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A LARGE SAMPLE TEST FOR THE LENGTH OF MEMORY OF STATIONARY SYMMETRIC STABLE RANDOM FIELDS VIA NONSINGULAR $\mathbb{Z}^d$-ACTIONS

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Abstract

Based on the ratio of two block maxima, we propose a large sample test for the length of memory of a stationary symmetric α-stable discrete parameter random field. We show that the power function converges to 1 as the sample-size increases to ∞ under various classes of alternatives having longer memory in the sense of Samorodnitsky (2004). Ergodic theory of nonsingular $\mathbb{Z}^d$-actions plays a very important role in the design and analysis of our large sample test.

Keywords: Long range dependence; stationary SαS random field; nonsingular group action; extreme value theory; statistical hypothesis testing

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1. Introduction and preliminaries

A random field $\boldsymbol{X} = \{X(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{Z}^d\}$ is called a stationary, *symmetric α-stable* (SαS) random field if every finite linear combination $\sum_{i=1}^{k} a_i X_{\boldsymbol{t}_i+\boldsymbol{s}}$ is an SαS random variable whose distribution does not depend on $\boldsymbol{s}$. Here we will consider the non-Gaussian case (i.e. $0 < \alpha < 2$) unless mentioned otherwise.

Long-range dependence is a very important property that has been observed in many real-life processes. By long-range dependence of the random field $\boldsymbol{X}$, we mean the dependence between the observations $X(\boldsymbol{t})$ which are well separated in $\boldsymbol{t}$. This concept was introduced in order to study the measurements of the water flow in the Nile river by the famous British hydrologist H. E. Hurst (see [8] and [9]). Most of the classical definitions of long-range dependence appearing in the literature are based on the second-order properties (e.g. covariance, spectral density, variance of partial sum, and so on) of stochastic processes. For example, one of the most widely accepted definitions of long-range dependence for a stationary Gaussian process is the following: we say that a stationary Gaussian process has *long-range dependence* (also known as *long memory*) if its correlation function is not summable. In the heavy-tails case, however, this definition becomes ambiguous because a correlation function may not even exist and even if it exists, it may not have enough information about the dependence structure of the process. For a detailed discussion on long range dependence, we refer the reader to [25] and the references therein.

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In the context of stationary SαS processes ($0 < \alpha < 2$), instead of looking for a substitute for a correlation function, in the seminal work [24], Samorodnitsky suggested a new approach for long-range dependence through a dichotomy in the long-run behaviour of the partial maxima. A partition of the underlying parameter space (formally defined later) has been suggested in the aforementioned reference which causes the dichotomy. This dichotomy has been studied for $d \geq 2$ in [23]. Phase transitions in many other probabilistic features of stationary SαS random fields have been connected to the same partition of the parameter space; see e.g. [6], [14], [16], [18], and [22].

The fact that the law of $\boldsymbol{X}$ is invariant under the group action of a shift transformation on the index set $\mathbb{Z}^d$ (stationarity) and certain rigidity properties of L^α spaces ($0 < \alpha < 2$) were used in [20] (for $d = 1$) and [21] (for $d \geq 2$) to show that there always exists an integral representation of the form

$$X(\boldsymbol{t}) \stackrel{\mathrm{D}}{=} \int_E c_{\boldsymbol{t}}(x)\bigg(\frac{\mathrm{d}m \circ \phi_{\boldsymbol{t}}}{\mathrm{d}m}(x)\bigg)^{1/\alpha} f \circ \phi_{\boldsymbol{t}}(x) M(\mathrm{d}x), \tag{1.1}$$

where M is an SαS random measure on a standard Borel space $(E, \mathcal{E})$ with σ-finite control measure m, $f \in L^\alpha(E, m)$ (a deterministic function), $\{\phi_{\boldsymbol{t}}\}$ is a nonsingular $\mathbb{Z}^d$-action on (E, m) (i.e. each $\phi_{\boldsymbol{t}} \colon E \to E$ is measurable and invertible, $\phi_{\boldsymbol{0}}$ is the identity map, $\phi_{\boldsymbol{t}_1} \circ \phi_{\boldsymbol{t}_2} = \phi_{\boldsymbol{t}_1 + \boldsymbol{t}_2}$ for all $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \mathbb{Z}^d$ and each $m \circ \phi_{\boldsymbol{t}}^{-1}$ is an equivalent measure of m), and $\{c_{\boldsymbol{t}}\}$ is a measurable cocycle for the nonsingular action $\{\phi_{\boldsymbol{t}}\}$ taking values in $\{+1, -1\}$ (i.e. each $c_{\boldsymbol{t}} \colon E \to \{+1, -1\}$ is measurable map such that, for all $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \mathbb{Z}^d$, $c_{\boldsymbol{t}_1+\boldsymbol{t}_2}(x) = c_{\boldsymbol{t}_2}(x) c_{\boldsymbol{t}_1}(\phi_{\boldsymbol{t}_2}(x))$ for all $x \in E$).

As a stationary SαS random field can be uniquely specified in terms of a function in $L^\alpha(E, m)$, a nonsingular action, and a cocycle, we consider the following parameter space for a stationary SαS random field:

$$\Theta = \{(f, \{\phi_{\boldsymbol{t}}\}, \{c_{\boldsymbol{t}}\}) \colon f \in L^\alpha(E, m), \{\phi_{\boldsymbol{t}}\} \text{ is a nonsingular action }, \{c_{\boldsymbol{t}}\} \text{ is a cocycle}\}.$$

Now based on the nonsingular action, we can obtain a decomposition of E (into two subsets) which is known as a Hopf decomposition as described below. A set W is called a wandering set for the nonsingular $\mathbb{Z}^d$-action $\{\phi_{\boldsymbol{t}}\}$ on (E, m) if $\{\phi_{\boldsymbol{t}}(W) \colon \boldsymbol{t} \in \mathbb{Z}^d\}$ is a pairwise disjoint collection of subsets of E. Following Proposition 1.6.1 of [1], we find that E can be decomposed into two disjoint and invariant (with respect to $\{\phi_{\boldsymbol{t}}\}$) subsets $\mathcal{C}$ and $\mathcal{D}$ such that for some wandering set $W \subset E$, $\mathcal{D} = \bigcup_{\boldsymbol{t} \in \mathbb{Z}^d} \phi_{\boldsymbol{t}}(W)$ and $\mathcal{C}$ does not have any wandering sets of positive measure. The subsets $\mathcal{C}$ and $\mathcal{D}$ are called the conservative and dissipative parts of $\{\phi_{\boldsymbol{t}}\}$, respectively. If $E = \mathcal{C}$ then we call the nonsingular $\mathbb{Z}^d$-action $\{\phi_{\boldsymbol{t}}\}$ conservative. If $E = \mathcal{D}$ then $\{\phi_{\boldsymbol{t}}\}$ is called dissipative. An example of a dissipative $\mathbb{Z}^d$-action is the shift action: take $E = \mathbb{R}^d$ (with m being the Lebesgue measure) and, for each $\boldsymbol{t} \in \mathbb{Z}^d$, define $\phi_{\boldsymbol{t}}(\boldsymbol{s}) = \boldsymbol{s} + \boldsymbol{t}$, $\boldsymbol{s} \in \mathbb{R}^d$. Section 3 contains examples of conservative $\mathbb{Z}^d$-actions. Roughly speaking, conservative actions tend to come back often while dissipative actions tend to move away.

Following [20], [21], and [23], and denoting the integrand in (1.1) by $f_{\boldsymbol{t}}(x)$,

$$X(\boldsymbol{t}) \stackrel{\mathrm{D}}{=} \int_{\mathcal{C}} f_{\boldsymbol{t}}(x) M(\mathrm{d}x) + \int_{\mathcal{D}} f_{\boldsymbol{t}}(x) M(\mathrm{d}x) =: X^{\mathcal{C}}(\boldsymbol{t}) + X^{\mathcal{D}}(\boldsymbol{t}), \qquad \boldsymbol{t} \in \mathbb{Z}^d, \tag{1.2}$$

where $\boldsymbol{X}^{\mathcal{C}} = \{X^{\mathcal{C}}(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{Z}^d\}$ and $\boldsymbol{X}^{\mathcal{D}} = \{X^{\mathcal{D}}(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{Z}^d\}$ are two independent stationary SαS random fields generated by conservative and dissipative nonsingular $\mathbb{Z}^d$-actions, respectively. It is important to note that the stationary SαS random field generated by a dissipative nonsingular $\mathbb{Z}^d$-action admits mixed moving average representation (see [28] and (1.3) below).

Based on the notion of partial block maxima, it was established in [23] and [24] that stationary SαS random fields generated by conservative actions have longer memory than those generated

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by a nonsingular action with a nontrivial dissipative part. This has formalized the intuition that 'conservative action keeps coming back' (i.e. the same value of the random measure M contributes to the observations $X(\boldsymbol{t})$ which are well separated in $\boldsymbol{t}$) and, hence, induces longer memory. Let, for all $n \in \mathbb{N}$,

$$\text{box}(n) = \{\boldsymbol{j} = (j_1, \dots, j_d) \in \mathbb{Z}^d : |j_i| \le n \text{ for } 1 \le i \le d\}$$

be the block containing the origin with size $(2n+1)^d$ in $\mathbb{Z}^d$. We define the partial block maxima for the stationary SαS random field $\boldsymbol{X}$ as

$$M_n = \max_{\boldsymbol{j} \in \text{box}(n)} |X(\boldsymbol{j})|, \qquad n \in \mathbb{N}.$$

The asymptotic behaviour of the partial block maxima M_n is related to the deterministic sequence

$$B_n = \left(\int_E \max_{\boldsymbol{j} \in \text{box}(n)} |f_{\boldsymbol{j}}(x)|^\alpha m(\mathrm{d}x) \right)^{1/\alpha}.$$

Note that, by Corollary 4.4.6 of [26], B_n is completely specified by the parameters associated to the SαS random field and does not depend on the choice of the integral representation. We will recall the results on rate of growth of $\{B_n\}$ from Proposition 4.1 of [23]. It is expected that the the rate of growth of B_n will be slower if the underlying group action is conservative. Indeed, if $\{\phi_{\boldsymbol{t}} : \boldsymbol{t} \in \mathbb{Z}^d\}$ is conservative then

$$\lim_{n\to\infty} \frac{1}{(2n+1)^{d/\alpha}} B_n = 0.$$

In the other case, we need the mixed moving average representation to describe the limit. A stable random field is called a mixed moving average (see [28]) if it is of the form

$$\boldsymbol{X} \stackrel{\mathrm{D}}{=} \left\{ \int_{W \times \mathbb{Z}^d} f(u, \boldsymbol{s} - \boldsymbol{t}) M(\mathrm{d}u, \mathrm{d}\boldsymbol{s}) : \boldsymbol{t} \in \mathbb{Z}^d \right\}, \tag{1.3}$$

where $f \in L^\alpha(W \times \mathbb{Z}^d, \nu \otimes l)$, l is the counting measure on $\mathbb{Z}^d$, ν is a σ-finite measure on a standard Borel space $(W, \mathcal{W})$, and the control measure m of M is equal to $\nu \otimes l$. It was shown in [20], [21], and [23] that a stationary SαS random field is generated by a dissipative action if and only if it is a mixed moving average with the integral representation (1.3). In this case,

$$\lim_{n\to\infty} \frac{1}{(2n+1)^{d/\alpha}} B_n = \left(\int_W (g(u))^\alpha \nu(\mathrm{d}u) \right)^{1/\alpha} \in (0, \infty), \tag{1.4}$$

where, for every $u \in W$,

$$g(u) = \max_{\boldsymbol{s} \in \mathbb{Z}^d} |f(u, \boldsymbol{s})|.$$

We will denote the right-hand side of (1.4) by $K_{\boldsymbol{X}}$ which depends solely on $\boldsymbol{X}$ and not on the integral representation.

Using the above facts, it has been established that if the SαS random field is not generated by the conservative action then

$$(2n+1)^{-d/\alpha} M_n \quad \Longrightarrow \quad C_\alpha^{1/\alpha} K_{\boldsymbol{X}} Z_\alpha, \tag{1.5}$$

where $K_{\boldsymbol{X}}$ is as above, Z_α is a standard Fréchet(α) random variable with distribution function

$$\mathbb{P}(Z_\alpha \le z) = \begin{cases} \mathrm{e}^{-z^{-\alpha}} & \text{if } z > 0, \\ 0 & \text{if } z \le 0, \end{cases}$$

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and

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin x\,\mathrm{d}x\right) = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi} & \text{if } \alpha = 1. \end{cases} \tag{1.6}$$

On the other hand, if the underlying group action is conservative then

$$(2n+1)^{-d/\alpha} M_n \xrightarrow{\mathbb{P}} 0. \tag{1.7}$$

See Theorem 4.3 of [23] and Theorem 4.1 of [24].

Note that the dichotomy between (1.5) and (1.7) can be justified by the intuitive reasoning that the longer memory prevents erratic changes in X_t causing the maxima to grow more slowly. In the Gaussian case, this phenomenon can be explained in the form of a comparison lemma; see, e.g. Corollary 4.2.3 of [13].

The effect of a transition from conservative to dissipative actions has been investigated for various other features of stationary SαS random fields. For example, the ruin probability of a negative drifted random walk with steps from a stationary ergodic stable processes was studied in [14]. It was observed that the ruin is more likely if the group action is conservative. The point processes associated to a stationary SαS random field were analysed in [18] (for $d = 1$) and [22] (for $d \geq 2$). It was observed that the point process converges weakly to a Poisson cluster process if the group action is not conservative and in the conservative case it does not remain tight due to the presence of clustering. The large deviation issues for point process convergence were addressed in [6], where different large deviation behaviour was observed depending on the ergodic theoretic properties of the underlying nonsingular actions.

Stationary SαS random fields have also been studied from a statistical perspective (see [10], [11], and [26]). Different inference problems associated to the long-range dependence for finite and infinite variance processes have been addressed in the literature; see, e.g. [2], [3], [5], [7], [15], [19], and [27] and the references therein. There are real-life data such as teletraffic data (see [4]) which exhibits heavy-tail phenomenon and long-range dependence. Motivated by all these works, the decomposition of the parameter space suggested in [23] and its effect on various probabilistic aspects of SαS random fields, a natural question comes to mind: *is it possible to design a hypothesis testing problem which will detect the presence of long memory in the observed stationary SαS random field?* In the following paragraph, we formulate the problem.

Motivated by [23] and [24] and the other related works mentioned above, we will consider the following decomposition of the parameter space Θ into Θ_0 and Θ_1. We define Θ_1 as

$$\Theta_1 = \{(f, \{\phi_t\}, \{c_t\}) \in \Theta \colon \{\phi_t\} \text{ is conservative}\}$$

and $\Theta_0 = \Theta \setminus \Theta_1$. In this paper our aim is to design a large sample statistical test in order to test

$$H_0 \colon \theta \in \Theta_0 \quad \text{against} \quad H_1 \colon \theta \in \Theta_1, \tag{1.8}$$

where $\theta = (f, \{\phi_t\}, \{c_t\})$ is the parameter associated to the observed stationary SαS random field defined by (1.1).

This paper is organized as follows. In Section 2 we will present a large sample test (based on the ratio of two appropriately scaled block maxima) in order to test H_0 against H_1 along with the asymptotics under both null and alternative hypotheses. In particular, our test will become consistent for a reasonably broad class of alternatives. Examples of such alternatives are given in Section 3 followed by numerical experiments in Section 4. Finally, proofs of our results are discussed in Section 5.

2. Proposed large sample test based on block maxima

Let $\{\boldsymbol{e}_i : 1 \le i \le d\}$ be the d unit vectors in $\mathbb{Z}^d$ such that the ith component of $\boldsymbol{e}_i$ is 1 and the other components are 0. Fix $0 < \varrho < 1$. Let

$$U_n = (2n+1)^{-d/\alpha} \max_{\boldsymbol{j} \in \mathrm{box}(n)} |X(\boldsymbol{j})| \quad \text{and} \quad V_n = (2[n^\varrho]+1)^{-d/\alpha} \max_{\boldsymbol{j} \in (2n+[n^\varrho])\boldsymbol{e}_1 + \mathrm{box}([n^\varrho])} |X(\boldsymbol{j})|.$$

In other words, U_n is the properly scaled block maxima for box(n) containing the origin as the centre and V_n is the properly scaled block maxima for shifted box$([n^\varrho])$ whose centre is sufficiently separated from box(n). To test hypotheses (1.8), we define the test statistic T_n as the ratio of two partial block maxima U_n and V_n, that is,

$$T_n = \frac{U_n}{V_n} = \left(\frac{2[n^\varrho]+1}{2n+1}\right)^{d/\alpha} \frac{\max_{\boldsymbol{j} \in \mathrm{box}(n)} |X(\boldsymbol{j})|}{\max_{\boldsymbol{j} \in (2n+[n^\varrho])\boldsymbol{e}_1 + \mathrm{box}([n^\varrho])} |X(\boldsymbol{j})|}.$$

We will derive the weak limit of the test statistic T_n under the null hypothesis with the help of following theorem.

Theorem 2.1. *Suppose that the stationary SαS random field* $\boldsymbol{X}$ *is generated by a nonconservative action and, hence, the dissipative part* $X^{\mathcal{D}}$ *admits a nontrivial moving average representation (1.3). Then*

$$(U_n, V_n) \quad \Longrightarrow \quad (Y_1, Y_2),$$

where the Y_i *are independent copies of* Y *with distribution function*

$$\mathbb{P}(Y \le y) = \begin{cases} \exp\{-C_\alpha K_{\boldsymbol{X}}^\alpha y^{-\alpha}\} & \text{if } y > 0, \\ 0 & \text{if } y \le 0, \end{cases}$$

with C_α *defined in (1.6).*

Corollary 2.1. *Under the assumptions of Theorem 2.1,* $T_n \Longrightarrow T$*, where* T *has the distribution function*

$$F_T(t) := \mathbb{P}(T \le t) = \frac{1}{1+t^{-\alpha}}. \tag{2.1}$$

Proof. Using the continuous mapping theorem and the fact that $Y_2 > 0$ almost surely, we obtain

$$T_n \quad \Longrightarrow \quad T := \frac{Y_1}{Y_2}.$$

The distribution of T will be derived using the joint distribution of Y_1 and Y_2. It is clear that the joint probability density function is

$$h_{Y_1,Y_2}(y_1, y_2) = (C_\alpha K_{\boldsymbol{X}}^\alpha \alpha)^2 (y_1 y_2)^{-\alpha-1} \mathrm{e}^{-C_\alpha K_X^\alpha (y_1^{-\alpha}+y_2^{-\alpha})}, \qquad y_1, y_2 > 0.$$

We follow a standard substitution procedure by setting $t = y_1 y_2^{-1}$ and $v = y_2$ which, in turn, gives us $y_1 = tv$ and $y_2 = v$. It is very easy to check that the associated modulus of the Jacobian of the transformation is v as $v > 0$. Hence, we obtain the joint distribution of (T, Y_2) as

$$h_{T,Y_2}(t, y_2) = (\alpha C_\alpha K_{\boldsymbol{X}}^\alpha)^2 t^{-\alpha-1} y_2^{-2\alpha-1} \mathrm{e}^{-C_\alpha K_X^\alpha y_2^{-\alpha}(1+t^{-\alpha})}, \qquad t > 0,\ y_2 > 0.$$

Now in order to obtain the distribution of T, we have to integrate on the whole range for y_2. Again using the standard substitution

$$z = y_2^{-\alpha}(1 + t^{-\alpha})C_\alpha K_X^\alpha,$$

we obtain

$$h_T(t) = \alpha \frac{t^{-\alpha-1}}{(1 + t^{-\alpha})^2} \int_0^\infty z^{2-1} \mathrm{e}^{-z} \, \mathrm{d}z = \frac{\alpha t^{-\alpha-1}}{(1 + t^{-\alpha})^2}, \qquad t > 0.$$

Hence, it is easy to see that (2.1) holds for all $t > 0$. □

We want to compute τ_β such that $\mathbb{P}(T < \tau_\beta) = \beta$. An easy computation yields that

$$\tau_\beta = \left(\frac{\beta}{1-\beta}\right)^{1/\alpha}. \tag{2.2}$$

Remark 2.1. Note that the distance between the two blocks is not visible in the asymptotics of T_n under the null hypothesis because the shorter memory (i.e. weaker dependence) is making the two blocks almost independent in the long run. Therefore, the asymptotic null distribution of the test statistic becomes rather simple (the ratio of two independent and identically distributed (i.i.d.) random variables as seen in Corollary 2.1) and the computation of the critical value (2.2) becomes very easy.

Remark 2.2. Even though our random field has many of unknown parameters (more precisely, the function $f \in L^\alpha(E, m)$, the cocycle $\{c_t\}_{t\in\mathbb{Z}^d}$, and the group action $\{\phi_t\}_{t\in\mathbb{Z}^d}$), only the underlying group action plays a role in the asymptotic test procedure described in this work. Even this parameter does not need to be explicitly estimated in our method of testing. Therefore, our test is free of any estimation procedure and all our asymptotic results work well without any additional correction making this test applicable to real-life situations.

In the following theorem we provide the asymptotics for the test statistic T_n for a very broad class of alternatives.

Theorem 2.2. *Let $\mathbf{X}$ be generated by a conservative $\mathbb{Z}^d$-action $\{\phi_t\}$. If there exists an increasing sequence of positive real numbers, $\{d_n\}$ such that*

$$d_n = n^{d/\alpha - \eta} L(n),$$

where $0 < \eta \le d/\alpha$, $L(n)$ is a slowly varying function of n, and $\{d_n^{-1} M_n\}_{n\ge1}$ and $\{d_n M_n^{-1}\}_{n\ge1}$ are tight sequences of random variables, then we have $T_n \xrightarrow{\mathbb{P}} 0$.

So we reject the null hypothesis H_0 against the class of alternatives considered in Theorem 2.2 if $T_n < \tau_\beta$. This provides a large sample level-β test for H_0 against H_1. Theorem 2.2 ensures that such a test is consistent. In the following section, we will discuss some examples which satisfy the conditions stated in above theorem. We also derive the empirical power in a few examples based on numerical experiments.

3. Important classes of alternatives

In this section we present a few important examples from the alternatives which satisfy the hypotheses of Theorem 2.2 and, hence, our test becomes consistent.

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Example 3.1. We consider a stationary SαS random field indexed by $\mathbb{Z}^2$, with the $\mathbb{Z}^2$-action $\{\phi_{(i,j)}\}_{(i,j)\in\mathbb{Z}^2}$ on $E = \mathbb{R}$ given by

$$\phi_{(i,j)}(x) = x + i + j\sqrt{2}, \qquad x \in \mathbb{R},$$

with m as Lebesgue measure on $\mathbb{R}$. From Example 6.3 of [23], it is clear that

$$\frac{1}{n^{1/\alpha}} M_n \quad \Longrightarrow \quad ((1+\sqrt{2})C_\alpha)^{1/\alpha} Z_\alpha.$$

Hence, Theorem 2.2 with $d_n = n^{1/\alpha}$ applies and we have

$$T_n \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

So the test rejects the null hypothesis H_0 if $T_n < \tau_\beta$ is consistent.

Example 3.2. Consider a random field which has an integral representation of the following form:

$$X(\boldsymbol{j}) = \int_{\mathbb{R}^{\mathbb{Z}^d}} g_{\boldsymbol{j}} \, \mathrm{d}M, \qquad \boldsymbol{j} \in \mathbb{Z}^d,$$

where M is an SαS random measure on $\mathbb{R}^{\mathbb{Z}^d}$ whose control measure m is a probability measure under which the projections $\{g_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ are i.i.d. random variables with finite absolute αth moment.

First we consider the case where under m, $\{g_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ are i.i.d. positive Pareto random variables with

$$m(g_{\boldsymbol{0}} > x) = \begin{cases} x^{-\gamma} & \text{if } x \geq 1, \\ 1 & \text{if } x < 1, \end{cases}$$

for some $\gamma > \alpha$. From Example 6.1 of [23], we obtain

$$B_n \sim c_{p,\gamma}^{1/\alpha} 2^{d/\gamma} n^{d/\gamma} \quad \text{as } n \to \infty$$

for some positive constant $c_{p,\gamma}$, and $B_n^{-1} M_n$ converges weakly to a Frechét random variable. So Theorem 2.2 applies with $d_n = n^{d/\gamma}$ and we have

$$T_n \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

Hence, the level-β test rejects H_0 when $T_n < \tau_\beta$, is consistent.

Now we consider the special case where under m, $\{g_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ is a sequence of i.i.d. standard normal random variables. Then $\{X_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbb{Z}^d}$ has the same distribution as the process $\{c_\alpha A^{1/2} G_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbb{Z}^d}$, where the $G_{\boldsymbol{j}}$ are i.i.d. standard Gaussian random variables, A is a positive $\alpha/2$-stable random variable independent of $\{G_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ with Laplace transform $\mathbb{E}(\mathrm{e}^{-tA}) = \mathrm{e}^{-t^{\alpha/2}}$, and $c_\alpha = \sqrt{2}(\mathbb{E}(|G_{\boldsymbol{0}}|^\alpha))^{1/\alpha}$; see [26, Section 3.7]. Then, from Example 6.1 of [23], we obtain

$$B_n \sim \sqrt{2d \log 2n}$$

such that

$$B_n^{-1} M_n \quad \Longrightarrow \quad A^{1/2},$$

which is a positive random variable.

So we can apply Theorem 2.2 with $d_n = \sqrt{2d \log 2n}$ to obtain

$$T_n \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

Hence, the level-β test that rejects H_0 if $T_n < \tau_\beta$ is consistent.

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Example 3.3. We will first review the basic notions and notation from [23]. Note that the group $R = \{\phi_{\boldsymbol{t}} : \boldsymbol{t} \in \mathbb{Z}^d\}$ of invertible nonsingular transformations on (E, m) is a finitely generated abelian group. Define the group homomorphism $\Phi \colon \mathbb{Z}^d \to R$ such that $\Phi(\boldsymbol{t}) = \phi_{\boldsymbol{t}}$ for all $\boldsymbol{t} \in \mathbb{Z}^d$. The kernel of this group homomorphism is $\ker(\Phi) = \{\boldsymbol{t} \in \mathbb{Z}^d : \phi_{\boldsymbol{t}} = \mathrm{id}_E\}$, where '$\mathrm{id}_E$' denotes the identity map on E. Being a subgroup of $\mathbb{Z}^d$, $\ker(\Phi)$ is a free abelian group. By the first isomorphism theorem of groups, we have

$$R \simeq \mathbb{Z}^d / \ker(\Phi)$$

Due to the structure theorem for finitely generated abelian groups (Theorem 8.5 of [12]), R can be written as the direct sum of a free abelian group $\bar{F}$ (the free part) and a finite abelian group $\bar{N}$ (the torsion part). So we obtain

$$R = \bar{F} \oplus \bar{N}.$$

We assume that $1 \leq \operatorname{rank}(\bar{F}) = p < d$. Since $\bar{F}$ is free, there exists an injective group homomorphism $\Psi \colon \bar{F} \to \mathbb{Z}^d$ such that $\Phi \circ \Psi = \mathrm{id}_{\bar{F}}$. Clearly, $F = \Psi(\bar{F})$ is a free subgroup of $\mathbb{Z}^d$ of rank p.

Note that F should be regarded as the effective index set and its rank p becomes the effective dimension of the random field. It was shown in [23] that

$$\frac{1}{(2n+1)^{p/\alpha}} M_n \implies \begin{cases} C_X Z_\alpha & \text{if } \{\phi_{\boldsymbol{t}}\}_{\boldsymbol{t} \in F} \text{ is not conservative,} \\ 0 & \text{if } \{\phi_{\boldsymbol{t}}\}_{\boldsymbol{t} \in F} \text{ is conservative.} \end{cases}$$

In the above setup, if $1 \leq p < d$, and $\{\phi_{\boldsymbol{t}}\}_{\boldsymbol{t} \in \mathbb{Z}^d}$ is not conservative, then using Theorem 2.2 with $d_n = (2n+1)^{p/\alpha}$, we obtain

$$T_n \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

Hence, the level-β test that rejects when $T_n < \tau_\beta$ is consistent.

4. Numerical experiments

In this section we consider some examples where the underlying group action is conservative. We will simulate the empirical power of the proposed test of level $\beta = 10\%$ in those particular cases. It will be clear from the tables below that if we use small values of ϱ, then the rejection will be very frequent and, hence, our test will become less reliable. On the other hand, a large value of ϱ results in fewer rejections and, hence, the power decreases for each fixed α. We will also observe that the empirical power decreases as α increases for every fixed ϱ. So it seems that we need to choose a smaller value of ϱ as α increases. So there is an inverse relation between ϱ and α. In all the examples, however, as n increases, the empirical power increases to 1 for all values of ϱ and α confirming the consistency of the proposed test.

Numerical Experiment 1. Consider the set up described in Example 3.2. For the purposes of the simulation, we consider the following alternative representation of the sub-Gaussian random field. Suppose that $\{G_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^2\}$ is a collection of i.i.d. standard Gaussian random variables and A is a positive $\alpha/2$-stable random variable independent of the collection $\{G_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^2\}$ with Laplace transform

$$\mathbb{E}(\mathrm{e}^{-tA}) = \mathrm{e}^{t^{\alpha/2}}.$$

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Let $c_\alpha = \sqrt{2}(\mathbb{E}(|G_0|^\alpha))^{1/\alpha}$. The sub-Gaussian random field has the same distribution as the collection of random variables $\{c_\alpha A^{1/2} G_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^2\}$. It easy to simulate the i.i.d. standard Gaussian random variables, and the random variable A is simulated following the method given in [29, p. 3]. In Tables 1–3, we present the results for the empirical power of the proposed test of level 10% based on the ratio of maxima taken over two disjoint blocks.

Numerical Experiment 2. In this example, we consider a stationary SαS random field $\{X(\boldsymbol{t}) : \boldsymbol{t} = (t_1, t_2, t_3) \in \mathbb{Z}^3\}$ admitting the following integral representation:

$$X(\boldsymbol{t}) = \int_{\mathbb{Z}} f_{(t_1,t_2,t_3)}(x) M(\mathrm{d}x) = \int_{\mathbb{Z}} f(x - t_1 + t_2) M(\mathrm{d}x), \tag{4.1}$$

where M is an SαS random measure on $\mathbb{Z}$ with a counting measure as the control measure and $f : \mathbb{Z} \to \mathbb{R}$ such that

$$f(u) = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this case, for each $\boldsymbol{t} = (t_1, t_2, t_3) \in \mathbb{Z}^3$,

$$\phi_{(t_1,t_2,t_3)}(x) = (x - t_1 + t_2),\ x \in \mathbb{Z}.$$

This is a special case of Example 3.3 and the effective dimension of the underlying group action is 1.

It is clear that for every fixed integer c, the random variables $X(\boldsymbol{t})$ are the same as long as $\boldsymbol{t} = (t_1, t_2, t_3)$ lies on the plane $t_1 - t_2 = c$. Also, as c runs over $\mathbb{Z}$, these random variables form an i.i.d. collection. Based on this observation, we simply simulate i.i.d. SαS random variables (following the method stated in [29, p. 3]) indexed by $\mathbb{Z}$ and use them appropriately for our test. In Tables 4–6 we present the results for the simulated empirical power of the proposed test conducted at 10% level of significance.

Numerical Experiment 3. Next, we consider another example of a stationary SαS random field admitting the integral representation (4.1) with $f : \mathbb{Z} \to \mathbb{R}$ such that

$$f(u) = \begin{cases} 1 & \text{if } u = 0, -1, \\ 0 & \text{otherwise.} \end{cases}$$

This example is similar to the previous one with the same effective dimension 1. In this case also, for each fixed $c \in \mathbb{Z}$, the collection $\{X(\boldsymbol{t}) : t_1 - t_2 = c\}$ consists of a single random variable. However, as c runs over $\mathbb{Z}$, these random variables no longer remain independent. Rather, they form a moving average process of order 1 with SαS innovations and unit coefficients. Using this observation, we simulate the random field easily. In Tables 7–9 we present the results of the simulated empirical power of the proposed test of level 10%.

Remark 4.1. For real data, we need to choose the blocksize (i.e. $\varrho \in (0, 1)$) before performing this test. Even though α and the best performing ϱ have an inverse relationship (as explained at the beginning of this section), it is observed in the above tables that $\varrho \approx 0.65$ seems to perform well for a broad class of alternatives. Therefore, in the absence of further knowledge, we prescribe $\varrho = 0.65$ to be used for our test.

TABLE 1: Empirical power for $\alpha = 0.7$ and 0.9.

ϱ	$\alpha = 0.7$			$\alpha = 0.9$		
	$n = 80$	$n = 90$	$n = 100$	$n = 80$	$n = 90$	$n = 100$
0.61	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.62	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.63	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.64	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.65	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.66	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.67	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.68	1.0000	1.0000	1.0000	0.9975	1.0000	1.0000
0.69	1.0000	1.0000	1.0000	0.9975	1.0000	1.0000
0.70	1.0000	1.0000	1.0000	0.9875	0.9975	1.0000

TABLE 2: Empirical power for $\alpha = 1.1$ and 1.3

ϱ	$\alpha = 1.1$			$\alpha = 1.3$		
	$n = 80$	$n = 90$	$n = 100$	$n = 80$	$n = 90$	$n = 100$
0.61	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.62	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.63	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.64	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.65	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.66	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.67	0.9975	1.0000	1.0000	1.0000	1.0000	1.0000
0.68	0.9950	1.0000	1.0000	0.9750	0.9875	0.9875
0.69	0.9850	1.0000	0.9975	0.9500	0.9575	0.9875
0.70	0.9600	0.9975	0.9900	0.8775	0.9375	0.9775

TABLE 3: Empirical power for $\alpha = 1.5$ and 1.7.

ϱ	$\alpha = 1.5$			$\alpha = 1.7$		
	$n = 80$	$n = 90$	$n = 100$	$n = 80$	$n = 90$	$n = 100$
0.61	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.62	1.0000	1.0000	1.0000	1.0000	1.0000	0.9975
0.63	0.9975	1.0000	1.0000	1.0000	1.0000	0.9950
0.64	0.9975	1.0000	1.0000	0.9925	1.0000	0.9950
0.65	0.9925	1.0000	1.0000	0.9875	0.9925	0.9975
0.66	0.9950	1.0000	0.9950	0.9650	0.9725	0.9875
0.67	0.9700	0.9825	0.9975	0.9125	0.9825	1.0000
0.68	0.9325	0.9650	0.9925	0.8700	0.9300	0.9575
0.69	0.8825	0.9150	0.9600	0.7625	0.8925	0.9175
0.70	0.7625	0.8650	0.9375	0.6800	0.8125	0.8725

TABLE 4: Empirical power for $\alpha = 0.7$ and 0.9.

ϱ	$\alpha = 0.7$			$\alpha = 0.9$		
	$n = 1000$	$n = 1500$	$n = 2000$	$n = 1000$	$n = 1500$	$n = 2000$
0.61	0.9750	0.9650	0.9750	0.9525	0.967 50	0.981 25
0.62	0.9550	0.9625	0.9800	0.9450	0.967 50	0.983 75
0.63	0.9475	0.9725	0.9750	0.9475	0.962 50	0.973 75
0.64	0.9525	0.9600	0.9675	0.9250	0.950 00	0.948 75
0.65	0.9275	0.9600	0.9325	0.9325	0.956 25	0.950 00
0.66	0.9175	0.9600	0.9525	0.9425	0.952 50	0.950 00
0.67	0.9225	0.9425	0.9325	0.9125	0.923 75	0.938 75
0.68	0.9100	0.9200	0.9300	0.9075	0.912 50	0.942 50
0.69	0.8875	0.9150	0.9275	0.9225	0.923 75	0.928 75
0.70	0.8800	0.9075	0.9200	0.9000	0.916 25	0.912 50

TABLE 5: Empirical power for $\alpha = 1.1$ and 1.3.

ϱ	$\alpha = 1.1$			$\alpha = 1.3$		
	$n = 1000$	$n = 1500$	$n = 2000$	$n = 1000$	$n = 1500$	$n = 2000$
0.61	0.955 00	0.972 50	0.972 50	0.961 25	0.981 25	0.982 50
0.62	0.952 50	0.970 00	0.976 25	0.952 50	0.955 00	0.973 75
0.63	0.942 50	0.971 25	0.967 50	0.931 25	0.951 50	0.968 75
0.64	0.945 00	0.967 50	0.965 00	0.932 50	0.956 25	0.956 25
0.65	0.925 00	0.952 50	0.971 25	0.937 50	0.958 75	0.963 75
0.66	0.917 50	0.943 75	0.946 25	0.917 50	0.938 75	0.943 75
0.67	0.911 25	0.942 50	0.953 75	0.903 50	0.935 00	0.966 25
0.68	0.911 25	0.941 25	0.935 00	0.880 00	0.931 25	0.950 00
0.69	0.895 00	0.903 75	0.950 00	0.903 75	0.935 00	0.933 75
0.70	0.863 75	0.887 50	0.926 25	0.875 00	0.880 00	0.916 25

TABLE 6: Empirical power for $\alpha = 1.5$ and 1.7.

ϱ	$\alpha = 1.5$			$\alpha = 1.7$		
	$n = 1000$	$n = 1500$	$n = 2000$	$n = 1000$	$n = 1500$	$n = 2000$
0.61	0.961 25	0.977 50	0.981 25	0.962 50	0.980 00	0.980 00
0.62	0.962 50	0.975 00	0.971 25	0.966 25	0.973 75	0.976 25
0.63	0.950 00	0.956 25	0.976 25	0.966 25	0.968 75	0.957 50
0.64	0.956 25	0.961 25	0.961 25	0.948 75	0.962 50	0.961 25
0.65	0.935 00	0.952 50	0.948 75	0.942 50	0.953 75	0.971 25
0.66	0.916 25	0.936 25	0.960 00	0.938 75	0.957 50	0.957 50
0.67	0.926 25	0.941 25	0.941 25	0.910 00	0.936 25	0.947 50
0.68	0.908 75	0.938 75	0.937 50	0.898 75	0.910 00	0.947 50
0.69	0.913 75	0.926 25	0.916 25	0.897 50	0.950 00	0.910 00
0.70	0.893 75	0.891 25	0.920 00	0.891 25	0.903 75	0.920 00

TABLE 7: Empirical power for $\alpha = 0.7$ and 0.9.

ϱ	$\alpha = 0.7$			$\alpha = 0.9$		
	$n = 1000$	$n = 1500$	$n = 2000$	$n = 1000$	$n = 1500$	$n = 2000$
0.61	0.967 50	0.968 75	0.970 00	0.957 50	0.970 00	0.976 25
0.62	0.958 75	0.952 50	0.972 50	0.955 00	0.962 50	0.983 75
0.63	0.955 00	0.952 50	0.980 00	0.953 75	0.956 25	0.975 00
0.64	0.955 00	0.952 50	0.967 50	0.948 75	0.963 75	0.958 75
0.65	0.920 00	0.947 50	0.962 50	0.946 25	0.946 25	0.956 25
0.66	0.918 75	0.938 75	0.948 75	0.926 25	0.940 00	0.961 25
0.67	0.923 75	0.937 50	0.943 75	0.935 00	0.940 00	0.951 25
0.68	0.917 50	0.918 75	0.922 50	0.918 75	0.918 75	0.940 00
0.69	0.900 00	0.922 50	0.937 50	0.907 50	0.926 25	0.925 00
0.70	0.895 00	0.905 00	0.916 25	0.881 25	0.892 50	0.915 00

TABLE 8: Empirical power for $\alpha = 1.1$ and 1.3.

ϱ	$\alpha = 1.1$			$\alpha = 1.3$		
	$n = 1000$	$n = 1500$	$n = 2000$	$n = 1000$	$n = 1500$	$n = 2000$
0.61	0.972 50	0.980 00	0.977 50	0.970 00	0.981 25	0.981 25
0.62	0.957 50	0.958 75	0.971 25	0.955 00	0.957 50	0.972 50
0.63	0.957 50	0.962 50	0.972 50	0.955 00	0.953 75	0.961 25
0.64	0.937 50	0.952 50	0.968 75	0.953 75	0.972 50	0.962 50
0.65	0.942 50	0.950 00	0.968 75	0.937 50	0.955 00	0.957 50
0.66	0.936 25	0.936 25	0.956 25	0.932 50	0.942 50	0.950 00
0.67	0.915 00	0.917 50	0.938 75	0.927 50	0.936 25	0.938 75
0.68	0.916 25	0.927 50	0.950 00	0.915 00	0.908 75	0.927 50
0.69	0.902 50	0.938 75	0.933 75	0.888 75	0.907 50	0.923 75
0.70	0.882 50	0.902 50	0.926 25	0.885 00	0.883 75	0.930 00

TABLE 9: Empirical power for $\alpha = 1.5$ and 1.7.

ϱ	$\alpha = 1.5$			$\alpha = 1.7$		
	$n = 1000$	$n = 1500$	$n = 2000$	$n = 1000$	$n = 1500$	$n = 2000$
0.61	0.978 75	0.970 00	0.982 50	0.961 25	0.978 75	0.972 50
0.62	0.957 50	0.980 00	0.972 50	0.950 00	0.981 25	0.952 50
0.63	0.958 75	0.973 75	0.966 25	0.956 25	0.976 25	0.951 25
0.64	0.953 75	0.960 00	0.963 75	0.938 75	0.962 50	0.958 75
0.65	0.947 50	0.963 75	0.965 00	0.951 25	0.955 00	0.963 75
0.66	0.921 25	0.940 00	0.963 75	0.936 25	0.971 25	0.941 25
0.67	0.913 75	0.955 00	0.960 00	0.901 25	0.948 75	0.928 75
0.68	0.900 00	0.930 00	0.948 75	0.918 75	0.948 75	0.920 00
0.69	0.902 50	0.926 25	0.930 00	0.903 75	0.922 50	0.916 25
0.70	0.876 25	0.905 00	0.936 25	0.868 75	0.886 25	0.916 25

5. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Without loss of generality, we will assume that $\boldsymbol{X}$ admits a moving average representation. This is because under our hypothesis, we can use the decomposition (1.2) with a nontrivial dissipative part and the conservative part does not contribute to the maxima after scaling. In particular, this means that

$$X(\boldsymbol{j}) = \int_W \int_{\mathbb{Z}^d} f(u, \boldsymbol{v} - \boldsymbol{j}) M(\mathrm{d}u, \mathrm{d}\boldsymbol{v}), \qquad \boldsymbol{j} \in \mathbb{Z}^d,$$

where M is an SαS random measure on $W \times \mathbb{Z}^d$ with the control measure $m = \nu \otimes l$ on $\mathcal{B}(W \times \mathbb{Z}^d)$, where l is a counting measure on $\mathbb{Z}^d$. Also $f \in L^\alpha(W \times \mathbb{Z}^d, \nu \otimes l)$. Let $\mathrm{box}(L) = \{\boldsymbol{j} \in \mathbb{Z}^d : |j_1| \le L, \dots, |j_d| \le L\}$ i.e. it is an L neighbourhood around the origin. Define

$$X(\boldsymbol{j}, L) = \int_W \int_{\mathbb{Z}^d} f(u, \boldsymbol{v} - \boldsymbol{j})\, \mathbf{1}_{W \times \mathrm{box}(L)}(w, \boldsymbol{v} - \boldsymbol{j}) M(\mathrm{d}u, \mathrm{d}\boldsymbol{v}) \tag{5.1}$$

for all positive integer, L, where $\mathbf{1}_A$ is the indicator function on the event A. Define

$$M_n(L) = \max\{|X(\boldsymbol{j}, L)| : \boldsymbol{j} \in \mathrm{box}(n)\},$$
$$\mathfrak{M}_n(L) = \max\{|X(\boldsymbol{j}, L)| : \boldsymbol{j} \in (2n + [n^\varrho])\boldsymbol{e}_1 + \mathrm{box}([n^\varrho])\}.$$

Fix $L \in \mathbb{N}$. It is important to observe that, as an easy consequence of Theorem 4.3 of [23], we have

$$\frac{1}{(2n+1)^{d/\alpha}} M_n(L) \quad \Longrightarrow \quad Y_1(L),$$

where $Y_1(L)$ is a positive random variable with distribution function

$$\mathbb{P}(Y_1(L) \le y) = \exp\{-C_\alpha K_X^\alpha(L) y^{-\alpha}\} \tag{5.2}$$

with

$$K_X^\alpha(L) = \int_W \sup_{\boldsymbol{j} \in \mathrm{box}(L)} |f(w, \boldsymbol{j})|^\alpha \nu(\mathrm{d}w). \tag{5.3}$$

Similar facts lead to the observation that $(2[n^\varrho] + 1)^{-d/\alpha} \mathfrak{M}_n$ converges weakly to a random variable with the same distribution as that of $Y_1(L)$. It is important to note that for all $n \ge 2L+1$, we have $\{X(\boldsymbol{j}, L) : \boldsymbol{j} \in \mathrm{box}(n)\}$ and $\{X(\boldsymbol{j}, L) : \boldsymbol{j} \in (2n+[n^\varrho])\boldsymbol{e}_1 + \mathrm{box}([n^\varrho])\}$ are independent random vectors, which follows from Theorem 3.5.3 of [26]. So M_n and $\mathfrak{M}_n$ are independent for all $n \ge 2L + 1$. Combining these facts, we obtain

$$\left(\frac{1}{(2n+1)^{d/\alpha}} M_n(L), \frac{1}{(2[n^\varrho]+1)^{d/\alpha}} \mathfrak{M}_n(L) \right) \quad \Longrightarrow \quad (Y_1(L), Y_2(L)),$$

where $Y_1(L)$ and $Y_2(L)$ are i.i.d. with law as specified in (5.2). It is easy to see that as $L \to \infty$, $K_{X(L)} \to K_X$. So we have

$$(Y_1(L), Y_2(L)) \quad \Longrightarrow \quad (Y_1, Y_2) \quad \text{as } L \to \infty.$$

Now it only remains to show that, for every fixed $\epsilon > 0$,

$$\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\frac{1}{(2n+1)^{d/\alpha}} |M_n - M_n(L)| + \frac{1}{(2[n^\varrho]+1)^{\varrho d/\alpha}} |\mathfrak{M}_n - \mathfrak{M}_n(L)| > \epsilon \right) = 0. \tag{5.4}$$

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To prove (5.4), it is enough to show that

$$\lim_{L\to\infty}\limsup_{n\to\infty}\mathbb{P}\bigg(\frac{1}{(2n+1)^{d/\alpha}}|M_n - M_n(L)| > \frac{\epsilon}{2}\bigg) = 0.$$

Recall that

$$B_n = \bigg(\int_E \max_{\boldsymbol{j}\in\mathrm{box}(n)}\Big|f(w,\boldsymbol{j})\Big|^\alpha m(\mathrm{d}w)\bigg)^{1/\alpha}$$

and define a new probability measure λ_n on $E = W\times\mathbb{Z}^d$ for every fixed n,

$$\frac{\mathrm{d}\lambda_n}{\mathrm{d}m}(w,\boldsymbol{j}) = B_n^{-\alpha}\max_{\boldsymbol{j}\in\mathrm{box}(n)}|f(w,\boldsymbol{j})|^\alpha.$$

Using Theorem 3.5.6 and Corollary 3.10.4 of [26], we know that, for $\boldsymbol{j}\in\mathrm{box}(n)$,

$$X(\boldsymbol{j}) \stackrel{\mathrm{D}}{=} C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\varepsilon_i\Gamma_i^{-1/\alpha}f(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j}), \qquad \boldsymbol{j}\in\mathrm{box}(n),$$

where C_α is a constant as specified in (1.6), $\{\varepsilon_i : i\ge 1\}$ is a collection of i.i.d. $\{\pm 1\}$-valued symmetric random variables, $\{\Gamma_i : i\ge 1\}$ is the collection of arrival times of the unit-rate Poisson process, and $\{(U_i^{(n)}, \boldsymbol{V}_i^{(n)}) : i\ge 1\}$ is a collection of i.i.d. $E = W\times\mathbb{Z}^d$-valued random variables with common law λ_n for every fixed n. It is straightforward to check that

$$X(\boldsymbol{j}, L) \stackrel{\mathrm{D}}{=} C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\varepsilon_i\Gamma_i^{-1/\alpha}f(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\,\mathbf{1}_{W\times\mathrm{box}(L)}(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j}), \qquad \boldsymbol{j}\in\mathrm{box}(n).$$

Now note that

$$\begin{aligned}
&\max_{\boldsymbol{j}\in\mathrm{box}(n)}|X(\boldsymbol{j})| - \max_{\boldsymbol{j}\in\mathrm{box}(n)}|X(\boldsymbol{j}, L)| \\
&\quad= \max_{\boldsymbol{j}\in\mathrm{box}(n)}\bigg|X(\boldsymbol{j}, L) + C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\varepsilon_i\Gamma_i^{-1/\alpha}f(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\,\mathbf{1}_{W\times(\mathrm{box}(L))^c} \\
&\qquad\qquad\times(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\bigg| - \max_{\boldsymbol{j}\in\mathrm{box}(n)}|X(\boldsymbol{j}, L)| \\
&\quad\le \max_{\boldsymbol{j}\in\mathrm{box}(n)}\bigg|C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\varepsilon_i\Gamma_i^{-1/\alpha}f(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\,\mathbf{1}_{W\times(\mathrm{box}(L))^c}(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\bigg|, \qquad (5.5)
\end{aligned}$$

where the last inequality follows from the fact that

$$\max_{\boldsymbol{j}\in\mathbb{Z}^d}(a_{\boldsymbol{j}} + b_{\boldsymbol{j}}) \le \max_{\boldsymbol{j}\in\mathbb{Z}^d}a_{\boldsymbol{j}} + \max_{\boldsymbol{j}\in\mathbb{Z}^d}b_{\boldsymbol{j}}$$

for two sequences $\{a_{\boldsymbol{j}} : \boldsymbol{j}\in\mathbb{Z}^d\}$ and $\{b_{\boldsymbol{j}} : \boldsymbol{j}\in\mathbb{Z}^d\}$ of positive real numbers. Also note that

$$\begin{aligned}
&\max_{\boldsymbol{j}\in\mathrm{box}(n)}|X(\boldsymbol{j})| - \max_{\boldsymbol{j}\in\mathrm{box}(n)}|X(\boldsymbol{j}, L)| \\
&\quad= \max_{\boldsymbol{j}\in\mathrm{box}(n)}|X(\boldsymbol{j})| - \max_{\boldsymbol{j}\in\mathrm{box}(n)}\bigg|X(\boldsymbol{j}) - C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\varepsilon_i\Gamma_i^{-1/\alpha}f(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j}) \\
&\qquad\qquad\times\mathbf{1}_{W\times(\mathrm{box}(L))^c}(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\bigg|
\end{aligned}$$

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$$\geq -\max_{\boldsymbol{j}\in \mathrm{box}(n)} \Big| C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} f(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\, \mathbf{1}_{W\times(\mathrm{box}(L))^c}(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j}) \Big| \tag{5.6}$$

using the fact that any two sequence of real numbers $\{a_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ and $\{b_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^d\}$ satisfy the following inequality:

$$\max_{\boldsymbol{j}\in\mathbb{Z}^d} |a_{\boldsymbol{j}}| - \max_{\boldsymbol{j}\in\mathbb{Z}^d} |a_{\boldsymbol{j}} - b_{\boldsymbol{j}}| \geq -\max_{\boldsymbol{j}\in\mathbb{Z}^d} |b_{\boldsymbol{j}}|.$$

Now combining the upper bound in (5.5) and the lower bound obtained in (5.6), we have

$$\Big| \max_{\boldsymbol{j}\in\mathrm{box}(n)} |X(\boldsymbol{j})| - \max_{\boldsymbol{j}\in\mathrm{box}(n)} |X(\boldsymbol{j}, L)| \Big| \leq \max_{\boldsymbol{j}\in\mathrm{box}(n)} |X^{(c)}(\boldsymbol{j}, L)|,$$

where

$$X^{(c)}(\boldsymbol{j}, L) = C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} f(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j})\, \mathbf{1}_{W\times(\mathrm{box}(L))^c}(U_i^{(n)}, \boldsymbol{V}_i^{(n)} - \boldsymbol{j}).$$

It is easy to verify that $\{X^{(c)}(\boldsymbol{j}, L) : \boldsymbol{j} \in \mathrm{box}(n)\}$ is a stationary SαS random field which admits a mixed moving average representation. Hence, we can again use Theorem 4.3 of [23] to obtain

$$\frac{1}{(2n+1)^{d/\alpha}} \max_{\boldsymbol{j}\in\mathrm{box}(n)} |X^{(c)}(\boldsymbol{j}, L)| \quad \Longrightarrow \quad C_\alpha^{1/\alpha} K_{\boldsymbol{X}}^{(c)}(L) Z_\alpha,$$

where Z_α is a Frechet random variable with distribution function $\mathbb{P}(Z_\alpha < x) = \mathrm{e}^{-x^{-\alpha}}$ and

$$K_{\boldsymbol{X}}^{(c)}(L) = \int_W \sup_{\boldsymbol{j}\in\mathbb{Z}^d\setminus\mathrm{box}(L)} |f(w, \boldsymbol{j})|^\alpha \nu(\mathrm{d}w).$$

Finally, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\bigg((2n+1)^{-d/\alpha} |M_n - M_n(L)| > \frac{\epsilon}{2} \bigg) \\
&\quad = \limsup_{n\to\infty} \mathbb{P}\bigg((2n+1)^{-d/\alpha} \Big| \max_{\boldsymbol{j}\in\mathrm{box}(n)} |X(\boldsymbol{j})| - \max_{\boldsymbol{j}\in\mathrm{box}(n)} |X(\boldsymbol{j}, L)| \Big| > \frac{\epsilon}{2} \bigg) \\
&\quad \leq \limsup_{n\to\infty} \mathbb{P}\bigg((2n+1)^{-d/\alpha} \max_{\boldsymbol{j}\in\mathrm{box}(n)} |X^{(c)}(\boldsymbol{j}, L)| > \frac{\epsilon}{2} \bigg) \\
&\quad = \mathbb{P}\bigg(C_\alpha^{1/\alpha} K_{\boldsymbol{X}}^{(c)}(L) Z_\alpha > \frac{\epsilon}{2} \bigg).
\end{aligned} \tag{5.7}$$

It is easy to see that as $L \to \infty$, $K_{\boldsymbol{X}}^{(c)}(L) \to 0$ and, hence, the expression in (5.7) vanishes. This completes the proof of (5.4) and the theorem. □

Proof of Theorem 2.2. From the fact that $\{d_n^{-1} M_n\}$ and $\{d_n M_n^{-1}\}$ are tight sequences, it follows using stationarity that $\{(d([n^\varrho]))^{-1} \mathfrak{M}_n\}$ and $\{d([n^\varrho]) \mathfrak{M}_n^{-1}\}$ are tight sequences, where $d(n) := d_n$. Note that, as a product of two tight sequences,

$$\frac{d([n^\varrho])}{d_n} \frac{M_n}{\mathfrak{M}_n} \tag{5.8}$$

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is also a tight sequence of random variables. Observe that

$$\frac{d([n^{\varrho}])}{d(n)} \sim n^{(d/\alpha-\eta)(\varrho-1)}\frac{L([n^{\rho}])}{L(n)} \quad \text{as } n \to \infty.$$

Note also that

$$T_n = \left(\frac{2[n^{\varrho}]+1}{2n+1}\right)^{d/\alpha}\frac{M_n}{\mathfrak{M}_n} \sim n^{d/\alpha(\varrho-1)}\frac{M_n}{\mathfrak{M}_n} \sim \frac{L(n)}{L([n^{\varrho}])}n^{\eta(\varrho-1)}\frac{d([n^{\varrho}])}{d(n)}\frac{M_n}{\mathfrak{M}_n},$$

from which the result follows since (5.8) is tight and

$$\frac{L(n)}{L([n^{\varrho}])}n^{\eta(\varrho-1)} \to 0$$

using Potter bounds (see, e.g. [17]). □

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