A CHARACTERIZATION OF CHAINABLE CONTINUA

J. B. FUGATE

1. Introduction. In this paper, certain results of Bing (**1**) and myself (**2**) are extended. It is well-known that a chainable compact metric continuum must be a-triodic (contain no triods), hereditarily unicoherent (the common part of each two subcontinua is connected), and each subcontinuum must be chainable. Our principal result states that a compact metric continuum M is chainable if and only if M is a-triodic, hereditarily unicoherent and each indecomposable subcontinuum of M is chainable. Some condition on the indecomposable subcontinua of M seems essential, if we consider the dyadic solenoid, S, which is indecomposable, a-triodic and hereditarily unicoherent. Indeed, each proper subcontinuum of S is an arc. However, S is not chainable, since it cannot be embedded in the plane.

2. Definitions and notation. *A chain* $\mathscr{E}$ is a finite collection $\{E_1, \ldots, E_m\}$ of open sets such that $E_i \cap E_j \neq \emptyset$ if and only if $|i - j| \leqq 1$. We frequently denote $\mathscr{E}$ by $E(1, m)$ and denote $\bigcup_{i=1}^{m} E_i$ by $E^*(1, m)$. The elements of $\mathscr{E}$ are called *links*; two links are *adjacent* if and only if they intersect. If non-adjacent links are a positive distance apart, $\mathscr{E}$ is said to be *taut*. If $E(1, m)$ and $F(1, j)$ are chains such that $E_i \cap F_j \neq \emptyset$ if and only if $i = m$ and $j = 1$, then the chain $\{E_1, \ldots, E_m, F_1, \ldots, F_n\}$ is denoted by $E(1, m) \oplus F(1, j)$. If $E(1, m)$ is a chain and S is an open set intersecting the common part of each pair of adjacent links, then the chain $\{E_1 \cap S, \ldots, E_m \cap S\}$ is denoted by $E(1, m) \cap S$. If $\epsilon > 0$, then $\mathscr{E}$ is an *ϵ-chain* if and only if each link of $\mathscr{E}$ has diameter less than ϵ. A compact metric continuum M is *ϵ-chainable* if and only if there is an ϵ-chain covering M; M is *chainable* (*snakelike*, *arclike*) if and only if for each $\epsilon > 0$, M is ϵ-chainable.

Finally, if $E(1, m)$ is a chain covering M, and K is a subcontinuum of M, then K *is contained exactly in the subchain* $E(j, l)$ (in symbols, $K \subset^e E^*(j, l)$) if and only if K is not contained in any proper subchain of $E(j, l)$ and

$$(\mathrm{Cl}(E^*(1, j - 1)) \cup \mathrm{Cl}(E^*(l + 1, m))) \cap K = \emptyset.$$

3. Terminal subcontinua. Given an $\epsilon > 0$, we must be able to cover M with an ϵ-chain. The basic idea is to decompose M into proper subcontinua A and B, and ϵ-chain each of these. We then fit the two chains together to obtain a chain covering M. The key to this fitting process is the concept of terminal subcontinuum.

Received November 28, 1967.

Definition 1. If M is a compact metric continuum and K is a subcontinuum of M, then K is a *terminal subcontinuum* of M if and only if for each pair L, N of subcontinua of M, each intersecting K, either $L \subset N \cup K$ or $N \subset L \cup K$. If K is degenerate, then K is a *terminal point* of M.

Remark 1. If the a-triodic, hereditarily unicoherent compact metric continuum M is the union of two of its proper subcontinua A and B, then each is a terminal subcontinuum of M. Moreover, $A \cap B$ is a terminal subcontinuum of A and of B. (This is proved as Claim 1 in the proof of (**2**, Theorem 1).)

Several important facts about terminal subcontinua are embodied in the following lemmas. Proofs of Lemmas 1 and 2 may be found in (**2**).

LEMMA 1. *Suppose that M is an a-triodic, hereditarily unicoherent compact metric continuum, K is a terminal subcontinuum of M, and $\mathscr{E} = E(1, m)$ is a chain covering M. Then there is a chain $\mathscr{G} = G(1, n)$ covering M and an integer s, $1 \leqq s \leqq n$, such that*
(1) *$\mathscr{G}$ is a refinement of $\mathscr{E}$,*
(2) *$K \subset^e G^*(s, n)$,*
(3) *if $\mathscr{E}$ is taut, so is $\mathscr{G}$.*

LEMMA 2. *If M is a-triodic, hereditarily unicoherent compact metric continuum and K is a subcontinuum of M, then K is a terminal subcontinuum of M if and only if for each subcontinuum P of M which intersects K, $K \cup P$ is irreducible between some pair of points, one of which belongs to K.*

LEMMA 3. *Suppose that M is an a-triodic, hereditarily unicoherent compact metric continuum, K is a terminal subcontinuum of M, each of A and B is a proper subcontinuum of K and $K = A \cup B$. Then at least one of A and B is a terminal subcontinuum of M.*

Proof. Suppose that the lemma fails. Since A is not a terminal subcontinuum of M, by Lemma 2 there is a subcontinuum R of M such that $R \cap A \neq \emptyset$ and
(†) $R \cup A$ is not irreducible between any pair of points, one of which belongs to A.

Clearly, $A \subset R$. Since R intersects the terminal continuum K, applying Lemma 2 again, we find that there are points $p \in R$ and $q \in K$ such that $R \cup K$ is irreducible from p to q. Moreover, $q \in B - A$, for if $q \in A$, then R is a subcontinuum of $R \cup K$ containing p and q; hence, $R \cup K = R$ and $R = R \cup A$ is irreducible from $p \in R$ to $q \in A$. This violates (†). Not only does $q \in B - A$, but $p \in R - K$, for if $p \in K$, then $p \in B$, otherwise we contradict (†). Then B is a proper subcontinuum of $R \cup K$ containing p and q and $R \cup K$ is reducible from p to q. This contradiction shows that $p \in R - K$.

In a similar fashion, there is a subcontinuum S of M such that $S \cap B \neq \emptyset$ and $S \cup B$ is not irreducible between any pair of points, one of which is in B.

Then $S \cup B = S$ and there are points $x \in S - K$, $y \in A - B$ such that $S \cup K$ is irreducible from x to y.

Since each of R and S is a continuum intersecting K, it follows from the definition of a terminal subcontinuum that either $R \subset S \cup K$ or $S \subset R \cup K$. We shall assume that $S \subset R \cup K$. Since $x \in S - K$, $x \in R - K$. From (†), $R = R \cup A$ is reducible from p to $y \in A$, and thus there is a proper subcontinuum L of R such that $p \in L$, $y \in L$. Thus $q \notin L$. Now $y \in L \cap K$, hence $L \cup K$ is a subcontinuum of $R \cup K$ containing p and q; thus, $L \cup K = R \cup K$ and $x \in L$. Since $q \in B - (L \cap (S \cup K)) \subset S, L \cap (S \cup K)$ is a proper subcontinuum of $S \cup K$ containing x and y and $S \cup K$ is reducible from x to y. This contradiction establishes the lemma.

LEMMA 4. *Suppose that M is an a-triodic, hereditarily unicoherent compact metric continuum and K is a terminal subcontinuum of M. Then there is a subcontinuum L of K such that*

(i) *L is a terminal subcontinuum of M;*

(ii) *L is irreducible with respect to* (i)*;*

(iii) *L is indecomposable or is a single point, a terminal point of M.*

Proof. If $B \subset M$, then B *has Property* P if and only if B is a terminal subcontinuum of M and $B \subset K$. We show that Property P is inductive.

Suppose that N is a decreasing sequence such that for each positive integer i, N_i is a continuum having Property P. Clearly, $\bigcap_{i=1}^{\infty} N_i$ is a continuum contained in K. If $\bigcap_{i=1}^{\infty} N_i$ does not have Property P, then $\bigcap_{i=1}^{\infty} N_i$ is not terminal for M. Thus there are subcontinua D and E of M, each intersecting $\bigcap_{i=1}^{\infty} N_i$, and neither is contained in the union of $\bigcap_{i=1}^{\infty} N_i$ and the other. Let

$$d \in D - \left(E \cup \left(\bigcap_{i=1}^{\infty} N_i\right)\right) \quad \text{and} \quad e \in E - \left(D \cup \left(\bigcap_{i=1}^{\infty} N_i\right)\right).$$

Since $M - \{d, e\}$ is open in M and contains $\bigcap_{i=1}^{\infty} N_i$, there is a positive integer j such that $N_j \subset M - \{d, e\}$. Thus, each of D and E intersect N_j and neither is contained in the union of N_j and the other. Hence, N_j is not a terminal subcontinuum of M. This is impossible; hence Property P is inductive.

Since K has Property P, there is a subcontinuum L of K such that L is irreducible with respect to Property P. This establishes (i) and (ii). According to Lemma 3, L cannot be decomposable, hence (iii) is established.

LEMMA 5. *Suppose that M is an a-triodic, hereditarily unicoherent compact metric continuum and K is an indecomposable terminal subcontinuum of M. Further, suppose that there is a subcontinuum A of M such that $A \cap K \neq \emptyset$, $K \not\subset A$, and $A \not\subset K$. Let D be the composant of K containing $A \cap K$. If B is a subcontinuum of M intersecting K, such that $B \not\subset K$ and $K \not\subset B$, then $B \cap K \subset D$.*

Proof. Suppose that there is a continuum B for which the conclusion fails. Since $B \cap K$ is a proper subcontinuum of K not contained in D, $B \cap K \cap$

$D = \emptyset$. Thus $A \cap K \cap B = \emptyset$. Now, K is a terminal subcontinuum of M; it follows that $A \subset B \cup K$ or $B \subset A \cup K$. Suppose that $A \subset B \cup K$. Since $A \not\subset K$, $A \cap B \neq \emptyset$. Then $(A \cup B) \cap K$ is a subcontinuum of K intersecting disjoint composants of K. Thus $(A \cup B) \cap K = K$. However, this means that K is the union of two proper subcontinua, $A \cap K$ and $B \cap K$. This contradicts the indecomposability of K and establishes the lemma.

Definition 2. Suppose that M is an a-triodic, hereditarily unicoherent compact metric continuum and K is an indecomposable terminal subcontinuum of M. If there exists a continuum A satisfying the hypothesis of Lemma 5, then the composant D is called the *accessible composant* of K. All other composants are *inaccessible*. If no such continuum A exists, then all composants of K are inaccessible. In either case, a point of an inaccessible composant of K is an *inaccessible point* of K.

Remark 2. Suppose that M is an a-triodic, hereditarily unicoherent compact metric continuum, K is an indecomposable terminal subcontinuum of M, and μ is an inaccessible point of K. It follows immediately from Lemma 5 that if R is a subcontinuum of M containing μ, then $R \subset K$ or $K \subset R$.

Definition 3. Suppose that M is a compact metric continuum and each of K and L is a terminal subcontinuum of M. K and L are *opposite* terminal subcontinua if and only if there are points $k \in K$ and $l \in L$ such that M is irreducible from k to l.

This notion is essentially a generalization of that of "opposite terminal points" found in (**1**). The following lemma extends (**2**, Theorem 14).

LEMMA 6. *Suppose that M is an a-triodic hereditarily unicoherent compact metric continuum, K is a non-degenerate indecomposable terminal subcontinuum of M, $\mathscr{E} = E(1, m)$ is a chain covering M, $\mathscr{F} = F(1, n)$ is a chain which refines $\mathscr{E}$ and covers K, and $\mu \in F_n \cap K$ is an inaccessible point of K. Then there is a chain $\mathscr{D} = D(1, t)$ covering M such that*

(i) *$\mathscr{D}$ refines $\mathscr{E}$;*

(ii) *$\mu \in D_t$.*

Proof. Suppose that the lemma is false. If $B \subset M$, then B *has Property* P if and only if B is a subcontinuum of M containing K, and no chain covering B satisfies (i) and (ii). We shall show that Property P is inductive. Suppose that J is a sequence such that for each positive integer i, J_i has Property P and $J_{i+1} \subset J_i$. Clearly, if $\bigcap_{i=1}^{\infty} J_i$ does not have Property P, then there is a chain $\mathscr{H}$ which refines $\mathscr{E}$, covers $\bigcap_{i=1}^{\infty} J_i$, and μ is in the last link of $\mathscr{H}$. Now $\mathscr{H}^*$ is an open set containing $\bigcap_{i=1}^{\infty} J_i$. Hence, there is a positive integer j such that $J_j \subset \mathscr{H}^*$. Thus, J_j does not have Property P. This contradiction shows that $\bigcap_{i=1}^{\infty} J_i$ has Property P and Property P is inductive.

Since M has Property P and K does not have Property P, there is a subcontinuum M' of M such that M' is irreducible with respect to having Property P. For notational convenience, we shall assume that $M' = M$. We may further assume that M is not contained in any proper subchain of $\mathscr{E}$. Since K is a terminal subcontinuum of M, it follows from Lemma 2 that there is a pair of points $p \in K$, $q \in M - K$ such that M is irreducible from p to q. Since the composant of M determined by p is dense in M, there is a proper subcontinuum N of M such that $K \subset N$ and N intersects each link of $\mathscr{E}$. Now N is a proper subcontinuum of M; hence, N does not have Property P and there is a chain $\mathscr{G} = G(1, b)$ covering N such that $\mathscr{G}$ refines $\mathscr{E}$ and $\mu \in G_b$. We may assume that no chain with fewer links than $\mathscr{G}$ has these properties.

We shall demonstrate that $G_1 \cap N \cap (E_1 \cup E_m) \neq \emptyset$. For, suppose this is not true. Then there is a link $E_r \in E(2, m - 1)$ such that $G_1 \cap N \subset E_r$. Now $\mathscr{G}^* \cap N$ intersects both E_1 and E_m. Let G_s be the first link of $\mathscr{G}$ which intersects $N \cap (E_1 \cup E_m)$. Clearly $1 < s$. For definiteness, let us suppose that $G_s \cap N \cap E_1 \neq \emptyset$. Let G_t denote the first link of $\mathscr{G}$ which intersects $N \cap E_m$. Since $G(s, t)$ is a refinement of $E(1, m)$ which intersects E_1 and E_m, some link of $G(s, t)$ is contained in E_r. Thus, E_r contains a link of $G(1, t)$ distinct from G_1.

Since G_t is the first link of $\mathscr{G}$ which intersects E_m, $G_t \subset E_{m-1}$. There is a link E_x of $\mathscr{E}$ such that $G^*(1, t) \subset E^*(x, m - 1)$ and $G^*(1, t)$ is not contained in any proper subchain of $E(x, m - 1)$. Since $G^*(1, t)$ intersects both E_1 and E_m, $1 \leqq x \leqq 2 \leqq r \leqq m - 1$. Define a new chain by

$$G^*(1, t - 1) \cap E(x, m - 2) \oplus G^*(1, t) \cap E_{m-1} \oplus G(t + 1, b).$$

Since $G_{t+1} \subset E_{m-1}$, this chain refines $\mathscr{E}$, covers N, and has μ in its last link. Moreover, this chain has fewer links than $\mathscr{G}$, since each link of $E(x, m - 1)$ contains at least one link of $\mathscr{G}$ and $E_r \in E(x, m - 1)$ contains at least two links of $\mathscr{G}$. This is contrary to the choice of $\mathscr{G}$, and hence $G_1 \cap (E_1 \cup E_m) \cap N \neq \emptyset$. We shall suppose that $G_1 \cap N \cap E_1 \neq \emptyset$.

Let V be an open set such that $\operatorname{Cl} V \subset (G_1 \cap E_1) - \{\mu\}$, and $V \cap N \neq \emptyset$. Let R denote the component of $M - V$ containing μ. Since each of N and R is a continuum intersecting K, either $R \subset N \cup K = N$ or $N \subset R \cup K$. We shall show that the last alternative is impossible. If $V \cap N \subset K$, then, since μ is an inaccessible point of K, it follows from Remark 2 that $R \subset K$. Since $N \not\subset K$, $N \not\subset R \cup K = K$. If $V \cap N \not\subset K$, then $V \cap N \not\subset R \cup K$ and $N \not\subset R \cup K$. In either event, $N \not\subset R \cup K$; hence $R \subset N \cup K = N$. Since R is a component of $M - V$ and N is covered by $\mathscr{G}$, no continuum in $M - V$ intersects both R and $(M - V) - \mathscr{G}^* = M - \mathscr{G}^*$. Thus, $M - V$ is the union of two disjoint closed sets, one containing $M - \mathscr{G}^*$, the other containing R. Using normality, we obtain disjoint open sets S and T such that $M - V \subset S \cup T$, $M - \mathscr{G}^* \subset T$ and $R \subset S$. Define a chain $\mathscr{D} = D(1, t)$ covering M as follows:

$$\mathscr{D} = E(m, 3) \cap T \oplus (E_2 \cap T) - \operatorname{Cl} V \oplus E_1 \cap T \oplus (G_1 \cap S) \cup V \oplus G(2, b) \cap S.$$

Since $\mathscr{D}$ has properties (i) and (ii), this concludes the proof.

LEMMA 7. *Suppose that D is an a-triodic hereditarily unicoherent compact metric continuum, K and L are disjoint opposite terminal subcontinua of D, and $\mathscr{E}'$ is an ϵ-chain covering D. Then there is a chain $\mathscr{F} = F(1, n)$ covering D such that*

(1) *$\mathscr{F}$ is a refinement of $\mathscr{E}'$;*

(2) *$F_1 \cap K \neq \emptyset$ and $F_n \cap L \neq \emptyset$;*

(3) *there are positive integers i and k with $K \subset^e F^*(1, i)$, $\quad L \subset^e F^*(k, n)$.*

Proof. We first wish to obtain a chain $\mathscr{F}'$ satisfying (1) and (2). Minor modifications of $\mathscr{F}'$ will then yield a chain $\mathscr{F}$ satisfying (1), (2), and (3). (The assumption that K and L are disjoint is not necessary, but it makes the proof of (3) easier.)

According to Lemma 1, there is an ϵ-chain $\mathscr{E} = E(1, m)$ which refines $\mathscr{E}'$, covers D, and $K \subset^e E^*(j, m)$. Applying this lemma again, we obtain a chain $\mathscr{G} = G(1, b)$ covering D such that $\mathscr{G}$ refines $\mathscr{E}$ and $L \subset^e G^*(a, b)$; we may assume that no chain with fewer links than $\mathscr{G}$ has these properties. If $E_1 \cap L \neq \emptyset$, we may take $\mathscr{F}' = \mathscr{E}$; similarly, if $G_1 \cap K \neq \emptyset$, we may take $\mathscr{F}' = \mathscr{G}$. Thus, we may assume that $E_1 \cap L = \emptyset$ and $G_1 \cap K = \emptyset$.

There is a link of $\mathscr{G}$ contained in E_1 and a link of $\mathscr{G}$ contained in E_m. An argument essentially the same as that given in the proof of Lemma 6 will show that $G_1 \subset E_1 \cup E_m$. (Substitute "links of $\mathscr{G}$ contained in $E_1 \cup E_m$" for "links of $\mathscr{G}$ intersecting $N \cap (E_1 \cup E_m)$".) Indeed, $G_1 \subset E_1$. For, suppose that $G_1 \not\subset E_1$, hence that $G_1 \subset E_m$. Now $G_1 \cap K = \emptyset$, and since $K \subset^e E^*(j, m)$, $(E_m - E_{m-1}) \cap K \neq \emptyset$. Hence, there is a link $G_r \in G(1, m)$ such that $G_r \subset E_m$. There is a positive integer z such that $G^*(1, r) \subset E^*(z, m)$ and $G^*(1, r) \not\subset E^*(z + 1, m)$. Then $E(z, m - 1) \cap G^*(2, r - 1) \oplus E_m \cap G^*(1, r) \oplus G(r + 1, b)$ is a proper consolidation of $\mathscr{G}$ since E_m contains G_1 and G_r. This violates the choice of $\mathscr{G}$ as being a chain with fewest links which has the desired properties. Thus $G_1 \subset E_1$.

Now D is irreducible from a point $k \in K$ to a point $l \in L$. Since the composant of D determined by l is dense in D, there is a proper subcontinuum N of D such that $l \in N$ and $N \cap G_1 \cap E_1 \neq \emptyset$. Since $k \notin N$, $N \cup L$ is a proper subcontinuum of D. We shall simply assume that $L \subset N$.

Let V be an open set intersecting N such that $\operatorname{Cl} V \subset G_1 \subset E_1$. Since $E_1 \cap L = \emptyset$ and $G_1 \cap K = \emptyset$, $V \cap (L \cup K) = \emptyset$. Since D is irreducible from k to l, no continuum in $D - V$ intersects both K and L. (If R is such a continuum, then $R \cup K \cup L$ is a proper subcontinuum of D containing k and l.) Hence, $D - V$ is the union of two disjoint closed sets, one containing K, and the other containing L. Thus, there are disjoint open sets S and T such that $M - V \subset S \cup T$, $K \subset S$ and $L \subset T$. Define a chain $\mathscr{F}' = F'(1, n)$ as follows:

$$\mathscr{F}' = E(m, 3) \cap S \oplus (E_2 \cap S) - \operatorname{Cl} V \oplus E_1 \cap S \oplus (G_1 \cap T) \cup V \oplus G(2, b) \cap T.$$

https://doi.org/10.4153/CJM-1969-040-x Published online by Cambridge University Press

Since $\mathscr{F}'$ refines $\mathscr{E}$, $\mathscr{F}'$ refines $\mathscr{E}'$. Moreover, $(F_1' - F_2') \cap R \neq \emptyset$, and $(F_n' - F_{n-1}') \cap L \neq \emptyset$.

Let Q and R be open sets such that $K \subset Q \subset \mathrm{Cl}Q \subset F'^*(1, i)$, $L \subset R \subset \mathrm{Cl}\, R \subset F'^*(k, n)$, and $\mathrm{Cl}\, Q \cap \mathrm{Cl}\, R = \emptyset$. We define the chain $\mathscr{F} = F(1, n)$ as follows: $F_p = F_p'$, if $p \neq i + 1$, $p \neq k - 1$. If $k - 1 \neq i + 1$, then

$$F_{i+1} = F_{i+1}' - \mathrm{Cl}\, Q \quad \text{and} \quad F_{k-1} = F_{k-1}' - \mathrm{Cl}\, R.$$

If $k - 1 = i + 1$, $F_{i+1} = F_{i+1}' - (\mathrm{Cl}\, Q \cup \mathrm{Cl}\, R)$. $\mathscr{F}$ satisfies (1) and (2) since $\mathscr{F}'$ does, and $\mathscr{F}$ satisfies (3) as well.

4. Principal results. In (**2**), the following theorem is proved.

THEOREM 1. *Suppose that M is an a-triodic hereditarily unicoherent compact metric continuum, $\epsilon > 0$, M is the union of two subcontinua A and B, A is ϵ-chainable and B is chainable. Then M is ϵ-chainable.*

This is a slightly strengthened version of (**2**, p. 466, Theorem 1), and the proof given there will go through with only very minor changes. Note, in particular, that if A is not simply ϵ-chainable but chainable, then M is chainable.

THEOREM 2. *A compact metric continuum M is chainable if and only if M is a-triodic, hereditarily unicoherent, and each indecomposable subcontinuum of M is chainable.*

Proof. Certainly, each of the three conditions is necessary for M to be chainable. Let us suppose, then, that the three conditions hold.

If M fails to be chainable, then there is an $\epsilon > 0$ such that no ϵ-chain covers M. Since the property of failing to be ϵ-chainable is inductive, there is a subcontinuum M' of M which is irreducible with respect to this property. We may assume that $M' = M$. Clearly, M is decomposable.

Case I. There is an indecomposable subcontinuum D of M such that $D^0 \neq \emptyset$.

Subcase Ia. D is a terminal subcontinuum of M.

By Lemma 2, M is irreducible between a pair of points, one of which belongs to D. Thus, $M - D$ is connected and $\mathrm{Cl}(M - D)$ is a continuum. Moreover, $\mathrm{Cl}(M - D)$ is a proper subcontinuum of M, since $D^0 \neq \emptyset$. Thus, M is the union of two proper subcontinua of M, $\mathrm{Cl}(M - D)$, which is ϵ-chainable, and D, which is chainable. Theorem 1 shows that $\mathrm{Cl}(M - D) \cup D = M$ is ϵ-chainable. This contradiction establishes the theorem for Subcase Ia.

Subcase Ib. D is not a terminal subcontinuum of M.

Then $\mathrm{Cl}(M - D)$ is not connected. For, if $\mathrm{Cl}(M - D)$ is connected, then by Remark 1, both $\mathrm{Cl}(M - D)$ and D are terminal subcontinua of M.

We shall show that $\mathrm{Cl}(M - D)$ has exactly two components. Suppose that X, Y, and Z are distinct components of $\mathrm{Cl}(M - D)$. Since D contains a limit

https://doi.org/10.4153/CJM-1969-040-x Published online by Cambridge University Press

point of each component of $M - D$, D certainly contains a limit point of each component of $\mathrm{Cl}(M - D)$. Since each of X, Y, and Z is closed, it follows that each must intersect D. Thus, $X \cup D$, $Y \cup D$, and $Z \cup D$ are three continua which intersect, no one of which is contained in the union of the other two. By (**3**, p. 440), their union is a triod. Since this is impossible, $\mathrm{Cl}(M - D)$ has exactly two components, X and Y.

Now X is a proper subcontinuum of M; hence, X is ϵ-chainable. Remark 1 shows that $D \cap X$ is a terminal subcontinuum of X. Applying Lemma 1, we obtain a taut ϵ-chain $\mathscr{E} = E(1, m)$ covering X and a positive integer j, $1 \leqq j \leqq m$, such that $D \cap X \subset^e E^*(j, m)$. In like fashion, there is a taut ϵ-chain $\mathscr{G} = G(1, t)$ covering Y and a positive integer s, $1 \leqq s \leqq t$, such that $D \cap Y \subset^e G^*(s, t)$. Since X and Y are disjoint closed sets, we may invoke the normality of M to assume that $\mathrm{Cl}\, G^* \cap \mathrm{Cl}\, \mathscr{E}^* = \emptyset$.

Since $D^0 \neq \emptyset$, each of $D \cap X$ and $D \cap Y$ is a proper terminal subcontinuum of D. Now D is indecomposable; hence, D is irreducible from $D \cap X$ to $D \cap Y$, i.e., $D \cap X$ and $D \cap Y$ are disjoint opposite terminal subcontinua of D. From Lemma 7, it follows that there is a taut ϵ-chain $\mathscr{F} = F(1, n)$ covering D and positive integers i and k such that $1 \leqq i < k - 2 < k \leqq n$, $X \cap D \subset^e F^*(k, n)$, and $Y \cap D \subset^e F^*(1, i)$. Moreover, since the links of $\mathscr{F}$ may be made as small as we please, we may assume that $F(k, n)$ is a closed refinement of $E(j, m)$, $F(1, i)$ is a closed refinement of $G(s, t)$, and

$$\mathrm{Cl}\,\mathscr{F}^* \cap (\mathrm{Cl}(E^*(1, j - 1)) \cup \mathrm{Cl}(G^*(1, s - 1))) = \emptyset;$$

see Figure 1.

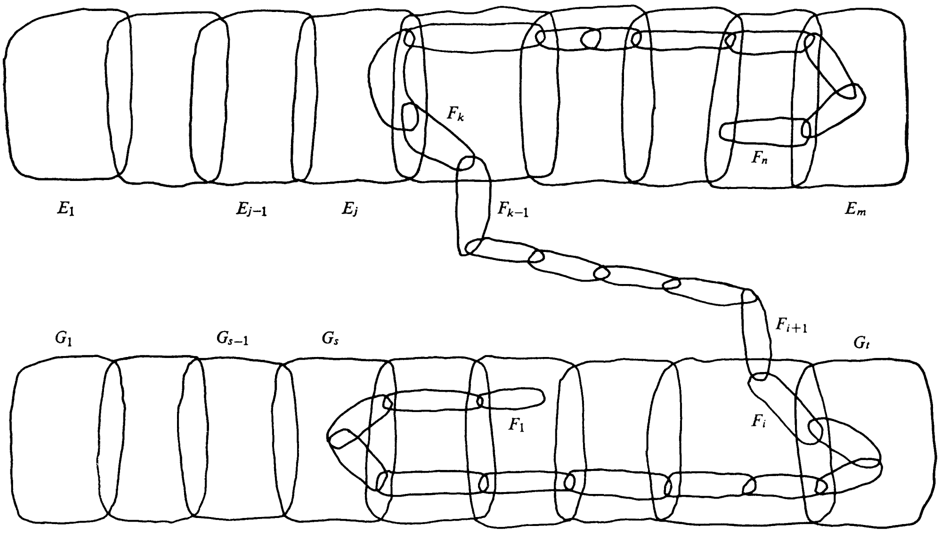

FIGURE 1

We now apply, to $\mathscr{E}$ and $\mathscr{F}$, the technique of "amalgamating two chains" used in the proof of Theorem 1, as given in (**2**). (Our construction shows that $\mathscr{E}$ and $\mathscr{F}$ satisfy (1)–(4) of (**2**, Lemma 4). Small modifications of $\mathscr{E}$ and $\mathscr{F}$, detailed in (**2**, p. 465) will ensure that they satisfy (5) and (6) as well.) This yields an ϵ-chain $\mathscr{H}$ covering $X \cup D$ such that the subchain of $\mathscr{H}$ containing $Y \cap D$ is precisely $F(1, i)$. Now we use this technique again, letting $\mathscr{H}$ play the role of $\mathscr{F}$, and $\mathscr{G}$ play the role of $\mathscr{E}$. This yields an ϵ-chain covering M and concludes the proof of Subcase Ib.

Case II. Each indecomposable subcontinuum of M has void interior.

Under this hypothesis, Bing has shown in the proof of (**1**, p. 658, Theorem 8) that there is a monotone upper semi-continuous decomposition J of M such that the decomposition space M/J is homeomorphic to [0, 1]. Let $f: M \to [0, 1]$ denote the projection map. Let $A = f^{-1}[0, \frac{1}{2}]$ and $B = f^{-1}[\frac{1}{2}, 1]$. Each of A and B is a proper subcontinuum of M, hence each is ϵ-chainable. By Remark 1, $A \cap B$ is a terminal subcontinuum of A and of B; in like fashion, $A \cap \mathrm{Cl}(B - A)$ is a terminal subcontinuum of A and $\mathrm{Cl}(B - A)$.

Claim 1. If Q is a subcontinuum of $\mathrm{Cl}(B - A)$ such that $Q \cap A \cap \mathrm{Cl}(B - A) \neq \emptyset$ and $Q \not\subset A \cap \mathrm{Cl}(B - A)$, then $A \cap \mathrm{Cl}(B - A) \subset Q$.

Suppose that there is a continuum Q satisfying the hypothesis but not the conclusion of Claim 1. Let $p \in (A \cap \mathrm{Cl}(B - A)) - Q$ and let V be an open set such that $p \in V$ and $V \cap Q = \emptyset$. Since $Q \not\subset A \cap \mathrm{Cl}(B - A)$, $Q \not\subset A$, and hence there is a point $t \in Q$ with $f(t) > \frac{1}{2}$. Now $p \in A \cap \mathrm{Cl}(B - A)$, hence there is a point $s \in V \cap f^{-1}(0, f(t)) \cap (B - A)$. Since $s \notin A$, $\frac{1}{2} < f(s) < f(t)$. Thus, each of $\mathrm{Cl}(B - A) \cap f^{-1}[\frac{1}{2}, f(s)]$ and Q is a subcontinuum of $\mathrm{Cl}(B - A)$ intersecting $A \cap \mathrm{Cl}(B - A)$, and neither is contained in the union of $A \cap \mathrm{Cl}(B - A)$ and the other ($t \in Q - (A \cup f^{-1}[\frac{1}{2}, f(s)])$, $p \in \mathrm{Cl}(B - A) \cap f^{-1}[\frac{1}{2}, f(s)]$, and $p \notin Q$). Thus, $A \cap \mathrm{Cl}(B - A)$ is not a terminal subcontinuum of $\mathrm{Cl}(B - A)$. This contradiction establishes Claim 1.

Lemma 4 yields a subcontinuum K of $A \cap \mathrm{Cl}(B - A)$ such that K is irreducible with respect to being a terminal subcontinuum of A. It follows that K is either a terminal point of A or a non-degenerate indecomposable continuum.

We shall show that K is a terminal subcontinuum of $\mathrm{Cl}(B - A)$. If this is not true, then there are subcontinua L and R of $\mathrm{Cl}(B - A)$, each intersecting K, and neither is contained in the union of K and the other. Since K is a terminal subcontinuum of A, $L \cup R \not\subset A$. Suppose that $L \not\subset A$. Then $L \not\subset A \cap \mathrm{Cl}(B - A)$, and from Claim 1, $A \cap \mathrm{Cl}(B - A) \subset L$. Since $R \not\subset L$, it follows that $R \not\subset A \cap \mathrm{Cl}(B - A)$, and thus $A \cap \mathrm{Cl}(B - A) \subset R$. Hence, R, L, and A are three continua which intersect, no one of which is contained in the union of the other two. Thus, their union is a triod. This contradiction shows that K is indeed a terminal subcontinuum of $\mathrm{Cl}(B - A)$.

Let $\mathscr{E}'$ and $\mathscr{F}'$ be ϵ-chains covering A and $\mathrm{Cl}(B - A)$, respectively. We now show that there are chains $\mathscr{E}$ and $\mathscr{F}$ covering A and $\mathrm{Cl}(B - A)$, respectively, such that $\mathscr{E}$ refines $\mathscr{E}'$, $\mathscr{F}$ refines $\mathscr{F}'$, and the last link of $\mathscr{E}$ intersects the last link of $\mathscr{F}$ in a point of K. If K is a terminal point of A, then it is also a terminal point of $\mathrm{Cl}(B - A)$, by the argument just given, and the existence of $\mathscr{E}$ and $\mathscr{F}$ follows immediately from Lemma 1. Thus, we may assume that K is a non-degenerate indecomposable continuum; hence, K is chainable. Let $\mathscr{G} = G(1, d)$ be a chain covering K which refines both $\mathscr{E}'$ and $\mathscr{F}'$. Let $\mu \in G_d$ be a point of K which is inaccessible from either A or $\mathrm{Cl}(B - A)$. (Since K has uncountably many disjoint composants, at most two of which are accessible from either A or $\mathrm{Cl}(B - A)$, such a point μ exists.) By Lemma 6, there is a chain $\mathscr{E} = E(1, x)$ which refines $\mathscr{E}'$, covers A, and $\mu \in E_x \cap K$. Similarly, there is a chain $\mathscr{F} = F(1, y)$ which refines $\mathscr{F}'$, covers $\mathrm{Cl}(B - A)$, and $\mu \in F_y \cap K$. Let U be an open set such that $\mu \in U$ and $\mathrm{Cl}\ U \subset E_x \cap F_y$.

Claim 2. No continuum in $M - U$ intersects both $A - U$ and $(M - U) - \mathscr{E}^* = M - \mathscr{E}^*$.

Suppose that there is such a continuum, N. Then N intersects $B - A$, since $N \not\subset \mathscr{E}^*$. Since N intersects A, N must intersect $A \cap \mathrm{Cl}(B - A)$. Thus, $N \cap \mathrm{Cl}(B - A)$ is a subcontinuum of $\mathrm{Cl}(B - A)$ intersecting $A \cap \mathrm{Cl}(B - A)$ and $B - A$. From Claim 1, it follows that $A \cap \mathrm{Cl}(B - A) \subset N \cap \mathrm{Cl}(B - A)$. However, $\mu \in K \subset A \cap \mathrm{Cl}(B - A)$, and $\mu \notin N$. This contradiction establishes Claim 2.

It follows that $M - U$ is the union of two disjoint closed sets, one containing $A - U$, the other containing $M - \mathscr{E}^*$. Using normality, we obtain disjoint open sets S and T such that $M - U \subset S \cup T$, $A - U \subset S$, and $M - \mathscr{E}^* \subset T$. An ϵ-chain covering M is given by

$$E(1, x - 2) \cap S \oplus (E_x \cap S) - \mathrm{Cl}\ U \oplus E_x \cap S \oplus (F_y \cap T) \cup U \oplus F(y - 1, 1) \cap T.$$

This establishes Theorem 2.

COROLLARY. *Suppose that M is an a-triodic compact plane continuum which does not separate the plane. Then M is chainable if and only if each indecomposable subcontinuum of M is chainable.*

Theorem 2 and its corollary are extensions of (**1**, p. 660, Theorem 11 and Corollary 2).

THEOREM 3. *Suppose that M is an a-triodic hereditarily unicoherent compact metric continuum which is the union of countably many chainable continua. Then M is chainable.*

Proof. Suppose that J is a sequence of chainable subcontinua of M such that $M = \bigcup_{i=1}^{\infty} J_i$. If N is an indecomposable subcontinuum of M, then

$N = \bigcup_{i=1}^{\infty}(N \cap J_i)$. Since, for each i, $N \cap J_i$ is a continuum and no indecomposable continuum is the union of countably many proper subcontinua, there is a positive integer l such that $N = N \cap J_l$. Thus, N is a subcontinuum of J_l; hence N is chainable. Since each indecomposable subcontinuum of M is chainable, we apply Theorem 2 and find that M is *vv* chainable.

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University of Kentucky,
Lexington, Kentucky