}$ is irreducible, *i.e.*, (TW) is satisfied.

We can apply Conjecture 1.2 to $\bar{\rho}_v$ for $v \in \Sigma$, where Σ is the infinite set of primes $v \mid \ell$ such that $v \notin S_K$, ℓ satisfies $(\star)$; $\bar{\rho}_v$ is absolutely irreducible and satisfies (TW). We have produced infinitely many f_v Hilbert modular eigenform, which by Theorem 3.4 are of parallel weight 2 and level dividing $\mathfrak{C}$. Their Fourier coefficients are defined over a ring $\mathcal{O}(v) \subset \mathcal{O}_K$, and at a prime $\lambda \mid v$, the associated Galois representation $\rho_{f_v,\lambda}$ is isomorphic to $\bar{\rho}_v$ modulo λ. Since the space of Hilbert modular form of fixed weight and with conductor dividing $\mathfrak{C}$ is finite dimensional (see [20, Theorem 6.1]), we can find at least one Hilbert modular eigenform f of parallel weight 2 and level dividing $\mathfrak{C}$ defined over some $\mathcal{O} \subset \mathcal{O}_K$ such that for infinitely many of the v above, the same property holds for $\rho_{f,\lambda}$ for $\lambda \mid v$. This implies that for all $w \notin S$, the congruence

$$a_w(f) \equiv a_w \mod \lambda \mid v$$

holds for infinitely many primes λ, and hence, $a_w(f) = a_w$, as required. ∎

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3.3 Proof of Theorem 1

Recall that, as in Section 1.2, the system $(\rho_v)_v$ is required to satisfy the following additional property:

($\mathcal{S}$.5) the field generated by a_w for every w is exactly K.

In other words, we have $K(f) = K$, where f is the Hilbert modular form for L produced in Proposition 3.1.

Proof Starting with our initial datum of Galois representations, we have produced a Hilbert modular form f for L. We can then apply Theorem 2.5, which gives an abelian variety A_f over L of dimension $[K : \mathbb{Q}]$ and an embedding of $\mathcal{O}_K$ into $\mathrm{End}(A)$. For all but finitely many $w \mid p$ at which A_f has good reduction

$$\det(1 - X\rho_{A_f,v}(\mathrm{Frob}_w)) = 1 - a_w(f)X + N_w X^2,$$

where v is a finite prime of K not dividing p and $\rho_{A_f,v}$ is the G_L-representation on $T_v(A_f)$, the v-adic Tate module of A_f. We have therefore produced an abelian variety A_f as stated in Theorems 1.A and 1.B.

We just need to stress that we do not require any conjectural statement in the case $n_L = 1$. Indeed, we can use Theorem 2.3 in place of Blasius' conjectural version, and Serre's conjecture is fully known thanks to the work of Khare–Wintenberger [27, Theorem 1.2]. ■

3.4 A Corollary

We rephrase the main results of the section as needed to prove Theorem 2.

Corollary 3.5 *Assume that, in the setting of Theorems 1.A and 1.B, we also have a representation*

$$\rho : G_L \longrightarrow GL_2(\mathcal{O}_K/\mathfrak{N}),$$

for some integral ideal $\mathfrak{N} \subset \mathcal{O}_K$, such that for all pairs (v, a), where v is a place of K and $a \in \mathfrak{N} - \{0\}$, such that v^a divides $\mathfrak{N}$, the reductions of ρ and ρ_v modulo v^a agree. Then there exists an n_K-dimensional abelian variety A/L with good reduction at all w outside S, and the action of G_L on $A[\mathfrak{N}]$ is given by ρ.

Proof Theorems 1.A and 1.B give an n_K-dimensional abelian variety A/L and, using the Néron–Ogg–Shafarevich criterion, we can see that it has good reduction at all w outside S. Finally, G_L acts on $A[\mathfrak{N}]$ via ρ, since the reduction modulo v^a of ρ and ρ_v agree. ■

4 Finite Descent Obstruction and Proof of Theorem 2

In this final section, we recall the finite descent obstruction for integral points, explaining how it relates to the system of Galois representations considered in the previous section. Using Theorem 1, we indeed produce an $\mathcal{O}_{L,S}$-point of integral

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models of twists of Hilbert modular varieties, which is what we require to prove Theorem 2.

4.1 Recap on the Integral Finite Descent Obstruction

Let Y/F be a smooth, geometrically connected variety (not necessarily proper) over a number field F. Let S be a finite set of places of F, and, as before, assume that S contains the archimedean places and all places of bad reduction of Y. Choose and fix a smooth model $\mathcal{Y}$ of Y over $\mathcal{O}_{F,S}$. In this section, we recall the definition of the set $\mathcal{Y}^{f-cov}$, which will correspond to the adelic points of $\mathcal{Y}$ that are unobstructed by all Galois covers. To make the paper self contained, we recall the discussion from [26, Section 2] (where the authors work with affine curves). We then recall that, for Hilbert modular varieties, a point unobstructed by finite covers admits an infinite tower of twists of covers with a compatible system of lifts of adelic points along the tower (following [26, Proposition 1]).

Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a map of $\mathcal{O}_{F,S}$-schemes, such that it becomes a Galois covering over $\overline{F}$. Such a map is called a geometrical Galois cover of $\mathcal{Y}$. Denote by $\mathrm{Tw}(\pi)$ the set of isomorphism classes of twists of π, *i.e.*, of maps $\pi' : \mathcal{X}' \to \mathcal{Y}$ that become isomorphic to π over $\overline{F}$. We have

$$\mathcal{Y}(\mathcal{O}_{F,S}) = \bigcup_{\pi' \in \mathrm{Tw}_0(\pi)} \pi'\big(\mathcal{X}'(\mathcal{O}_{F,S})\big),$$

where $\mathrm{Tw}_0(\pi)$ is a suitable finite subset of $\mathrm{Tw}(\pi)$ (for a more detailed discussion, we refer the reader to [38, pp. 105, 106]), and $\pi' : \mathcal{X}' \to \mathcal{Y}$ is a twist of π. In what follows, w denotes a place of F.

Definition 4.1 We define $\mathcal{Y}^{f-cov}(\mathcal{O}_{F,S}) = \mathcal{Y}^{f-cov}$ as the set of $(P_w)_w \in \prod_{w\notin S} \mathcal{Y}(\mathcal{O}_{F_w})$ such that, for each geometrical Galois cover π, we can write

$$P_w = \pi'(Q_w), \quad \forall w \notin S$$

for some $\pi' \in \mathrm{Tw}_0(\pi)$ and $(Q_w)_w \in \prod_{w\notin S} \mathcal{X}'(\mathcal{O}_{F_w})$.

Proposition 4.1 *A point $(P_w)_w$ lies in $\mathcal{Y}^{f-cov}$ if and only if, for each geometrical Galois cover $\pi : \mathcal{X} \to \mathcal{Y}$, we can choose a twist $\pi' : \mathcal{X}' \to \mathcal{Y}$ and a point $(P_w)_\pi \in \prod_{w\notin S} \mathcal{X}'(\mathcal{O}_{F_w})$ lifting $(P_w)_w$ in a compatible way (i.e., if π_1, π_2 are Galois covers and π_2 dominates π_1, then π_2' dominates π_1' and $(P_w)_{\pi_2'}$ maps to $(P_w)_{\pi_1'}$).*

A few words to justify the equivalence between the two definitions are needed. This is explained in [26, Proposition 1] for curves, and it relies on results from [39] (notably [39, Lemma 5.7][3]). In [39], Stoll works with projective varieties and their rational points, but what he says still holds true for the integral points of non-projective varieties. Once such differences are taken into account, the proof works in the same way in our setting.

[3]It is actually better to refer to the corrected version of [39] available on the author's website (www.mathe2.uni-bayreuth.de/stoll/papers/Errata-FiniteDescent-ANT.pdf).

Clearly, we have $\mathcal{Y}(\mathcal{O}_{F,S}) \subset \mathcal{Y}^{f-cov}$, and so if $\mathcal{Y}^{f-cov}$ is empty, then $\mathcal{Y}(\mathcal{O}_{F,S})$ has to be empty as well. What can be said when $\mathcal{Y}^{f-cov}$ contains a point?

Definition 4.2 If $\mathcal{Y}^{f-cov}(\mathcal{O}_{F,S}) \neq \emptyset$ implies that $\mathcal{Y}(\mathcal{O}_{F,S})$ is non-empty, we say that the S-integral finite descent obstruction is the only obstruction for the existence of S-integral points.

From now on, we specialise to the case of Hilbert modular varieties (and their twists).

4.2 Integral Points on Hilbert Modular Varieties

Recall the notation from Section 2.2.1. Let $Y_K(\mathfrak{N})$ be the n_K-dimensional $\mathbb{Q}$-scheme described in Section 2.2.1 and let N be the integer such that $\mathfrak{N} \cap \mathbb{Z} = (N)$. The set of twists of $\pi : Y_K(\mathfrak{N}) \to Y_K(1)$ over a number field F corresponds to the set of Galois representations $\rho : G_F \to \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{N})$ whose determinant is the cyclotomic character $\chi : G_F \to (\mathbb{Z}/N\mathbb{Z})^\times$. Moreover, a point $x \in Y_K(1)(F)$ lifts to a F-rational point of the twist of $Y_K(\mathfrak{N})$ corresponding to a representation ρ, if and only if ρ describes the action of G_F on the $\mathfrak{N}$-torsion of the underlying abelian variety A_x (as an $\mathcal{O}_K$-module).

Using the moduli interpretation of Section 2.2.1, we can construct a model of $Y_K(\mathfrak{N})$ over $\mathbb{Z}$, which is smooth over $\mathbb{Z}[1/b]$, for some natural number b, divisible by N. To be more precise,b depends on the level structure and the discriminant of K. Fixing such a model, which we denote by $\mathcal{Y}_K(\mathfrak{N})$, we can talk about $\mathcal{O}_{F,S}$-points of $Y_K(\mathfrak{N})$, for any number field F and set of places S containing the archimedean places and the ones dividing b. Such $\mathcal{O}_{F,S}$-points then correspond to abelian varieties (with some extra structure), having good reduction outside S. Recall that N is assumed to be bigger than 3, since it is important to have a fine moduli space. For example, the affine line is the moduli space of elliptic curves and has plenty of $\mathbb{Z}$-points, even though there are no elliptic curves defined over $\mathbb{Z}$.

We are ready to study the finite descent obstruction for the $\mathcal{O}_{L,S}$-points of $\mathcal{Y}_K(\mathfrak{N})$ and its twists, where L is a totally real field (the fact that L is totally real will be used only in the next section). Let

$$\rho : G_L \longrightarrow \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{N})$$

be a representation whose determinant is the cyclotomic character. Assume that S contains the places w of ramification of ρ. Under this assumption, arguing as above, we can consider $\mathcal{Y}_\rho$ to be the S-integral model of the twist of $\mathcal{Y}_K(\mathfrak{N})$ corresponding to ρ. From now on, we assume that $\mathcal{Y}_\rho(\mathcal{O}_{L,S})$ is non-empty. The next lemma relates a point $(P_w)_w \in \mathcal{Y}_\rho^{f-cov}$ to a system of Galois representations as considered in the previous section.

Lemma 4.2 *A point $(P_w)_w \in \mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$ corresponds to the following data:*

- *for each finite place v of K a representation $\rho_v : G_L \to GL_2(K_v)$;*
- *for each finite place w of L such that $w \notin S$ an abelian variety A_w/L_w of dimension n_K, with good reduction and $\mathcal{O}_K$-multiplication;*

https://doi.org/10.4153/S0008439520000569 Published online by Cambridge University Press

satisfying:

- *for every place v in K the action of G_{L_w} on $T_v(A_w)$ is given by the restriction of ρ_v to the decomposition group at w;*
- *for all pairs (v, a) such that v^a divides $\mathfrak{N}$, the reductions of ρ and ρ_v modulo v^a agree.*

Moreover, every such system satisfies the first four conditions of ($\mathcal{S}$).

Proof We first check, using Lemma 4.1, that an unobstructed point corresponds to a system of Galois representations as described above, and then we show that every such system enjoys the desired properties.

Thanks to Proposition 4.1, we can fix a compatible system of lifts of $(P_w)_w$ on $\mathcal{Y}_\rho^{f-cov}(O_{L,S})$. In particular, for each $\mathfrak{M}$ divisible by $\mathfrak{N}$, we obtain a twist $\mathcal{Y}_K(\mathfrak{M})'$ of $\mathcal{Y}_K(\mathfrak{M})$ and a compatible family of points $(P_w)_{\mathfrak{M}}$ of $\mathcal{Y}_K(\mathfrak{M})'$ lifting $(P_w)_w$. We remark here that the latter compatible family of points depends on $(P_w)_w$. Indeed, a priori, we cannot simply lift ρ to mod $\mathfrak{M}$ coefficients.

By the interpretation of $\mathcal{Y}_K(\mathfrak{M})'$ as moduli space of abelian varieties discussed above, the point $(P_w)_{\mathfrak{M}}$ corresponds to an abelian variety A_w/L_w of dimension n_K, with good reduction and O_K-multiplication and prescribed $\mathfrak{M}$-torsion. The other conditions are easily checked as at the end of proof of [26, Theorem 2].

The fact that the action of G_{L_w} on $T_v(A_w)$ is given by the restriction of ρ_v to the decomposition group at w ensures that ($\mathcal{S}$.1) and ($\mathcal{S}$.2) are satisfied. Moreover, since A_w has good reduction, $A_w[v] \simeq \bar{\rho}_v$ is a finite flat group scheme over O_{K_w} for all $w \mid \ell$ if $v \mid \ell$; this implies that ($\mathcal{S}$.3) also holds.

Finally, we need to show that ($\mathcal{S}$.4) is satisfied. With the three conditions above, one can show, as in the proof of Proposition 3.1, that the conductor of $\bar{\rho}_v$ divides a fixed ideal $\mathfrak{C}$ of L. If w is such that A_w/L_w is supersingular at v, then $A_w[v] \simeq \bar{\rho}_v$ is absolutely irreducible. If there existed infinitely many v such that $\bar{\rho}_v$ is absolutely reducible, we could then write the semisemplification of $\bar{\rho}_v$ as direct sum of ϕ and $\chi_\ell\phi^{-1}$, for some character ϕ. Since $\bar{\rho}_v \simeq A_w[v]$ for $w \mid p$, we then have that A_w is ordinary, and hence ϕ is unramified at w. We also know that the conductor of ϕ divides $\mathfrak{C}$; hence, if $w \equiv 1 \mod \mathfrak{C}$, we have

$$a_w(A_w) := \mathrm{Tr}\big(\mathrm{Frob}_w, T_v(A_w)\big) \equiv \chi_\ell(\mathrm{Frob}_w) + 1 \mod v.$$

Since $\chi_\ell(\mathrm{Frob}_w) = p^{[L_w:\mathbb{Q}_p]}$, we showed that if we had infinitely many v such that $\bar{\rho}_v$ is absolutely reducible, we would find $a_w(A_w) = p^{[L_w:\mathbb{Q}_p]} + 1$. Since the Weil bound says that $|a_w| \leq 2\sqrt{p^{[L_w:\mathbb{Q}_p]}}$, we reached a contradiction. ∎

Remark 4.3 As discussed above, for any number field F, we have a map from $Y_K(F)$ to systems of Galois representations satisfying ($\mathcal{S}$.1)–($\mathcal{S}$.4). Thanks to Faltings [18, Satz 6], this map has finite fibres. Indeed, if two points give rise to the same system, the two corresponding abelian varieties have the same locus of bad reduction, which we denote by S. It follows from the Shafarevich conjecture that Shimura varieties of abelian type have only finitely many $O_{F,S}$-points. For more details, we refer the reader to [43, Theorem 3.2(A)].

We are now ready to prove the main theorem about descent obstruction for Hilbert modular varieties

4.3 Proof of Theorem 2

We do not treat the cases $n_L = 1$ and $n_L > 1$ separately, but, as in the proof of Theorem 1, we emphasize that we do not need any conjectural statement in the case $n_L = 1$, since we have Shimura's unconditional result, Theorem 2.3.

Proof Thanks to Lemma 4.2, a point in $\mathcal{Y}_\rho^{f-cov}(\mathcal{O}_{L,S})$, which is assumed to be non-empty, gives rise to a compatible system of representations of G_L, denoted by $\{\rho_v\}_v$. Let E be the the subfield of K generated by $\mathrm{tr}(\rho_v(\mathrm{Frob}_w))$ for all w. If $E = K$, Corollary 3.5 produces an $\mathcal{O}_{L,S}$-abelian variety A with $\mathcal{O}_K$-multiplication, such that G_L acts on $A[\mathfrak{N}]$ via ρ. To conclude the proof, we just need to see how A corresponds to a point $P \in \mathcal{Y}_\rho(\mathcal{O}_{L,S})$. The only issue that is not clear from the quoted corollary is whether A is principally $\mathcal{O}_K$-polarised, but this follows from the discussion in Section 2.4.

If E is strictly contained in K, *i.e.*, condition ($\mathcal{S}$.5) is not satisfied, we consider S_E, the Hilbert modular variety associated with E and of level $\mathfrak{N} \cap \mathcal{O}_E$. For the same reason as above, the system $\{\rho_v\}$ corresponds to a $\mathcal{O}_{L,S}$-point P of the twist by ρ of S_E. The embedding $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_2 \hookrightarrow \mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_2$ induces a map of Shimura varieties

$$r : S_E \longrightarrow Y_K(\mathfrak{N}),$$

and therefore on their twists by ρ. Via r, we can regard P as an $\mathcal{O}_{L,S}$-point of $\mathcal{Y}_\rho$, which completes the proof of the theorem. The only difference is that the abelian variety constructed in this case is not primitive. This concludes the proof of Theorem 2. ■

Acknowledgement It is a pleasure to thank Toby Gee for helpful discussions about the weight part of Serre's conjecture and the Taylor–Wiles assumption. We thank Matteo Tamiozzo for his comments on an earlier draft of this paper. We would also like to express our gratitude to the anonymous referees whose comments improved the paper.

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