

ACTUARIAL VALUATION OF PERISHABLE INVENTORY SYSTEMS

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The aim of this article is to derive the income and cost functionals required to determine the actuarial value of certain types of perishable inventory system. In the basic model, the arrival times of the items to be stored and the ones of the demands for those items form independent Poisson processes. The shelf lifetime of every item is finite and deterministic. Every demand is for a single item and is satisfied by the oldest item on the shelf, if available. The price of an item depends on its shelf age. For an actuarial valuation, it is important to know the distribution of the total value of the items in the system and the expected (discounted) total income and cost generated by the system when in steady state. All of these functionals are determined explicitly. As extensions of the original model, we also deal with the case of batch arrivals and general renewal interdemand times; in both cases, closed-form solutions are obtained.

1. INTRODUCTION

We consider a perishable inventory system in which both the arrival and the demand are independent Poisson processes. Our interest is in the *actuarial value* of the system,

defined as the value of items in stock plus the total expected future discounted net income generated by the system. Fresh stock arrives to the system according to a Poisson process with rate λ . Without loss of generality, we assume that the lifetime of each item is one time unit. If an item has not been consumed by demand before its expiration date, it is discarded. The demand arrival times form a Poisson process with rate μ . Demands are satisfied on a FIFO basis (i.e., oldest items are issued first), and the return for a sold item is assumed to depend on its age at issuing time. A demand that arrives when the system is empty is lost.

This problem can arise in several contexts. One would be to value a concern that sells or supplies perishable inventories, such as a blood bank. Another would be to determine the value of a particular perishable product. Inventory valuation is straightforward for nonperishables, but it is more subtle for perishables, since some items may ultimately satisfy demands and some may perish. The methodology of this article provides a means of valuing a specific line of perishable goods.

This above-described basic model as well as several extensions have been studied in previous articles (see, e.g., [3,8–12]) in which also various applications to real-life systems such as blood banks, food storage places, and so forth are exhibited. Here, we are interested in determining the stationary economic value of the system, which is the combination of the stationary expected purchase value of the items present in the system and the net income generated in the future.

The purchase cost of a fresh item is fixed as c monetary units. On the other hand, the price of each item while aging on the shelf decreases. Let $R(x)$ be its price at shelf age x . We suppose that $R(x)$ is nonincreasing on $(0,1]$ and $R(0) > c$. It is natural but not necessary to assume that $R(1) = 0$. Based on R , we will determine the stationary value of the items in the system.

Regarding future profits and cost, we introduce discounting by the discount factor $\beta > 0$. The net expected gain of the system is composed of four functionals:

1. The total expected income from selling items to demands
2. The total expected purchase cost
3. The total expected penalty cost due to outdatings of items
4. The total expected penalty cost due to unsatisfied demands

Our main tool for determining the eventual expected value of the items on the shelf and the functionals 1–4 is the *virtual outdating time* (VOT) process $\mathbf{W} = \{W(t): t \geq 0\}$. First define the *extended age* process $\mathbf{X} = \{X(t): t \geq 0\}$. If at time t the system is not empty, then $X(t)$ is the age of the oldest item present on the shelf; if the system is empty, then $-X(t)$ is the time until the next item arrival. Now, set $W(t) = 1 - X(t)$. Clearly, $W(t)$ can be interpreted as the time from t until the next outdating (removal from the shelf) of an item provided that there are no demand arrivals until then. \mathbf{W} is a regenerative Markov process with state space $[0, \infty)$. Whenever \mathbf{W} hits zero, an item reaches shelf age 1 and is therefore outdated. Then, the second oldest item becomes the oldest one or, in case the system is now empty, \mathbf{W} jumps above 1, indicating the time until the item arriving next would be outdated if it is not used to satisfy some demand. The time intervals between two visits at zero can be taken as

the regenerative cycles of \mathbf{W} . Any demand arriving as long as $\mathbf{W} > 1$ will not be satisfied and is lost. For more details, see [10]. A typical sample path of \mathbf{W} is depicted in Figure 1.

A little reflection shows that \mathbf{W} can also be described as the workload (virtual waiting time) process of a special $M/M/1$ queuing system. This queue has arrival rate μ , service rate λ , and impatient customers in the sense that any of them that would have to wait for more than one time unit do not enter the line; moreover, the idle periods are deleted, so that whenever the system becomes empty a new customer immediately enters. This interpretation is often helpful.

The models in this article are based on the assumption that the arrival times of the perishable items are random and form a renewal process. We do not consider controlling the input by ordering policies. The study of such policies and their optimization is the subject of the second main strand of perishable inventory theory (see, e.g., [5–7] and the more recent papers [2,4,14] where further references can be found).

The article is organized as follows. First, we derive the stationary distribution of \mathbf{W} in Section 2. In Section 3, we express the Laplace transform (LT), and the expected value, of the price of the current inventory and the functionals 1–4 in terms of this distribution. In Sections 4 and 5, we consider two generalizations of the basic model: batch demands of random i.i.d. sizes and general interarrival times for the demands. The first extension leads to the consideration of a modified Markovian queuing system with phase-type service times; the second one can be treated by a certain transformation of \mathbf{W} including time reversal that leads to an $M/G/1$ -type finite dam. In both cases, all functionals of interest can be computed in a closed form involving infinite series of convolutions.

2. STATIONARY DISTRIBUTION OF \mathbf{W}

Being a regenerative process with finite expected cycle length, \mathbf{W} possesses a stationary distribution. By level-crossing theory (see, e.g., [1]), the stationary density $f(x)$ is given as the rate of downcrossings of level x . Since in steady state this rate is

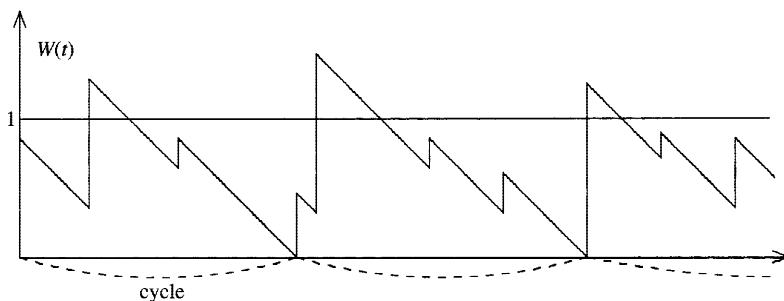


FIGURE 1. A typical sample path of \mathbf{W} .

equal to that of upcrossings of x , we immediately arrive at the Pollaczek–Khintchine integral equation

$$f(x) = \begin{cases} \mu \int_0^x e^{-\lambda(x-\omega)} f(\omega) d\omega + f(0)e^{-\lambda x}, & x \leq 1 \\ \mu \int_0^1 e^{-\lambda(x-\omega)} f(\omega) d\omega + f(0)e^{-\lambda x}, & x > 1. \end{cases} \tag{2.1}$$

Indeed, the arrival rate of upward jumps is μ , and when starting from $\omega \in (0, \min[x, 1])$ the probability to jump above x is $e^{-\lambda(x-\omega)}$; the term $f(0)e^{-\lambda x}$ accounts for jumps to (x, ∞) just after outdatings (i.e., at hittings of zero). It follows from (2.1) that f is of the form

$$f(x) = \begin{cases} k_1 e^{-(\lambda-\mu)x}, & x \leq 1 \\ k_2 e^{-\lambda x}, & x > 1 \end{cases} \tag{2.2}$$

for certain constants k_1 and k_2 . As f is continuous on $(0, \infty)$, we have $k_2 = k_1 e^\mu$. By the normalizing condition $\int_0^\infty f(x) dx = 1$, we find that

$$k_1 = \begin{cases} \frac{\lambda(\lambda - \mu)}{\lambda - \mu e^{-(\lambda-\mu)}} & \text{if } \lambda \neq \mu \\ \frac{\lambda}{\lambda + 1} & \text{if } \lambda = \mu \end{cases} \tag{2.3}$$

$$k_2 = \begin{cases} \frac{\lambda(\lambda - \mu)}{\lambda^{-\mu} - \mu e^{-\lambda}} & \text{if } \lambda \neq \mu \\ \frac{\lambda e^\lambda}{\lambda + 1} & \text{if } \lambda = \mu. \end{cases} \tag{2.4}$$

By (2.2), $k_1 = f(0)$, so that, by level-crossing theory,

$$k_1 \stackrel{def}{=} \lambda^*$$

is the long-run average rate of outdatings per unit time.

Remark: It is straightforward to show that the outdating process (i.e., the arrival times of outdatings), as well as the unsatisfied demand process, are delayed renewal processes. In fact, these two renewal processes are dual to each other in the sense that the law of the latter is equal to that of the former if λ and μ are interchanged. Let μ^* be the rate of the unsatisfied demand process. That means that $1/\mu^*$ is the mean time between unsatisfied demands. On the one hand, we have, by renewal theory,

$$\mu^* = \mu - \lambda + \lambda^* \tag{2.5}$$

because, on the average, the net input is equal to the net output (the long-run time average of satisfied demands coincides with the long-run time average of arriving items that have not been outdated). On the other hand, we have, by PASTA,

$$\mu^* = \mu \left[1 - \int_1^\infty f(x) dx \right]. \tag{2.6}$$

It can be easily verified that the right-hand sides of (2.5) and (2.6) are equal.

3. THE ACTUARIAL VALUATION

We now assume that the system is in steady state. Let V be the total value of all items present in the system. We start by determining the LT of V . Let W be the steady state variable of the VOT process; note that W has density f . If $W > 1$, the shelf is empty. If $W \leq 1$, then $1 - W$ is the age of the currently oldest item, which is the time elapsed since its arrival. In this case, let N be the number of items that have arrived after the oldest item and denote by $T_1 \leq \dots \leq T_N$ the times elapsed at their arrivals since the oldest item has entered the system, so that their ages are $1 - W - T_N, \dots, 1 - W - T_1$. The current prices of the items on the shelf are then $R(1 - W), R(1 - W - T_1), \dots, R(1 - W - T_N)$, respectively. Hence,

$$V = \begin{cases} 0, & W > 1 \\ \sum_{j=0}^N R(1 - W - T_j), & W \leq 1, \end{cases} \tag{3.1}$$

where $T_0 = 0$. By conditioning on N and W , we obtain

$$E(e^{-\alpha V}) = E \left(E \left(\exp \left\{ -\alpha \sum_{j=0}^N R(1 - W - T_j) \right\} 1_{\{W \leq 1\}} \mid N, W \right) \right). \tag{3.2}$$

Since the item arrival process is Poisson, the conditional joint distribution of T_1, T_2, \dots, T_N , given N and W , is the same as that of the order statistics $0 < U_{(1)} \leq \dots \leq U_{(N)} < 1 - W$ derived from an independent sample U_1, \dots, U_N of size N of the uniform distribution on the interval $(0, 1 - W)$. Obviously,

$$\sum_{j=0}^N R(1 - W - U_{(j)}) = \sum_{j=0}^N R(1 - W - U_j).$$

Let U_u be a random variable having the uniform distribution on $(0, u)$ and let

$$G(\alpha, u) = E(e^{-\alpha R(u - U_u)}) = \frac{1}{u} \int_0^u e^{-\alpha R(y)} dy.$$

Then, (3.2) yields

$$E(e^{-\alpha V}) = E(e^{-\alpha R(1 - W)}) G(\alpha, 1 - W)^N 1_{\{W \leq 1\}}.$$

Given $W = w$, N has a Poisson distribution with parameter $\lambda(1 - w)$. Thus,

$$\begin{aligned}
 E(e^{-\alpha V}) &= P(W > 1) + \int_0^1 \exp\{-\alpha R(1 - w)\} \\
 &\quad \times \sum_{n=0}^{\infty} G(\alpha, 1 - w)^n e^{-\lambda(1-w)} \frac{(\lambda(1 - w))^n}{n!} f(w) dw \\
 &= k_1 e^\mu e^{-\lambda} + k_1 \int_0^1 e^{-\alpha R(1-w) - \lambda(1-w)[1 - G(\alpha, 1-w)] - (\lambda - \mu)w} dw, \tag{3.3}
 \end{aligned}$$

where the second step follows from (2.2) and k_1 is given in (2.3). Equation (3.3) provides an explicit expression for the LT of V . In particular, the expected total price of the items on the shelf is

$$E(V) = k_1 \int_0^1 e^{-(\lambda - \mu)w} \left(R(1 - w) + \lambda \int_0^{1-w} R(y) dy \right) dw. \tag{3.4}$$

Remark: One possible assumption on the price function R is that it is related to some deterioration rate function $r(x) \geq 0$ of the shelf age x via the relation

$$R(x) = \begin{cases} p \left(1 - \int_0^x r(y) dy \right), & 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the sale price of a fresh item is $R(0) = R(0+) = p (> c)$. It is natural to assume that $\int_0^1 r(x) dx = 1$, which means that an outdated item is of no value. If this integral is smaller than one, the scrap value of outdated items has to be taken into account, as it will decrease the penalty due to outdatedings.

Now let us turn to the functionals characterizing the long-run profits and cost of the inventory system. Let $\mathbf{N}_1 = \{N_1(t) : t \geq 0\}$, $\mathbf{N}_2 = \{N_2(t) : t \geq 0\}$, $\mathbf{M}_1 = \{M_2(t) : t \geq 0\}$, and $\mathbf{M}_2 = \{M_2(t) : t \geq 0\}$ be the counting processes of item arrivals, demand arrivals, outdatedings, and unsatisfied demands, respectively, and let $\beta > 0$ be the discount factor. Then, \mathbf{N}_1 and \mathbf{N}_2 are Poisson processes (of rates λ and μ , respectively) and \mathbf{M}_1 and \mathbf{M}_2 are stationary renewal processes with interarrival rates $\lambda^* = k_1$ and $\mu^* = \lambda - \mu + k_1$, respectively. Let $S_1 < S_2 < \dots$ be the item arrival times; they are consecutive sums of i.i.d. $\exp(\lambda)$ -distributed random variables. The total expected discounted purchase cost of items is

$$cE \left(\int_0^\infty e^{-\beta t} dN_1(t) \right) = c \sum_{n=1}^\infty E(e^{-\beta S_n}) = \frac{c\lambda}{\beta}. \tag{3.5}$$

By PASTA, the total expected discounted income from selling the items to incoming demands is

$$E(R(1 - W)1_{\{W \leq 1\}})E\left(\int_0^\infty e^{-\beta t} dN_2(t)\right) = \frac{E(R(1 - W)1_{\{W \leq 1\}})\mu}{\beta}. \tag{3.6}$$

Now, let a be the penalty on the outdated of an item and let $\tau_1 < \tau_2 < \dots$ be the interoutdating times. Then, τ_2, τ_3, \dots have the same distribution function, say F , while τ_1 has the density $\lambda^*(1 - F)$. Denote by $\psi(\alpha)$ the LT of τ_2 . Clearly, τ_1 has LT $\lambda^*(1 - \psi(\alpha))/\alpha$. The total expected penalties on outdatings is, therefore,

$$\begin{aligned} aE\left(\int_0^\infty e^{-\beta t} dM_1(t)\right) &= a \sum_{n=1}^\infty E(e^{-\beta \sum_{i=1}^n \tau_i}) \\ &= a \sum_{n=1}^\infty \frac{\lambda^*(1 - \psi(\beta))^n}{\beta} \psi(\beta)^{n-1} \\ &= \frac{ak_1}{\beta}. \end{aligned} \tag{3.7}$$

Finally, let b be the penalty on an unsatisfied demand. Proceeding as above, we find that the total expected discounted penalties on unsatisfied demands are given by

$$bE\left(\int_0^\infty e^{-\beta t} dM_2(t)\right) = \frac{b(\lambda - \mu + k_1)}{\beta}. \tag{3.8}$$

All functionals of interest have now been computed.

Examples: Let $\lambda \neq \mu$. In the following, we need the formulas

$$P(W > 1) = k_1 e^{\mu - \lambda}, \tag{3.9}$$

$$E(W1_{\{W \leq 1\}}) = \frac{k_1}{(\lambda - \mu)^2} [1 - e^{-(\lambda - \mu)} - (\lambda - \mu)e^{-(\lambda - \mu)}], \tag{3.10}$$

$$E(W^2 1_{\{W \leq 1\}}) = \frac{k_1}{(\lambda - \mu)^3} [2 - ((\lambda - \mu)^2 + 1)e^{-(\lambda - \mu)}], \tag{3.11}$$

which follow from (2.2).

- (a) Consider the price function associated with $r(x) \equiv 1$. Then, $R(y) = p(1 - y)$ and we get

$$\begin{aligned} E(R(1 - W)1_{\{W \leq 1\}}) &= pE(W1_{\{W \leq 1\}}) \\ &= \frac{pk_1}{(\lambda - \mu)^2} [1 - e^{-(\lambda - \mu)} - (\lambda - \mu)e^{-(\lambda - \mu)}]. \end{aligned}$$

A quick calculation shows that $G(\alpha, u) = (\alpha p u)^{-1}(e^{-\alpha p(1-u)} - e^{-\alpha p})$, and (3.3) yields

$$E(e^{-\alpha V}) = k_1 e^{\mu-\lambda} + k_1 \int_0^1 \exp\{-(\alpha p + \lambda - \mu)w - \lambda[1 - w - (\alpha p)^{-1}(e^{-\alpha p w} - e^{-\alpha p})]\} dw.$$

(b) Let $r(y) = 2y$. Then, $R(y) = p(1 - y^2)$, so that

$$E(R(1 - W)1_{\{W \leq 1\}}) = p(1 - E((1 - W)^2 1_{\{W \leq 1\}})) = p[P(W > 1) + 2E(W1_{\{W \leq 1\}}) - E(W^2 1_{\{W \leq 1\}})],$$

and the terms on the right-hand side are given by (3.9)–(3.11). Furthermore, we obtain $G(\alpha, u) = u^{-1} \int_0^u e^{\alpha p(y^2-1)} dy$. Thus, by (3.3),

$$E(e^{-\alpha V}) = k_1 e^{\mu-\lambda} + k_1 \int_0^1 \exp\left\{-(\alpha p(2w - w^2) - (\lambda - \mu)w) - \lambda\left[1 - w - \int_0^{1-w} e^{\alpha p(y^2-1)} dy\right]\right\} dw.$$

4. BATCH DEMANDS

We now extend the original model by assuming that demands arrive in batches of random sizes K_1, K_2, \dots . The K_i are i.i.d. random variables with common distribution $P(K_i = k) = p_k, k = 1, 2, \dots$, and generating function $K(z) = \sum_{k=1}^{\infty} p_k z^k$. We suppose that each demand can be either

- (i) fully satisfied, which happens if there are at least as many items on the shelf as demanded,
- (ii) partially satisfied, if more is demanded than available, in which case the shelf is emptied,
- (iii) or fully unsatisfied, if the shelf is found empty.

(We might have to consider an additional actuarial functional if the penalty on partially satisfied demands is different from that of fully unsatisfied ones.)

Formulas (3.4)–(3.8) for the actuarial functionals still hold, but the law of W is no longer the one given in (2.2) because the jump sizes are no more exponential with rate λ . A demand of k items, if fully satisfiable, takes away the k oldest items in the system. Since the interarrival times are i.i.d. and $\exp(\lambda)$ -distributed, the jump of W generated by this demand is composed of k exponential phases. However, if a demand is only partially satisfied, W upcrosses level 1, and by the lack-of-memory property of the jump size distribution, the overshoot above level 1 is $\exp(\lambda)$ -distributed (independent of everything that has happened in the past); note that this overshoot is equal to the time until the next item arrival. Also, just after a moment of outdating (when W hits zero), the second oldest item becomes the new oldest one,

and the upward jump is again $\exp(\lambda)$ -distributed. It is now clear that the level-crossing equation for the stationary density of \mathbf{W} is given by

$$f(x) = \begin{cases} \mu \int_0^x [1 - H(x - \omega)]f(\omega) d\omega + f(0)e^{-\lambda x}, & x \leq 1 \\ \tilde{k}e^{-\lambda x}, & x > 1, \end{cases} \tag{4.1}$$

where H is the distribution function having the density

$$h(y) = \sum_{k=1}^{\infty} p_k \frac{e^{-\lambda y}(\lambda y)^{k-1}\lambda}{(k-1)!}.$$

Hence,

$$1 - H(y) = 1 - \sum_{k=0}^{\infty} \frac{e^{\lambda y}(\lambda y)^k}{k!} \left(\sum_{n=k+1}^{\infty} p_k \right)$$

and

$$\begin{aligned} \tilde{k} &= \mu e^{\lambda} \int_0^1 [1 - H(1 - \omega)]f(\omega) d\omega + f(0) \\ &= \mu e^{\lambda} \int_0^1 \left(1 - \sum_{k=0}^{\infty} \frac{e^{\lambda(1-\omega)}(\lambda(1-\omega))^k}{k!} \left(\sum_{n=k+1}^{\infty} p_k \right) \right) f(\omega) d\omega + f(0). \end{aligned}$$

On $[0,1]$, we can write (4.1) as

$$f(x) = f(0)e^{-\lambda x} + \left(\frac{\mu}{\lambda E(K)} h_e * f \right)(x), \tag{4.2}$$

where h_e denotes the equilibrium density associated with h and $*$ denotes convolution. Note that $\lambda E(K)$ is the mean associated with h . Define $\rho = \mu/(\lambda E(K))$ and let $h_e^{*n} = h_e * \dots * h_e$ (n -fold convolution). Solving for f in (4.2) for $x \in [0,1]$, we obtain

$$f(x) = f(0)e^{-\lambda x} \sum_{n=0}^{\infty} \rho^n h_e^{*n}(x). \tag{4.3}$$

By the continuity of $f(x)$ at $x = 1$, we get

$$\tilde{k} = f(0) \sum_{n=0}^{\infty} \rho^n h_e^{*n}(1),$$

and by the normalizing condition $\int_0^{\infty} f(x) dx = 1$, we find the constant $f(0)$ to be

$$f(0) = \left(\int_0^{\infty} e^{-\lambda x} \sum_{n=0}^{\infty} \rho^n h_e^{*n}(x) dx + \lambda^{-1} \sum_{n=0}^{\infty} \rho^n h_e^{*n}(1)e^{-\lambda} \right)^{-1}.$$

Substituting $f(0)$ in (4.3), we have $f(x)$.

Example: In the case that K is geometrically distributed (i.e., $p_k = (1 - \theta)^{k-1}\theta$ for some $\theta \in (0,1)$), no convolution series are necessary. It is straightforward to show that for all $x \geq 0$,

$$h(x) = \lambda\theta e^{-\lambda\theta x}.$$

Then, (4.1) transforms into

$$f(x) = \begin{cases} \mu \int_0^x e^{-\lambda\theta(x-\omega)} f(\omega) d\omega + f(0)e^{-\lambda x}, & x \leq 1 \\ \tilde{k}e^{-\lambda x}, & x > 1 \end{cases}$$

for some constant \tilde{k} . Assume that $\lambda\theta \neq \mu$. Solving for $f(x)$, we get

$$f(x) = \begin{cases} \frac{\mu}{\lambda + \mu} f(0)e^{-(\lambda\theta-\mu)x} + \frac{\lambda}{\lambda + \mu} f(0)e^{-\lambda(\theta+1)x}, & x \leq 1 \\ \tilde{k}e^{-\lambda x}, & x > 1. \end{cases}$$

The two constants \tilde{k} and $f(0)$ can be computed using the continuity of $f(x)$ at $x = 1$ (i.e., $f(1) = f(1+)$) and the normalization equation $\int_0^\infty f(x) dx = 1$. We find that

$$f(0) = \left(\frac{\mu(1 - e^{-(\lambda\theta-\mu)})}{(\lambda + \mu)(\lambda\theta - \mu)} + \frac{\lambda(1 - e^{-\lambda(\theta+1)})}{(\lambda + \mu)\lambda(\theta + 1)} + \frac{e^\lambda}{\lambda} \right)^{-1},$$

$$\tilde{k} = e^\lambda f(0).$$

If $\lambda\theta = \mu$, we obtain the solution by taking limits.

5. RENEWAL DEMAND ARRIVAL TIMES

We now consider the case that the arrival times of the demands form a general renewal process with distribution function A for the interarrival times. Let $1/\mu = \int_0^\infty x dA(x)$. In this $G/M/1$ -type model, the VOT process \mathbf{W} is, in general, not Markov but, of course, still regenerative. Equation (2.1) holds so that

$$E(e^{-\alpha V}) = \Pr(W > 1) + E(e^{-\alpha(R(1-W))-\lambda(1-W)(1-G(\alpha,1-W))} 1_{\{W \leq 1\}})$$

$$= \int_1^\infty f(w) dw + \int_0^1 e^{-\alpha(R(1-w))-\lambda(1-w)(1-G(\alpha,1-w))} f(w) dw, \tag{5.1}$$

where the *time-average* density $f(x) = f_{G/M/1}(x)$ satisfies the Pollaczek–Khintchine integral equation

$$f_{G/M/1}(x) = \begin{cases} \mu \int_0^x e^{-\lambda(x-\omega)} \hat{f}(\omega) d\omega + \hat{f}(0)e^{-\lambda x}, & x \leq 1 \\ \mu \int_0^1 e^{-\lambda(x-\omega)} \hat{f}(\omega) d\omega + \hat{f}(0)e^{-\lambda x}, & x > 1. \end{cases} \tag{5.2}$$

Here, $\hat{f}(\cdot)$ is the *Palm* density, which is different from $f_{G/M/1}(x)$ because PASTA is not applicable for the $GI/M/1$ model. Therefore, $f_{G/M/1}(x)$ has to be computed by other means. Our approach is based on a *time-reversal* technique, which has been described rigorously in previous papers (e.g., [1,13]). The $GI/M/1$ model is transformed into an $M/G/1$ dam whose stationary density can be derived by level-crossing arguments. We proceed in steps.

1. The VOT process \mathbf{W} can be interpreted as the *virtual waiting time* (VWT) process of a special variant of the $GI/M/1$ queue in which the idle periods are deleted and arriving customers do not enter the system if they have to wait in line for more than one time unit. Recall Figure 1, which shows a typical sample path, but note that the times between jumps now have the general distribution function A .
2. Reverse time by the following:
 - (i) Replacing any positive jump by a linear increasing piece of trajectory with slope 1 on an interval whose length is equal to the jump size
 - (ii) Replacing the pieces of \mathbf{W} between jumps by a negative jump whose size is equal to the interjump time

The reversed process $\mathbf{E} = \{E(t): t \geq 0\}$ is called the *elapsed waiting time* process associated with the same $GI/M/1$ queue, because $E(t)$ can be interpreted as the time elapsed since the arrival of the customer being served at time t . Note that \mathbf{E} is a “conditional” Markov process in the sense that conditional on $\mathbf{E} = x \leq 1$, \mathbf{E} possesses the Markov property but not conditional on $\mathbf{E} = x > 1$. The times between the (negative) jumps are independent and $\exp(\lambda)$ -distributed, whereas the jump sizes are i.i.d., with common distribution function A and mean $1/\mu$. The sample path of \mathbf{E} corresponding to that of \mathbf{W} in Figure 1 is displayed in Figure 2.

\mathbf{E} is a regenerative process for which each cycle is terminated by a hitting of level 0.

3. Now, define the process $\mathbf{D} = \{D(t): t \geq 0\}$ by $D(t) = \max[1 - E(t), 0]$. The process \mathbf{D} is the finite dam process with constant release rule of the $M/G/1$

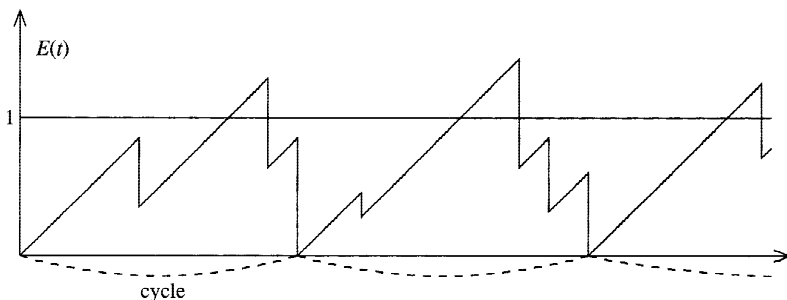


FIGURE 2. The sample path of \mathbf{E} derived from that of \mathbf{W} in Figure 1.

type; the sample path of **D** derived from that of **E** in Figure 2 is given in Figure 3.

Clearly, the stationary distribution of **D** has an atom at 0. Let π be the probability of this atom and let $f_{M/G/1}(x)$ be the stationary density of **D** on $(0, \infty)$. By level-crossing theory, $f_{M/G/1}(x)$ exists, and by time reversal [13], we have, for all $x > 0$,

$$f_{M/G/1}(x) = f_{G/M/1}(1 - x). \tag{5.3}$$

The Pollaczeczek–Khintchine integral equation for $f_{M/G/1}(x)$ reads as

$$f_{M/G/1}(x) = \lambda \int_0^x [1 - A(x - w)] f_{M/G/1}(w) dw + \lambda \pi [1 - A(x)], \quad 0 < x \leq 1. \tag{5.4}$$

Introducing $a_e(x) = \mu[1 - A(x)]$, the equilibrium density associated with A , we can rewrite (5.4) as

$$f_{M/G/1}(x) = (\rho a_e * f_{M/G/1})(x) + \rho \pi a_e(x), \quad 0 < x \leq 1. \tag{5.5}$$

Solving for $f_{M/G/1}$ in (5.5), we find that

$$f_{M/G/1}(x) = \pi \sum_{n=1}^{\infty} \rho^n a_e^{*(n)}(x). \tag{5.6}$$

From the normalizing condition

$$\int_0^1 f_{M/G/1}(x) dx = 1 - \pi,$$

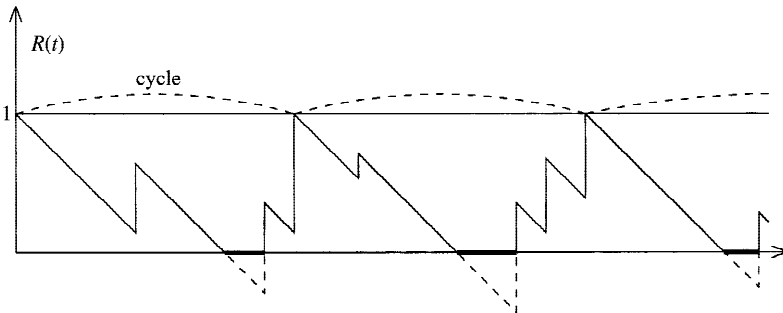


FIGURE 3. The sample path of $D(t) = \max[1 - E(t), 0]$.

it follows that

$$\pi = \left(1 + \int_0^1 \sum_{n=1}^{\infty} \rho^n a_e^{*(n)}(x) dx \right)^{-1}. \tag{5.7}$$

By (5.3), (5.6), and (5.7), we obtain $f_{M/G/1}$ and $f_{G/M/1}$. The Laplace transform of V is given by

$$\begin{aligned} E(e^{-\alpha V}) &= \Pr(W > 1) + E(e^{-\alpha(R(1-W))-\lambda(1-W)(1-G(\alpha,1-W))} 1_{\{W \leq 1\}}) \\ &= \pi + \pi \int_0^1 \sum_{n=1}^{\infty} e^{-\alpha(R(1-w))-\lambda(1-w)(1-G(\alpha,1-w))} \rho^n a_e^{*(n)}(1-w) dw \\ &= \pi + \pi \int_0^1 \sum_{n=1}^{\infty} e^{-\alpha p(R(w))-\lambda w(1-G(\alpha,w))} \rho^n a_e^{*(n)}(w) dw. \end{aligned}$$

Another possible extension is to the case of general i.i.d. interarrival times with distribution function B , say, for the items (keeping the Poisson arrival assumptions for the demands). In this $M/G/1$ -type model, \mathbf{W} is a Markov process whose stationary density f satisfies (2.1) and is given by

$$f(x) = \begin{cases} \mu \int_0^x [1 - B(x-w)]f(w) dw + f(0)[1 - B(x)], & x \leq 1 \\ \mu \int_0^1 [1 - B(x-w)]f(w) dw + f(0)[1 - B(x)], & x > 1, \end{cases} \tag{5.8}$$

which is the direct generalization of (2.1). However, (3.3) cannot be easily extended because the conditional distribution of the items on the shelf, given their number, is more complicated than in the Poisson case.

References

1. Boxma, O.J., Perry, D., & Stadje, W. (2001). Clearing models for $M/G/1$ queues. *Queueing Systems* 38: 287–306.
2. Kalpakam, S. & Shanti, S. (2000). A perishable system with modified $(S - 1, S)$ policy and arbitrary processing times. *Computers and Operations Research* 28: 453–471.
3. Kaspi, H. & Perry, D. (1983). Inventory system of perishable commodities. *Advances in Applied Probability* 15: 674–685.
4. Lian, Z. & Liu, L. (1999). A discrete-time model for perishable inventory systems. *Annals of Operations Research* 87: 103–116.
5. Nahmias, S. (1982). Perishable inventory systems: A review. *Operations Research* 30: 680–708.
6. Nahmias, S. & Schmidt, C.P. (1985). $(S - 1, S)$ policies for perishable inventory. *Management Science* 31: 719–728.
7. Nahmias, S. & Schmidt, C.P. (1986). An application of the theory of weak convergence to the dynamic perishable inventory problem with discrete demand. *Mathematics of Operations Research* 11: 62–69.
8. Perry, D. (1999). An optional sampling control of shelf life for a perishable inventory system. *Operations Research* 47: 966–973.

9. Perry, D. & Stadjé, W. (1999). Perishable inventory systems with impatient demands. *Mathematical Methods of Operations Research* 50: 77–90.
10. Perry, D. & Stadjé, W. (2000). An inventory system for perishable items with by-products. *Mathematical Methods of Operations Research* 51: 287–300.
11. Perry, D. & Stadjé, W. (2000). Inventory systems for goods with censored random lifetimes. *Operations Research Letters* 27: 21–27.
12. Perry, D. & Stadjé, W. (2001). Disasters in Markovian inventory systems for perishable items. *Advances in Applied Probability* 33: 61–75.
13. Perry, D., Stadjé, W., & Zacks, S. (2001). Busy period analysis for $M/G/1$ and $G/M/1$ -type queues with restricted accessibility. *Operations Research Letters* 27: 163–174.
14. Tekin, E., Gürler, Ü., & Berk, E. (2001). Age-based vs. stock level control policies for a perishable inventory system. *European Journal of Operational Research* 134: 309–329.