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INTRINSIC AGING AND CLASSES OF NONPARAMETRIC DISTRIBUTIONS

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We develop a general framework for understanding the nonparametric (aging) properties of nonnegative random variables through the notion of intrinsic aging. We also introduce some new notions of aging. Many classical and more recent results are special cases of our general results. Our general framework also leads to new results for existing notions of aging, as well as many results for our new notions of aging.

1. INTRODUCTION AND SUMMARY

Consider a nonnegative absolutely continuous random variable Y. The random variable Y could be the time to default in a credit risk (e.g., Ammann [1]), the lifetime of a reliability system (e.g., Barlow and Proschan [2]), or the demand for an item in a supply chain (e.g., Porteus [15]). Nonparametric (aging) properties of the distribution function of this random variable often play a crucial role in characterizing the optimal

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operational policies associated with the random variable. For example, the pricing of a default swap in credit risk, the replacement policy for a reliability system, or setting the price-only contract in a supply chain can depend on the nonparametric properties of the distribution of Y.

In this article we develop a general framework for studying nonparametric classes of random variables, based on the cumulative hazard rate and the notion of intrinsic aging (e.g., Çinlar and Ozekici [5] and Çinlar, Shaked, and Shanthikumar [6]). Let X be a nonnegative absolutely continuous random variable. Suppose X is the intrinsic life of a reliability system. The actual lifetime T of this system will depend on how the intrinsic age is accumulated over the calendar time. For example, under extreme conditions the system will age faster than under milder conditions. Suppose the intrinsic age of the system at time t is $\phi(t)$ ($\phi(0) = 0$). Then

$$T = \inf\{t : \phi(t) \geq X; t \in \mathcal{R}_+\}.$$

We give a very general result (Theorem 5.3) that allows us to relate aging properties of random intrinsic lifetimes to aging properties of actual lifetimes, given the appropriate conditions on ϕ. It also allows us to generate classes of distributions starting with a canonical element of the class. Our approach unifies many existing results, leads to new results for existing notions of aging, and suggests new notions of aging. In this article almost all of the results are new; any result that is not new is given with a reference.

Applications of our results span a number of areas of applied probability. In classical reliability theory, there are many important and well-understood aging notions for a nonnegative absolutely continuous random variable Y. These notions relate to the hazard, or failure, rate, and to the residual life, $Y_R(y) = \{Y - y | Y > y\}$. More recently, the importance of a different notion of aging for income distributions (see Belzunce, Candel, and Ruiz [3,4]) and pricing problems (see Lariviere and Porteus [10]) has been recognized. In the pricing context, Y is the random valuation of a customer for a product, so $\bar{F}(p) = P(Y > p)$ is the probability that a random customer will buy the product at price p and $p\bar{F}(p)$ is the expected revenue for price p. It turns out (see Remark 2.1 in Section 2) that the aging notion that is appropriate for this application is based on the proportional, or length-biased, failure rate of the random variable Y, defined by $l(t) = th(t)$, where $h(t) = f(t)/\bar{F}(t)$ is the usual hazard rate of Y and $f(t)$ is the density of Y. The proportional failure rate was introduced by Singh and Maddala [19] in the context of modeling income distributions. For the pricing model, $l(p)$ is the elasticity of demand. We study the scaled conditional life, $Y_{SC}(y) \stackrel{d}{=} \{Y | Y > y\}/y$ (called the left proportional residual income by Belzunce et al. [4]). This is more relevant for pricing problems than the residual life $Y_R(y)$ because it more directly relates to the elasticity. In particular, $P(Y_R(y) > a)$ represents the proportion of the market willing to pay at least a out of those willing to pay at least y, whereas $P(Y_{SC}(y) > a)$ represents the proportion willing to pay at least $a\%(\times 100)$ more than y, among those willing to pay at least y. The latter is more directly related to the elasticity. Among other results, we show that if Y has increasing proportional failure rate (IPFR), then $Y_{SC}(y)$ is decreasing in y in the hazard rate sense. We also introduce the complementary notions of scaled hazard rate and scaled residual life and show that to have a unique optimizer for an inspection problem for incoming components, Y must have increasing scaled hazard rate, where Y is the quantity of components inspected before the first defect in an arbitrary order of inspection. We show how different notions of aging lead to different properties for the residual life, the scaled conditional life, and the scaled residual life. We also introduce two new notions of aging that imply IPFR and we study their closure properties.

In Section 2 we give extensive background material. We first define hazard rate, cumulative hazard rate, and residual lifetime and then extend these notions (Section 2.1). Our approach depends heavily on notions of aging determined by properties of the cumulative hazard rate function and preserved depending on properties of the intrinsic aging function, so we define a number of properties of functions that can be considered extensions of convexity (Section 2.2). We then define various notions of aging, based on the properties of the cumulative hazard rate (Section 2.3). Finally, we recall some definitions of various stochastic orders and define some new orders. In Section 3 we give a unified approach to show which aging properties of intrinsic lifetimes imply other aging properties of actual lifetimes. In Section 4 we connect our notions of aging (from Section 2.3) to our new notions of residual lifetimes (from Section 2.1). In Section 5 we give conditions on the intrinsic aging property such that properties of the intrinsic lifetime are preserved for the actual lifetime. We show that recent results for IPFR random variables are consequences of our general results. Finally, in Section 6 we show how classes of random variables with our new aging properties can be generated from primitive random variables such as exponential or Pareto random variables.

Throughout the article, the terms "increasing," "positive," and so forth are used in the nonstrict sense.

2. PRELIMINARIES AND DEFINITIONS

2.1. Various Notions of Hazard Rates and Residual Lifetimes

The following notation will be used. For any random variable Y, we denote its distribution function by F_Y and its survival function by $\bar{F}_Y$. When Y is absolutely continuous, we denote its density function by f_Y. For a nonnegative random variable Y ($F_Y(0) = 0; \lim_{y\to\infty} F_Y(y) = 1$), we define $a_Y = \inf\{y : F_Y(y) > 0\}$ and $b_Y = \sup\{y : F_Y(y) < 1\}$. Note that b_Y could be infinite, and $a_Y \geq 0$. Throughout we will consider only nonnegative random variables and we generally suppose they are absolutely continuous. However, to preserve some duality properties, we also permit mixtures of continuous random variables with the constant (degenerate random variable) 0; that is, Y will have a density f_Y on (a_Y, ∞), and if $a_Y = 0$, Y may also have a point mass at 0, $F_Y(0) = P\{Y = 0\} > 0$.

2.1.1. Residual Life and Hazard Rate (e.g., Barlow and Proschan [2]). The residual life of a random variable Y is defined as

$$Y_R(y) \stackrel{d}{=} \{Y - y | Y > y\}, \qquad a_Y \le y < b_Y,$$

and the hazard rate (also called the failure rate) of Y is given by

$$h_Y(y) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\{Y_R(y) \le \Delta)\}, \qquad a_Y < y < b_Y.$$

Observe that $a_{Y_R(y)} = 0, b_{Y_R(y)} = b_Y - y$ for all $a_Y < y < b_Y$. Additionally, $h_{Y_R(y)}(t) = h_Y(y+t)$ for $a_Y < y < b_Y$ and $0 < t < b_Y - y$.

The cumulative hazard function H_Y is

$$H_Y(y) = -\log\{\bar{F}_Y(y)\},\ y \in (a_Y, b_Y).$$

Thus, the failure rate, or hazard rate, of Y is

$$h_Y(y) = \frac{d}{dy} H_Y(y) = \frac{f_Y(y)}{\bar{F}_Y(y)}, \qquad y \in (a_Y, b_Y).$$

We use the following convention: $h_Y(y) = \infty$ (and $H_Y(y) = \infty$) if $\bar{F}_Y(y) = 0$; that is, for finite b_Y, we set $h_Y(y) = \infty,\ y \ge b_Y$. We also define the average failure (or hazard) rate function $\bar{h}_Y$ by

$$\bar{h}_Y(y) = \frac{1}{y}\left[H(0) + \int_0^y h_Y(z)dz\right] = \frac{1}{y} H_Y(y), \qquad y \in \mathcal{R}_+$$

2.1.2. Conditional Shortfall and Reverse Hazard Rate (e.g., Chandra and Roy [7]). The conditional shortfall, or inactivity time, of a random variable Y is defined as

$$Y_S(y) \stackrel{d}{=} \{y - Y | Y \le y\}, \qquad a_Y \le y < b_Y,$$

and the reverse hazard rate of Y is given by

$$r_Y(y) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\{Y_S(y) \le \Delta)\}, \qquad a_Y < y < b_Y.$$

Observe that $a_{Y_S(y)} = 0,\ b_{Y_S(y)} = y - a_Y$ for all $a_Y < y < b_Y$. We also have that $h_{Y_S(y)}(t) = r_Y(y-t)$ for $a_Y < y < b_Y$ and $0 < t < y - a_Y$.

We note here that the analysis of this article can be applied to shortfalls and reverse hazard rate, where we think of time as running backward. Thus, we could define a cumulative reverse hazard rate as $R(x) = \int_{b_Y}^x r(t)\,dt$ and a reverse intrinsic aging function $\bar{\phi}(t)$ as the intrinsic age at $b_Y - t$; the results we have for H and ϕ could be applied to R and $\bar{\phi}$. The analysis is tedious and the results have less application than those for hazard rates, so we omit it.

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2.1.3. Scaled Conditional Life and Proportional Failure Rate (e.g., Belzunce et al. [3]). The scaled conditional life of a random variable Y is the total life relative to the current age, conditioned on the current age, and is given by

$$Y_{SC}(y) \stackrel{d}{=} \frac{1}{y}\{Y|Y > y\} = (y + Y_R(y))/y, \qquad a_Y < y < b_Y.$$

The scaled conditional life is called the left proportional residual income by Belzunce et al. [3]. The proportional failure rate is

$$l_Y(y) = \lim_{\Delta\downarrow 0} \frac{1}{\Delta} P\{Y_{SC}(y) \le 1 + \Delta)\} = y h_Y(y), \qquad a_Y < y < b_Y.$$

Observe that $P\{Y_{SC}(y) \le 1\} = 0$, so $a_{Y_{SC}(y)} = 1$ for all $a_Y < y < b_Y$. Additionally, for $a_Y < y < b_Y$, $h_{Y_{SC}(y)}(t) = y h_Y(yt) = (1/t) l_Y(yt)$ for $1 \le t < b_Y/y$. The proportional failure rate is also known as the generalized failure rate of Y, as defined by Lariviere [8] (also see Lariviere and Porteus [10]). Because the usual failure rate is not a special case of the generalized failure rate, we prefer the term "proportional failure rate."

In the context of income distributions, Singh and Maddala ([19] p. 964) say of the proportional failure rate that "at any income, it measures the odds against advancing further to higher incomes in a proportionate sense." They argue that empirically income levels tend to constant proportional failure rate at high income levels.

Remark 2.1: As mentioned in Section 1, an application of the proportional failure rate is to pricing (Lariviere [9]), where $l_Y(p)$ is the elasticity of demand and where the optimal price p^* is such that $l_Y(p^*) = 1$. There will be a unique revenue maximizing price if Y is IPFR; that is, if it is increasing in proportional failure rate ($l_Y(t)$ is increasing in t) and if $\lim_{x\downarrow a_Y} l_Y(x) \le 1$ and $\lim_{x\uparrow b_Y} l_Y(x) > 1$, where (a_Y, b_Y) is the support of Y. Note that $Y_{SC}(p)$ can be interpreted as a consumer surplus factor at price p: It is a random customer's valuation of the product relative to price p, given the customer's valuation is at least p.

2.1.4. Scaled Residual Life and Scaled Hazard Rate (for finite b_Y). The scaled residual life of a random variable Y is the remaining life relative to the maximal possible remaining life, conditioned on the current age, and is given by

$$Y_{SR}(y) \stackrel{d}{=} \frac{\{Y - y|Y > y\}}{b_Y - y} = \frac{Y_R(y)}{b_Y - y}, \qquad a_Y < y < b_Y.$$

The scaled hazard rate is

$$s_Y(y) = \lim_{\Delta\downarrow 0} \frac{1}{\Delta} P\{Y_{SR}(y) \le \Delta)\} = (b_Y - y) h_Y(y), \qquad a_Y < y < b_Y.$$

Observe that $P\{0 \le Y_{SR}(y) \le 1\} = 1$, so $a_{Y_{SR}(y)} = 0$ and $b_{Y_{SR}(y)} = 1$ for all $a_Y < y < b_Y$. Additionally, $h_{Y_{SR}(y)}(t) = (b_Y - y) h_Y(y + t(b_Y - y))$ for $a_Y < y < b_Y$ and $0 < t < 1$.

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The scaled hazard rate arises in the following inspection context. Consider a company that receives shipments of size b_Y of some material from its supplier, where the material is either continuous, as in fabric, or b_Y is large. Let Y be the amount of material that is inspected before the first defect is found. The company chooses an inspection level $y \in [a_Y, b_Y]$ and it inspects an amount $\min\{y, Y\}$ of the batch. (In practice, a_Y will probably be zero.) If $Y \leq y$, it accepts the inspected portion of the batch and returns the uninspected portion, an amount $b_Y - Y$, to the supplier. If $Y > y$, the company accepts the full batch. The (scaled) profit to the company per unit of inspected material (after inspection costs) is 1 and the (scaled) profit per unit of uninspected material is $c < 1$. (If $c \geq 1$, there is no point in inspecting.) The company wishes to choose y to maximize its profit, $\Pi(y) = E\min\{y, Y\} + c(b_Y - y)P\{Y > y\}$. We have the following.

Theorem 2.1: *If $s_Y(y)$ is strictly increasing, then there is a unique y^* that maximizes the company's total return.*

Proof: We take the derivative of the profit function:

$$\Pi(y) = \int_0^y \bar{F}_Y(x)\,dx + c(b_Y - y)\bar{F}(y),$$

$$\frac{d}{dy}\Pi(y) = \bar{F}_Y(y) - c\bar{F}_Y(y) - c(b_Y - y)f_Y(y)$$

$$= \bar{F}_Y(y)[1 - c - cs_Y(y)].$$

If $s_Y(a_Y) \geq (1-c)/c$, then $y^* = a_Y$, and it is optimal to accept the whole lot with minimal inspection. If $s_Y(a_Y) < (1-c)/c$ and $\lim_{y\to b_Y} s_Y(y) \leq (1-c)/c$, then $y^* = b_Y$ and it is optimal to keep inspecting until the first defect is found or the entire lot is inspected. Finally, if $s_Y(a_Y) < (1-c)/c < \lim_{y\to b_Y} s_Y(y)$, because $s_Y(y)$ is strictly increasing, then there is a unique $y^* \in (a_Y, b_Y)$. ■

The scaled hazard rate and scaled residual life can also be applied to reliability problems. Consider a single-component system with a warm standby that is replaced every b_Y time units, where we wish to maximize the proportion of time that the overall system is available. During a cycle of length b_Y, if the original component fails at some time $0 < y < b_Y$ and the warm standby has not failed, we will prefer a warm standby with a smaller scaled hazard rate (i.e., a larger scaled residual life). Another application is to insurance problems where we are concerned about the total remaining liability relative to the maximal liability for a customer that has made claims so far totaling y dollars.

2.2. Classes of Functions

We will use the following classes of functions. Unless otherwise specified, for all basic definitions in this article, one can refer to either Marshall and Olkin [12], Ross [16], or Shaked and Shanthikumar [18].

Let $\eta : [a, b] \Rightarrow \mathcal{R}_+$ be a positive increasing function, for $0 \leq a < b < \infty$. We write

$$\eta'(x) = \frac{d}{dx}\eta(x), \qquad x \in (a, b),$$

and assume that the derivative is well defined on (a, b). For the classes of distribution functions that we consider later, we will relate properties of distributions to properties of cumulative hazard H, as defined in Section 2.1, and the intrinsic aging function ϕ. Note that H and ϕ are increasing, and for intrinsic aging functions, $a = 0$.

Superadditive [Subadditive]: η is superadditive (subadditive) on (a, b) if $\eta(x) + \eta(y) \leq [\geq]\ \eta(x + y)$, $x, y \in (a, b)$. We denote this by $\eta \in$ SupA [SubA].

Star-Shaped [Anti-Star-Shaped]: η is star-shaped [anti-star-shaped] on (a, b) if $(1/x)\eta(x)$ is increasing [decreasing] in x for $x \in (a, b)$. We denote this by $\eta \in$ SS [AntiSS].

Convex [Concave]: η is convex [concave] on (a, ∞) if $\eta'(x)$ is increasing [decreasing] in x for $x \in (a, b)$. We denote this by $\eta \in$ CX [CV].

Convex in Log Scale [Concave in Log Scale]: η is convex in log scale [concave in log scale] on (a, b) if $x\eta'(x)$ is increasing [decreasing] in x for $x \in (a, b)$; that is, $\eta(e^x)$ is convex [concave]. We denote this by $\eta \in$ CX(Log) [CV(Log)].

Log Convex [Log Concave]: η is log convex [log concave] on (a, b) if $\eta'(x)/\eta(x)$ is increasing [decreasing] in x for $x \in (a, b)$; that is, $\log \eta(x)$ is convex [concave]. We denote this $\eta \in$ LogCX [LogCV].

Log Convex in Log Scale [Log Concave in Log Scale]: η is log convex in log scale [log concave in log scale] on (a, b) if $x\eta'(x)/\eta(x)$ is increasing [decreasing] in x for $x \in (a, b)$; that is, $\log \eta(e^x)$; is convex [concave]. We denote this by $\eta \in$ LogCX(Log) [LogCV(Log)].

For completeness, we also include the following definition that we use later to define random variables that have an increasing or decreasing scaled hazard rate (ISFR or DSFR).

Scaled Convex [Scaled Concave]: η is Scaled Convex [Scaled Concave] on (a, b) if $(b - x)\eta'(x)$ in increasing [decreasing] in x for $x \in (a, b)$. We denote this $\eta \in$ ScCX [ScCV].

Because we only consider increasing functions $\eta : (a, b) \rightarrow \mathcal{R}_+$, we insert an I and write $\eta \in$ ISupA and so forth and we sometimes abuse notation and write $\eta(x) \in$ ISupA and so forth. We also only consider functions where either $a = 0$ or, if $a > 0$, $\eta(a) = 0$, and we extend the domain of η to $(0, \infty)$ by defining $\eta(x) = 0$ for $0 \leq x \leq a$ and $\eta(x) = \infty$ for $x \geq b$. Note that we could have $a = 0$ and $\eta(0) > 0$. However, in some cases we also want to require $a = 0$ and $\eta(0) = 0$, in which case we write $\eta \in \text{ISupA}_0$ for example, and so forth. When we restrict ourselves to functions $\eta : (a, b) \rightarrow \mathcal{R}_+$ where $b < \infty$, we write, for example, $\eta \in \text{ICX}^b$. Sometimes we also

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want functions such that $\lim_{x \to b} \eta(x) = \infty$ (where we could have $b = \infty$), in which case we write, for example, $\eta \in$ ICX-∞. Finally, when we write in Lemma 2.2 that, $\text{ICX}_0 \Rightarrow \text{ISS}_0$, we mean $\eta \in \text{ICX}_0 \Rightarrow \eta \in \text{ISS}_0$ (i.e., $\text{ICX}_0 \subseteq \text{ISS}_0$).

Lemma 2.2:

$$
\begin{array}{lclcl}
\text{ICX}_0 & \Rightarrow & \text{ISS}_0 & \Rightarrow & \text{ISupA}_0, \\
\text{ICV}_0 & \Rightarrow & \text{IAntiSS}_0 & \Rightarrow & \text{ISubA}_0, \\
\text{ILogCX} & \Rightarrow & \text{ILogCX(Log)} & \Rightarrow & \text{ICX(Log)}, \\
\text{ILogCX} & \Rightarrow & \text{ICX} & \Rightarrow & \text{ICX(Log)}, \\
\text{ICV(Log)} & \Rightarrow & \text{ILogCV(Log)} & \Rightarrow & \text{ILogCV}, \\
\text{ICV(Log)} & \Rightarrow & \text{ICV} & \Rightarrow & \text{ILogCV}, \\
\text{IScCX}^b & \Rightarrow & \text{ICX}^b & & \\
\text{ICV}^b & \Rightarrow & \text{IScCV}^b, & & \\
 & & \text{ILogCX}_0\text{-}\infty = \emptyset. & &
\end{array}
$$

Proof: The subset relations are not difficult to show; the proof is omitted. We show that $\text{ILogCX}_0\text{-}\infty = \emptyset$. We need to show that there does not exist a function $\eta : \mathcal{R}_+ \to \mathcal{R}_+$ such that $\eta(0) = 0$, η is increasing without bound, and $\log \eta$ is convex; that is, we want to show that there does not exist a function $\nu : \mathcal{R}_+ \to \mathcal{R}$ such that $\nu(0) = -\infty$ and ν is increasing without bound and convex, where $\nu(x) = \log \eta(x)$. Suppose such a function ν does exist and pick $a_2 > a_1 > 0$ such that $-\infty < y_1 := \nu(a_1) < y_2 := \nu(a_2) < \infty$. Such a_1 and a_2 exist because we assumed well-defined derivatives, and hence continuity, in (a, b). Then, by convexity, $\nu(0) \geq (y_1 a_2 - y_2 a_1)/(a_2 - a_1) > -\infty$, which is a contradiction. ■

The fact that $\text{ILogCX}_0\text{-}\infty = \emptyset$ means that if $\eta \in$ ILogCX-∞, then $\eta(0) > 0$. There is an analogous result for ICV(Log) which will have implications for DPFR random variables as defined in the sequel. In particular, although $\text{ICV(Log)}_0\text{-}\infty \neq \emptyset$, there is no function $\eta \in$ ICV(Log)-∞ such that $\eta(0+) > 0$.

Lemma 2.3: $\eta \in \text{ICV(Log)}_0\text{-}\infty \Rightarrow \exists a > 0$ *such that* $\eta(x) = 0$ *for* $0 \leq x \leq a$.

Lemma 2.3 will follow from the last part Lemma 2.2 once we define inverse functions and complementary sets.

For any increasing function $\phi : \mathcal{R}_+ \to \mathcal{R}_+$, define its right inverse by

$$\psi(t) = \phi^{-1}(t) = \inf\{x : \phi(x) \geq t; x \geq 0\}.$$

Two classes $\mathcal{B}$ and $\mathcal{B}^C$ of functions satisfy the complementary property if

$$\phi \in \mathcal{B} \Leftrightarrow \psi \in \mathcal{B}^C.$$

Note that $\mathcal{B}$ is closed under composition if and only if $\mathcal{B}^C$ is closed under composition.

Note that ILogCX-∞^C = ICV(Log)-∞, so $\eta \in$ ICV(Log)-$\infty \Leftrightarrow \eta^{-1} \in$ ILogCX-∞, which means, from the last part Lemma 2.2, that $\eta^{-1}(0) = a > 0$, so $\eta(a) = 0$, and because η is increasing, Lemma 2.3 follows.

We use the following notation for composition: $\eta \circ \nu$, where $(\eta \circ \nu)(x) = \eta(\nu(x))$. For two classes of functions $\mathcal{B}_1$ and $\mathcal{B}_2$, let $\mathcal{C}(B_1, \mathcal{B}_2)$ be the (largest) set of compatible input functions defined as follows:

$$\nu \in \mathcal{C}(\mathcal{B}_1, \mathcal{B}_2) \Leftrightarrow \eta \circ \nu \in \mathcal{B}_2 \quad \text{for all } \eta \in \mathcal{B}_1,$$
$$\nu \notin \mathcal{C}(\mathcal{B}_1, \mathcal{B}_2) \Leftrightarrow \exists\, \eta \in \mathcal{B}_1 : \eta \circ \nu \notin \mathcal{B}_2.$$

Intuitively, if we start with a function in $\mathcal{C}(\mathcal{B}_1, \mathcal{B}_2)$ and input it into a function from $\mathcal{B}_1$, the output is a function from $\mathcal{B}_2$. We say a set $\mathcal{A}$ is compatible with $\mathcal{B}_1$ and $\mathcal{B}_2$ if $\mathcal{A} \subseteq \mathcal{C}(\mathcal{B}_1, \mathcal{B}_2)$; for example, ICX(Log) $\subseteq \mathcal{C}$(ICX, ICX(Log)). If $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$, we write $\mathcal{C}(\mathcal{B}) = \mathcal{C}(\mathcal{B}, \mathcal{B})$ and call it the (largest) set of compatible functions. We say a set $\mathcal{A}$ is compatible with $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{C}(\mathcal{B})$; for example, ICV $\subseteq \mathcal{C}$(ILogCV). If $\mathcal{B}$ is closed under composition, then $\mathcal{B} \subseteq \mathcal{C}(\mathcal{B})$. The following lemma is not hard to show.

LEMMA 2.4:

(i) The classes ISupA_0, ISubA_0, ISS_0, IAntiSS_0, ICX_0, ICV_0, ICX, ICV, ICV(Log), ILogCX, ILogCX(Log), *and* ILogCV(Log) *are closed under composition. The classes* ICX(Log), ILogCV, IScCX^{b}, *and* IScCV^{b} *are not.*

(ii)

$$\text{ICX} \subseteq \mathcal{C}(\text{ILogCX}),$$
$$\text{ICV} \subseteq \mathcal{C}(\text{ILogCV}),$$
$$\text{ILogCX(Log)} \subseteq \mathcal{C}(\text{ICX(Log)}),$$
$$\text{ILogCV(Log)} \subseteq \mathcal{C}(\text{ICV(Log)}).$$

(iii)

$$\text{ICX(Log)} \subseteq \mathcal{C}(\text{ILogCX}, \text{ILogCX(Log)}),$$
$$\text{ICX(Log)} \subseteq \mathcal{C}(\text{ICX}, \text{ICX(Log)}),$$
$$\text{IScCX} \subseteq \mathcal{C}(\text{ICX}, \text{IScCX}),$$
$$\text{IScCV} \subseteq \mathcal{C}(\text{ICV}, \text{IScCV}).$$

(iv)

$$\text{ISupA}^C = \text{ISubA}, \qquad \text{ISS}^C = \text{IAntiSS},$$
$$\text{ICX}^C = \text{ICV}, \qquad \text{ILogCX(Log)}^C = \text{ILogCV(Log)},$$
$$\text{ICX(Log)}^C = \text{ILogCV}, \qquad \text{ILogCX}^C = \text{ICV(Log)}.$$

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PROOF: We show that ILogCX is closed under composition and ILogCV is not. The other results can be shown with fairly similar arguments. Suppose $f, g \in$ ILogCX, so $f, g \in$ ICX. We need to show that

$$\frac{f'(g(x))g'(x)}{f(g(x))}$$

is increasing, but this follows because $f'(g(x))/f(g(x))$ and $g'(x)$ are both increasing and positive. The same argument does not work for ILogCV because ICV $\Rightarrow$ ILogCV, not the other way around. For example, $f(x) = g(x) = e^x \in$ ILogCV, but $\exp(\exp(x))$ is not. ∎

We also have (refer to Lemma 2.4 (iii)) that ICV(Log) $\subseteq \mathcal{C}$(ILogCV, ILogCV(Log)), but this is weaker than saying ICV(Log) $\subseteq \mathcal{C}$(ILogCV) (from Lemma 2.4 (ii)); ICV(Log) $\subseteq \mathcal{C}$(ICV, ICV(Log)) is implied by the fact that ICV(Log) is closed under composition (Lemma 2.4 (i)).

2.3. Classes of Distributions Based on Aging

New Better (Worse) Than Used: A random variable Y (or its distribution function) is said to have the new better than used (NBU) property if $\bar{F}_Y(x)\bar{F}_Y(y) \geq \bar{F}_Y(x+y)$ for any $x, y \in \mathcal{R}_+$. Equivalently, $H_Y \in$ ISupA$_0$-∞. We denote this by $Y \in$ NBU. The new worse than used (NWU) property is similarly defined, with $H_Y \in$ ISubA-∞. (Note that $\lim_{x\to\infty} H_Y(x) = \lim_{x\to\infty}[-\log \bar{F}_Y(x)] = -\log(0) = \infty$. Also note that NWU random variables have infinite support.)

Increasing (Decreasing) Failure Rate on Average: A random variable Y (or its distribution function) is said to have the increasing failure rate on the average (IFRA) property, $Y \in$ IFRA, if $\bar{h}_Y(y)$ is increasing in y. Equivalently, $H_Y \in$ ISS$_0$-∞. The DFRA property is similarly defined, with $H_Y \in$ IAntiSS-∞.

Increasing (Decreasing) Failure Rate: A random variable Y is said to have the increasing failure rate (IFR) property, $Y \in IFR$, if $h_Y(y)$ is increasing in y. Equivalently, $H_Y \in$ ICX$_0$-∞. The DFR property is similarly defined, with $H_Y \in$ ICV-∞.

Increasing (Decreasing) Reverse Failure Rate: A random variable Y (or its distribution function) is said to have the increasing reverse failure rate (IRF) property, $Y \in$ IRF, if $r_Y(y)$ is increasing in y. The DRF property is similarly defined.

Increasing (Decreasing) Proportional Failure Rate: A random variable Y is said to have the increasing proportional failure rate (IPFR) property, $Y \in$ IPFR, if $l_Y(y)$ is increasing in y. Equivalently, $H_Y \in$ ICX(Log)-∞. The DPFR property is similarly defined, with $H_Y \in$ ICV(Log)$_0$-∞. From Lemma 2.3, for $Y \in$ DPFR, we must have $a_Y > 0$ (and $b_Y = \infty$). The IPFR property is studied, in the context

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of the hazard rate stochastic order, in Ma [11] and in Examples 2.A.6, 7.C.6, and 7.C.6a of Marshall and Olkin [13, pp. 54 and 226].

We now introduce two additional classes based on the class of functions ILogCX-∞ and ILogCX(Log)-∞, which will also (along with IFR) imply the IPFR property.

Increasing (Decreasing) Failure Rate Relative to Cumulative Hazard Rate: A random variable Y is said to have the increasing failure rate relative to cumulative hazard rate (IFR/C) property, $Y \in$ IFR/C, if $h_Y(y)/H_Y(y)$ is increasing in y (equivalently, $H_Y \in$ ILogCX-∞). The DFR/C property is similarly defined, with $H_Y \in$ ILogCV-∞. Log concavity of H_Y is mentioned in Sengupta and Nanda [17], who showed that $H_Y \in$ ILogCV-$\infty \Rightarrow F_Y \in$ ILogCV. From the remark after Lemma 2.2, $Y \in$ IFR/C $\Rightarrow a_Y = 0$ and $H_Y(0) > 0$.

Increasing (Decreasing) Failure Rate Relative to Average Hazard Rate: A random variable Y is said to have the increasing failure rate relative to average hazard rate (IFR/A) property, $Y \in$ IFR/A, if $h_Y(y)/\bar{h}_Y(y)$ is increasing in y (equivalently, $H_Y \in$ ILogCX(Log)-∞). The DFR/A property is similarly defined, with $H_Y \in$ ILogCV(Log)-∞.

Finally, we introduce a class based on the scaled hazard rate.

Increasing (Decreasing) Scaled Hazard Rate: A random variable Y is said to have the increasing scaled failure rate (ISFR) property, $Y \in$ ISFR, if $s_Y(y) = (b_Y - y)h_Y(y)$ is increasing in y (i.e. $H_Y \in$ IScCX^b). The DSFR property is similarly defined, with $H_Y \in$ IScCV^b.

2.4. Various Stochastic Orders

We say that X is greater than Y in the usual stochastic (st) sense, $X \geq_{\text{st}} Y$, if $\bar{F}_X(t) \geq \bar{F}_Y(t)$ for all t. Thus, when we say later that, for example, $Y_R(y)$ is increasing in the usual stochastic sense in y, we mean $Y_R(y_1) \leq_{\text{st}} Y_R(y_2)$ for all $y_1 \leq y_2$ (i.e., $\bar{F}_{Y_R(y)}(t)$ is increasing in y for all t). We use the notation $Y_R(y) \uparrow_{\text{st}}$ to mean $Y_R(y)$ is increasing in the usual stochastic sense in y.

We say that X is greater than Y in the hazard rate (hr) sense, $X \geq_{\text{hr}} Y$, if $h_X(t) \leq h_Y(t)$ for all t, so $Y_R(y)$ increasing in the hazard rate sense in y ($Y_R(y) \uparrow_{\text{hr}}$) means $h_{Y_R(y)}(t)$ is decreasing in y for all t. (This is at first confusing — a decreasing hazard rate corresponds to increasing in the hazard rate sense. This is because larger hazard rates are associated with stochastically smaller random variables.)

Similarly, X is greater than Y in the PFR (FR/C, FR/A, rh) sense if $l_X(t) \leq l_Y(t)$ ($h_X(t)/H_X(t) \leq h_Y(t)/H_Y(t), h_X(t)/\bar{h}_X(t) \leq h_Y(t)/\bar{h}_Y(t), r_X(t) \geq r_Y(t)$) for all t. Therefore, $Y_R(y)$ increasing in the PFR (FR/C, FR/A) sense in y (i.e., $Y_R(y) \uparrow_{\text{PFR}}$ ($Y_R(y) \uparrow_{\text{FR/C}}$, $Y_R(y) \uparrow_{\text{FR/A}}$)), means $l_{Y_R(y)}$ ($h_{Y_R(y)}(t)/H_{Y_R(y)}(t)$, $h_{Y_R(y)}(t)/\bar{h}_{Y_R(y)}(t)$) is decreasing in y for all t and $Y_R(y)$ increasing in the rh sense in y; that is, $Y_R(y) \uparrow_{\text{rh}}$, means $r_{Y_R(y)}(t)$ is increasing in y for all t.

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Finally, X is greater than Y in the likelihood ratio sense, $X \geq_{\rm lr} Y$, if $f_X(t)/f_Y(t)$ is increasing in t for $t \in [a_X, b_X] \cup [a_Y, b_Y]$. Additionally, a random variable Y is ILR (DLR) if $f_Y \in$ LogCV ($f_Y \in$ LogCX). It is well known that $X \geq_{\rm lr} Y \Rightarrow X \geq_{\rm hr} Y$ and $X \geq_{\rm rh} Y$, and that

$$Y \in \text{DLR} \Leftrightarrow Y_R(y) \uparrow_{\rm lr} \Rightarrow Y_R(y) \uparrow_{\rm hr} \text{ and } Y_R(y) \uparrow_{\rm rh},$$

$$Y \in \text{ILR} \Leftrightarrow Y_R(y) \downarrow_{\rm lr} \Rightarrow Y_R(y) \downarrow_{\rm hr} \text{ and } Y_R(y) \downarrow_{\rm rh}.$$

The following lemma is immediate.

LEMMA 2.5: *$X \geq_{\rm hr} Y \Leftrightarrow X \geq_{\rm PFR} Y$ and $X \geq_{\rm FR/C} Y \Leftrightarrow X \geq_{\rm FR/A} Y$.*

3. RELATING CLASSES OF DISTRIBUTIONS

It is well known and easy to show that

$$\text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU},$$

$$\text{DFR} \Rightarrow \text{DFRA} \Rightarrow \text{NWU},$$

$$\text{ILR} \Rightarrow \text{IFR} \Rightarrow \text{IPFR}, \text{ILR} \Rightarrow \text{IRF}, \text{DLR} \Rightarrow \text{DFR}, \text{DLR} \Rightarrow \text{DRF},$$

where, for instance, by IFR $\Rightarrow$ IFRA, we mean $Y \in$ IFR $\Rightarrow Y \in$ IFRA (i.e., IFR $\subseteq$ IFRA). It is also possible to have random variables that are both IPFR and DFR (e.g., the Weibull and gamma distributions with shape parameter ≤ 1).

We extend these relationships for our new classes of distributions.

LEMMA 3.1:

$$\begin{array}{lllll}
\text{IFR/C} & \Rightarrow & \text{IFR/A} & \Rightarrow & \text{IPFR}, \\
\text{IFR/C} & \Rightarrow & \text{IFR} & \Rightarrow & \text{IPFR}, \\
\text{ISFR} & \Rightarrow & \text{IFR}, & & \\
\text{DPFR} & \Rightarrow & \text{DFR/A} & \Rightarrow & \text{DFR/C}, \\
\text{DPFR} & \Rightarrow & \text{DFR} & \Rightarrow & \text{DFR/C}, \\
 & & \text{DFR} & \Rightarrow & \text{DSFR},
\end{array}$$

PROOF: The result follows from Lemma 2.2 and from the cumulative hazard function characterizations of aging properties, for example, $Y \in$ IFR/A $\Leftrightarrow H_Y \in$ ILogCX(Log)-∞, $Y \in$ IPFR $\Leftrightarrow H_Y \in$ ICX(Log)-∞, and so forth. ■

Since $H_Y \in$ ILogCV-$\infty \Rightarrow F_Y \in$ LogCV $\Leftrightarrow Y \in$ DRF [17], we also have DFR/C $\Rightarrow$ DRF.

4. AGING PROPERTIES OF RESIDUAL AND CONDITIONAL LIFETIMES

Properties for a random variable Y also have implications for the random variables introduced earlier, such as the residual life of Y, the scaled conditional life, the scaled residual life, and so forth.

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4.1. Residual Life

First, let us consider the residual life, $Y_R(y) \stackrel{d}{=} \{Y - y | Y > y\}$, with $a_Y \le y < b_Y$. Then, for $0 < t < b_Y - y$,

$$H_{Y_R(y)}(t) = -\log \bar{F}_{Y_R(y)}(t) = -\log \bar{F}_Y(y+t) + \log \bar{F}_Y(y) = H_Y(y+t) - H_Y(y),$$
$$h_{Y_R(y)}(t) = h_Y(y+t).$$

We have the following lemma. Part (i) and most of part (iv) are well known and are included for completeness. The new part of (iv), $Y \in$ IFR (DFR) $\Leftrightarrow Y_R(y) \downarrow_{\text{PFR}}$ ($Y_R(y) \uparrow_{\text{PFR}}$), is easy to show. Note that we cannot have $Y_R(y) \in$ DPFR from Lemma 2.3 because $a_{Y_R(y)} = 0$.

LEMMA 4.1:

(i) $Y \in \mathcal{S} \Leftrightarrow Y_R(y) \in \mathcal{S}\ \forall y \in [a_Y, b_Y)$ *for* $\mathcal{S} =$ IFR, DFR, ILR, DLR.

(ii) $Y \in$ IPFR $\Rightarrow Y_R(y) \in$ IPFR $\forall y \in [a_Y, b_Y)$, *and if* $a_Y = 0$, $Y \in$ IPFR $\Leftrightarrow Y_R(y) \in$ IPFR $\forall y \in [a_Y, b_Y)$.

(iii) $Y_R(y) \uparrow_{\text{FR/C}}$ ($Y_R(y) \downarrow_{\text{FR/C}}$) $\Leftrightarrow Y_R(y) \uparrow_{\text{FR/A}}$ ($Y_R(y) \downarrow_{\text{FR/A}}$), $y \in [a_Y, b_Y)$.

(iv) $Y \in$ IFR(DFR) $\Leftrightarrow Y_R(y) \downarrow_{\text{hr}}$ ($Y_R(y) \uparrow_{\text{hr}}$) $\Leftrightarrow Y_R(y) \downarrow_{\text{PFR}}$ ($Y_R(y) \uparrow_{\text{PFR}}$) $\Leftrightarrow Y_R(y) \downarrow_{\text{st}}$ ($Y_R(y) \uparrow_{\text{st}}$).

PROOF: Part (iii) is immediate from Lemma 2.5.

For part (ii), suppose $Y \in$ IPFR; that is, $xh_Y(x)$ is increasing in x (i.e., $h_Y(x) + xh'_Y(x) \ge 0$ for all $x \in [a_Y, b_Y)$). (To keep the arguments simple, we assume the cumulative hazard function is twice differentiable; that is, the hazard rate function is differentiable. The extension to the nondifferentiable case is straightforward.) Then $h_Y(t+y) + (t+y)h'_Y(t+y) = h_{Y_R(y)}(t) + th'_{Y_R(y)}(t) + yh'_Y(t+y) \ge 0$ for all t and y such that $t + y \in [a_Y, b_Y)$. If $h'_Y(t+y) \le 0$, then $h_{Y_R(y)}(t) + th'_{Y_R(y)}(t) \ge h_{Y_R(y)}(t) + th'_{Y_R(y)}(t) + yh'_Y(t+y) \ge 0$. If $h'_Y(t+y) \ge 0$, so $h'_{Y_R(y)}(t) \ge 0$, then $h_{Y_R(y)}(t) + th'_{Y_R(y)}(t) \ge 0$ because all of the terms are positive. Therefore, $th_{Y_R(y)}(t)$ is increasing in t for all $y \in [a_Y, b_Y)$ and $t \in [0, b_Y - y)$.

The reverse implication when $a_Y = 0$ is immediate. ■

As noted in Lemma 4.1(iv), $Y_R(y) \downarrow_{\text{st}}$ ($Y_R(y) \uparrow_{\text{st}}$) $\Rightarrow Y_R(y) \downarrow_{\text{hr}}$ ($Y_R(y) \uparrow_{\text{hr}}$), even though hazard rate ordering is stronger than stochastic ordering. We now show a similar result: that $Y_R(y) \downarrow_{\text{rh}}$ ($Y_R(y) \uparrow_{\text{rh}}$) $\Rightarrow Y_R(y) \downarrow_{\text{lr}}$ ($Y_R(y) \uparrow_{\text{lr}}$), even though likelihood ratio ordering is stronger than reverse hazard ordering. So, for example, $Y_R(y) \downarrow_{\text{rh}}$ ($Y_R(y) \uparrow_{\text{rh}}$) $\Rightarrow Y_R(y) \downarrow_{\text{hr}}$ ($Y_R(y) \uparrow_{\text{hr}}$), although the reverse is not true. Also note that although

$$Y \in \text{ILR} \Leftrightarrow Y_R(y) \in \text{ILR} \Leftrightarrow Y_R(y) \downarrow_{\text{lr}},$$
$$Y \in \text{IFR} \Leftrightarrow Y_R(y) \in \text{IFR} \Leftrightarrow Y_R(y) \downarrow_{\text{hr}},$$

we do not have the same relationships for the reverse hazard rate: $Y \in$ IRF $\not\Leftrightarrow Y_R(y) \in$ IRF and $Y \in$ IRF $\not\Leftrightarrow Y_R(y) \uparrow_{\text{rh}}$. (This is because $Y_R(y)$ is "forward looking" and the reverse hazard rate is "backward looking." $Y_S(y)$ is backward looking; see Lemma 4.3.) Because $Y \in \text{ILR} \Rightarrow Y \in \text{DRF}$, a consequence of the following lemma is that $Y_R(y) \downarrow_{\text{rh}} \Rightarrow Y \in \text{DRF}$ even though the reverse is not true.

LEMMA 4.2: *Suppose $b_Y = \infty$. Then* $Y_R(y) \uparrow_{\text{rh}}$ $(Y_R(y) \downarrow_{\text{rh}}) \Leftrightarrow Y_R(y) \uparrow_{\text{lr}}$ $(Y_R(y) \downarrow_{\text{lr}})$ $(\Leftrightarrow Y \in \text{DLR(ILR)})$.

PROOF: From [18] we have that $Y_R(y) \downarrow_{\text{lr}}$ is equivalent to saying that for all α, u, β : $0 \le \alpha \le u \le \beta < \infty$ and for all $y_1, y_2 : a_Y \le y_1 \le y_2 < \infty$, we have

$$P\{Y_R(y_1) > u | Y_R(y_1) \in [\alpha, \beta]\} \ge P\{Y_R(y_2) > u | Y_R(y_2) \in [\alpha, \beta]\}$$

and this inequality is equivalent to

$$\begin{aligned} P\{Y_R(y_1+\alpha) > u-\alpha | Y_R(y_1+\alpha) \le \beta-\alpha\} &\ge P\{Y_R(y_2+\alpha) \\ > u-\alpha | Y_R(y_2+\alpha) \le \beta-\alpha\}. \end{aligned} \tag{1}$$

Additionally, $Y_R(y) \downarrow_{\text{rh}}$ is equivalent to saying that for all $u', \beta' : 0 \le u' \le \beta' < \infty$ and for all $y_1', y_2' : a_Y \le y_1' \le y_2' < \infty$, we have

$$P\{Y_R(y_1') > u' | Y_R(y_1') \le \beta'\} \ge P\{Y_R(y_2') > u' | Y_R(y_2') \le \beta'\}. \tag{2}$$

That $Y_R(y) \downarrow_{\text{lr}} \Rightarrow Y_R(y) \downarrow_{\text{rh}}$ follows immediately with $\alpha = 0, u = u', \beta = \beta', y_1 = y_1'$, and $y_2 = y_2'$. Suppose $Y_R(y) \downarrow_{\text{rh}}$ and fix any α, u, β, y_1, and y_2 such that $0 \le \alpha \le u \le \beta < \infty$ and $a_Y \le y_1 \le y_2 < \infty$, and let $u' = u - \alpha$, $\beta' = \beta - \alpha$, $y_1' = y_1 + \alpha$, and $y_2' = y_2 + \alpha$. Note that $0 \le u' \le \beta' < \infty$ and $y_2' \ge y_1' \ge a_Y$, so Eq. (1) follows from Eq. (2). ■

Recall that the conditional shortfall is $Y_S(y) \stackrel{d}{=} \{y - Y | Y \le y\}$, with $a_Y < y < b_Y$, and $h_{Y_S(y)}(t) = r_Y(y-t)$. It is well known and easy to see the following lemma holds.

LEMMA 4.3: *For $a_Y < y < b_Y$, $Y \in$* IRF (DRF) $\Leftrightarrow Y_S(y) \in$ DFR (IFR) $\Leftrightarrow Y_S(y) \downarrow_{\text{hr}}$ $(Y_S(y) \uparrow_{\text{hr}})$.

4.2. Scaled Conditional Life

Now we consider the scaled conditional life, $Y_{SC}(y) \stackrel{d}{=} (1/y)\{Y | Y > y\}$, with $a_Y < y < b_Y$. For $t \ge 1$,

$$\begin{aligned} H_{Y_{SC}(y)}(t) &= -\log \bar{F}_{Y_{SC}(y)}(t) = -\log \bar{F}_Y(yt) + \log \bar{F}_Y(y) = H_Y(yt) - H_Y(y), \\ h_{Y_{SC}(y)}(t) &= y h_Y(yt), \\ l_{Y_{SC}(y)}(t) &= t h_{Y_{SC}(y)}(t) = yt h_Y(yt) = l_Y(yt), \end{aligned}$$

and for $0 \le t < 1$, $H_{Y_{SC}(y)}(t) = h_{Y_{SC}(y)}(t) = l_{Y_{SC}(y)}(t) = 0$.

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LEMMA 4.4: *For $a_Y < y < b_Y$, we have the following:*

(i) $Y \in \mathcal{S} \Leftrightarrow Y_{SC}(y) \in \mathcal{S}$ *for* $\mathcal{S} =$ IFR, IPFR.

(ii) $Y_{SC}(y) \in$ IFR/A $\forall y \in [a_Y, b_y) \Rightarrow Y \in$ IFR/A.

(iii) $Y_{SC}(y) \uparrow_{\mathrm{FR/C}}$ $(Y_{SC(y)} \downarrow_{\mathrm{FR/C}}) \Leftrightarrow Y_{SC}(y) \uparrow_{\mathrm{FR/A}}$ $(Y_{SC}(y) \downarrow_{\mathrm{FR/A}})$.

(iv) $Y \in$ IPFR (DPFR) $\Leftrightarrow Y_{SC}(y) \downarrow_{\mathrm{hr}}$ $(Y_{SC}(y) \uparrow_{\mathrm{hr}})$.

(v) $Y_{SC}(y) \in$ IFR/A $\Rightarrow Y_{SC}(y) \downarrow_{\mathrm{FR/A}}$ *(or* $Y_{SC}(y) \downarrow_{\mathrm{FR/C}}$).

PROOF: Part (i) follows immediately from $h_{Y_{SC}(y)}(t) = yh_Y(yt)$ and $l_{Y_{SC}(y)}(t) = l_Y(yt)$, and part (iii) follows from Lemma 2.5. Part (iv) is also immediate, because the fact that $h_{Y_{SC}(y)}(t) = yh_Y(yt) = (1/t)yth_Y(yt)$ is increasing (decreasing) in y for all $t > 0$ $\Leftrightarrow l_Y(x) = xh_Y(x)$ is increasing (decreasing) in x (i.e., $Y \in$ IPFR (DPFR)).

For part (ii), suppose $Y_{SC}(y) \in$ IFR/A (and $a_Y > 0$; otherwise the result is immediate); that is, $H_{Y_{SC}(y)} \in$ ILogCX(Log) or $th_{Y_{SC}(y)}(t)/H_{Y_{SC}(y)}(t) = yth_Y(yt)/[H_Y(yt) - H_Y(y)]$ is increasing in t, so

$$[yh_Y(ty) + ty^2h'_Y(ty)][H_Y(ty) - H_Y(y)] \geq ty^2[h_Y(ty)]^2 \ \forall y \in (a_Y,\ b_Y), t \in (1,\ b_Y/y).$$

From this, we have $[h_Y(x) + xh'_Y(x)] \geq 0$ and $[h_Y(x) + xh'_Y(x)]H_Y(x) \geq x[h_Y(x)]^2$ for all $x \in (a_Y, b_Y)$ (i.e., $H_Y \in$ ILogCX(Log)), so we have (ii). We also have that $th_{Y_{SC}(y)}(t)/H_{Y_{SC}(y)}(t) = yth_Y(yt)/[H_Y(yt) - H_Y(y)]$ is increasing in y because

$$[th_Y(ty) + t^2yh'_Y(ty)][H_Y(ty) - H_Y(y)] \geq t^2y[h_Y(ty)]^2 - yth_Y(yt)h_Y(y),$$

so the first part of (v) follows. The second part of (v) is immediate from Lemma 2.5. ■

4.3. Scaled Residual Life

Now let us consider the scaled residual life, $Y_{SR}(y) \stackrel{d}{=} \{Y - y|Y > y\}/b_Y - y$, and the scaled hazard rate, $s_Y(y) = (b_Y - y)h_Y(y)$, with $a_Y < y < b_Y < \infty$. We have, for $0 \leq t \leq 1$,

$$\begin{aligned}
H_{Y_{SR}(y)}(t) &= -\log\, \bar{F}_{Y_{SR}(y)} = -\log\, \bar{F}_Y(y + t(b_Y - y)) + \log\, \bar{F}_Y(y) \\
&= H_Y(g(y,t)) - H_Y(y), \\
h_{Y_{SR}(y)}(t) &= (b_Y - y)h_Y(y + t(b_Y - y)) = (b_Y - y)h_Y(g(y,t)), \\
h'_{Y_{SR}(y)}(t) &= (b_Y - y)^2 h_Y(y + t(b_Y - y)), \\
\frac{d}{dy} h_{Y_{SR}(y)}(t) &= (b_Y - y)(1-t)h'_Y(b_Y t + (1-t)y) - h_Y(b_Y t + (1-t)y) \\
&= (b_Y - g(y,t))h'_Y(g(y,t)) - h_Y(g(y,t)) = s'_Y(g(y,t)),
\end{aligned}$$

with $g(y,t) = b_Y t + (1-t)y = y + t(b_Y - y)$, which is positive and increasing in both y and t. We have the following lemma.

LEMMA 4.5: *For $a_Y \le y < b_Y$, we have the following:*

(i) $Y \in \text{IFR} \Leftrightarrow Y_{SR}(y) \in \text{IFR}$.

(ii) $Y \in \text{IPFR} \Rightarrow Y_{SR}(y) \in \text{IPFR}\ \ \forall y \in [a_Y, b_Y)$; *and if* $a_Y = 0$, $Y \in \text{IPFR} \Leftrightarrow Y_{SR}(y) \in \text{IPFR}\ \forall y \in [a_Y, b_Y)$.

(iii) $Y \in \text{ISFR(DSFR)} \Rightarrow Y_{SR}(y) \downarrow_{\text{hr}}\ (Y_{SR}(y) \uparrow_{\text{hr}})$.

PROOF: Part (i) is immediate because $h_{Y_{SR}(y)}(t) = (b_Y - y)h_Y(g(y,t))$.

For (ii), suppose $Y \in \text{IPFR}$ (i.e., $xh_Y(x)$ is increasing in x); that is, $h_Y(x) + xh'_Y(x) \ge 0$ for all $x \in [a_Y, b_Y)$. Then $(b_Y - y)h_Y(y + t(b_Y - y)) + (b_Y - y)(y + t(b_Y - y))h'_Y(y + t(b_Y - y)) = h_{Y_{SR}(y)}(t) + th'_{Y_{SR}(y)}(t) + y(b_Y - y)h'_Y(y + t(b_Y - y)) \ge 0$ for all t and y such that $y + t(b_Y - y) \in [a_Y, b_Y)$. If $h'_Y(y + b_Y(t - y)) \le 0$, then $h_{Y_{SR}(y)}(t) + th'_{Y_{SR}(y)}(t) \ge h_{Y_{SR}(y)}(t) + th'_{Y_{SR}(y)}(t) + y(b_Y - y)h'_Y(y + t(b_Y - y)) \ge 0$. If $h'_Y(y + b_Y(t - y)) \ge 0$, so $h'_{Y_{SR}(y)}(t) \ge 0$, then $h_{Y_{SR}(y)}(t) + th'_{Y_{SR}(y)}(t) \ge 0$ because all of the terms are positive. Therefore, $th_{Y_{SR}(y)}(t)$ is increasing in t for all $y \in [a_Y, b_Y)$ and $t \in [0, b_Y - y)$. The reverse implication when $a_Y = 0$ is immediate.

Part (iii) is immediate from $dh_{Y_{SR}(y)}(t)/dy = s'_Y(g(y,t))$. ■

5. RELATING THE AGING PROPERTIES OF THE INTRINSIC LIFE AND THE ACTUAL LIFE

Let X be a nonnegative absolutely continuous random variable representing the intrinsic life of a reliability system. The actual lifetime T of this system will depend on how the intrinsic age is accumulated over the calendar time. Suppose the intrinsic age of the system at time t is $\phi(t)$ (with ϕ increasing). Then

$$T = \inf\{t : \phi(t) \ge X; t \in \mathcal{R}_+\} =: \phi^{-1}(X)$$

and $X = \phi(T)$.

The following results are well known.

COROLLARY 5.1:

$$\begin{aligned}
X \in \text{NBU} \quad &\Leftrightarrow \quad T \in \text{NBU}\ \textit{for all}\ \phi \in \text{ISupA},\\
X \in \text{NWU} \quad &\Leftrightarrow \quad T \in \text{NWU}\ \textit{for all}\ \phi \in \text{ISubA},\\
X \in \text{IFRA} \quad &\Leftrightarrow \quad T \in \text{IFRA}\ \textit{for all}\ \phi \in \text{ISS},\\
X \in \text{DFRA} \quad &\Leftrightarrow \quad T \in \text{DFRA}\ \textit{for all}\ \phi \in \text{IAntiSS},\\
X \in \text{IFR} \quad &\Leftrightarrow \quad T \in \text{IFR}\ \textit{for all}\ \phi \in \text{ICX},\\
X \in \text{DFR} \quad &\Leftrightarrow \quad T \in \text{DFR}\ \textit{for all}\ \phi \in \text{ICV}.
\end{aligned}$$

We give a general result for which the above is a special case, as well as new results for IPFR random variables and for our new notions of aging.

It is easy to show the following key lemma.

LEMMA 5.2: $H_T(t) = H_X(\phi(t))$.

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For a class of functions $\mathcal{B}$, let $\mathcal{F}_{\mathcal{B}}$ be the family of random variables Y such that $H_Y \in \mathcal{B}$. For example, $\mathcal{F}_{\text{ICX(Log)}} = \text{IPFR}$. Let I be the identity function $I(x) = x$ for all x. The next theorem is immediate from the definition of compatible functions (Section 2.2) and Lemma 5.2.

THEOREM 5.3: *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two classes of functions and let $\mathcal{A} \subseteq \mathcal{C}(B_1, \mathcal{B}_2)$.*

(i) $X \in \mathcal{F}_{\mathcal{B}_1} \Rightarrow T \in \mathcal{F}_{\mathcal{B}_2}$ for all $\phi \in \mathcal{A}$ (i.e., $\phi^{-1}(X) \in \mathcal{F}_{\mathcal{B}_2}$ for all $\phi^{-1} \in \mathcal{A}^C$).

(ii) If $I \in \mathcal{A}$, then

$$X \in \mathcal{F}_{\mathcal{B}_1} \Leftrightarrow T \in \mathcal{F}_{\mathcal{B}_2} \quad \textit{for all } \phi \in \mathcal{A}$$
$$(\textit{i.e.}, \phi^{-1}(X) \in \mathcal{F}_{\mathcal{B}_2} \textit{ for all } \phi^{-1} \in \mathcal{A}^C).$$

(iii) If $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$ (so $\mathcal{C}(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{C}(\mathcal{B})$) and if $I \in \mathcal{A}$, then

$$X \in \mathcal{F}_{\mathcal{B}} \Leftrightarrow T \in \mathcal{F}_{\mathcal{B}} \quad \textit{for all } \phi \in \mathcal{A} \ (\textit{i.e.}, \phi^{-1}(X) \in \mathcal{F}_{\mathcal{B}} \textit{ for all } \phi^{-1} \in \mathcal{A}^C).$$

From Lemmas 2.4(ii) and 2.4(iii) we have the following corollary.

COROLLARY 5.4:

$$\begin{array}{lcl}
X \in \text{IPFR} & \Leftrightarrow & T \in \text{IPFR } \textit{for all } \phi \in \text{ILogCX(Log)}, \\
X \in \text{DPFR} & \Leftrightarrow & T \in \text{DPFR } \textit{for all } \phi \in \text{ILogCV(Log)}, \\
X \in \text{IFR/C} & \Leftrightarrow & T \in \text{IFR/C } \textit{for all } \phi \in \text{ICX}, \\
X \in \text{DFR/C} & \Leftrightarrow & T \in \text{DFR/C } \textit{for all } \phi \in \text{ICV}, \\
X \in \text{IFR} & \Rightarrow & T \in \text{IPFR } \textit{for all } \phi \in \text{ICX(Log)}, \\
X \in \text{IFR/C} & \Rightarrow & T \in \text{IFR/A } \textit{for all } \phi \in \text{ICX(Log)}, \\
X \in \text{IFR} & \Rightarrow & T \in \text{ISFR } \textit{for all } \phi \in \text{IScCX}, \\
X \in \text{DFR} & \Rightarrow & T \in \text{DSFR } \textit{for all } \phi \in \text{IScCV}-\infty.
\end{array}$$

The following is an immediate corollary of Theorem 5.3.

COROLLARY 5.5: *If $\mathcal{B}$ is closed under composition, then*

$$X \in \mathcal{F}_{\mathcal{B}} \Rightarrow T \in \mathcal{F}_{\mathcal{B}} \quad \textit{for all } \phi \in \mathcal{B} \ (\textit{i.e.}, \phi^{-1}(X) \in \mathcal{F}_{\mathcal{B}} \text{ for all } \phi^{-1} \in \mathcal{B}^C).$$

If also $I \in \mathcal{B}$, then

$$X \in \mathcal{F}_{\mathcal{B}} \Leftrightarrow T \in \mathcal{F}_{\mathcal{B}} \quad \textit{for all } \phi \in \mathcal{B} \ (\textit{i.e.}, \phi^{-1}(X) \in \mathcal{F}_{\mathcal{B}} \textit{ for all } \phi^{-1} \in \mathcal{B}^C).$$

Corollary 5.1 then follows, along with Corollary 5.6, using Lemma 2.4(i).

COROLLARY 5.6:

$$\begin{array}{lcl}
X \in \text{IFR/A} & \Leftrightarrow & T \in \text{IFR/A } \textit{for all } \phi \in \text{ILogCX(Log)}, \\
X \in \text{DFR/A} & \Leftrightarrow & T \in \text{DFR/A } \textit{for all } \phi \in \text{ILogCV(Log)}, \\
X \in \text{IFR/C} & \Leftrightarrow & T \in \text{IFR/C } \textit{for all } \phi \in \text{ILogCX}, \\
X \in \text{DFR/C} & \Leftrightarrow & T \in \text{DFR/C } \textit{for all } \phi \in \text{ILogCV}.
\end{array}$$

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The results of [14] (the first two parts below) are also a corollary of Corollary 5.4.

COROLLARY 5.7:

(i) $X \in \text{IPFR} \Leftrightarrow X^a \in \text{IPFR}$ *for all* $a \geq 0$.

(ii) $X \in \text{IPFR} \Leftrightarrow aX \in \text{IPFR}$ *for all* $a \geq 0$.

(iii) $X \in \text{IPFR} \Rightarrow \log_a X \in \text{IPFR}$ *for all* $a \geq 1$.

(iv) $X \in \text{IPFR} \Leftrightarrow \exp\{(\log X)^a\} \in \text{IPFR}$ *for all* $0 \leq a \leq 1$.

PROOF: For (i) we use Theorem 5.3 with $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B} = \text{ICX(Log)}$ and $\mathcal{A} = \{\phi : \phi(x) = x^{1/a}, a \geq 0\} = \{\phi : \phi(x) = x^a, a \geq 0\} \subseteq \text{ILogCX(Log)}$. The other parts follow similarly. ■

Lariviere [9] showed that $X \in \text{IPFR} \Leftrightarrow \log(X) \in \text{IFR}$. This is a special case of the following theorem, which follows from Theorem 5.3.

THEOREM 5.8: *For two classes of functions* $\mathcal{B}_1$ *and* $\mathcal{B}_2$, *if* $\phi \in \mathcal{C}(B_1, \mathcal{B}_2)$ *and* $\phi^{-1} \in \mathcal{C}(B_2, \mathcal{B}_1)$, *then*

$$X \in \mathcal{F}_{\mathcal{B}_1} \Leftrightarrow T \in \mathcal{F}_{\mathcal{B}_2}.$$

COROLLARY 5.9:

(i) $X \in \text{IPFR} \Leftrightarrow \log(X) \in \text{IFR}$.

(ii) $X \in \text{IFR/A} \Leftrightarrow \log(X) \in \text{IFR/C}$.

PROOF: Let $\phi(x) = e^x$ for $x \geq 0$ (so the system immediately ages to 1 when it is put into operation) and $\phi^{-1}(x) = 0$ for $0 \leq x \leq 1$, $\phi^{-1}(x) = \log(x)$ for $1 \leq x < \infty$. It is easy to check that $\phi(x) \in \mathcal{C}(\text{ILogCX(Log)}, \text{ILogCX})$ and $\phi^{-1}(x) \in \mathcal{C}(\text{ILogCX}, \text{ILogCX(Log)})$. ■

6. GENERATING CLASSES OF RANDOM VARIABLES

Our approach also shows that we can characterize classes of random variables in terms of functions of exponential random variables.

THEOREM 6.1: *For a class of functions* $\mathcal{B}$,

$$X \in \mathcal{F}_B \Leftrightarrow X = \psi(Z) \quad \textit{for some } \psi \in \mathcal{B}^C,$$

where Z *is exponentially distributed with rate* 1, $Z \sim \exp(1)$.

PROOF: This follows from Lemmas 5.2 and 2.4(iv), with $\phi = \psi^{-1} = H_X \in \mathcal{B}$, and the fact that H_Z is the identity function, $H_Z(t) = I(t) = t$. ■

COROLLARY 6.2: *Let* $Z \sim \exp(1)$. *Then we have the following:*

$$
\begin{array}{lll}
X \in \text{NBU} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ISubA}, \\
X \in \text{NWU} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ISupA}, \\
X \in \text{IFRA} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{IAntiSS}, \\
X \in \text{DFRA} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ISS}, \\
X \in \text{IFR} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ICV}, \\
X \in \text{DFR} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ICX}, \\
X \in \text{IPFR} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ILogCV}, \\
X \in \text{DPFR} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ILogCX}, \\
X \in \text{IFR/C} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ICV(Log)}, \\
X \in \text{DFR/C} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ICX(Log)}, \\
X \in \text{IFR/A} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ILogCV(Log)}, \\
X \in \text{DFR/A} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ILogCX(Log)}.
\end{array}
$$

To characterize IPFR and DPFR random variables, it is more natural, by an observation of [19], to use $Z \sim \text{Pareto}(1,1) \in \text{IPFR} \cap \text{DPFR}$, which has $l_Z(t) = 1$ for $t \geq 1 = a_Z$, and $H_Z(t) = \log(t)$ for $t \geq 1$, $H_Z(t) = 0$ for $0 \leq t \leq 1$. Note that exponential random variables are not DPFR. Also note that $Z = e^X$, where $X \sim \exp(1)$, which is consistent with Corollary 6.2, because $e^x \in \text{ILogCX} \cap \text{ILogCV}$.

COROLLARY 6.3: *Let* $Z \sim \text{Pareto}(1,1)$. *Then we have the following:*

$$
\begin{array}{lll}
X \in \text{IPFR} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ILogCV(Log)}, \\
X \in \text{DPFR} & \Leftrightarrow & X = \psi(Z) \textit{ for some } \psi \in \text{ILogCX(Log)}.
\end{array}
$$

PROOF: This follows from Lemma 5.2, with $\phi(t) = \psi^{-1}(t) = e^{H_X(t)}$, $t \geq 0$, and $H_Z(t) = \log(t)$, $t \geq 1$. ■

Note that the scaled conditional life of a Pareto random variable is also Pareto, $Z_{SC}(x) \stackrel{d}{=} (1/x)\{Z|Z > x\} \sim \text{Pareto}(1,1)$. This is analogous to the fact that the residual life of an exponential random variable is exponential. As noted earlier, from Lemma 2.3, for $Y \in \text{DPFR}$, $a_Y > 0$ (and $b_Y = \infty$), which is true for the Pareto random variable.

Now consider IFR/C and DFR/C random variables. Because $\text{ILogCX}_0\text{-}\infty = \emptyset$ from Lemma 2.2, we know that for an IFR/C random variable X, we must have $H_X(0) > 0$. Thus, exponential random variables are not IFR/C. Let Z be defined by $H_Z(z) = e^z$ for $z \geq 0$; that is, $F_Z(0) = P\{Z = 0\} = 1 - 1/e$ and $h(z) = e^z$ for $z > 0$. Thus, Z is a mixture of the constant 0 and a continuous random variable, with $f(z) = e^z e^{-e^z}$ for $z > 0$. Then $Z \in \text{IFR/C} \cap \text{DFR/C}$. Note that $Z = 0I\{X < 1\} + [\log(X)|X \geq 1]$ where $X \sim \exp(1)$, which is consistent with Corollary 6.2, because $\log(x), x \geq 1 \in \text{ICX(Log)} \cap \text{ICV(Log)}$.

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COROLLARY 6.4: *Let Z be such that $H_Z(z) = e^z$ for $z \geq 0$. Then we have the following:*

$$X \in \mathrm{IFR/C} \quad \Leftrightarrow \quad X = \psi(Z) \text{ for some } \psi \in \mathrm{ICV},$$
$$X \in \mathrm{DFR/C} \quad \Leftrightarrow \quad X = \psi(Z) \text{ for some } \psi \in \mathrm{ICX}.$$

PROOF: This follows from Lemma 5.2, with $\phi(t) = \psi^{-1}(t) = \log(H_X(t)), t \geq 0$. ■

Finally, let us consider ISFR and DSFR random variables. Such random variables have finite support. Let $U \sim \mathrm{unif}(0, 1)$. Then $s_U(t) = (1-t)h_U(t) = (1-t)/(1-t) = 1$, so $U \in \mathrm{ISFR} \cap \mathrm{DSFR}$. Also note that the scaled residual life for a uniform random variable is again uniform, $U_{SR}(u) \stackrel{\mathrm{d}}{=} \{U - u | U > u\}/(1-u) \sim \mathrm{unif}(0, 1)$.

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