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ON UNIMODALITY OF THE LIFETIME DISTRIBUTION OF COHERENT SYSTEMS WITH FAILURE-DEPENDENT COMPONENT LIFETIMES

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Abstract

We study the conditions for unimodality of the lifetime distribution of a coherent system when the ordered component lifetimes in the system are described by generalized order statistics. Results for systems with independent and identically distributed lifetimes of components are included in this setting. The findings are illustrated with some examples for different types of systems. In particular, coherent systems with strictly bimodal density functions are presented in the case of independent standard uniform distributed lifetimes of components. Furthermore, we use the results to derive a sharp upper bound on the expected system lifetime in terms of the mean and the standard deviation of the underlying distribution.

Keywords: Unimodality; bimodality; signature; coherent system; sequential order statistics; generalized order statistics; variation diminishing property; bounds

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1. Introduction

The study of unimodality for lifetime distributions of particular system structures dates back to the beginnings of modern reliability theory. Barlow and Proschan (1966) and Alam (1972) established conditions for unimodal distributions of the lifetime of k-out-of-n systems with independent and identically distributed (i.i.d.) component lifetimes (see also Huang and Ghosh (1982) and Dharmadhikari and Joag-Dev (1988)). Following this, Sabnis and Nair (1997) generalized Alam's result to the setting of coherent systems. Moreover, the results on k-out-of-n systems have also been extended in another direction. For the subset of k-out-of-n systems, the system lifetime coincides with that of a fixed order statistic among the ordered component lifetimes. A unifying framework for a number of models for ordered data including the usual order statistics is provided by generalized order statistics introduced by Kamps (1995) (see also Kamps (2016)). Results on unimodality in this larger model of generalized order statistics were presented in Cramer (2004), Cramer *et al.* (2004), Chen *et al.* (2009), Alimohammadi and Alamatsaz (2011), and Alimohammadi *et al.* (2016).

In this paper we study the unimodality of the lifetime distribution of coherent systems with failure-dependent component lifetimes. A univariate continuous distribution function G is called unimodal with a mode at $m \in \mathbb{R}$ if G is convex on $(-\infty, m)$ and concave on (m, ∞).

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Consider the distribution function for a coherent system consisting of n components with failure-dependent lifetimes $X_1^*, \dots, X_n^*$ based on continuous distribution functions $F_1, \dots, F_n$. Let ϕ denote the associated coherent life function of the system (see Esary and Marshall (1970) or Barlow and Proschan (1975, p. 12)) and let $\boldsymbol{s} = (s_1, \dots, s_n)$ denote the associated signature (see Samaniego (1985) and Samaniego (2007)). Then the system lifetime $T = \phi(X_1^*, \dots, X_n^*)$ can be expressed as (see Burkschat (2009) and Navarro and Burkschat (2011))

$$F_T(t) = \sum_{i=1}^{n} s_i \mathbb{P}(X_{i:n}^* \le t), \qquad t \in \mathbb{R},$$

where $X_{1:n}^* \le \cdots \le X_{n:n}^*$ denote the sequential order statistics based on $F_1, \dots, F_n$ (see Kamps (1995), Cramer and Kamps (2001), and Cramer (2016)). Due to the construction of sequential order statistics, the exchangeable random variables $X_1^*, \dots, X_n^*$ describe component lifetimes in an environment where it is possible that component failures affect the performance of remaining intact components (see, e.g. Hollander and Peña (1995), Aki and Hirano (1997), Burkschat (2009), and Navarro and Burkschat (2011)).

In what follows, we assume that the distribution functions $F_1, \dots, F_n$ possess proportional hazard rates. Therefore, we impose the condition that

$$F_i(t) = 1 - (1 - F(t))^{\alpha_i}, \qquad t \in \mathbb{R},$$

with parameters $\alpha_1, \dots, \alpha_n > 0$ and an absolutely continuous distribution function F with density function f. Moreover, we assume that the density f is positive on an interval (a, b), where $0 \le a < b \le \infty$, and 0 otherwise. However, the results remain valid also for $-\infty \le a < b \le \infty$.

If no change in the underlying distributions takes place, i.e. in the $\alpha_1 = \cdots = \alpha_n = 1$ case, then the random variable T describes the lifetime of a coherent system with signature $\boldsymbol{s} = (s_1, \dots, s_n)$ and i.i.d. component lifetimes with distribution function F. Under the general assumption of proportional hazard rates, the sequential order statistics have the same joint distribution as generalized order statistics with model parameters $\gamma_i = (n-i+1)\alpha_i$, $1 \le i \le n$, and the above survival function of the system lifetime can be expressed as (see Burkschat and Navarro (2018))

$$\mathbb{P}(T > t) = \overline{q}(\overline{F}(t)), \qquad t \in \mathbb{R}, \tag{1.1}$$

with the survival function $\overline{F} = 1 - F$ and the distortion function

$$\overline{q}(x) = 1 - \sum_{r=1}^{n} s_r F_{*,r}(1 - x), \qquad x \in [0, 1],$$

where $F_{*,1}, \dots, F_{*,n}$ denote the distribution functions of uniform generalized order statistics with parameters $\gamma_1, \dots, \gamma_n$. In particular, the lifetime T has the density

$$f_T(t) = \sum_{r=1}^{n} s_r f_{*,r}(F(t)) f(t), \qquad t \in (a, b),$$

with the corresponding densities $f_{*,1}, \dots, f_{*,n}$ of the uniform generalized order statistics. In the particular case of an underlying standard uniform distribution with $F(x) = x$, $x \in [0, 1]$, we will denote the density function of the system lifetime by

$$g_{\boldsymbol{s},\boldsymbol{\gamma}}(u) = \sum_{r=1}^{n} s_r f_{*,r}(u), \qquad u \in (0, 1), \tag{1.2}$$

with the signature $\boldsymbol{s} = (s_1, \ldots, s_n)$ and the parameter vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ with entries $\gamma_i = (n - i + 1)\alpha_i$, $1 \le i \le n$. In order to evaluate the distribution function and the density of the system lifetime, corresponding representations for uniform generalized order statistics given in Kamps and Cramer (2001) and Cramer and Kamps (2003) can be utilized.

The paper is organized as follows. In Section 2 we present our main results on unimodality (and bimodality) for coherent systems in the present set-up. Section 3 contains some examples of system types, where unimodality and bimodality can be verified with our means. Finally, an application yielding sharp upper bounds on the expected system lifetime is given in Section 4.

In this paper, the terms increasing and decreasing are used in the weak sense, i.e. a function is called increasing (decreasing) if it is nondecreasing (nonincreasing).

2. Conditions for uni- and bimodality

In this section we study the unimodality of the density function f_T for the system lifetime given in the introduction. We call a density function g, which is positive on (a, b) and 0 otherwise, unimodal with mode m if there is some $m \in [a, b]$ such that g is increasing on (a, m) and decreasing on (m, b). Clearly, the corresponding distribution function is unimodal. If, additionally, $m \in (a, b)$ and g is strictly increasing on (a, m) and strictly decreasing on (m, b), then g is called strictly unimodal (otherwise g can be just increasing or decreasing). Moreover, g is called log-concave if $\ln g$ is concave on (a, b). It is well known that a log-concave density function is also unimodal.

Unimodality has been studied by Sabnis and Nair (1997) for coherent systems with i.i.d. component lifetimes. The authors obtained an extension of a result by Alam (1972) in which the density function of the usual order statistics was considered. A corresponding result for systems with failure-dependent components is given in the following theorem.

Theorem 2.1. *Let* $1/f$ *be convex on* (a, b). *If* $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ *is log-concave then the density function* f_T *is unimodal.*

Proof. The function $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is positive and differentiable on (0, 1) (see Cramer *et al.* (2004)). Now, the result can be proved along the lines of the proof of Sabnis and Nair (1997, Theorem 2.1) (see also Cramer (2004, Theorem 2.8)) by considering (1.1) and observing that the derivative $\overline{q}'(u) = g_{\boldsymbol{s},\boldsymbol{\gamma}}(1 - u)$, $u \in (0, 1)$, is log-concave if and only if $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is log-concave. This completes the proof. □

Since log-concavity of the density function $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ of the system lifetime for an underlying standard uniform distribution is relevant in order to achieve unimodality of the corresponding density for general F, we study the weaker condition of unimodality of $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ in greater detail. Moreover, by deriving conditions for bimodality, we are able to show that there exist coherent systems such that the log-concavity property of $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is not satisfied (see Section 3). Bimodal density functions are defined analogously to the unimodal density functions.

In what follows we will use the variation diminishing property of $f_{*,1}, \ldots, f_{*,n}$ with arbitrary parameters $\gamma_1, \ldots, \gamma_n$ (see Bieniek (2007, Theorems 1–3 and Corollary 1)). For a fixed set of parameters $\gamma_1, \ldots, \gamma_n > 0$, we denote $\gamma_{1:n} = \min(\gamma_1, \ldots, \gamma_n)$ and

$$\ell = \max\{1 \le j \le n \colon \gamma_j = \gamma_{1:n}\}.$$

In other words, γ_ℓ is the last repetition of the smallest element of the sequence $\gamma_1, \ldots, \gamma_n$. Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be any fixed sequence of coefficients. We are interested in sign changes

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of the linear combination

$$H_{\boldsymbol{a}}(u) = \sum_{r=1}^{n} a_r f_{*,r}(u), \qquad u \in (0,1).$$

Theorem 2.2. *For* $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, *define*

$$k = \min\{1 \le j \le n : a_j \ne 0\}, \quad m = \max\{1 \le j \le n : a_j \ne 0\},$$

i.e. a_k *is the first nonzero element of* $\boldsymbol{a}$ *and* a_m *is the last nonzero element of* $\boldsymbol{a}$.

(i) *The number of zeroes of* $H_{\boldsymbol{a}}$ *in* $(0,1)$ *does not exceed the number of sign changes in the sequence* $\boldsymbol{a}$ *after deletion of zeroes.*

(ii) *The first sign of* $H_{\boldsymbol{a}}$ *coincides with the sign of* a_k.

(iii) *The last sign of* $H_{\boldsymbol{a}}$ *coincides with the sign of*

$$A_n(\boldsymbol{a}) = a_m + \sum_{j=\max(k,\ell)}^{m-1} a_j \prod_{i=j+1}^{m} (\gamma_i - \gamma_\ell).$$

(iv) *If* $\gamma_1 \ge \dots \ge \gamma_n$ *then the last sign of* $H_{\boldsymbol{a}}$ *is the same as the sign of* a_m.

(v) *The above statements hold if* $(0,1)$ *is replaced with* (a,b) *and* $f_{*,1}, \dots, f_{*,n}$ *with* $\hat{f}_{*,1}, \dots, \hat{f}_{*,n}$, *where* $\hat{f}_{*,r} = f_{*,r} \circ W$ *and* $W\colon (a,b) \to (0,1)$ *is some strictly increasing continuous distribution function.*

We aim at finding the conditions on $s_1, \dots, s_n$ and $\gamma_1, \dots, \gamma_n$ which ensure strict unimodality or at worst bimodality of $g_{s,\gamma}$ (see (1.2)). To this aim, we study the sign changes of the derivative $g'_{s,\gamma}$. We have, for $1 \le r \le n$ (see Cramer *et al.* (2004)),

$$f'_{*,r}(u) = \frac{1}{1-u}[\gamma_r f_{*,r-1}(u) - (\gamma_r - 1) f_{*,r}(u)],$$

adopting the convention $f_{*,0} \equiv 0$. Therefore, a straightforward computation reveals that $g'_{s,\gamma}$ can be expressed as a linear combination of $f_{*,1}, \dots, f_{*,n}$:

$$g'_{s,\gamma}(u) = \frac{1}{1-u} \sum_{r=1}^{n} b_r f_{*,r}(u), \tag{2.1}$$

where

$$b_r = \begin{cases} s_{r+1}\gamma_{r+1} - s_r(\gamma_r - 1) & \text{for } 1 \le r < n, \\ -s_n(\gamma_n - 1) & \text{for } r = n. \end{cases} \tag{2.2}$$

We consider the important case of i.i.d. component lifetimes first. This corresponds to the situation where the successive failures in the system are modeled by the usual order statistics.

Theorem 2.3. *Assume the case of i.i.d. component lifetimes, i.e.* $\gamma_r = n - r + 1$ *for* $r = 1, \dots, n$.

(i) *Let* $n \ge 2$. *If the sequence* $s_1, \dots, s_n$ *is increasing (decreasing) with* $s_1 < (>)\, s_n$ *then* $g_{s,\gamma}$ *is strictly increasing (decreasing).*

(ii) *Let $n \geq 3$. If the signature $\boldsymbol{s}$ is strictly unimodal, i.e. there exists $1 < k < n$ such that*

$$s_1 \leq s_2 \leq \cdots \leq s_k, \qquad s_k \geq s_{k+1} \geq \cdots \geq s_n \tag{2.3}$$

with at least one strict inequality in each block, then $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is strictly unimodal.

(iii) *Let $n \geq 5$. If the signature $\boldsymbol{s}$ is strictly bimodal, i.e. there exist $1 < k < l < m < n$ such that*

$$s_1 \leq s_2 \leq \cdots \leq s_k, \qquad s_k \geq s_{k+1} \geq \cdots \geq s_l,$$
$$s_l \leq s_{l+1} \leq \cdots \leq s_m, \qquad s_m \geq s_{m+1} \geq \cdots \geq s_n$$

with at least one strict inequality in each block, then $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is either strictly unimodal or strictly bimodal.

Proof. By assumption, $\gamma_r = n - r + 1$ and, consequently,

$$b_r = \begin{cases} (s_{r+1} - s_r)(n - r) & \text{for } 1 \leq r < n, \\ 0 & \text{for } r = n. \end{cases} \tag{2.4}$$

Moreover, we clearly have $\gamma_1 > \cdots > \gamma_n$ so Theorem 2.2(iv) is applicable. By (2.4) we have, in general, $b_n = 0$.

In case (i), it follows that $b_1, \ldots, b_{n-1} \geq (\leq)\, 0$ with at least one positive (negative) element among the b_r. Thus, according to Theorem 2.2(i), $g'_{\boldsymbol{s},\boldsymbol{\gamma}}$ has no zeroes in $(0, 1)$ and, therefore, Theorem 2.2(ii) yields that $g'_{\boldsymbol{s},\boldsymbol{\gamma}}$ is positive (negative) on $(0, 1)$.

In case (ii), if (2.3) holds then

$$b_1, \ldots, b_{k-1} \geq 0, \qquad b_k, \ldots, b_{n-1} \leq 0$$

with at least one positive element in the former block, and at least one negative in the latter. By Theorem 2.2(i), there is at most one sign change of $g'_{\boldsymbol{s},\boldsymbol{\gamma}}$. But, by Theorem 2.2(ii), the first sign of $g'_{\boldsymbol{s},\boldsymbol{\gamma}}$ is positive, and by part (iv) of the same theorem, the last sign of $g'_{\boldsymbol{s},\boldsymbol{\gamma}}$ is negative. Therefore, $g'_{\boldsymbol{s},\boldsymbol{\gamma}}$ is first positive and finally negative ($+-$ for short), so $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ itself is strictly unimodal.

In case (iii), similar arguments show that if $\boldsymbol{s}$ is bimodal then $g'_{\boldsymbol{s},\boldsymbol{\gamma}}$ is either $+-$ or $+-+-$. This yields the assertion. □

Remark 2.1. In the above proof we have shown that for i.i.d. component lifetimes, the variability of $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is at worst the same as that of the signature $\boldsymbol{s}$.

Remark 2.2. The majority of coherent systems have unimodal signatures and therefore satisfy the assumptions of Theorem 2.3(ii). However, there are also systems that fulfill the conditions of Theorem 2.3(i) and 2.3(iii). Examples of coherent systems with increasing signature entries are the parallel systems for arbitrary $n \geq 2$ and the systems with signature vector $(0, 0, \frac{1}{2}, \frac{1}{2})$ in the case of $n = 4$ components (see Samaniego (2007, Table 3.2, System 17)) as well as $(0, 0, \frac{3}{10}, \frac{3}{10}, \frac{2}{5})$ and $(0, 0, 0, \frac{2}{5}, \frac{3}{5})$ in the case of $n = 5$ components (see Navarro and Rubio (2010, Table 2, Systems 176 and 179)). Signatures with decreasing entries are obtained by considering the corresponding dual systems (see Samaniego (2007, Theorem 3.3)). An example of a bimodal signature vector in the $n = 5$ case is provided by Jasiński *et al.* (2009); namely, $(0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}, 0)$. Note that the density $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ of this system is strictly unimodal in the i.i.d. case. An extension to higher orders yielding strictly bimodal densities can be found in Section 3.

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Now let us consider the situation of arbitrary $\alpha_1, \ldots, \alpha_n > 0$ which corresponds to the general setting of failure-dependent component lifetimes. Then $\gamma_r = \alpha_r(n - r + 1)$ holds and we define the vector $\boldsymbol{b} = (b_1, \ldots, b_n)$ with the entries given by (2.2). A general condition for strict unimodality of $g_{s,\gamma}$ is stated in the following theorem. The proof proceeds analogously to that of Theorem 2.3. Moreover, other cases can be obtained similarly by arguing with the results on the variation diminishing property in Theorem 2.2.

Theorem 2.4. *Let $n \geq 2$ and $\gamma_1, \ldots, \gamma_n > 0$ with $A_n(\boldsymbol{b}) < 0$. If there exists $k \in \{2, 3, \ldots, n\}$ such that*

$$b_1, \ldots, b_{k-1} \geq 0, \qquad b_k, \ldots, b_n \leq 0$$

with at least one strict inequality in each block, then $g_{s,\gamma}$ is strictly unimodal. For $\gamma_1 \geq \cdots \geq \gamma_n > 0$, the condition $A_n(\boldsymbol{b}) < 0$ can be dropped.

Remark 2.3. By considering signature vectors with one entry equal to 1 and the remaining ones equal to 0, results for k-out-of-n systems can be obtained. In our setting these just correspond to densities of single generalized order statistics. Unimodality in this case has been discussed in detail by Cramer *et al.* (2004) and Bieniek (2007) (see also the recent results on strong unimodality in Alimohammadi *et al.* (2016)).

Example 2.1. Consider a situation where the load on an n-component system is evenly distributed among the remaining units after each failure. Then, at the rth stage, the (constant) overall load on the system is described by $\gamma_r = n$ and the individual load on the intact components by $\alpha_r = n/(n - r + 1)$ (see, e.g. Balakrishnan *et al.* (2011, Example 1) and Burkschat and Navarro (2013, Remark 2.3)). In particular, the failure times can be also interpreted as n-records (see Dziubdziela and Kopociński (1976)). It follows that

$$b_r = \begin{cases} (s_{r+1} - s_r)n + s_r & \text{for } 1 \leq r < n, \\ -s_n(n-1) & \text{for } r = n. \end{cases}$$

For example, assume that $\boldsymbol{s}$ is increasing with $s_1 < s_n$. Then, for some $1 \leq r_0 < n$, $b_1 = \cdots = b_{r_0-1} = 0$, $b_r > 0$, $r = r_0, \ldots, n-1$, and $b_n < 0$. Hence, the density $g_{s,\gamma}$ is strictly unimodal according to Theorem 2.4. Note that in the setting without load sharing, i.e. for i.i.d. component lifetimes, the corresponding density is strictly increasing under this assumption.

If the location of the mode of unimodal signature entries $s_1, \ldots, s_n$ and the behavior of the model parameters $\gamma_1, \ldots, \gamma_n$ match in some way, then the following condition can be checked.

Corollary 2.1. *Assume that the signature $\boldsymbol{s}$ is strictly unimodal so that (2.3) holds for some $k \in \{2, \ldots, n-1\}$. If $A_n(\boldsymbol{b}) < 0$ and*

$$\gamma_r - \gamma_{r+1} \begin{cases} \leq 1 & \text{for } 1 \leq r < k, \\ \geq 1 & \text{for } k \leq r < n, \end{cases}$$

then $g_{s,\gamma}$ is strictly unimodal.

So far, we have studied the unimodality of $g_{s,\gamma}(x)$, $x \in (0, 1)$, which coincides with the density function $f_T(x) = g_{s,\gamma}(F(x))f(x)$ of the system lifetime in the case of an underlying uniform distribution on $[0, 1]$. In the remainder of the section, we want to determine other distributions F for which the above approach may be effectively applied. Note that

$$f_T'(x) = g_{s,\gamma}'(F(x))(F'(x))^2 + g_{s,\gamma}(F(x))F''(x), \qquad x \in (a, b).$$

Using (1.2) and (2.1), we easily see that

$$f_T'(x) = \frac{(F'(x))^2}{1-F(x)} \sum_{r=1}^{n} b_r f_{*,r}(F(x)) + F''(x) \sum_{r=1}^{n} s_r f_{*,r}(F(x)), \qquad x \in (a,b).$$

In order to apply the variation diminishing property, we need to have

$$F''(x) = c\frac{(F'(x))^2}{1-F(x)} \quad \text{for some } c \in \mathbb{R}.$$

However, Bieniek (2008, Lemma 2.1) states that all lifetime distributions (with $F(0) = 0$) that satisfy this relation have a distribution function of the form $F(x) = W_\alpha(\lambda x)$ for some $\lambda > 0$, $\alpha \in \mathbb{R}$, and $c = \alpha - 1$, where W_α denotes the distribution function of the generalized Pareto distribution defined by

$$W_\alpha(x) = \begin{cases} 1-(1-\alpha x)^{1/\alpha} & \text{for } x \geq 0 \text{ if } \alpha < 0, \\ 1-(1-\alpha x)^{1/\alpha} & \text{for } 0 \leq x \leq 1/\alpha \text{ if } \alpha > 0, \\ 1-\mathrm{e}^{-x} & \text{for } x \geq 0 \text{ if } \alpha = 0. \end{cases}$$

Therefore, if $F = W_\alpha$ then setting $\hat{f}_{*,r} = f_{*,r} \circ W_\alpha$, we have

$$f_T'(x) = \frac{(W_\alpha'(x))^2}{1-W_\alpha(x)} \sum_{r=1}^{n} d_r \hat{f}_{*,r}(x),$$

where

$$d_r = \begin{cases} s_{r+1}\gamma_{r+1} - s_r(\gamma_r - \alpha) & \text{for } 1 \leq r < n, \\ -(\gamma_n - \alpha)s_n & \text{for } r = n. \end{cases} \tag{2.5}$$

Due to Theorem 2.2(v), results on the unimodality of this density can be obtained analogously to Theorems 2.3 and 2.4 by examining the sign changes in the sequence $d_1, \ldots, d_n$.

Example 2.2. Consider the choice of model parameters as in Example 2.1 in the case of an underlying exponential distribution ($\alpha = 0$). Then

$$d_r = \begin{cases} n(s_{r+1} - s_r) & \text{for } 1 \leq r < n, \\ -ns_n & \text{for } r = n. \end{cases}$$

Hence, if the signature $\boldsymbol{s}$ is strictly unimodal then the density function f_T is also strictly unimodal on $(0, \infty)$.

Remark 2.4. The coefficients d_r in (2.5) illustrate the considerable influence of the underlying distribution F on the monotonicity behavior of the density f_T irrespective of the modality properties of $g_{\boldsymbol{s},\boldsymbol{\gamma}}$. For convenience, let $\gamma_1 \geq \cdots \geq \gamma_n$. For arbitrary signatures vectors $(s_1, \ldots, s_n)$, all d_r become nonnegative if $\alpha > 0$ is chosen appropriately large. Since at least one of the entries $s_1, \ldots, s_n$ is positive, it follows that for the choice $F(x) = W_\alpha(\lambda x)$, $0 \leq x \leq 1/(\lambda\alpha)$ with suitable $\alpha > 0$, the density f_T is then strictly increasing on the interval $(0, 1/(\lambda\alpha))$. Moreover, for sufficiently small $\alpha < 0$, f_T is strictly decreasing on $(0, \infty)$ if $s_1 > 0$, and strictly unimodal if $s_1 = 0$, since the signature has no internal zeroes (see Ross *et al.* (1980, Theorem 2) and D'Andrea and De Sanctis (2015)).

3. Illustrative examples

In this section we consider more elaborate examples to illustrate our findings. In the first example, we present the construction of a class of systems that possesses strictly bimodal density functions $g_{s,\gamma}$ of the system lifetimes in the setting of i.i.d. component lifetimes distributed according to a uniform distribution on [0, 1]. The example is of particular interest since Jasiński *et al.* (2009) conjectured that in this setting every density function $g_{s,\gamma}$ is at most unimodal. As a consequence, we provide a counterexample to this conjecture. Moreover, the considered systems provide examples for density functions $g_{s,\gamma}$ that obviously do not satisfy the log-concavity assumption in Theorem 2.1.

Example 3.1. Consider a coherent system with $n \geq 5$ components which is described by the following minimal path sets:

$$P_1 = \{2, \dots, n\}, \qquad P_k = \{1, k\}, \quad k \in \{2, \dots, n\}.$$

Then the structure function ϕ of this system can be expressed as, for instance,

$$\phi(x_1, \dots, x_n) = \max(\min(x_2, \dots, x_n), \min(x_1, x_2), \min(x_1, x_3), \dots, \min(x_1, x_n))$$

with $x_1, \dots, x_n \in \{0, 1\}$. This system was considered by Jasiński *et al.* (2009) in the $n = 5$ case. Then the corresponding signature vector $(0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}, 0)$ is strictly bimodal. We will now show that, for $n \geq 5$, the signature vector $\boldsymbol{s} = (s_1, \dots, s_n)$ of the system has the entries

$$s_1 = 0, \qquad s_2 = \frac{2}{n}, \qquad s_j = \frac{1}{n}, \quad j \in \{3, \dots, n-2\}, \qquad s_{n-1} = \frac{2}{n}, \qquad s_n = 0.$$

In particular, the system has a strictly bimodal signature vector for every $n \geq 5$. In order to derive the above signature, we make use of a representation from Boland (2001) (see also Marichal *et al.* (2011)):

$$s_i = a_{n-i+1} - a_{n-i}, \qquad i \in \{1, \dots, n\}, \tag{3.1}$$

where $a_0 = 0$ and

$$a_i = \frac{\#\text{ path sets of size } i \text{ for the system}}{\binom{n}{i}}, \qquad i \in \{1, \dots, n\}.$$

In the present setting, we obtain

$$\#\text{ path sets of size } i \text{ for the system} = \begin{cases} 0, & i = 1, \\ n-1, & i = 2, \\ \binom{n-1}{i-1}, & i = 3, \dots, n-2, \\ \binom{n}{n-1}, & i = n-1, \\ 1, & i = n. \end{cases}$$

For instance, the numbers in the third case can be obtained by subtracting the number $\binom{n-1}{i}$ of subsets of the path set P_1 that have i elements from the total number $\binom{n}{i}$ of possible path sets of size i. Then, it follows that

$$a_1 = 0, \qquad a_2 = \frac{2}{n}, \qquad a_i = \frac{i}{n}, \quad i \in \{3, \dots, n-2\}, \qquad a_{n-1} = 1, \qquad a_n = 1,$$

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and using (3.1) this yields the claimed entries of the signature. Now we study the density function $g_{s,\gamma}$ of the system lifetime distribution in the case of i.i.d. component lifetimes (i.e. $\gamma_r = n - r + 1$ for $1 \le r \le n$) following a standard uniform distribution. Then the density function of the system lifetime T is given by

$$g_{s,\gamma}(t) = \sum_{i=1}^{n} i s_i \binom{n}{i} t^{i-1}(1-t)^{n-i}, \qquad t \in (0,1).$$

By substituting the known entries of the signature, it follows that

$$g_{s,\gamma}(t) = 1 - (1-t)^{n-1} - t^{n-1} + (n-1)t(1-t)^{n-2} + (n-1)t^{n-2}(1-t), \qquad t \in (0,1).$$

Since the function is symmetric about $\frac{1}{2}$, it has a local extremum there. For $n = 5$ and 6, this is a maximum point, since then $g'_{s,\gamma}(t)$ is equal to $4(1-2t)(5t^2 - 5t + 2)$ and $10(1-2t)^3$, respectively. Otherwise, $\frac{1}{2}$ is a minimum point, since

$$g''_{s,\gamma}\left(\frac{1}{2}\right) = \frac{16(n-1)(n-2)(n-6)}{2^n} > 0 \quad \text{for } n \ge 7.$$

Therefore, due to $g_{s,\gamma}(0) = g_{s,\gamma}(1) = 0$, it follows that $g_{s,\gamma}$ has at least one local maximum in each interval $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. Due to Theorem 2.3(iii), the density $g_{s,\gamma}$ can have at most two modes and, consequently, it is strictly bimodal for every $n \ge 7$.

Remark 3.1. Note that, in this example, $g_{s,\gamma} = \varphi_{1,n-1:n}$, where

$$\varphi_{r,s:n} = \frac{1}{n}\left((r+1) f_{r+1:n} + \sum_{i=r+2}^{s-1} f_{i:n} + (n-s+1) f_{s:n}\right)$$

with

$$f_{i:n}(t) = i\binom{n}{i} t^{i-1}(1-t)^{n-i}, \qquad t \in (0,1),$$

as defined by Bieniek (2016), while deriving optimal bounds on expectations of Winsorized means based on i.i.d. samples.

In the following, we cover two generalizations of k-out-of-n:F systems obtained from systemwise redundancy and componentwise redundancy. Each of the resulting systems consist of $2n$ components. We consider the particular choice of model parameters

$$\gamma_j = 2n - j + 1 + \beta(j-1), \qquad j \in \{1, \dots, 2n\},$$

with $\beta \in [0,1]$. Note that $\gamma_1 \ge \cdots \ge \gamma_{2n}$ and $\alpha_1 \le \cdots \le \alpha_{2n}$ hold. The parameters can be interpreted as follows. For $\beta = 1$, i.e. $\gamma_j = 2n$, the total load on the system remains unchanged after every failure (see Example 2.1). For $\beta = 0$, i.e. $\gamma_j = 2n - j + 1$, there is no load sharing and we are in the classical situation of i.i.d. component lifetimes. In these systems, the load of the failed components is not distributed among the surviving components. For $\beta \in (0,1)$, a fixed proportion β of the individual initial load on each component is still imposed on the system after every failure. In particular, the parameters

$$\alpha_j = 1 + \beta \cdot \frac{j-1}{2n-j+1}, \qquad j \in \{1, \dots, 2n\},$$

describe the adjusted individual load on every intact unit before the jth failure under the assumption that the system load is divided evenly among these units. Moreover, the overall load γ_j on the system is strictly decreasing, while the load α_j on the individual components is strictly increasing with each loss of a unit. Observe that the model parameters can be also written as a convex combination

$$\gamma_j = \beta \cdot 2n + (1-\beta) \cdot (2n - j + 1)$$

of both extreme cases described above.

Example 3.2. Consider the signature $(s_1, \dots, s_{2n})$ of a k-out-of-n:F system with systemwise redundancy (see Samaniego (2007, Theorem 4.7)):

$$s_{2k+r} = \frac{\binom{n-1}{k-1}\binom{n}{k+r}}{\binom{2n-1}{2k+r-1}}, \qquad r \in \{0, \dots, n-k\},$$

and $s_i = 0$ for $i \in \{1, \dots, 2k-1\} \cup \{n+k+1, \dots, 2n\}$. Then the distribution of the lifetime of the corresponding system based on sequential order statistics with model parameters $\gamma_j = 2n - j + 1 + \beta(j-1)$, $j = 1, \dots, 2n$, and underlying standard uniform distribution has a unimodal density function $g_{s,\boldsymbol{\gamma}}$ for $1 \le k \le n$ and $\beta \in [0, 1]$. The density is strictly unimodal except for the case of $k = n$ and $\beta = 0$, where the density is strictly increasing.

The derivation proceeds as follows. According to Theorem 2.4, we have to study the signs of the nonzero elements of the sequence

$$s_2\gamma_2 - s_1(\gamma_1 - 1), \dots, s_{2n}\gamma_{2n} - s_{2n-1}(\gamma_{2n-1} - 1), -s_{2n}(\gamma_{2n} - 1). \tag{3.2}$$

At first, let $k = n$. Then $s_{2n} = 1$, $s_r = 0$, $r = 1, \dots, 2n-1$, and so we obtain

$$s_{2n}\gamma_{2n} - s_{2n-1}(\gamma_{2n-1} - 1) = s_{2n}\gamma_{2n} = 1 + \beta(2n-1) > 0,$$
$$-s_{2n}(\gamma_{2n} - 1) = -\beta(2n-1) \le 0.$$

The other entries in sequence (3.2) are 0. Consequently, for $\beta > 0$, the density is strictly unimodal due to Theorem 2.4. Note that the density is strictly increasing on $(0, 1)$ for $\beta = 0$. Now let $n \ge 2$ and $k < n$. It can be shown, after some calculation, that

$$s_{2k+r+1}\gamma_{2k+r+1} - s_{2k+r}(\gamma_{2k+r} - 1) < 0$$

if and only if $-(2(n-k)-r)(2(n-k)+r(n-1)) < \beta[(2k-2)(n-k)+k(2n-1)r+nr^2+r]$.

The expression on the left-hand side of the preceding inequality is strictly negative. Moreover, the expression in brackets on the right-hand side is nonnegative. Thus, the inequality is valid for every $\beta \in [0, 1]$. Therefore, we obtain

$$s_{2k}\gamma_{2k} - s_{2k-1}(\gamma_{2k-1} - 1) = s_{2k}\gamma_{2k} > 0,$$
$$s_{2k+r+1}\gamma_{2k+r+1} - s_{2k+r}(\gamma_{2k+r} - 1) < 0, \qquad r \in \{0, \dots, n-k-1\},$$
$$s_{n+k+1}\gamma_{n+k+1} - s_{n+k}(\gamma_{n+k} - 1) = -s_{n+k}(\gamma_{n+k} - 1) < 0$$

with $\gamma_{n+k} - 1 = n - k + \beta(n-k-1) > 0$ and all the remaining entries of sequence (3.2) are 0. By applying Theorem 2.4, we obtain the claimed result.

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Example 3.3. The signature $(s_1, \ldots, s_{2n})$ of a k-out-of-n:F system with componentwise redundancy (see Samaniego (2007, Theorem 4.8)) is given by

$$s_{2k+r} = \frac{\binom{n-1}{k-1}\binom{n-k}{r}}{\binom{2n-1}{2k+r-1}} \cdot 2^r, \qquad r \in \{0, \ldots, n-k\},$$

and $s_i = 0$ for $i \in \{1, \ldots, 2k-1\} \cup \{n+k+1, \ldots, 2n\}$. The density function $g_{s,\gamma}$ of the lifetime distribution of the corresponding system based on sequential order statistics with model parameters $\gamma_j = 2n - j + 1 + \beta(j-1)$, $j = 1, \ldots, 2n$, and underlying standard uniform distribution is unimodal for $1 \le k \le n$ and $\beta \in [0, 1]$. The density is strictly unimodal except for the case $k = n$ and $\beta = 0$, where the density is strictly increasing.

At first, note that, for $k = n$, the system coincides with the one from Example 3.2. In both situations the system fails if and only if all components fail. Thus, the corresponding results can be transferred. Now let $n \ge 2$ and $k < n$. By arguing analogously as in Example 3.2, we have

$$s_{2k}\gamma_{2k} - s_{2k-1}(\gamma_{2k-1} - 1) = s_{2k}\gamma_{2k} > 0,$$
$$s_{n+k+1}\gamma_{n+k+1} - s_{n+k}(\gamma_{n+k} - 1) = -s_{n+k}(\gamma_{n+k} - 1) < 0.$$

Therefore, we have to examine the signs in the sequence

$$s_{2k+r+1}\gamma_{2k+r+1} - s_{2k+r}(\gamma_{2k+r} - 1), \qquad r \in \{0, \ldots, n-k-1\}. \tag{3.3}$$

It is sufficient to prove that

$$u(r) := (r+1)(2(n-k) - r)\left(\frac{s_{2k+r+1}}{s_{2k+r}}\gamma_{2k+r+1} - (\gamma_{2k+r} - 1)\right)$$

is decreasing for $r \in \{0, \ldots, n-k-1\}$, since the signs in the sequence $u(0), u(1), \ldots, u(n-k-1)$ are the same as in sequence (3.3). Observe that, for the $k = n-1$ case, nothing remains to be shown. Therefore, we assume that $k \le n-2$ and check the nonnegativity of the function

$$v(r) := u(r) - u(r+1) \quad \text{for } r \in [0, n-k-2].$$

It can be shown, after some lengthy calculations, that

$$\begin{aligned} v(r) = {} & 3(\beta - 1)r^2 + (12(\beta - 1)k + 4n + 3\beta - 1)r + 12(\beta - 1)k^2 - 4\beta k(n-1) \\ & + 12nk - 2(n+k) + 2\beta. \end{aligned}$$

Observe that, due to $\beta \le 1$, the function v is concave on $[0, n-k-2]$. Thus, it is sufficient to show that $v(0) \ge 0$ and $v(n-k-2) \ge 0$. Interpreting $v(0)$ as a continuous function of n, i.e. $w_1(n) := v(0)$, $n \ge k+2$, the corresponding derivative satisfies $w_1'(n) = (10 - 4\beta)k + 2(k-1) > 0$. Therefore, we obtain

$$v(0) = w_1(n) \ge w_1(k+2) = 4\beta k(2k-1) + 16k + 4(k-1) + 2\beta > 0.$$

Defining analogously $w_2(n) := v(n-k-2)$, $n \ge k+2$, we have

$$w_2'(n) = 6\beta\left(n - \tfrac{3}{2}\right) + 2\beta k + 2(n+k) + 1 > 0.$$

Thus, it follows that $v(n-k-2) = w_2(n) \ge w_2(k+2) = w_1(k+2) > 0$, which finally yields the assertion.

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4. Application: optimal bounds on the mean lifetime of a system

In this section we present an important application of the results on unimodality of the density function of the system lifetime in the standard uniform distribution setting given in the preceding sections. Let T denote the lifetime of a system consisting of failure-dependent components with signature $\boldsymbol{s} = (s_1, \dots, s_n)$, parameter vector $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$, and underlying distribution function F. Concerning the model parameters $\gamma_r = \alpha_r(n-r+1)$, $1 \le r \le n$, we may assume without loss of generality that $\alpha_1 = 1$ (otherwise choose $\widetilde{F} = 1-(1-F)^{\alpha_1}$ as the underlying distribution function). Then F describes the behavior of the components before the first failure occurs.

For arbitrary distribution functions F with finite mean μ and finite positive variance σ^2, we provide a sharp upper bound on the expected system lifetime T of the form

$$\mathbb{E}(T) \le \sigma B_{\boldsymbol{s},\boldsymbol{\gamma}} + \mu, \tag{4.1}$$

where the real number $B_{\boldsymbol{s},\boldsymbol{\gamma}}$ is obtained by applying a well-known projection method introduced by Gajek and Rychlik (1996). We only outline the procedure below; see Rychlik (2001) for an extensive explanation. The bound can be used to obtain an estimate of the expected system lifetime, when only information on the mean and standard deviation of the underlying distribution is available. The derivation proceeds as follows. Note that

$$\mathbb{E}(T) = \int_0^1 F^{-1}(u) g_{\boldsymbol{s},\boldsymbol{\gamma}}(u)\,\mathrm{d}u,$$

where $F^{-1}(u) = \inf\{t \in \mathbb{R} \colon F(t) \ge u\}$, $u \in (0,1)$, denotes a generalized inverse of F and $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is given by (1.2). In particular, from $\mu = \int_0^1 F^{-1}(u)\,\mathrm{d}u$ and $\int_0^1 g_{\boldsymbol{s},\boldsymbol{\gamma}}(u)\,\mathrm{d}u = 1$, it follows that

$$\mathbb{E}(T) - \mu = \int_0^1 (F^{-1}(u) - \mu)(g_{\boldsymbol{s},\boldsymbol{\gamma}}(u) - c)\,\mathrm{d}u \quad \text{for every } c \in \mathbb{R}.$$

For square-integrable $g_{\boldsymbol{s},\boldsymbol{\gamma}}$, the above mentioned projection approach in combination with the Cauchy–Schwarz inequality yields

$$\mathbb{E}(T) - \mu \le \bigg(\int_0^1 (F^{-1}(u) - \mu)^2\,\mathrm{d}u\bigg)^{1/2} \bigg(\int_0^1 (\overline{g}_{\boldsymbol{s},\boldsymbol{\gamma}}(u) - c)^2\,\mathrm{d}u\bigg)^{1/2}, \tag{4.2}$$

where $\overline{g}_{\boldsymbol{s},\boldsymbol{\gamma}}$ denotes the projection of $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ onto the convex cone of square-integrable increasing functions on $[0,1]$. The square integrability of $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ can be characterized by using the following result (see Cramer (2003, Lemma 4.2.2)). The sufficient condition can be also found in Cramer *et al.* (2002b). Let $\gamma_{1:r} = \min(\gamma_1, \dots, \gamma_r)$.

Lemma 4.1. *Let* $\lambda > 1$. *Then,*

$$\int_0^1 f_{*,r}^{\lambda}(u)\,\mathrm{d}u < \infty \quad \Longleftrightarrow \quad \gamma_{1:r} > 1 - \frac{1}{\lambda}.$$

Consequently, for a system signature of the form $(s_1, \dots, s_r, 0, \dots, 0)$ with $s_r > 0$, it can be concluded that $g_{\boldsymbol{s},\boldsymbol{\gamma}}$ is square-integrable if and only if $\gamma_{1:r} > \frac{1}{2}$.

Furthermore, since $\int_0^1 \overline{g}_{\boldsymbol{s},\boldsymbol{\gamma}}(u)\,\mathrm{d}u = 1$ (see Rychlik (2001, Lemma 2, p. 30)), the right-hand side of (4.2) is minimized for $c = 1$. Therefore, we arrive at (4.1) with

$$B_{\boldsymbol{s},\boldsymbol{\gamma}} = \bigg(\int_0^1 \overline{g}^{\,2}_{\boldsymbol{s},\boldsymbol{\gamma}}(u)\,\mathrm{d}u - 1\bigg)^{1/2}. \tag{4.3}$$

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Following the approach of Moriguti (1953), the right derivative of the greatest convex minorant of the antiderivative

$$G_{s,\gamma}(x) = \int_0^x g_{s,\gamma}(u)\,\mathrm{d}u, \qquad x \in [0, 1],$$

of $g_{s,\gamma}$ is considered in order to obtain the projection $\overline{g}_{s,\gamma}$. We conclude (see, e.g. Cramer *et al.* (2002a)) that $\overline{g}_{s,\gamma} \equiv 1$ if $g_{s,\gamma}$ is decreasing, $\overline{g}_{s,\gamma} = g_{s,\gamma}$ if $g_{s,\gamma}$ is increasing, and $\overline{g}_{s,\gamma}(\cdot) = g_{s,\gamma}(\min(\cdot, v_*))$, where v_* is the unique solution of

$$(1-u)g_{s,\gamma}(u) = 1 - G_{s,\gamma}(u), \qquad u \in (0, 1), \tag{4.4}$$

if $g_{s,\gamma}$ is strictly unimodal on (0, 1) and $g_{s,\gamma}(0) < 1$. By (numerically) evaluating (4.3) for given $\overline{g}_{s,\gamma}$, sharp bounds in (4.1) are obtained. Distributions that attain these bounds may be constructed by analogy to the corresponding results of Cramer *et al.* (2002a).

Example 4.1. Let T be the lifetime of a k-out-of-n:F system with componentwise redundancy (or systemwise redundancy) as in Example 3.3 (or Example 3.2) with model parameters $\gamma_j = 2n - j + 1 + \beta(j-1)$, $1 \le j \le 2n$. Then, for every F with mean $\mu \in \mathbb{R}$ and variance $0 < \sigma^2 < \infty$, we have the bound $\mathbb{E}(T) \le \sigma B_{s,\gamma} + \mu$ with

$$B^2_{s,\gamma} = \int_0^1 g^2_{s,\gamma}(u)\,\mathrm{d}u - 1 \quad \text{for } k = n,\ \beta = 0$$

(i.e. for strictly increasing $g_{s,\gamma}$), and

$$B^2_{s,\gamma} = \int_0^{v_*} g^2_{s,\gamma}(u)\,\mathrm{d}u + (1 - v_*)g^2_{s,\gamma}(v_*) - 1, \tag{4.5}$$

where v_* solves (4.4), otherwise. For numerical evaluation of the occurring integral, suitable representations of the density function $g_{s,\gamma}$ can be derived, for instance, from Cramer and Kamps (2003).

Example 4.2. Let T be the lifetime of the system with i.i.d. component lifetimes considered in Example 3.1. We integrate it in the present setting by choosing the model parameters $\gamma_j = n - j + 1$, $1 \le j \le n$. For $n \in \{5, 6\}$, the density $g_{s,\gamma}$ is strictly unimodal. Consequently, the projection $\overline{g}_n$ satisfies

$$\overline{g}_{s,\gamma}(u) = \begin{cases} g_{s,\gamma}(u) & \text{for } 0 < u \le v_*, \\ g_{s,\gamma}(v_*) & \text{for } v_* \le u < 1, \end{cases}$$

with the solution v_* of (4.4). For $n \ge 7$, the density is strictly bimodal, but the projection still fulfills the preceding specification (see Bieniek (2016, Section 2) and Remark 3.1). Therefore, the corresponding constant $B_{s,\gamma}$ can be determined as in (4.5). We conclude this example with numerical computations of $B_{s,\gamma} = B_n$, say, for $5 \le n \le 20$ (see Table 1). Note that the values of the corresponding optimal bounds decrease as n increases.

TABLE 1: Optimal bounds for $5 \le n \le 20$.

n	B_n	n	B_n	n	B_n	n	B_n
5	0.240 89	9	0.167 72	13	0.136 01	17	0.117 39
6	0.214 75	10	0.157 77	14	0.130 54	18	0.113 82
7	0.195 02	11	0.149 40	15	0.125 68	19	0.110 56
8	0.179 84	12	0.142 24	16	0.121 32	20	0.107 56

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