Can. J. Math. Vol. 47 (4), 1995 pp. 841–851

DISTANCE FROM PROJECTIONS TO NILPOTENTS

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ABSTRACT. The distance from an arbitrary rank-one projection to the set of nilpotent operators, in the space of $k \times k$ matrices with the usual operator norm, is shown to be $\sec(\pi/(k+2))/2$. This gives improved bounds for the distance between the set of all non-zero projections and the set of nilpotents in the space of $k \times k$ matrices. Another result of note is that the shortest distance between the set of non-zero projections and the set of nilpotents in the space of 3×3 matrices is $\sqrt{(3-\sqrt{5})/2}$.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$ with the usual norm

$$\|A\| = \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| = 1\}$$

for all $A \in \mathcal{B}(\mathcal{H})$. Consider the problem of finding the shortest distance between the set of non-zero orthogonal projections (which we denote by $\mathcal{P}(\mathcal{H})$) and the set of nilpotent operators (which we denote by $\mathcal{N}(\mathcal{H})$). (Recall that an operator $P \in \mathcal{B}(\mathcal{H})$ is in $\mathcal{P}(\mathcal{H})$ if and only if $P^2 = P = P^*$ and an operator $N \in \mathcal{B}(\mathcal{H})$ is in $\mathcal{N}(\mathcal{H})$ if and only if $N^j = 0$ for some whole number j.)

When $\mathcal{H} = C^2$, as shown in [1],

$$P_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

are a closest pair, and $\|P_2 - N_2\| = 1/\sqrt{2}$.

When the dimension of $\mathcal{H}$ is 3, the best previously known bound on this distance is $2/3$, which is achieved by the projection

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

and the nilpotent

$$N = \begin{pmatrix} 0 & 2/3 & 2/3 \\ 0 & 0 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}.$$

In [3], it is shown, using a quantitative spectral continuity theorem, that the distance between the set of non-zero projections (all projections referred to in this paper are orthogonal projections) on a space $\mathcal{H}$ and the set of nilpotents on the same space is always at least $1/2$, and if $\mathcal{H}$ is infinite dimensional, the distance is shown to be $1/2$.

Received by the editors December 27, 1993.
AMS subject classification: Primary: 47A58; secondary: 15A99.

Here we are mainly concerned with a related problem of computing the distance from the set of rank-one projections to the set of nilpotents. This problem is interesting in its own right, and gives an elegant solution. The answers we obtain to this question also allow us to show that the distance from the set of nonzero projections to the set of nilpotents in $\mathcal{B}(C^3)$ is $\sqrt{(3-\sqrt{5})/2}$, and gives better bounds for the shortest distance from the set of non-zero projections to the set of nilpotents in $\mathcal{B}(C^k)$. This allows us to disprove a conjecture of Herrero concerning these distances.

Let us begin with definitions and notation for the two main sequences of distances that concern us.

DEFINITION. Let

$$\delta_k = \inf\{\|P-N\| : P \in \mathcal{P}(C^k),\ N \in \mathcal{N}(C^k)\}$$

and

$$\nu_k = \inf\{\|P-N\| : P \in \mathcal{P}(C^k), \operatorname{rank} P = 1,\ N \in \mathcal{N}(C^k)\}$$

for $k = 1, 2, \ldots$.

As mentioned, the sequence δ_k has been extensively studied, and although values are not known, a number of bounds on the sequence are known (see [4], [7]). The best bound known for large values of k was obtained by Herrero [4] using Berg's technique and is

$$1/2 < \delta_k \leq 1/2 + \sin\big(\pi/[(k+1)/2]\big)$$

for $k \geq 3$ (Here, and throughout this paper, [ ] denotes the greatest integer function.). So δ_k decreases to $1/2$ as k goes to infinity. A sketch of the background information about δ_k can be found in Chapter 2 of [5].

Clearly $\delta_k \leq \nu_k$ for all values of k since we restrict the set of projections over which the infimum is taken when computing ν_k. Also, for all values of k there exists at least one rank-one projection P_k and one nilpotent N_k such that $\nu_k = \|P_k - N_k\|$. (Since $\nu_k \leq 1$ and all projections have norm equal to 1, a nilpotent as close to the set of projections as the 0 nilpotent can have norm no greater than 2, so we need only consider such nilpotents. But both the set of rank-one projections and the set of nilpotents of norm no greater than 2 are compact in $\mathcal{B}(C^k)$, so the infimum is attained.) Similarly, for each k, there exists a projection and nilpotent which are at a distance of δ_k.

We shall now state the main theorem of this paper. Its proof shall be given, interspersed with some asides on related topics, throughout the remainder of this paper.

THEOREM 1. *For $k = 1, 2, \ldots$, the shortest distance from the set of rank-one projections to the set of nilpotents in the space $\mathcal{B}(C^k)$ is*

$$\nu_k = \sec\big(\pi/(k+2)\big)/2.$$

As we begin to prove the main theorem, we give a number of alternate definitions of ν_k.

Both the set of rank-one projections and the set of nilpotents are invariant under unitary conjugation. That is, if $P \in \mathcal{P}(\mathcal{H})$, $N \in \mathcal{N}(\mathcal{H})$ and U is a unitary operator in $\mathcal{B}(\mathcal{H})$ then $UPU^* \in \mathcal{P}(\mathcal{H})$ and $UNU^* \in \mathcal{N}(\mathcal{H})$. Also, such unitary conjugation does not change norms, $\|A\| = \|UAU^*\|$ for all $A \in \mathcal{B}(\mathcal{H})$ and unitary $U \in \mathcal{B}(\mathcal{H})$. Since, in fact, all rank-one projections are unitarily equivalent, for $k = 1, 2, \ldots,$

$$\nu_k = \inf\{\|P - N\| : N \in \mathcal{N}(C^k)\}$$

where P is an arbitrary fixed rank-one projection in $\mathcal{B}(C^k)$. Also, since every nilpotent operator is unitarily equivalent to one whose matrix, with respect to the standard basis, is strictly upper-triangular, we have that for $k = 1, 2, \ldots,$

$$\nu_k = \inf\{\|P - T\| : P \in \mathcal{P}(C^k), \operatorname{rank} P = 1,\ T \in \mathcal{T}(C^k)\},$$

where $\mathcal{T}(C^k)$ denotes the set of operators in $\mathcal{B}(C^k)$ which are strictly upper triangular with respect to the standard basis. It is this definition of ν_k that will be the most useful.

A slight digression on rank-one operators is now in order. If $A \in \mathcal{B}(\mathcal{H})$ has rank one, then $\ker^{\perp} A$ is a one dimensional closed subspace of $\mathcal{H}$. Choose a non-zero vector $y \in \ker^{\perp} A$ and let $x = Ay/\|y\|^2$ then,

$$A(z) = \langle z, y \rangle x$$

since this is obviously true for vectors z in $\ker A$ or $\ker^{\perp} A$. A standard notation for this rank one operator is $A = x \otimes y^*$, and $\|x \otimes y^*\| = \|x\|\,\|y\|$. When A is also a projection, then $\ker^{\perp} A = \operatorname{Range} A$ and so if $e \in \ker^{\perp} A$ is chosen to be a unit vector, $A = e \otimes e^*$.

Using the above notation, we can give another equivalent definition of ν_k.

$$\begin{aligned}\nu_k &= \inf\{\|e \otimes e^* - T\| : e \in C^k, \|e\| = 1, T \in \mathcal{T}(C^k)\} \\ &= \inf_{\substack{e \in C \\ \|e\|=1}} \left\{\inf\{\|e \otimes e^* - T\| : T \in \mathcal{T}(C^k)\}\right\}.\end{aligned}$$

In order to compute ν_k, we first must compute the distance of a rank-one projection to the space of strictly upper triangular operators. The following theorem, a version of the Arveson Distance Formula, allows us to do this very thing.

THEOREM 2 [6]. *Let $\{e_i\}_{i=1}^k$ denote the standard basis for C^k and let $P_0 = 0$ and $P_i = \sum_{j=1}^i e_j \otimes e_j^*$ for $i = 1, \ldots, j$ denote the orthogonal projection onto the i-dimensional subspace spanned by $\{e_j\}_{j=1}^i$. Then, for an arbitrary operator $A \in \mathcal{B}(C^k)$, the distance from A to the set of operators whose matrix with respect to the standard basis is strictly upper triangular is*

$$\operatorname{dist}\bigl(A, \mathcal{T}(C^k)\bigr) = \inf\{\|A - T\| : T \in \mathcal{T}(C^k)\} = \max_{1 \le i \le k} \|P_{i-1}^{\perp} A P_i\|. \qquad \blacksquare$$

Applying this,

$$\begin{aligned}\nu_k &= \inf_{\substack{e\in C^k\\ \|e\|=1}} \max_{1\le i\le k} \|P_{i-1}^{\perp} e\otimes e^* P_i\| \\ &= \inf_{\substack{e\in C^k\\ \|e\|=1}} \max_{1\le i\le k} \|P_{i-1}^{\perp} e\otimes (P_i e)^*\| \\ &= \inf_{\substack{e\in C^k\\ \|e\|=1}} \max_{1\le i\le k} \|P_{i-1}^{\perp} e\|\,\|P_i e\|.\end{aligned}$$

Now, if we set $\alpha_i(e) = \|P_i e\|^2$ then since $1 = \|e\|^2 = \|P_i e\|^2 + \|P_i^{\perp} e\|^2$ for all $i = 1,2,\ldots,k$, we have that $1-\alpha_{i-1}(e) = \|P_{i-1}^{\perp} e\|^2$. Hence,

$$\nu_k = \inf_{\substack{e\in C^k\\ \|e\|=1}} \max_{1\le i\le k} \sqrt{\big(1-\alpha_{i-1}(e)\big)\alpha_i(e)}.$$

Note that any sequence of the form

$$0 = \alpha_0 \le a_1 \le \cdots \le a_{k-1} \le a_k = 1$$

can be achieved as $\{\alpha_i(e)\}_{i=1}^k$ for some unit vector $e \in C^k$, namely

$$e = (\sqrt{a_1}, \sqrt{a_2-a_1}, \sqrt{a_3-a_2}, \ldots, \sqrt{a_{k-1}-a_{k-2}}, \sqrt{a_k-a_{k-1}})$$

and hence

$$\nu_k = \inf_{0=a_0\le a_1\le\cdots\le a_{k-1}\le a_k=1} \max_{1\le i\le k} \sqrt{(1-a_{i-1})a_i}.$$

To compute this we need the following.

LEMMA 3. *The above infimum is achieved when all the terms* $(1-a_{i-1})a_i$ *are equal and hence when* $a_i = a_1/(1-a_{i-1})$ *for* $i = 1,2,\ldots,k$, *so*

$$\nu_k = \inf\{\sqrt{a_1} : 0 = a_0 \le a_1 \le \cdots \le a_{k-1} \le a_k = 1, \textit{ and } a_1 = a_1/(1-a_{i-1}) \textit{ for } i = 1,2,\ldots k\}.$$

PROOF. Suppose that we are given a nondecreasing sequence $\{a_i\}_{i=1}^k$ where all the terms $(1-a_{i-1})a_i$ are not equal and choose j between 1 and k to be the largest index at which the maximum is achieved. Then the maximum of the k terms is $(1-a_{j-1})a_j$. With no loss of generality we may assume that

$$(1-a_{j-1})a_j > (1-a_j)a_{j+1}.$$

(If it occurs that $j = k$, replace the sequence $\{a_i\}_{i=1}^k$ by the sequence $\{1-a_{k-i}\}_{i=1}^k$ and repeat above. If again $j = k$ then the terms are all equal.)

Also, we may assume that $a_j > a_{j-1}$, since if $a_j = a_{j-1}$ we obtain from the above inequality (and the fact that $a_j \neq 1$) that $a_j > a_{j+1}$ which is a contradiction.

Note that

$$\frac{a_j}{1-a_j} > \frac{a_{j+1}}{1-a_{j-1}}$$

and the function $t \longrightarrow t/(1-t)$ is an increasing function on the interval $[0,1]$, we can choose a number a_j' such that $a_j' < a_j$, $a_j' \geq a_{j-1}$ and

$$\frac{a_j'}{1-a_j'} > \frac{a_{j+1}}{1-a_{j-1}}.$$

Define a new nondecreasing sequence $\{b_i\}_{i=1}^k$ by

$$b_i = \begin{cases} a_j', & \text{if } i = j; \\ a_i, & \text{otherwise.} \end{cases}$$

Then either

$$\max_{1\leq i\leq k}(1-b_{i-1})b_i < \max_{1\leq i\leq k}(1-a_{i-1})a_i$$

or the largest index at which the maximum of the terms $(1-b_{i-1})b_i$ occurs is now less than j, and we may repeat this procedure until the maximum is decreased, or the terms are all equal. ∎

If we set

$$f_t(z) = \frac{t}{1-z}$$

then a sequence recursively defined by $a_i = a_1/(1-a_{i-1})$ for $i = 1,2,\ldots,k$ can alternatively be described by $a_1 = t$ and $a_j = f_t^{(j)}(0)$. Hence,

$$\nu_k = \inf\{\sqrt{t} : t \geq 0,\, f_t^{(k)}(0) = 1\}$$

or equivalently

$$\nu_k^2 = \inf\{t \geq 0 : f_t^{(k)}(0) = 1\}.$$

As mentioned earlier, it has been shown that $\delta_k > 1/2$ for all $k = 1,2,\ldots$. Since $\nu_k \geq \delta_k$, this implies that $\nu_k > 1/2$ as well. This can be seen directly from the above formula for ν_k. If, for some $k = 1,2,\ldots$, we have $\nu_k \leq 1/2$ then the above formula gives that $f_t^{(k)}(0) = 1$ for some t with $0 \leq t \leq 1/4$. But it is easily checked that for such t, we always have that

$$f_t(x) \leq \frac{1-\sqrt{1-4t}}{2} \quad \text{whenever} \quad 0 \leq x \leq \frac{1-\sqrt{1-4t}}{2}.$$

Note that $f_t^{(k)}(0)$ is a rational function of t so we can define

$$\frac{p_{k-1}(t)}{q_{k-1}(t)} = f_t^{(k)}(0)$$

for $k = 1,2,\ldots$, then $p_0(t) = t$, $q_0(t) = 1$ and

$$\begin{aligned}\frac{p_k(t)}{q_k(t)} &= f_t^{(k+1)}(0) = f_t\big(f_t^{(k)}(0)\big) = f_t\left(\frac{p_{k-1}(t)}{q_{k-1}(t)}\right) \\ &= \frac{t}{1-\frac{p_{k-1}(t)}{q_{k-1}(t)}} = \frac{tq_{k-1}(t)}{q_{k-1}(t)-p_{k-1}(t)}\end{aligned}$$

https://doi.org/10.4153/CJM-1995-043-3 Published online by Cambridge University Press

so we obtain the recursive formulas:

$$p_k(t) = tq_{k-1}(t) \quad \text{and} \quad q_k(t) = q_{k-1}(t) - p_{k-1}(t).$$

Combining these we get the recursive formula $q_k(t) = q_{k-1}(t) - tq_{k-2}(t)$ for $q_k(t)$, with initial conditions $q_{-1}(t) = 1$ and $q_0(t) = 1$.

The condition $f_t^{(k)}(0) = 1$ becomes $p_{k-1}(t) = q_{k-1}(t)$ or, by the recursive formula for $q_k(t)$, $q_k(t) = 0$. Therefore,

$$\nu_k^2 = \inf\{t \geq 0 : q_k(t) = 0, \text{ where } q_k(t) = q_{k-1}(t) - tq_{k-2}(t),\ q_{-1}(t) = 1,\ q_0(t) = 1\},$$

so we have determined ν_k, up to finding the smallest positive real zeroes of a sequence of inductively defined polynomials. Later, we will derive formulae for $q_k(t)$ and prove the general form of ν_k stated in Theorem 1, but first, let us see what is readily evident.

Clearly, the smallest zero of $q_1(t) = 1-t$ is 1, so $\nu_1 = 1$. Of course, this is trivial since this is the distance, in the space $\mathcal{B}(C^1) = C^1$ from the set of rank one projections ($\{1\}$) to the set of nilpotents ($\{0\}$). Also, $q_2(t) = q_1(t) - tq_0(t) = 1 - t - t = 1 - 2t$, which has smallest zero $1/2$, so $\nu_2 = 1/\sqrt{2}$ which agrees with the example given earlier. A little elementary computation shows that $q_3(t) = q_2(t) - tq_1(t) = 1 - 2t - 3t^2$ has smallest zero $(3 - \sqrt{5})/2$ so ν_3, the shortest distance from the set of rank-one projections to the set of nilpotents in $\mathcal{B}(C^3)$, is $\sqrt{(3 - \sqrt{5})/2}$. We can show even more in this case.

COROLLARY 4. *The shortest distance from the set of all non-zero projections to the set of nilpotents in the space of* 3×3 *matrices (i.e.* $\mathcal{B}(C^3)$*) is*

$$\delta_3 = \sqrt{\frac{3 - \sqrt{5}}{2}}.$$

PROOF. As shown above, the shortest distance from the set of rank-one projections to the set of nilpotents in $\mathcal{B}(C^3)$ is $\sqrt{(3 - \sqrt{5})/2}$, so to complete the proof we need only show that the shortest distance from the set of non-zero projections to the set of nilpotents can only be achieved by rank-one projections. Suppose $P \in \mathcal{P}(C^3)$ and has rank greater than 1. Then, if $N \in \mathcal{N}(C^3)$

$$2 \leq \operatorname{rank} P = \operatorname{trace}(P) = \operatorname{trace}(P - N) \leq \|\operatorname{trace}\|\,\|P - N\|,$$

since all nilpotents have trace 0. But, the trace has norm 3 on $\mathcal{B}(C^3)$ so $\|P-N\| \geq 2/3 > \sqrt{(3 - \sqrt{5})/2}$ for all nilpotents and all projections of rank greater than 1. ∎

This corollary tells us that, at least in $\mathcal{B}(C^3)$, the non-zero projections closest to the nilpotents (which have trace 0) have trace as small as possible. This heuristic guide leads to the following conjecture.

CONJECTURE. *For all $k = 1, 2, \ldots$, $\delta_k = \nu_k$, that is, a closest projection to the set of nilpotents in $\mathcal{B}(C^k)$ is of rank one.*

For larger values of k, an imitation of the proof of Corollary 5 gives that a closest projection to the set of nilpotents in $\mathcal{B}(C^k)$ can have rank no greater than $k/2$.

Returning to the main thread, the analysis of ν_k, our next step shall be to develop non-inductive formulae for the polynomials $q_k(t)$. Evidently, $q_k(t)$ is a polynomial of degree $[(k+1)/2]$ with integer coefficients. A straightforward induction proof shows that

$$q_k(t) = \sum_{n=0}^{\infty} \binom{k-n+1}{n} (-t)^n$$

where

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!\,(a-b)!}, & \text{if } a \geq b; \\ 0 & \text{otherwise.} \end{cases}$$

(Note that the n-th term in the sum is zero for $n > [(k+1)/2]$.) This formula shows that $q_k(t)$ has no negative zeroes; however, the following alternate description of $q_k(t)$ will better allow us to obtain exact values for ν_k for all values of k. These formulae for $q_k(t)$ are obtained by generalizing a common method of deriving values of the Fibonacci sequence.

The inductive formula $q_k(t) = q_{k-1}(t) - tq_{k-2}(t)$, where $q_{-1} = 1$ and $q_0 = 1$ can be rewritten as the 2×2 matrix equation with polynomial entries

$$\begin{aligned} \begin{pmatrix} q_k(t) \\ q_{k-1}(t) \end{pmatrix} &= \begin{pmatrix} 1 & -t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{k-1}(t) \\ q_{k-2}(t) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -t \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

The matrix

$$\begin{pmatrix} 1 & -t \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $(1 \pm \sqrt{1-4t})/2$, so, as long as $t \neq 1/4$, it can be diagonalized. Doing so,

$$\begin{pmatrix} 1 & -t \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \frac{1+\sqrt{1-4t}}{2} & 0 \\ 0 & \frac{1-\sqrt{1-4t}}{2} \end{pmatrix} P^{-1}$$

where

$$P = \begin{pmatrix} 1+\sqrt{1-4t} & 1-\sqrt{1-4t} \\ 2 & 2 \end{pmatrix}$$

and

$$P^{-1} = \frac{1}{4\sqrt{1-4t}} \begin{pmatrix} 2 & -1+\sqrt{1-4t} \\ -2 & 1+\sqrt{1-4t} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} q_k(t) \\ q_{k-1}(t) \end{pmatrix} = P \begin{pmatrix} \left(\frac{1+\sqrt{1-4t}}{2}\right)^k & 0 \\ 0 & \left(\frac{1-\sqrt{1-4t}}{2}\right)^k \end{pmatrix} P^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so by multiplying and considering the first coordinate of the resulting vector, we obtain that

$$q_k(t) = \frac{1}{\sqrt{1-4t}}\left(\left(\frac{1+\sqrt{1-4t}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{1-4t}}{2}\right)^{k+2}\right)$$

for all $t \neq 1/4$.

We shall be interested in the case where $t > 1/4$, (since we know that $\nu_k > 1/2$, there are no zeroes less than or equal to $1/4$), so to accent the complex nature, we rewrite

$$q_k(t) = \frac{-i}{\sqrt{4t-1}}\left(\left(\frac{1+i\sqrt{4t-1}}{2}\right)^{k+2} - \left(\frac{1-i\sqrt{4t-1}}{2}\right)^{k+2}\right).$$

Finding the smallest real zero of $q_k(t)$ is equivalent to finding the smallest positive zero of

$$p_k(y) = \frac{-i}{y}\left(\left(\frac{1+iy}{2}\right)^{k+2} - \left(\frac{1-iy}{2}\right)^{k+2}\right)$$

(setting $y = \sqrt{4t-1}$, we have that $p_k(y) = q_k(t)$ for $t \geq 1/4$).

Now $p_k(y) = 0$ for some $y > 0$ if and only if

$$(1+iy)^{k+2} = (1-iy)^{k+2}.$$

Let $\rho_{k+2} = e^{2\pi i/(k+2)}$ then

$$\left(\frac{1+iy}{1-iy}\right)^{k+2} = 1 \quad \text{implies that} \quad \frac{1+iy}{1-iy} = \rho_{k+2}^n$$

for some integer n. Solving y we obtain that $p_k(y) = 0$ when

$$\begin{aligned} y &= \frac{1}{i}\frac{\rho_{k+2}^n - 1}{\rho_{k+2}^n + 1} = \frac{1}{i}\frac{\rho_{k+2}^{n/2}(\rho_{k+2}^{n/2} - \rho_{k+2}^{-n/2})}{\rho_{k+2}^{n/2}(\rho_{k+2}^{n/2} + \rho_{k+2}^{-n/2})} \\ &= \frac{\rho_{k+2}^{n/2} - \rho_{k+2}^{-n/2}}{2i}\,\frac{2}{\rho_{k+2}^{n/2} + \rho_{k+2}^{-n/2}} = \frac{\sin\big(\pi n/(k+2)\big)}{\cos\big(\pi n/(k+2)\big)} = \tan\big(\pi n/(k+2)\big) \end{aligned}$$

for some integer n. Thus, the smallest positive zero of $p_k(y)$ occurs when $n = 1$. Since $y^2 = 4t - 1$, the smallest zero of $q_k(t)$ which is greater than $1/4$ is

$$r_k = \frac{\tan^2\big(\pi/(k+2)\big) + 1}{4} = \frac{\sec^2\big(\pi/(k+2)\big)}{4}$$

so

$$\nu_k = \sqrt{r_k} = \frac{\sec\big(\pi/(k+2)\big)}{2}.$$

This ends the proof of Theorem 1. ∎

An immediate corollary of Theorem 1 is the following result concerning infinite dimensional spaces.

COROLLARY 5. *For $\mathcal{H}$ an infinite dimensional Hilbert space, ν_∞, the distance from the set of all rank-one projections to the set of all nilpotents in the space $\mathcal{B}(H)$ is* $1/2$.

PROOF. This distance is greater than or equal to $1/2$ since, as mentioned in the preamble to Theorem 1, the distance from the set of all non-zero projections to the set of nilpotents is $1/2$. Clearly, however, by considering each C^k as a subspace of $\mathcal{H}$, we have that $\nu_\infty \leq \nu_k$ for all k. Hence

$$\nu_\infty \leq \lim_{k\to\infty} \nu_k = \lim_{k\to\infty} \sec\big(\pi/(k+2)\big)/2 = 1/2$$

and the corollary is proven. ∎

There is a quantity η_k, which is often considered in conjunction with δ_k, defined by

$$\eta_k = \{\|P-N\| : P \in \mathcal{P}(\mathcal{H}),\ N \in \mathcal{N}(\mathcal{H}),\ N^k = 0\}$$

where $\mathcal{H}$ is a separable infinite dimensional Hilbert space. It is apparent that $\eta_k \leq \delta_k \leq \nu_k$. In [5], Herrero conjectured that $\eta_k = \delta_k$ and a conjecture was also made about the bounds on η_k. He theorized that the sequence $\{k(\eta_k - 1/2)\}_{k=1}^{\infty}$ has a limit as k goes to infinity and that

$$\pi/2 < \lim_{k\to\infty} k(\eta_k - 1/2) < 2\pi.$$

Theorem 1 disproves the last part of the above conjecture, since it implies that

$$\begin{aligned}\limsup_{k\to\infty} k(\eta_k - 1/2) &\leq \limsup_{k\to\infty} k(\nu_k - 1/2)\\ &= \lim_{k\to\infty} k\left(\frac{\sec\big(\pi/(k+2)\big)}{2} - \frac{1}{2}\right) = 0\end{aligned}$$

contradicting the conjectured lower bound.

Actually finding the closest pairs $\{P_k, N_k\}$ which achieve the shortest distance in $\mathcal{B}(C^k)$ is problematic. One of the cornerstones in developing the formulae for ν_k was the Arveson Distance Formula. This gives a formula for the distance from a fixed operator A to the upper triangular operators. However, to explicitly find an upper triangular operator which is closest to A one must compute the square roots of a number of positive operators. Rather than do this we make a few reasonable assumptions which are borne out. If $\{P_k, N_k\}$ is a closest pair, then $P_k - N_k$ is an operator of norm ν_k. It seems reasonable that this operator be ν_k times a unitary, since otherwise, it might be possible to change P_k or N_k in some direction and decrease the distance between them. When this assumption is correct, starting from P_k, and assuming that $P_k - N_k$ is a multiple of a unitary, one is able to construct N_k by solving a few linear equations. In the process of doing so, we note that the unitary has a very special form. With respect to the basis in which N_k is strictly upper triangular (which we have chosen to be the standard basis), the matrix of this unitary has all the entries which are above the first superdiagonal equal to zero.

THEOREM 6. *For k a fixed positive integer, $\{a_i\}_{i=0}^k$ a nondecreasing sequence of positive numbers satisfying $a_0 = 0$, $a_1 = \nu_k^2$ and $a_i = a_1/(1 - a_{i-1})$ for $i = 1, 2, \ldots, k$, and $\{z_i\}_{i=1}^k$ a sequence of complex numbers of modulus 1, let*

$$e_k = (z_1\sqrt{a_1 - a_0}, z_2\sqrt{a_2 - a_1}, \ldots, z_k\sqrt{a_k - a_{k-1}})$$

and $P_k = e_k \otimes e_k^*$ so that P_k is the rank-one projection with matrix entries, with respect to the standard basis,

$$(P_k)_{i,j} = z_i z_j^* \sqrt{(a_i - a_{i-1})(a_j - a_{j-1})}$$

and N_k the nilpotent operator with matrix entries, with respect to the standard basis,

$$(N_k)_{i,j} = \begin{cases} 0, & \text{if } i \geq j; \\ z_i z_k^* \sqrt{\frac{a_j - a_{j-1}}{a_i - a_{i-1}}}(a_{i-1} + 1), & \text{if } i = j - 1; \\ z_i z_j^* \sqrt{(a_i - a_{i-1})(a_j - a_{j-1})}, & \text{if } i \leq j - 2. \end{cases}$$

then $\{P_k, N_k\}$ is a closest pair, and $P_k - N_k = U_k$, where U_k is ν_k times a unitary and has entries

$$(U)_{i,j} = \begin{cases} z_i z_j^* \sqrt{(a_i - a_{i-1})(a_j - a_{j-1})}, & \text{if } i \geq j; \\ -z_i z_j^* \sqrt{\frac{a_j - a_{j-1}}{a_i - a_{i-1}}}(\frac{a_1}{a_j}), & \text{if } i = j - 1; \\ 0 & \text{if } i \leq j - 2. \end{cases}$$

PROOF. In the proof of Theorem 1 it was determined that the rank-one projections which are closest to the set of nilpotents must be of the above form. It is clear that P_k is a projection and that N_k is a nilpotent. It is left as an exercise to check that U_k is ν_k times a unitary and that $P_k - N_k = U_k$. ∎

Note that it is not clear whether all closest pairs are of the above form; however, it seems reasonable that this be so. If every closest pair $\{P_k, N_k\}$ is such that their difference is a multiple of a unitary, then, by the uniqueness of the above construction, all closest pairs must be of the form in Theorem 5.

To save the reader from tedious calculation, note that

$$P_3 = \begin{pmatrix} \frac{3-\sqrt{5}}{2} & \sqrt{\frac{-11+5\sqrt{5}}{2}} & \frac{3-\sqrt{5}}{2} \\ \sqrt{\frac{-11+5\sqrt{5}}{2}} & -2+\sqrt{5} & \sqrt{\frac{-11+5\sqrt{5}}{2}} \\ \frac{3-\sqrt{5}}{2} & \sqrt{\frac{-11+5\sqrt{5}}{2}} & \frac{3-\sqrt{5}}{2} \end{pmatrix}$$

and

$$N_3 = \begin{pmatrix} 0 & \sqrt{\frac{-1+\sqrt{5}}{2}} & \frac{3-\sqrt{5}}{2} \\ 0 & 0 & \sqrt{\frac{-1+\sqrt{5}}{2}} \\ 0 & 0 & 0 \end{pmatrix}$$

are a closest pair in $\mathcal{B}(C^3)$ (and, by Corollary 4, a closest pair among all non-zero projections and all nilpotents as well). Also,

$$P_3 - N_3 = \begin{pmatrix} \frac{3-\sqrt{5}}{2} & \sqrt{-2+\sqrt{5}} & 0 \\ \sqrt{\frac{-11+5\sqrt{5}}{2}} & -2+\sqrt{5} & \sqrt{-2+\sqrt{5}} \\ \frac{3-\sqrt{5}}{2} & \sqrt{\frac{-11+5\sqrt{5}}{2}} & \frac{3-\sqrt{5}}{2} \end{pmatrix}$$

is $\sqrt{(3-\sqrt{5})/2}$ times a unitary.

In $\mathcal{B}(C^4)$, a projection P_4 and a nilpotent N_4 which achieve the shortest distance between the set of rank-one projections and set of nilpotents are

$$P_4 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad N_4 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$P_4 - N_4 = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3\sqrt{2}} & 0 & 0 \\ \frac{1}{3\sqrt{2}} & \frac{1}{6} & \frac{-1}{2} & 0 \\ \frac{1}{3\sqrt{2}} & \frac{1}{6} & \frac{1}{6} & \frac{-2}{3\sqrt{2}} \\ \frac{1}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix}$$

which is a $1/\sqrt{3}$ times a unitary.

ACKNOWLEDGEMENTS. The author wishes to thank Heydar Radjavi, Peter Fillmore and Yong Zhong for many helpful discussions.

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