ON NON-NEGATIVE SPECTRUM IN BANACH ALGEBRAS

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1. Introduction

Let A be a complex Banach algebra with an identity 1. In this note we study the subset Γ of A consisting of all $g \in A$ such that the spectrum of g, $sp(g)$, contains at least one non-negative real number. Clearly Γ is not, in general, a semi-group with respect to either addition or multiplication. However, Γ is an instance of a subset Q of A with the following properties, where $\rho(f)$ denotes the spectral radius of f (**4**, p. 30).

(*a*) If $f \in Q$ and $t \geqq 0$, then $tf \in Q$.

(*b*) If $f \in Q$ and $\rho(f) < 1$, then $f(1-f)^{-1} \in Q$.

(*c*) The distance of Q from -1 is larger than zero.

(*d*) If $f \in Q$ and $\rho(f) < 1$, then $1-f \in Q$.

(*e*) If $f \in Q$ and $\rho(f) < 1$, then $(1-f)^{-1} \in Q$.

(*f*) If $f \in Q$ and $t > 0$, then $t+f \in Q$.

For our purpose (the characterisation of Γ) properties (*d*), (*e*) and (*f*) are not as useful as (*b*). To see this consider the Banach algebra B of all complex-valued functions on the compact Hausdorff space E in the sup norm. The subset Q of all $f \in B$ where $\operatorname{Re} f(t) \geqq 0$, for all $t \in E$, is a closed subset of B satisfying properties (*a*), (*c*), (*d*), (*e*) and (*f*). Here Q neither contains nor is contained in Γ. On the other hand, we shall see that $Q \subset \Gamma$ if Q satisfies (*a*), (*b*) and (*c*).

Condition (*b*) can be restated in the language of quasi-inverses (**4**, p. 16). For if g' denotes the quasi-inverse of g, then $f(1-f)^{-1} = -f'$.

2. On the properties (a), (b) and Γ

Theorem 1. *Let Q be a subset of A with properties* (*a*) *and* (*b*). *Then either Q is contained in Γ or -1 lies in the closure of Q.*

Proof. Notice that, for a complex number z, $|z/(1-z)| < 1$ if and only if $\operatorname{Re}(z) < \frac{1}{2}$. Now let D_n be the closed disc in the complex plane with centre at $n/(n^2-1)$ and radius $1/(n^2-1)$, $n = 2, \ldots$. Then, for these values of n,

$$|z/(1-nz)| < 1 \text{ if and only if } z \notin D_n.$$

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In these terms we define a set G in the complex plane by

$$G=\{z\colon |z|<1 \text{ and } |z/(1-nz)|<1, \quad n=1,2,\ldots\}$$

and can readily picture G graphically. Observe that if $z\in G$ then $z/(1-z)\in G$.

We suppose that -1 is not in the closure of Q and must show that, for each $f\in Q$, $sp(f)$ contains a non-negative number.

To this end we show first that if $g\in Q$ and $sp(g)\subset G$, then g has no inverse in A. We define by induction a sequence $\{g_n\}$ starting with $g_1=g(1-g)^{-1}$. Note that $g_1\in A$ and $sp(g_1)\subset G$. Then, setting $g_{n+1}=g_n(1-g_n)^{-1}$, we see that every $g_n\in Q$ and $sp(g_n)\subset G$. We show, by induction, that $(1-ng)^{-1}$ exists in A and $g_n=g(1-ng)^{-1}$, $n=1,2,\ldots$. This is certainly true for $n=1$. Assuming this fact for n we consider

$$1-g_n=[1-(n+1)g](1-ng)^{-1}. \tag{1}$$

Then

$$1-(n+1)g=(1-g_n)(1-ng)$$

is the product of two invertible elements. Moreover, by (1), and the induction hypothesis, we get

$$g_{n+1}=g(1-ng)^{-1}(1-g_n)^{-1}=g(1-(n+1)g)^{-1}.$$

Now that we have $g(1-ng)^{-1}\in Q$ for each positive integer n we use (a) to get

$$ng_n=g(n^{-1}-g)^{-1}\in Q. \tag{2}$$

If $g^{-1}\in A$, then, from (2), we see that -1 is in the closure of Q. Therefore g^{-1} fails to exist, as claimed.

Next let $f\in Q$. Suppose that $sp(f)$ is disjoint with $[0,\infty]$. As $sp(f)$ is compact there is a number α, $0<\alpha<\pi/2$, so that $sp(f)$ is disjoint with the wedge W of complex numbers of the form $z=r\exp(i\theta)$, $-\alpha\leqq\theta\leqq\alpha$ and $0\leqq r<\infty$. Moreover, $sp(af)$ is disjoint with W for all $a>0$.

Elementary computations show that D_n is contained in the interior of W for all $n=2,3,\ldots$ such that $n>csc(\alpha)$. Let N be the smallest of these integers. Note that $|z|\geqq(n+1)^{-1}$ for all $z\in D_n$. Therefore, if we choose $b>0$ so that $\|bf\|<(N+1)^{-1}$, we see that $sp(bf)$ is also disjoint with D_j, $j\leqq N$. This ensures that $sp(bf)\subset G$. But then, as shown above, f^{-1} does not exist or $0\in sp(f)$. This contradicts our assumption that $[0,\infty]$ is disjoint with $sp(f)$ and completes the proof.

Suppose $g\notin\Gamma$. Since $sp(g)$ is compact there is an open set V in the complex plane containing $sp(g)$ and disjoint with $[0,\infty)$. By (**4**, Theorem 1.6.16), $sp(h)\subset V$ if $h\in A$ is sufficiently close to g. Consequently Γ is closed in A and we deduce the following result from our theorem.

Corollary 1. *Γ is the unique maximal element in the collection of closed subsets of A with the properties* (a), (b) *and* (c).

These results also hold for a real Banach algebra A as can be seen by considering the complexification (**4**) of A.

Following Bonsall (**1**) (see also **2**) we call a subset B of A a *semi-algebra* if, whenever f, $g \in B$ and t is a non-negative scalar, we have $f+g \in B$, $fg \in B$ and $tf \in B$.

Corollary 2. *Any closed semi-algebra B in A either contains -1 or is contained in Γ.*

Proof. Let $f \in B$, $\rho(f) < 1$. Then, inasmuch as

$$f(1-f)^{-1} = \sum_{n=1}^{\infty} f^n$$

we see that $f(1-f)^{-1} \in B$. Hence B satisfies (b) and Theorem 1 applies. Corollary 2 was obtained in an entirely different way by Civin and White (**3**, p. 242).

Bonsall (**1**) and Brown (**2**) study type 0 semi-algebras (semi-algebras B which have the additional property that $(1+f)^{-1} \in B$ whenever $f \in B$). In this case B has the following property.

$$sp(f) \cap (-\infty, 0) \text{ is void for each } f \in B. \tag{3}$$

Property (3) is related to our earlier properties.

Proposition. *Let Q be any subset of A with properties* (a) *and* (b). *Let J be the set of all $g \in Q$ such that $sp(g)$ intersects* $(-\infty, 0)$ *vacuously. Then J has properties* (a) *and* (b).

Proof. Consider $g \in J$ with $\rho(g) < 1$ and let A_0 be a maximal closed subalgebra of A containing f. We let Φ denote the carrier space of A_0 and use (**4**, Theorem 1.6.14).

Set $h = g(1-g)^{-1}$. If there exists some λ, $-\infty < \lambda < 0$, $\lambda \in sp(h)$ then, for some $\phi \in \Phi$ we have also

$$\hat{g}(\phi)/(1-\hat{g}(\phi)) = \lambda.$$

From this we get $\hat{g}(\phi) = \lambda - \lambda\hat{g}(\phi)$. We cannot have $\lambda = -1$. If $\lambda < -1$, then

$$\hat{g}(\phi) = \lambda(1+\lambda) > 1$$

contrary to $\rho(g) < 1$. If $-1 < \lambda < 0$ we get $g(\phi) < 0$ contrary to $g \in J$. Therefore $h \in J$.

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