Math. Struct. in Comp. Science (2006), *vol.* 16, *pp.* 223–229. © 2006 Cambridge University Press
doi:10.1017/S0960129506005184 Printed in the United Kingdom

Quotients of countably based spaces are not closed under sobrification

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Received 4 September 2005; revised 22 October 2005

Dedicated to Klaus Keimel on the occasion of his 65th birthday

In this note we show that *quotients of countably based spaces* (qcb spaces) and *topological predomains*, as introduced by M. Schröder and A. Simpson, are not closed under sobrification. As a consequence, replete topological predomains need not be sober, that is, in general, repletion is not given by sobrification. Our counterexample also shows that a certain tentative 'equaliser construction' of repletion fails for qcb spaces.
Our results also extend to the more general class of core compactly generated spaces.

1. Background

In Schröder (2003) and Simpson (2003) M. Schröder and A. Simpson introduced the categories **QCB** (quotients of countably based spaces) and **PreDom** (topological predomains) as frameworks for denotational semantics that also contain most classical spaces such as, for example, separable Banach spaces. One easily shows that if a T_0 space is a quotient of a countably based space, then it can also be obtained as the quotient of a countably based T_0 space, that is, as a quotient of a subspace of Scott's $\mathscr{P}\omega$. As from both the topological and the semantical point of view it is reasonable to restrict attention to T_0 spaces, we will do so in the rest of the paper. Thus **QCB** is defined as the category whose objects are T_0 quotients of countably based T_0 spaces and whose morphisms are the continuous maps. Subsequently, we refer to the objects of **QCB** as *qcb spaces*.

In Schröder (2003), qcb spaces were characterised as those sequential T_0 spaces X for which there exists a *countable pseudobase*, that is, a countable subset $\mathscr{B}$ of $\mathscr{P}(X)$ such that for every sequence (x_n) converging to x and open neighbourhood U of x, there exists a $B \in \mathscr{B}$ with $x \in B \subseteq U$ and x_n eventually in B.

It was shown in Menni and Simpson (2002) and Schröder (2003) that **QCB** is cartesian closed. As **QCB** contains the Sierpiński space Σ (with underlying set $\{\bot, \top\}$ and $\{\bot\}$ as its only non-open subset), the open subsets of X may be identified with **QCB**-morphisms from X to Σ. It has been shown by J. Lawson (see Escardó *et al.* (2004, Theorem 4.7))

[†] Research partially supported by NSF grant DMS-0405216.
[‡] Research partially supported by EC Working Group APPSEM-II IST-2001-38957.

that the exponential Σ^X in **QCB** is homeomorphic to $(\mathcal{O}(X), \subseteq)$ endowed with its Scott topology. Accordingly, from now on we will use $\mathcal{O}$ to denote the covariant functor $\Sigma^{(-)}$. Note that $\mathcal{O}(f) = f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ for continuous maps $f : X \to Y$.

It was observed in Menni and Simpson (2002) and Simpson (2003) that **QCB** is equivalent (see Battenfeld (2004) for a proof) to the category **ExPer**($\mathscr{P}\omega$) of *extensional pers* over Scott's graph model $\mathscr{P}\omega$ (see Phoa (1992) for a definition and discussion of **ExPer**($\mathscr{P}\omega$)). This equivalence provides further evidence for the naturalness of the notion of qcb space and has the consequence that qcb spaces form a model of polymorphic λ-calculus.

Since qcb spaces do not in general enjoy the completeness properties required for denotational semantics, A. Simpson, in Simpson (2003), introduced the notion of a *topological predomain*, that is, a qcb space X that has suprema of ω-chains with respect to its information ordering $\sqsubseteq$ and in which all open sets are also Scott open (with respect to $\sqsubseteq$). It was stated in Simpson (2003) that topological predomains also have arbitrary directed suprema with respect to $\sqsubseteq$ (see Battenfeld *et al.* (2006) for a proof). In a sense, however, this form of completeness is somewhat *ad hoc*. But much earlier M. Hyland and P. Taylor had already introduced (independently in Hyland (1991) and Taylor (1991)) the notion of *repleteness*, which is formally quite similar to the notion of sobriety (see Johnstone (1982)). The setting of Hyland (1991) and Taylor (1991) is much more general than qcb spaces. This generality, however, is not required for our purposes, so we recall the notion of repleteness for the particular case of qcb spaces only.

As we have already mentioned, the notion of repleteness looks very similar to the notion of sobriety. Recall that a space X is sober iff for every T_0 space Y a continuous map $e : X \to Y$ is a homeomorphism whenever $\mathcal{O}(e) = e^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an order isomorphism. Sober spaces form a full reflective subcategory of (T_0) spaces. We write $\eta_X : X \to \mathsf{Sob}(X)$ for the reflection map and note that $\mathcal{O}(\eta_X) : \mathcal{O}(\mathsf{Sob}(X)) \to \mathcal{O}(X)$ is an order isomorphism and that η_X is one-to-one if X is a T_0 space. Analogously, a qcb space X is called *replete* iff a map $e : X \to Y$ in **QCB** is a homeomorphism whenever $\mathcal{O}(e) = e^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an order isomorphism, that is, an isomorphism in **QCB**. Replete qcb spaces form a full reflective subcategory of **QCB**, and for every $X \in$ **QCB** the reflection map $r_X : X \to R(X)$ is one-to-one, $\mathcal{O}(r_X)$ is an order isomorphism and thus $\mathsf{Sob}(r_X)$ is a homeomorphism. Despite this analogy, the construction of repletion is much more complicated than that of sobrification (see Hyland (1991), Taylor (1991) and Streicher (1999)).

We now discuss the relation between sobriety and repleteness. From the above definitions, it is clear that a sober qcb space is also replete. It can be shown that every replete qcb space is a topological predomain (see Hyland (1991) and Taylor (1991)). An example of a non-replete topological predomain is the non-sober dcpo (with its Scott topology) introduced in Johnstone (1981) whose sobrification coincides with its repletion (since sobrification adds a single new point that can be obtained as the limit of a sequence of point filters). Motivated by these observations, one might hope that for qcb spaces repletion is given by sobrification. One easily sees that this is equivalent to qcb spaces being closed under sobrification, which was raised as Question 6.1 in Simpson (2003). In the next section we construct a qcb space whose sobrification is no longer qcb, and thus give a negative answer to Simpson's question.

2. A qcb space whose sobrification is not qcb

We will construct a relatively simple replete qcb space X whose sobrificaton $\mathsf{Sob}(X)$ is not sequential, and thus *a fortiori* not qcb.

The underlying set of X is $\mathbb{N} \times \mathbb{N}$. We write π_0 and π_1 for first and second projection, respectively. For $p = (n,m) \in X$ and $f : \{i \in \mathbb{N} \mid i > n\} \to \mathbb{N}$, let $U(p,f) = \{p\} \cup \{(i,j) \in \mathbb{N}^2 \mid i > n \text{ and } j \geqslant f(i)\}$. Note that $p \in U(p,f)$. A subset U of X is called **open** iff for every $p \in U$ there is an f with $U(p,f) \subseteq U$. Obviously, we have $U(p, \max(f,g)) = U(p,f) \cap U(p,g)$. Moreover, if $q \in U(p,f)$, then $U(q,g) \subseteq U(p,f)$ for some g. Thus, sets of the form $U(p,f)$ are open, and for every $q \in U(p_1,f_1) \cap U(p_2,f_2)$ there is a g with $U(q,g) \subseteq U(p_1,f_1) \cap U(p_2,f_2)$. Thus, open sets are closed under finite intersections. Since open sets are also closed under arbitrary unions, they form a topology on X. Moreover, sets of the form $U(p,f)$ provide a basis for this topology.

It is easy to see that X is a T_1 space. Thus, the specialisation order on X is discrete. Accordingly, the space X is a topological predomain provided it is sequential and has a countable pseudobase, and hence is a qcb space.

Lemma 2.1. For every $A \subseteq X$ and $p \in X \setminus A$ we have $p \in \bar{A}$ iff $A \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite for some $i > \pi_0(p)$.

Proof. Let $A \subseteq X$ and $p \in X \setminus A$.

For the forward direction, suppose that $p \in \bar{A}$ and $A \cap \big(\{i\} \cap \mathbb{N}\big)$ is finite for all $i > \pi_0(p)$. Then there exists an f with $A \cap U(p,f) = \varnothing$, which contradicts $p \in \bar{A}$ since $U(p,f)$ is an open neighbourhood of p.

For the reverse direction, suppose $A \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite for some $i > \pi_0(p)$. In order to show that $p \in \bar{A}$, suppose U is an open neighbourhood of p. Then there exists f with $U(p,f) \subseteq U$. Then, since $A \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite, there exists a $j \geqslant f(i)$ with $(i,j) \in A$. Thus, $(i,j) \in A \cap U(p,f) \subseteq A \cap U$, as desired. □

Lemma 2.2. X is a Fréchet space, that is, for every $p \in \bar{A}$ there is a sequence (a_n) in A converging to p. Thus, X is, in particular, also a sequential space.

Proof. Suppose $p \in \bar{A}$. Without loss of generality, assume that $p \notin A$. Then, by Lemma 2.1, there exists an $i > \pi_0(p)$ such that $A \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite. Let $\phi : \mathbb{N} \to \mathbb{N}$ be strictly increasing with $\{(i, \phi(n)) \mid n \in \mathbb{N}\} = A \cap \big(\{i\} \times \mathbb{N}\big)$. Let $a_n := (i, \phi(n)) \in A$. We show that $(a_n) \to p$.

Suppose U is an open neighbourhood of p. Then $U(p,f) \subseteq U$ for some f. Let $n_0 \in \mathbb{N}$ with $\phi(n_0) \geqslant f(i)$. Then, for all $n \geqslant n_0$, we have $\phi(n) \geqslant f(i)$, and thus $a_n = (i, \phi(n)) \in U(p,f) \subseteq U$. □

Let $\mathscr{B}_0$ be the collection of all $V_{i,n} := \{(i,j) \mid j \geqslant n\}$ with $i, n \in \mathbb{N}$. We define $\mathscr{B}$ as the set of all finite unions $B = B_1 \cup \ldots \cup B_n$ with $B_i \in \mathscr{B}_0 \cup \big\{\{x\} \mid x \in X\big\}$ and $B \cap \big(\{i_0\} \times \mathbb{N}\big)$ finite if $B \cap \big(\{i\} \times \mathbb{N}\big) = \varnothing$ for all $i < i_0$.

Lemma 2.3. $\mathscr{B}$ is a countable pseudobase for X.

Proof. Obviously, $\mathscr{B}$ is countable since $\mathscr{B}_0$ and X are both countable.

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To show that $\mathscr{B}$ is a pseudobase for X, suppose that (p_n) converges to p and U is an open neighbourhood of p. Then $U(p,f) \subseteq U$ for some f. For $i > \pi_0(p)$, let $I_i = \{n \in \mathbb{N} \mid p_n \in V_{i,f(i)}\}$.

Next we show that almost all I_i are empty. To show a contradiction, suppose this were not the case. Then there exists a subsequence (q_n) of (p_n) with $\pi_0(q_n) > \pi_0(p)$ and $\pi_0(q_n) < \pi_0(q_{n+1})$ for all $n \in \mathbb{N}$. Let $g : \{i \in \mathbb{N} \mid i > \pi_0(p)\} \to \mathbb{N}$, such that $q_n \notin U(p,g)$ for all $n \in \mathbb{N}$. Then $U(p,g)$ is an open neighbourhood of p containing no q_n, which is impossible since (q_n) converges to p.

Let $i_0 \in \mathbb{N}$ with $I_i = \varnothing$ for $i \geqslant i_0$. Then $B = \{p\} \cup \bigcup_{\pi_0(p)<j<i_0} V_{j,f(j)} \in \mathscr{B}$, so $B \subseteq U(p,f) \subseteq U$, and (p_n) is eventually in B, as desired. □

Thus, since X is a sequential space with a countable pseudobase, the space X is qcb.

Lemma 2.4. The irreducible closed subsets of X are the singleton subsets and X itself.

Proof. As non-empty open subsets of X are closed under finite intersections, they form a complete prime filter in $\mathcal{O}(X)$, and thus X is an irreducible closed subset of X. As X is a T_1 space, the singleton sets are all closed, and thus also irreducible closed.

Suppose C is an irreducible closed subset of X different from X. By Lemma 2.1 if $C \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite, then for all $j < i$ and $n \in \mathbb{N}$, we have $(j,n) \in \overline{C \cap \big(\{i\} \times \mathbb{N}\big)} \subseteq \bar{C} = C$. Thus $C \cap \big(\{i\} \times \mathbb{N}\big)$ is infinite for only finitely many i since otherwise $X = C$.

Thus, precisely one of the following two conditions holds:

(1) $C \cap (\{i\} \times \mathbb{N})$ is finite for all $i \in \mathbb{N}$.

(2) There is a greatest $i \in \mathbb{N}$ with $C \cap (\{i\} \times \mathbb{N})$ infinite.

In case (1) every point of C is isolated in the subspace C. Thus C cannot be irreducible closed unless C is a singleton.

In case (2) every point of the infinite set $C \cap \big(\{i\} \times \mathbb{N}\big)$ is isolated in the subspace C. But as irreducible closed sets contain at most one isolated point, this is impossible. □

Thus $\mathsf{Sob}(X) = X \cup \{\infty\}$ where ∞ stands for the irreducible closed set X. The non-empty open sets of $\mathsf{Sob}(X)$ are those of the form $U \cup \{\infty\}$ where $U \in \mathcal{O}(X) \setminus \{\varnothing\}$.

Lemma 2.5. As a subset of $\mathsf{Sob}(X)$, the set X is sequentially closed but not closed with respect to the sober topology. Thus, the space $\mathsf{Sob}(X)$ is not sequential, and hence not qcb.

Proof. Obviously, X is not closed in $\mathsf{Sob}(X)$ since $\infty \in \bar{X} \setminus X$. Nevertheless, X is a sequentially closed subset of $\mathsf{Sob}(X)$, which can be seen as follows.

To show a contradiction, suppose (x_n) is a sequence in X converging to ∞ in $\mathsf{Sob}(X)$. As ∞ is in the closure of $S = \{x_n \mid n \in \mathbb{N}\}$, it is impossible for $S \cap \big(\{i\} \times \mathbb{N}\big)$ to be finite for all $i \in \mathbb{N}$. Thus, there exists an $i \in \mathbb{N}$ with $S \cap \big(\{i\} \times \mathbb{N}\big)$ infinite. But then $U = \{(j,k) \in \mathbb{N}^2 \mid i < j\} \cup \{\infty\}$ is an open neighbourhood of ∞ in $\mathsf{Sob}(X)$ such that infinitely many elements of S, namely those of $S \cap \big(\{i\} \times \mathbb{N}\big)$, are not in U, which contradicts our assumption that (x_n) converges to ∞. □

Thus, we have verified that X is qcb but its sobrification $\mathsf{Sob}(X)$ is not.

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Theorem 2.6. The space X is a replete qcb space but not sober.

Proof. Let $r_X : X \to R(X)$ be the reflection map from X to its repletion $R(X)$. Since $\mathsf{Sob}(r_X)$ is an isomorphism and $\eta_{R(X)} \circ r_X = \mathsf{Sob}(r_X) \circ \eta_X$, we have $\eta_X = i \circ r_X$ for $i = \mathsf{Sob}(r_X)^{-1} \circ \eta_{R(X)}$. Since $R(X)$ is a T_0 space, the map $\eta_{R(X)}$ is one-to-one. Thus i is also one-to-one. Since $\mathcal{O}(r_X)$ and $\mathcal{O}(\eta_X)$ are both isomorphisms, it follows that $\mathcal{O}(i)$ is an isomorphism as well. As i is also one-to-one, it follows that $i : R(X) \to \mathsf{Sob}(X)$ is a subspace embedding. Since η_X factors through i, we have either $R(X) = X$ or $R(X) = \mathsf{Sob}(X)$. As $R(X)$ is sequential but $\mathsf{Sob}(X)$ is not, it follows that $R(X) = X$, that is, that X is replete.

Thus X is a replete qcb space that is not sober by Lemma 2.5. □

A. Simpson has pointed out to us that our counterexample can also be used to answer some open questions about the category **CCG** of *core compactly generated spaces* that was introduced in Day (1972) and further investigated in Escardó *et al.* (2004). Problem 9-5 of Escardó *et al.* (2004) asks:

(a) Are the core compactly generated spaces closed under sobrification?
(b) Is the core compactly generated topology of a sober space also sober?

The answer to both questions is no.

ad (a) Escardó *et al.* (2004, Theorem 6.10) shows that **CCG** contains **QCB** as the full subcategory of those core compactly generated spaces having a countable $\ll$-pseudobase (see Escardó *et al.* (2004) for a definition). Obviously, the space X is in **CCG** since it is in **QCB**. If $\mathsf{Sob}(X)$ were in **CCG** as well, $\mathsf{Sob}(X)$ would be a subspace of $\mathcal{O}^2(\mathsf{Sob}(X)) \cong \mathcal{O}^2(X)$. From Escardó *et al.* (2004) we know that **QCB** is closed under subspaces in **CCG**, and thus $\mathsf{Sob}(X)$ would be a qcb space, which is in contradiction with Lemma 2.5.

ad (b) It is easy to check that $\{B \cup \{\infty\} \mid B \in \mathcal{B}\}$ is a countable $\ll$-pseudobase for $\mathsf{Sob}(X)$. By Escardó *et al.* (2004, Corollary 6.6), the core compactly generated topology on $\mathsf{Sob}(X)$ coincides with its sequentialisation $\mathsf{Seq}(\mathsf{Sob}(X))$. Since, by (the proof of) Lemma 2.5, the point ∞ is isolated in $\mathsf{Seq}(\mathsf{Sob}(X))$, the subset $X \subseteq \mathsf{Seq}(\mathsf{Sob}(X))$ is irreducible closed, but, obviously, not the closure of a singleton set. Thus, the space $\mathsf{Seq}(\mathsf{Sob}(X))$ is not sober.

Obviously, these arguments apply to all the cartesian closed categories of spaces considered in Escardó *et al.* (2004) as long as they contain all T_1 qcb spaces. Thus sobriety appears to be fundamentally incompatible with cartesian closedness.

We conclude this section by showing that $\mathcal{O}(X)$ is sober, though X is not. To do this, we will need the following lemma.

Lemma 2.7. Let $\mathcal{B}$ be the countable pseudobase for X as introduced before Lemma 2.3. For every $B \in \mathcal{B}$ the set $[B] = \{U \in \mathcal{O}(X) \mid B \subseteq U\}$ is a Scott open filter in $\mathcal{O}(X)$. Moreover, every $\mathcal{U} \in \mathcal{O}^2(X)$ is the union of all $[B]$ with $B \in \mathcal{B}$ and $[B] \subseteq \mathcal{U}$, from which it follows that $\mathcal{O}(X)$ is countably based and thus qcb.

Proof. It is easy to see that the elements of $\mathcal{B}$ are compact subsets of X, and thus $[B] = \{U \in \mathcal{O}(X) \mid B \subseteq U\}$ is a Scott continuous filter in $\mathcal{O}(X)$. As $\mathcal{B}$ is closed under

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finite unions, the set $\mathscr{B}_1 = \{[B] \mid B \in \mathscr{B}\}$ is closed under finite intersections, and thus provides a countable basis for $\mathcal{O}(X)$ (since, as shown in Schröder (2003), if $\mathscr{B}$ is a countable pseudobase for X, then $\{[B_1] \cup \ldots \cup [B_n] \mid B_1, \ldots, B_n \in \mathscr{B}\}$ is a countable pseudobase for $\Sigma^X = \mathcal{O}(X)$). Thus, we have $\mathscr{U} = \bigcup\{[B] \mid B \in \mathscr{B}, [B] \subseteq \mathscr{U}\}$ for every $\mathscr{U} \in \mathcal{O}^2(X)$. □

3. Failure of the 'equaliser construction' of repletion

In the first half of the 1990's several people suggested that for arbitrary X its repletion $R(X)$ might be given by the equaliser $E(X)$ of the maps $\eta_{\mathcal{O}^2(X)}, \mathcal{O}^2(\eta_X) : \mathcal{O}^2(X) \to \mathcal{O}^4(X)$, where for arbitrary Y the map $\eta_Y : Y \to \mathcal{O}^2(Y)$ sends $y \in Y$ to its neighbourhood filter $\eta_Y(y) = \{U \in \mathcal{O}(Y) \mid y \in U\}$. Since for $\mathscr{U} \in \mathcal{O}^2(X)$ and $\Phi \in \mathcal{O}^3(X)$ we have $\eta_{\mathcal{O}^2(X)}(\mathscr{U})(\Phi) = \Phi(\mathscr{U})$ and $\mathcal{O}^2(\eta_X)(\mathscr{U})(\Phi) = \mathscr{U}(\mathcal{O}(\eta_X)(\Phi)) = \mathscr{U}(\Phi \circ \eta_X)$, the equaliser $E(X)$ is the regular subject of $\mathcal{O}^2(X)$ consisting of those $\mathscr{U} \in \mathcal{O}^2(X)$ such that $\Phi(\mathscr{U}) = \mathscr{U}(\Phi \circ \eta_X)$ for all $\Phi \in \mathcal{O}^3(X)$.

We will show that, for our space X from Section 2, the equaliser $E(X)$ contains the element $\exists = \{U \in \mathcal{O}(X) \mid U \neq \varnothing\} \in \mathcal{O}^2(X)$, from which it will follow that $E(X)$ is different from $R(X) = X$. To do this, we will need the following lemma.

Lemma 3.1. The closure of $\{\eta_X(x) \mid x \in X\}$ in $\mathcal{O}^2(X)$ contains $\exists$ as an element.

Proof. For $n, i \in \mathbb{N}$, consider the sets

$$F_{i,n} = \{U \in \mathcal{O}(X) \mid (\exists p \in U.\, \pi_0(p) < i) \wedge (\forall j \geqslant n.\, (i,j) \in U)\},$$

which are easily shown to be elements of $\mathcal{O}^2(X)$ and to satisfy $F_{i,n} \subseteq \eta_X(i,n)$. Thus, all $F_{i,n}$ lie in $\overline{\{\eta_X(x) \mid x \in X\}}$. Since for all $i \in \mathbb{N}$ the sequence $(F_{i,n})_{n\in\mathbb{N}}$ is increasing, its union $G_i = \bigcup_{n\in\mathbb{N}} F_{i,n}$ is in $\overline{\{\eta_X(x) \mid x \in X\}}$ as well. One easily shows that we have $G_i = \{U \in \mathcal{O}(X) \mid \exists p \in U.\, \pi_0(p) < i\}$, and thus $\exists = \bigcup_{i\in\mathbb{N}} G_i$. Thus, as the sequence (G_i) is increasing, it follows that $\exists \in \overline{\{\eta_X(x) \mid x \in X\}}$. □

Theorem 3.2. The set $E(X) = \{\mathscr{U} \in \mathcal{O}^2(X) \mid \forall \Phi \in \mathcal{O}^3(X).\, \Phi(\mathscr{U}) = \mathscr{U}(\Phi \circ \eta_X)\}$ contains $\exists$ as an element. Thus η_X is not the equaliser of $\eta_{\mathcal{O}^2(X)}$ and $\mathcal{O}^2(\eta_X)$, that is, the equaliser construction of repletion fails for X.

Proof. To show that $\exists \in E(X)$, we have to show that $\Phi(\exists) = \exists(\Phi \circ \eta_X)$ for all $\Phi \in \mathcal{O}^3(X)$. Let $\Phi \in \mathcal{O}^3(X)$. Suppose $\exists(\Phi \circ \eta_X) = \top$. Then $\eta_X(x) \in \Phi$ for some $x \in X$. Thus $\Phi(\exists) = \top$ since $\eta_X(x) \subseteq \exists$ and Φ preserves the specialisation order. For the reverse direction, suppose $\Phi(\exists) = \top$, that is, $\exists \in \Phi$. Then, by Lemma 3.1, we have $\eta_X(x) \in \Phi$ for some $x \in X$. Thus $\Phi(\eta_X(x)) = \top$, from which it follows that $\exists(\Phi \circ \eta_X) = \top$, as desired.

It is easy to check that η_X equalises $\eta_{\mathcal{O}^2(X)}$ and $\mathcal{O}^2(\eta_X)$. But since $\exists \in E(X)$ and $\exists \notin \{\eta_X(x) \mid x \in X\}$, the map η_X is not the equaliser of $\eta_{\mathcal{O}^2(X)}$ and $\mathcal{O}^2(\eta_X)$. □

Thus we have shown that all known attempts to simplify the construction of repletion do not work for **QCB** and topological predomains, that is, for realisability models over $\mathscr{P}\omega$. The same holds for function realisability since $\mathbf{ExPer}(\mathbb{N}^{\mathbb{N}}) \simeq \mathbf{ExPer}(\mathscr{P}\omega)$, as shown in Bauer (2002). Whether our counterexample can be adapted to number realisability remains a task for future investigations.

https://doi.org/10.1017/S0960129506005184 Published online by Cambridge University Press

Acknowledgements

We gratefully acknowledge discussions with Alex Simpson, Ingo Battenfeld, M. Schröder and Jimmie Lawson. Moreover, we thank Alex Simpson for pointing out some mistakes in an earlier version of this paper, and for his detailed suggestions for improvement. Finally, we thank Klaus Keimel for establishing the contact between the two authors of this paper.

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