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Forbidden Hypermatrices Imply General Bounds on Induced Forbidden Subposet Problems

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We prove that for every poset P, there is a constant C_P such that the size of any family of subsets of $\{1,2,\dots,n\}$ that does not contain P as an induced subposet is at most

$$C_P\binom{n}{\lfloor n/2\rfloor},$$

settling a conjecture of Katona, and Lu and Milans. We obtain this bound by establishing a connection to the theory of forbidden submatrices and then applying a higher-dimensional variant of the Marcus–Tardos theorem, proved by Klazar and Marcus. We also give a new proof of their result.

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1. Introduction

We are interested in the largest family of subsets of $[n] := \{1,2,\dots,n\}$ avoiding certain subposets.

1.1. Forbidden weak subposets

Let P be a finite poset, and $\mathcal{F}$ be a family of subsets of $[n]$. We say that P is contained in $\mathcal{F}$ as a *weak* subposet if there is an injection $\alpha : P \to \mathcal{F}$ satisfying $x_1 <_p x_2 \Rightarrow \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. $\mathcal{F}$ is called P-free if P is not contained in $\mathcal{F}$ as a weak subposet. We define the corresponding extremal function as

$$La(n,P) := \max\{|\mathcal{F}| \mid \mathcal{F} \text{ is } P\text{-free}\}.$$

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The linearly ordered poset on k elements, $a_1 < a_2 < \cdots < a_k$, is called a chain of length k, and is denoted by P_k. Using this notation, the well-known theorem of Sperner can be stated as

$$La(n,P_2) = \binom{n}{\lfloor n/2 \rfloor}.$$

Erdős extended Sperner's theorem by showing that $La(n,P_k)$ is equal to the sum of the $k-1$ largest binomial coefficients of order n. Notice that, since any poset P is a weak subposet of a chain of length $|P|$, Erdős's theorem implies that

$$La(n,P) \leqslant (|P|-1)\binom{n}{\lfloor n/2 \rfloor} = O\left(\binom{n}{\lfloor n/2 \rfloor}\right).$$

Subsequently many authors, including Katona and Tarján [18] and Griggs, Lu and Li [11, 12] (to name just a few), studied various other posets; see the recent survey by Griggs and Li [10] for an excellent account of all the posets that have been studied. Let $h(P)$ denote the height (maximum length of a chain) of P. One of the first general results is due to Bukh, who showed that if T is a finite poset whose Hasse diagram is a tree of height $h(T) \geqslant 2$, then

$$La(n,T) = (h(T)-1)\binom{n}{\lfloor n/2 \rfloor}(1+O(1/n)).$$

For general posets, it is natural to conjecture[1] that

$$\lim \frac{La(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}$$

exists, and equals the maximum number of complete consecutive middle levels of the Boolean lattice whose union is P-free. Most notoriously, this conjecture is open for the *diamond* D_2, the poset on four elements with the relations $a < b, c < d$ where b and c are incomparable, for which the best known bound is

$$(2.20711 + o(1))\binom{n}{\lfloor n/2 \rfloor}$$

(see [14]). A weaker version of the conjecture for general posets was obtained in a series of papers by Burcsi and Nagy [3], Chen and Li [5], and Grósz, Methuku and Tompkins [13], who showed that

$$La(n,P) = O\left(h(P)\log\left(\frac{|P|}{h(P)}\right)\right)\binom{n}{\lfloor n/2 \rfloor}.$$

1.2. Forbidden induced subposets

We say that P is contained in $\mathcal{F}$ as an *induced* subposet if and only if there is an injection $\alpha : P \to \mathcal{F}$ satisfying $x_1 <_p x_2 \Leftrightarrow \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. $\mathcal{F}$ is called induced-P-free if P is not contained in $\mathcal{F}$ as an induced subposet. We define the corresponding extremal function as

$$La^{\#}(n,P) := \max\{|\mathcal{F}| \mid \mathcal{F} \text{ is induced-}P\text{-free}\}.$$

[1] This conjecture motivated much of the early work of Katona and his co-authors, though it has not been explicitly stated. The first appearance seems to be in [2], and a couple of months later in [12].

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Despite considerable progress made on forbidden weak subposets, little is known about forbidden induced subposets (except for P_k, where the weak and induced containment are equivalent). The first result of this type is due to Carroll and Katona [4], who showed that

$$La^{\#}(n,V_2) = \binom{n}{\lfloor n/2 \rfloor}(1+o(1)).$$

Later Katona [16] showed that

$$La^{\#}(n,V_{r+1}) = \binom{n}{\lfloor n/2 \rfloor}(1+o(1)), \quad \text{for any } r \geqslant 1.$$

Boehnlein and Jiang [1] generalized this by extending Bukh's result to induced containment, proving that

$$La^{\#}(n,T) = (h(T)-1)\binom{n}{\lfloor n/2 \rfloor}(1+o(1)).$$

Recently, Patkós [24] determined the asymptotic behaviour of $La^{\#}(n,P)$ for all complete two-level posets and some complete multilevel posets.

However, no non-trivial general upper bound was known for $La^{\#}(n,P)$. A few years ago Katona [17], and recently Lu and Milans [21], independently conjectured that the analogue of Erdős's bound holds for induced posets as well, namely,

$$La^{\#}(n,P) = O\left(\binom{n}{\lfloor n/2 \rfloor}\right).$$

The main result of our paper is to prove their conjecture for all posets P.

Theorem 1.1. *For every poset P,*

$$La^{\#}(n,P) \leqslant C_P\binom{n}{\lfloor n/2 \rfloor}$$

for some constant C_P that depends on P.

It is interesting to note that the constant C_P in our upper bound on $La^{\#}(n,P)$ does not depend on $h(P)$, which appears in the upper bounds on $La(n,P)$ (as noted in the previous section), but on the *dimension* of P. The *dimension* of a poset P is the least integer d for which there exists t linear orderings, $<_1,\dots,<_d$, of the elements of P such that, for every x and y in P, $x <_P y$ if and only if $x <_i y$ for all $1 \leqslant i \leqslant d$. Just as in the non-induced case, one might conjecture that

$$\lim \frac{La^{\#}(n,P)}{\binom{n}{\lfloor n/2 \rfloor}}$$

exists, and equals the maximum number of complete consecutive middle levels of the Boolean lattice whose union is induced P-free.

To establish our theorem, we use a method that is a certain generalization of the so-called *circle method* of Katona [15]. That is, if $\mathcal{F}$ is a family of subsets which is induced P-free, we define some special families $\mathcal{Q}$ of subsets of $[n]$, and double-count all pairs $(\mathcal{Q},F)$ such that $F \in \mathcal{F}$ and $F \in \mathcal{Q}$. What is novel in our paper is that we associate a d-dimensional $0-1$ hypermatrix with $\mathcal{Q}$, and we establish a connection to the theory of forbidden submatrices. Then, using this connection, we calculate the number of pairs $(\mathcal{Q}, F)$ for a fixed $\mathcal{Q}$. This is made more precise in the proof of Proposition 1.2, for which we first need to introduce some notation.

A d-dimensional *hypermatrix* is an $n_1 \times \cdots \times n_d$ sized ordered array. For short, we refer to such a hypermatrix as a *d-matrix*. So a vector is a 1-matrix and a matrix is a 2-matrix. Moreover, we simply say that a d-matrix is of size n^d if $n_1 = \cdots = n_d = n$. We refer to the entries of a d-matrix M as $M(\underline{i})$, where $\underline{i} = (i_1, \ldots, i_d)$ and $1 \leqslant i_j \leqslant n_j$ for every $j \in [d]$. In this paper we only deal with d-matrices whose entries are all 0 and 1. We let $|M|$ denote the number of 1s in a d-matrix M. We say that a d-matrix M *contains* a d-matrix A if it has a d-submatrix $M' \subset M$ that is of the same size as A such that $A(\underline{i}) = 1 \Rightarrow M'(\underline{i}) = 1$. If M does not contain A, then we say that M is *A-free*. We define the corresponding extremal function as

$$ex_d(n_1 \times \cdots \times n_d, A) := \max\{|M| \mid M \text{ is an } A\text{-free } d\text{-matrix of size } n_1 \times \cdots \times n_d\},$$

and if $n_1 = \cdots = n_d = n$, we use $ex_d(n,A) := ex_d(n_1 \times \cdots \times n_d, A)$ for convenience. Notice that $ex_1(n,A) = \min\{n, |A| - 1\}$ and $ex_2(n,A)$ is the well-studied forbidden submatrix problem: see [8, 26].

We also need to generalize the notion of a permutation matrix to higher dimensions. We say that a d-matrix M of size k^d is a *permutation d-matrix* if $|M| = k$ and it contains exactly one 1 in each axis-parallel hyperplane. In other words, for every $j \in [d]$ and $1 \leqslant i_j \leqslant k$ there is a unique $\underline{i} = (i_1, \ldots, i_d)$ such that $M(\underline{i}) = 1$.

From the definition of poset dimension, we get that for every poset P of size k whose dimension is d, there is a unique permutation d-matrix M_P of size k^d that *represents* P in the following sense. The 1-entries of M_P are in bijection with the elements of P such that, if the element $M_P(\underline{i})$ is in bijection with $p \in P$ and the element $M_P(\underline{i}')$ is in bijection with $p' \in P$, then $p < p' \Leftrightarrow \forall j\ i_j < i'_j$. This M_P can be constructed as follows. Consider the d linear orderings, $<_1, \ldots, <_d$, of the elements of P such that, for every x and y in P, $x <_P y$ if and only if $x <_i y$ for all $1 \leqslant j \leqslant d$. For each $p \in P$ the coordinates of the associated 1-entry of M_P are $\underline{i} = (i_1, \ldots, i_d)$, where i_j is the position of p in the linear ordering $<_j$.

The following is our key proposition establishing a connection to the theory of forbidden submatrices.

Proposition 1.2.

$$La^{\#}(n,P) \leqslant C_d \frac{ex_d(n, M_P)}{n^{d-1}} \binom{n}{\lfloor n/2 \rfloor}$$

for every d-dimensional poset P for some constant C_d that depends on d.

We note that in the special case $d = 2$ we get

$$La^{\#}(n,P) \leqslant (1 + o(1)) \frac{ex_2(n, M_P)}{n} \binom{n}{\lfloor n/2 \rfloor},$$

as n tends to infinity in the above statement.

We can combine Proposition 1.2 with the following theorem, which is a higher-dimensional variant of the Marcus–Tardos theorem [22] about forbidden submatrices.

Theorem 1.3 (Klazar–Marcus [19]). *If a d-dimensional $0-1$ hypermatrix of size $n \times \cdots \times n$ does not contain a given d-dimensional permutation hypermatrix of size $k \times \cdots \times k$, then it has at most $C_k n^{d-1}$ non-zero elements for some constant C_k that depends on k.*

Notice that Theorem 1.1 follows from Proposition 1.2 and Theorem 1.3. We only learned after the first version of our manuscript that Theorem 1.3 was proved earlier by Klazar and Marcus [19]. Surprisingly, our proof of Theorem 1.3 is different from their proof and appears to be a bit shorter, perhaps due to the use of the Loomis–Whitney inequality [20].

The organization of the rest of the paper is as follows. In Section 2 we prove our main result, Proposition 1.2, which establishes the connection between the two theories. In Section 3 we give a new proof of Theorem 1.3. Finally, we make some concluding remarks in Section 4.

2. Proof of Proposition 1.2

Define a *permutation d-partition* $\mathcal{Q} := Q_1|Q_2|\cdots|Q_d$ to be an ordered partition of a permutation of the elements of $[n]$ into d parts $Q_1, Q_2, \ldots, Q_d$. We denote the ith element of Q_j by $Q_j(i)$. The set of the form

$$Q_j[i) := \{Q_j(1), \ldots, Q_j(i-1)\}$$

is called a *prefix* of Q_j, and the set of the form

$$\mathcal{Q}[\underline{i}) := \bigcup_{j=1}^{d} Q_j[i_j)$$

is called a *prefix union* of $\mathcal{Q}$.

An example of a permutation 3-partition is $\mathcal{Q} = 142|5|3$, and $\mathcal{Q}[3,1,2) = \{1,3,4\}$ is a prefix union of $142|5|3$. Notice that since the order of the parts is respected, we consider, say, $\mathcal{Q}' = 5|142|3$ as a different permutation 3-partition, but of course the prefixes of $\mathcal{Q}$ and $\mathcal{Q}'$ are the same. Some parts might also be empty in a permutation 3-partition, such as $142||53$.

The total number of possible permutation d-partitions is $(n+d-1)!/(d-1)!$, by taking all permutations of the elements of $[n]$ and the $d-1$ separators.

We are now ready to start the proof of Proposition 1.2. Let $\mathcal{F}$ be an induced-P-free family of subsets of $[n]$. We double-count pairs $(\mathcal{Q}, F)$, where $F \in \mathcal{F}$ and F is a prefix union of $\mathcal{Q}$. First, let us fix a set $F \in \mathcal{F}$ and calculate the number of permutation d-partitions $\mathcal{Q}$ such that F is a prefix union of $\mathcal{Q}$.

Lemma 2.1. *Given $F \subset [n]$ and $d \in \mathbb{N}$, there are exactly*

$$\frac{(n+2d-2)!}{((d-1)!)^2} \binom{n+2d-2}{|F|+d-1}^{-1}$$

permutation d-partitions $\mathcal{Q}$ of $[n]$ such that F is a prefix union of $\mathcal{Q}$.

Proof. Permute the elements of F and $d-1$ separators '$|$' in $(|F|+d-1)!/(d-1)!$ ways. Each such permutation is of the form $L_1|L_2|\cdots|L_d$. Also permute the elements of $[n] \setminus F$ and

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$d-1$ separators '$|$' in $(n-|F|+d-1)!/(d-1)!$ ways. Each such permutation is of the form $R_1|R_2|\cdots|R_d$.

Now, we concatenate $L_1|L_2|\cdots|L_d$ and $R_1|R_2|\cdots|R_d$ as $L_1R_1|L_2R_2|\cdots|L_dR_d$ to obtain a permutation d-partition for which F is a prefix union. Since

$$\frac{(|F|+d-1)!}{(d-1)!}\cdot\frac{(n-|F|+d-1)!}{(d-1)!}=\frac{(n+2d-2)!}{((d-1)!)^2}\binom{n+2d-2}{|F|+d-1}^{-1},$$

the proof is complete. □

The following property about monotonicity for matrices avoiding submatrices will be useful.

Proposition 2.2. *If $\forall i: m_i \leqslant n_i$, then*

$$ex_d(n_1\times\cdots\times n_d, A)\leqslant\frac{n_1}{m_1}\times\cdots\times\frac{n_d}{m_d}\,ex_d(m_1\times\cdots\times m_d, A).$$

Proof. Let M be an A-free d-matrix of size $n_1\times\cdots\times n_d$. Any M' d-submatrix of M is also A-free. If M' is of size $m_1\times\cdots\times m_d$, then $|M'|\leqslant ex_d(m_1\times\cdots\times m_d, A)$. Averaging over all submatrices of this size, the statement follows, as any entry of M has probability

$$\frac{m_1}{n_1}\times\cdots\times\frac{m_d}{n_d}$$

of being in a submatrix. □

Now, let us fix a $\mathcal{Q}=Q_1|Q_2|\cdots|Q_d$ and calculate the number of sets $F\in\mathcal{F}$ such that F is a prefix union of $\mathcal{Q}$.

Lemma 2.3. *Let P be a poset of dimension d, and let M_P be the d-dimensional permutation matrix that represents P. Given an induced-P-free family $\mathcal{F}$ of subsets of $[n]$ and a permutation d-partition $\mathcal{Q}$ of $[n]$, there exist at most*

$$\left(\frac{n+d}{nd}\right)^d ex_d(n, M_P)$$

sets $F\in\mathcal{F}$ such that F is a prefix union of $\mathcal{Q}$.

Proof. We first associate a d-matrix $M_{\mathcal{Q}}$ of size $(|Q_1|+1)\times\cdots\times(|Q_d|+1)$ with $\mathcal{Q}$, where $|Q_j|$ denotes the length of Q_j. This is done by setting $M_{\mathcal{Q}}(\underline{i})=1$ if the prefix union $\mathcal{Q}[\underline{i}]\in\mathcal{F}$ and $M_{\mathcal{Q}}(\underline{i})=0$ otherwise. Now consider the permutation d-matrix M_P of size $|P|^d$ that represents P. Notice that $\mathcal{Q}[\underline{i}']\subset\mathcal{Q}[\underline{i}]$ if and only if $\forall j: i'_j\leqslant i_j$ and equality can hold only if $\underline{i}=\underline{i}'$. From this it follows that if $M_{\mathcal{Q}}$ contains M_P, then the same relations hold in $\mathcal{F}$ and thus $\mathcal{F}$ contains an induced copy of P, which is impossible. Therefore $M_{\mathcal{Q}}$ is M_P-free. Using Proposition 2.2, we have that the number of sets $F\in\mathcal{F}$ such that F is a prefix union of $\mathcal{Q}$ is

$$|M_{\mathcal{Q}}|\leqslant\frac{(|Q_1|+1)\times\cdots\times(|Q_d|+1)}{n^d}\,ex_d(n,M_P)\leqslant\left(\frac{n+d}{nd}\right)^d ex_d(n,M_P).$$ □

Now combining the lower and upper bounds that we get from Lemmas 2.1 and 2.3 for the number of pairs $(\mathcal{Q},F)$, such that $\mathcal{Q}$ is a permutation d-partition of $[n]$ and $F\in\mathcal{F}$ is a prefix union of $\mathcal{Q}$, we obtain

$$\frac{(n+2d-2)!}{((d-1)!)^2}\sum_{F\in\mathcal{F}}\binom{n+2d-2}{|F|+d-1}^{-1}\leqslant\frac{(n+d-1)!}{(d-1)!}\left(\frac{n+d}{nd}\right)^d ex_d(n,M_P).$$

Using

$$(n+2d-2)!\geqslant(n+d-1)!(n+d)^{d-1}$$

and multiplying by $((d-1)!)^2$ on both sides, we obtain

$$|\mathcal{F}|\binom{n+2d-2}{\lfloor n/2\rfloor+d-1}^{-1}\leqslant\sum_{F\in\mathcal{F}}\binom{n+2d-2}{|F|+d-1}^{-1}\leqslant\frac{(d-1)!}{d^{d-1}}\left(\frac{n+d}{nd}\right)\frac{ex_d(n,M_P)}{n^{d-1}}.$$

Now using that

$$\binom{n+2d-2}{\lfloor n/2\rfloor+d-1}\leqslant 4^{d-1}\binom{n}{\lfloor n/2\rfloor},$$

we have proved Proposition 1.2. □

3. Proof of Theorem 1.3

The proof is similar to the proof of Marcus and Tardos [22], except that we use induction on d and n, just like Klazar and Marcus [19]. However, surprisingly, even though both the Klazar–Marcus proof and our proof are a very natural generalization of the Marcus–Tardos proof, they are still quite different. Below we present our proof.

We prove by induction on d and n that any d-matrix of size n^d not containing some permutation d-matrix of size k^d has at most $C_{k,d}n^{d-1}$ non-zero elements (k is fixed throughout the proof and $C_{k,d}$ is a constant that depends on k and d). As $d\leqslant k$, our final constant $C_k=C_{k,k}$. The statement trivially holds for $d=1$, for all $n\geqslant 1$. We will show that it is true for d and n.

Let M be a d-matrix of size n^d and let A be a permutation d-matrix of size k^d. If S is a d-matrix, let $Proj_iS$ denote the $(d-1)$-matrix obtained by orthogonally projecting S to the hyperplane orthogonal to the ith axis. Notice that $Proj_iA$ is a permutation $(d-1)$-matrix of size k^{d-1}.

We partition M into smaller d-matrices of size s^d called *blocks* (for convenience, suppose that n is divisible by s) in the following way. We partition $[n]$ into intervals $I_1<I_2<\cdots<I_{n/s}$ each of length s. For any $\underline{b}=(b_1,\ldots,b_d)$ with $b_i\in\{I_1,I_2,\ldots,I_{n/s}\}$, we define the block

$$S_{\underline{b}}:=\{M(\underline{a})\mid\underline{a}=(a_1,\ldots,a_d)\text{ and }a_i\in I_i\}.$$

An *i-blockcolumn* is a series of blocks parallel to the ith axis, that is, $\{S_{\underline{b}}\mid b_i=I_1,\ldots,I_{n/s}\}$, and an *i-column* is simply a series of matrix elements parallel to the ith axis, that is, $\{M(\underline{a})\mid a_i=1,2,\ldots,n\}$. A block S is called *i-wide* if $Proj_iS$ contains $Proj_iA$ as a $(d-1)$-submatrix. Using induction on the dimension, if this is not the case, then

$$|Proj_iS|\leqslant C_{k,d-1}s^{d-2}=O(s^{d-2}).$$

If a block is not i-wide for any $i=1,\ldots,d$, we call it *thin*.

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For the induction, we also need to use the following inequality of Loomis and Whitney.

Lemma 3.1 (Loomis–Whitney [20]).

$$|S|^{d-1} \leqslant \Pi_{i=1}^{d} |Proj_i S|.$$

If a block S is thin, then using the above inequality and that $|Proj_i S| \leqslant C_{k,d-1} s^{d-2}$ (for all $1 \leqslant i \leqslant d$), we get

$$|S| \leqslant (C_{k,d-1} s^{d-2})^{d/(d-1)} = O(s^{d-1-1/(d-1)}) = o(s^{d-1}).$$

The number of i-wide blocks in an i-blockcolumn is at most $(k-1)\binom{s^{d-1}}{k}$, because if $Proj_i A$ were to occur k times, in the same k i-columns, then we could 'build' a copy of A from them (here we use that A is a permutation d-matrix).

We define the d-matrix M' of size $(n/s)^d$ as $M'_{\underline{b}} = 1$ if and only if the block $S_{\underline{b}}$ is thin. As M' must also be A-free, we get the following bound by induction on d and n, where k is fixed, that is,

$$\begin{aligned}
|M| &\leqslant \sum_{S \text{ is thin}} |S| + \sum_{i=1}^{d} \sum_{\substack{BC \text{ is an} \\ i\text{-blockcolumn}}} \sum_{\substack{S \in BC \text{ is} \\ i\text{-wide}}} |S| \\
&\leqslant \sum_{S \text{ is thin}} o(s^{d-1}) + \sum_{i=1}^{d} \sum_{\substack{BC \text{ is an} \\ i\text{-blockcolumn}}} \sum_{\substack{S \in BC \text{ is} \\ i\text{-wide}}} s^d \\
&\leqslant |M'| o(s^{d-1}) + \sum_{i=1}^{d} \sum_{\substack{BC \text{ is an} \\ i\text{-blockcolumn}}} (k-1)\binom{s^{d-1}}{k} s^d \\
&\leqslant C_{k,d} \left(\frac{n}{s}\right)^{d-1} o(s^{d-1}) + d\left(\frac{n}{s}\right)^{d-1} (k-1)\binom{s^{d-1}}{k} s^d
\end{aligned}$$

which for a sufficiently large s, is less than

$$(1-\delta) C_{k,d} n^{d-1} + s^{dk} n^{d-1} \leqslant C_{k,d} n^{d-1}$$

for some $\delta > 0$. With a more precise calculation, we can upper-bound $C_{k,d}$ by

$$k^{k^d((d+1)!)^2} = 2^{k^{\Theta(d)}}.$$

This is much weaker than the bound achieved by Klazar and Marcus [19], which gives $C_k = 2^{O_d(k \log k)}$. Very recently it has been proved by Geneson and Tian [9] that $C_k = 2^{O_d(k)}$.

4. Concluding remarks

Lu and Milans [21] have proposed the following strengthening of Theorem 1.1.

Problem 4.1 (Lu–Milans [21]). *For every poset P, there is a constant C such that for every induced-P-free $\mathcal{F}$ we have*

$$\sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1} \leqslant C.$$

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We could not establish this conjecture, as our approach only gives that

$$\sum_{F\in\mathcal{F}} \binom{n+2d-2}{|F|+d-1}^{-1} = O(1).$$

However, very recently this conjecture has been verified by Méroueh [23].

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