Macroeconomic Dynamics, **1**, 1997, 518–548. Printed in the United States of America.

STRUCTURAL NONLINEAR CONTINUOUS-TIME MODELS IN ECONOMETRICS

CLIFFORD R. WYMER

Economic theory indicates the need for nonlinear structural models to study medium-term and long-run dynamic behavior of an economy. This paper argues that economic systems can be better specified and estimated using differential-equation rather than difference-equation systems and briefly reviews the estimators of continuous models. This approach of specifying structural models on the basis of economic theory and institutional structure explicitly and then testing the underlying hypothesis to verify the structural form is contrasted with a general-to-specific approach of successively more restricted VARMAX processes. Previous analyses of stability about the steady state or fixed point in phase space are extended to more general attractors to allow an investigation of complexity in economic systems. The critical dependence of some attractors, and particularly strange attractors, on parameter values emphasizes the need for consistent, efficient estimation. A structural approach provides a rigorous alternative to using single time series to determine whether economic systems exhibit aperiodic or chaotic dynamical behavior.

Keywords: Nonlinear Structural Models, Continuous-Time Models, VARMAX Process, Differential-Equation Systems, Strange Attractors

1. INTRODUCTION

This paper discusses the need for nonlinear structural models to study the medium-term and long-run behavior of economic systems and reviews recent developments in the analysis of nonlinear models. This work on the continuous-time approach to econometrics is directed toward the development and testing of economic theory, particularly in the macroeconomic field and in commodity and financial markets, and the study of the implications of that theory. It provides an integrated approach to the specification, estimation, and analysis of economic models incorporating several features: the use of relatively small, highly overidentified models based on economic theory and specified as differential-equation systems, the use of full-information maximum likelihood or Gaussian estimators to estimate discrete models that are stochastically equivalent to the differential-equation system, and derivation of the properties of these models. This approach is discussed by Wymer

I wish to thank William Barnett, Kieran Donaghy, and an unknown referee for their comments. This paper was presented at the Economics and Finance Conference, Washington University, St. Louis, Missouri, 1995. Address correspondence to: Clifford R. Wymer, 6 Muritai Road, Takapuna, Auckland 9, New Zealand; e-mail: wymer@ihug.co.nz.

 1365-1005/97 $9.00 + .10

(1993, 1995, 1996), who gives extensive reference; its advantages are reviewed by Gandolfo (1993).

The aim of this research is to allow the study of more realistic models, based on economic theory, to obtain a better understanding of economic behavior, and especially dynamic behavior. The models are often such that some long-run behavior, such as a steady state, can be studied analytically although, of course, this depends on the complexity of the model. The structural properties of models that have strange attractors, for example, depend crucially on parameter values. Models are generally too complex to allow short-to medium-term behavior to be studied analytically but it is possible to derive results that, although numerical, provide information of a nature similar to that which can be obtained from very small theoretical models.

In the physical sciences and in engineering, the properties of complex systems can be investigated using values of parameters found either by direct measurement or from experiments. This is seldom possible in economics, and so, simultaneous estimation is required. Although parameter values of economic models may be assumed, the relevance of these models and the theory on which they are based and their implications for economic behavior and policy cannot be substantiated without estimation.

A characteristic of the continuous models is that an observationally equivalent discrete model can be derived independently of the observation interval of the sample being used for estimation so that the differential-equation model can be specified independently of the observation interval. The discrete model then can be estimated using discrete data. Because the parameters of the discrete model are those of the underlying continuous model and are estimated subject to all of the restrictions inherent in the continuous model and in the derivation of the discrete models, these estimators provide asymptotically unbiased and efficient estimates of the continuous model.

Although the continuous-time estimators discussed here have been developed within the past 30 years, the advantages of using differential-equation models were foreseen; Koopmans (1950), for example, argued in favor of such models and Marschak (1950) stated that "if proper mathematical treatment of [continuous] stochastic models can be developed, such models promise to be a more accurate and more flexible tool for inference in economics than the discrete models used heretofore." The mathematical and statistical theory on which these estimators are based has a longer history and is discussed by Bergstrom (1988). Many of the key papers in which these estimators and their properties were developed are reprinted in Bergstrom (1976, 1990).[1]

The long-run properties of these models, their dynamic and structural stability and the question of whether they have fixed-point (steady state) or more complex attractors in the neighborhood of the parameter estimates can be examined using eigensystem analysis and by calculating the Lyapunov exponents of the system. If the system has a strange attractor, it will exhibit aperiodic or chaotic behavior. A distinction must be made between dissipative and conservative systems; although a

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

Hamiltonian system, for example, does not have an attractor (but see note 7), it can exhibit chaotic behavior in the neighborhood of the Hamiltonian orbit, but this is really local turbulence and hence of a different nature from that which arises from strange attractors. It is not unusual for some parts of economic models to be based on Hamiltonian optimization, but learning processes, adaptive adjustment to partial equilibrium when there is risk and transaction costs, and price, wage, interest, and exchange-rate behavior are not Hamiltonian and could give rise to a dissipative system. Usually a model will be known to be Hamiltonian by construction.

Although the size and complexity of the models means that much of the analysis must be numerical, many of the results are of a more qualitative nature. Thus a great deal of information can be compressed into relatively few numbers. For example, calculation of the eigensystem and the Lyapunov exponents of a model gives a great deal of information about the medium- and long-run properties of a model. So does looking at the largest partial derivatives of the eigenvalues with respect to the parameters; although there may be several hundred or many more partial derivatives, often only the largest few (which can be seen immediately) will be relevant.

During the past 20 years, there has been an increasing divergence between economic theory on one hand and estimation of economic models on the other. Although economic theorists continue to develop structural models, the models are seldom subjected to rigorous testing or verification; the hypotheses incorporated within these models often requires full-information procedures because the same parameters may occur in several equations. The estimators discussed here are designed to estimate a wide range of such models. On the other hand, much of the research in econometrics is directed toward time-series analysis using variable autoregressive, moving-average error with exogenous variables (VARMAX) processes, often in the context of a single equation. This research is essentially atheoretical and is founded in the belief that it is possible to go from the general to the specific, that is, to derive an economically plausible structural model from the unrestricted VARMAX estimates. Section 3.3 shows that, if the system generating the data is continuous, this methodology is invalid even for linear systems. The implications of a continuous system for structural VAR processes also are discussed in Section 3.3.

The need for nonlinear structural models is discussed in Section 2 and the estimation of these models in Section 3; Section 3 also comments on the implications of the continuous-time research for other econometric methodologies. Analysis of the dynamics and the long-run properties of these models is discussed in Section 4. A suite of computer programs has been developed to implement most of the estimators and analytical methods discussed in this paper.

2. STRUCTURAL MODELS

The rationale for using continuous-time models in econometrics is founded on the assumption that the underlying economic system is continuous and dynamic.

Although individual decisions may be made at regular or irregular intervals, the information on which those decisions are based is likely to be largely continuous and the decisions themselves are likely to concern a broadly continuous path of production, consumption, and transactions in commodities and financial assets. The lags in this process may be much shorter than the observation period of data used for estimation. Because macroeconomic behavior is the result of the action and interaction of individual economic agents, aggregation of the microvariables across sectors or markets will produce macrovariables that will tend to be continuous, so that the macroeconomic process can be treated as continuous. Although some of these variables are observed almost continuously, most data available for economic analysis are obtained from observations of the continuous trajectory of these variables at discrete intervals, ranging from daily or even less to quarterly or annual.

From an econometric point of view, the distinction between continuous and ordinary discrete models is important.[2] In a discrete model the stochastic errors in successive observations are usually assumed to be independent but that assumption is valid only if the lags in the system are integral multiples of the time unit of the discrete model. Because most economic systems do not have a natural time unit and because the minimum lag in any macrosystem will be much smaller than the observation period, this assumption cannot be justified in models involving aggregate variables.

An important feature of continuous-time models is that they can be estimated using a discrete model that is satisfied by the observations generated by the differential-equation system irrespective of the observation interval of the sample so that the properties of the parameters of the differential-equation system may be derived from the sampling properties of the discrete model. This allows a more satisfactory treatment of distributed lag processes and of the disturbances in the model. In particular, the minimum lag in the economic system can be much smaller than the observation interval and tend to zero, and the lag functions of an aggregate model may be specified in a way that allows the length of the lag to be estimated rather than assumed. Thus a continuous-time model, unlike ordinary discrete models, can be specified and analyzed independently of the observation interval of the sample to be used for estimation, and the forecasting interval is also independent of the observation interval.

Another feature of these estimators is that they allow stock–flow models to be handled correctly, because they expressly recognize that although variables such as stocks and prices can be measured instantaneously, other variables such as flows or averages cannot; those variables are observed only as an integral over the observation period. Because many behavioral functions in economics involve the interaction between stocks and flows, it is essential that these be treated correctly; this cannot be done in an ordinary discrete model.

The ability to estimate these models is important. Economic theory uses structural models to formalize some aspect of economic behavior under a set of assumptions chosen, at least in part, to allow an analytical solution to the model. The

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

properties of the model and its implications then can be investigated. Although these models may capture the essence of some part of economic behavior, the underlying assumptions, by necessity, are often very restrictive and the complexity of the model is strictly limited. This may prevent it from being a plausible representation of economic behavior. Once a theoretical model moves from a static to a dynamic formulation, perhaps because of intertemporal rather than atemporal optimization, the introduction of expectations or owing to information, decision, or production lags, purely qualitative analysis becomes strictly limited by the difficulty or impossibility of obtaining analytical solutions to the model. For instance, once a model involves more than two or three first-order differential equations, or a second-order one, it becomes difficult to obtain useful analytical results and such results rapidly become impossible. Thus much of economic theory, particularly dynamic theory, must necessarily use models that are very small. Simplifications may include assuming away certain feedbacks or assuming some aspects of the economy always to be in equilibrium. Although this may be useful for studying some properties of the model, those properties are conditional on the assumptions being made and often these are not realistic. Such models, however, can provide the foundation for a more complete system.

These models are often highly nonlinear.[3] Not only does economic theory lead to nonlinear functions, but the use of a nonlinear model may be necessary to represent certain features of an economy or market as it allows dynamic behavior, including the possibility of limit cycles, which is precluded by a linear system. Nonlinearities may be due to the basic theoretical structure, such as the form of a utility or production function that is assumed to be underlying the economic system or due to the institutional or market structure in which the utility or production functions are imbedded. Behavioral functions derived, implicitly or explicitly, by the optimization of some objective function subject to constraint will have a functional form determined by the constrained objective function and will often by subject to homogeneity, symmetry, and curvature conditions as in Donaghy and Richard (1993, 1995).[4] The institutional or market structure also may be nonlinear, and so, the dynamic model must be able to incorporate such functions and rigidities. These nonlinearities may have significant implications both in the long run and in the adjustment from one state to another.

Theoretical models must be validated by appropriate testing if they are to provide a basis for further investigation. Moreover, the properties of even some of the simplest models vary qualitatively depending on the values of the parameters. The study of structural stability of complex dynamic systems shows that their asymptotic behavior is not generic and may depend crucially on the values of parameters. Thus it is necessary to obtain estimates of these parameters to allow the theory to be more fully developed and to provide a guide to the implications of the theory for economic agents.

These models are likely to be heavily overidentified, with parameters often occurring nonlinearly within and across equations. The aim in developing more sophisticated estimators is to allow these functional forms and restrictions on the parameters to be maintained during estimation so that the estimates of the model are consistent with its theoretical formulation.

For the purposes of this paper and for simplicity, but without loss of generality, it is assumed that the economic system can be represented by a recursive system of mixed second-order mixed stock–flow nonlinear differential equations

$$\begin{aligned} D^2\boldsymbol{y}_1(t) &= \psi_1\{D\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \boldsymbol{y}_1(t), D\boldsymbol{z}(t), \boldsymbol{z}(t), \boldsymbol{\theta}\} + \boldsymbol{u}_1(t), \\ D\boldsymbol{y}_2(t) &= \psi_2\{D\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \boldsymbol{y}_1(t), D\boldsymbol{z}(t), \boldsymbol{z}(t), \boldsymbol{\theta}\} + \boldsymbol{u}_2(t), \end{aligned} \tag{1}$$

where the $\boldsymbol{y}_i(t)$ are vectors of m_i continuous endogenous variables with $\boldsymbol{y}(t) = [\boldsymbol{y}_1(t) \quad \boldsymbol{y}_2(t)]'$, $\boldsymbol{z}(t)$ is a vector of n continuous exogenous variables, and $\boldsymbol{\theta}$ a vector of p parameters. D is the differential operator d/dt and the ψ_i are continuous and differentiable functions. It is assumed that the $\boldsymbol{u}_i(t)$ are vectors of white-noise disturbances so that the integral $\zeta(t) = \int_0^t \boldsymbol{u}(s)\,ds$ is a homogeneous random process with uncorrelated increments. A rigorous definition of the disturbances is given by Bergstrom (1983). The variables $\boldsymbol{y}(t)$ and $\boldsymbol{z}(t)$ may be stocks or flows; this distinction becomes important for estimation. In this model, the $\boldsymbol{y}_1(t)$ are called second-order variables in that this is the highest order in which $\boldsymbol{y}_1(t)$ appears in the model; similarly, the $\boldsymbol{y}_2(t)$ are called first-order variables. Some equations may be identities, in which case the corresponding $\boldsymbol{u}(t)$ are zero. Also, the model may include zero-order equations, that is, equations defining zero-order variables, which need not be recursive although in many models they are also likely to have a causal interpretation. For the purposes of this paper and for simplicity, it is assumed below that any zero-order equations have been eliminated from the model.[5]

Defining additional variables such as $D\boldsymbol{y}_i(t) = \boldsymbol{y}_j(t)$, (1) can be written as a general first-order model of the form

$$D\boldsymbol{y}^*(t) = \psi^*\{\boldsymbol{y}^*(t), \boldsymbol{z}^*(t), \boldsymbol{\theta}\} + \boldsymbol{u}^*(t), \tag{2}$$

where $\boldsymbol{y}^*(t) = [D\boldsymbol{y}_1(t) \;\; \boldsymbol{y}_2(t) \;\; \boldsymbol{y}_1(t)]'$ is the vector of $m^* = 2m_1 + m_2$ endogenous variables, $\boldsymbol{z}^*(t) = [D\boldsymbol{z}(t) \;\; \boldsymbol{z}(t)]'$ and $\boldsymbol{u}^*(t) = [\boldsymbol{u}_1(t) \;\; \boldsymbol{u}_2(t) \;\; 0]'$. System (1) can be extended immediately to mixed rth-order mixed stock–flow systems and these can be reduced in the same way by the definition of additional variables to the first-order system (2).

This framework can be extended to allow boundary-point constraints to be imposed so that, for example, some dynamic equations can define forward-looking variables. A simple example is a model that includes the present value of a future income stream. Let the (discounted) income stream be given by $f[\boldsymbol{y}(t), t]$ so that its present value is given by

$$\boldsymbol{y}_p(t) = \int_t^{T_h} f\{\boldsymbol{y}(s), s\}\,ds,$$

where T_H is the required horizon and $\boldsymbol{y}_p(T_H) = 0$. Although $\boldsymbol{y}_p(t)$ is not observed, it is defined by the dynamic equation $D\boldsymbol{y}_p(t) = -f[\boldsymbol{y}(t), t]$ with the end- or boundary-point condition $\boldsymbol{y}_p(T_H) = 0$ and hence may be included as one of the endogenous variables $\boldsymbol{y}(t)$ in the model. Hamiltonian systems, rational expectations processes, some forms of game theory models, and models including other forms of boundary constraint such as the European Union Maastricht conditions can be represented in a similar way.

3. ESTIMATION

To provide an outline of the properties of the estimators of these models, assume that the general nonlinear model is a recursive system of first-order stochastic differential equations of the form

$$d\boldsymbol{y}(t) = \psi\{\boldsymbol{y}(t), \boldsymbol{z}(t), \boldsymbol{\theta}\}\, dt + \zeta(dt), \tag{3}$$

where $\boldsymbol{y}(t), \boldsymbol{z}(t), \boldsymbol{\theta}$, and $\psi\{\boldsymbol{y}(t), \boldsymbol{z}(t), \boldsymbol{\theta}\}$ are defined as in (1); this is the same as (1) when $m_1 = 0$ but can be extended to the general system (2). The term $\zeta(dt)$ is a vector of white-noise innovations such that $E[\zeta(dt)] = 0$, $E[\zeta(\mathrm{d}t)\zeta'(dt)] = |dt|\Omega$, where Ω is a positive definite matrix of order m, and $E[\zeta_i(\Delta_1)\zeta_j'(\Delta_2)] = 0$ for any disjoint sets Δ_1 and Δ_2. Some equations may be identities with the corresponding elements of $\zeta(\mathrm{d}t)$ and Ω zero; in this case the submatrix of Ω corresponding to stochastic equations will be positive definite. For instance, if the first-order model (3) has been derived from a higher-order system such as (1), then

$$E[\zeta^*(dt)\zeta^{*\prime}(dt)] = |dt|\Omega^*, \quad \text{where } \Omega^* = \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix}$$

is a matrix of order m^* and Ω is of order $m_1 + m_2$.

Although the parameters of the nonlinear model may be estimated directly using either an approximate discrete or exact estimator, as discussed below, the costs are high and the properties of the exact estimator are not well known and have to be inferred heuristically. On the other hand, the estimators of linear models are well developed and their asymptotic properties well known, so that the nonlinear model can be approximated by a Taylor-series expansion about some appropriate point, such as the sample mean, or some path, such as the steady state, to give a linearized model that then can be estimated subject to all of the restrictions inherent in the nonlinear model and in the linearization. This provides estimates of the parameters $\boldsymbol{\theta}$ of the nonlinear model and those estimates can be used for hypothesis testing, analysis, and forecasting. The approximation error inherent in the linearization means, however, that even if the white-noise innovations or their integrals are Gaussian, this may not be true of the disturbances in the linearized model. These issues are discussed by Wymer (1993b).

The exact discrete estimator of linear (or linearized) models is discussed in Section 3.1 and the corresponding estimator for nonlinear models, which uses numerical integration, is discussed in Section 3.2. Approximate discrete estimators, which provide a simpler form of the discrete model and are hence easier and faster

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

to estimate, are not discussed in this paper; these estimators are discussed by Wymer (1993, 1996) who provides full references.

3.1. Linear Estimators

The linear, or linearized, model corresponding to (3) can be written

$$d\boldsymbol{y}(t) = A(\boldsymbol{\theta})\boldsymbol{y}(t)\,dt + B(\boldsymbol{\theta})\boldsymbol{z}(t)\,dt + \zeta(dt), \tag{4}$$

where the elements of the matrices A and B are functions of $\boldsymbol{\theta}$, and in the linearized case, the point or path around which the system has been linearized. The solution to (4) is then

$$\boldsymbol{y}(t) = e^{\delta A(\boldsymbol{\theta})}y(t-\delta) + \int_{t-\delta}^{t} e^{(t-s)A(\boldsymbol{\theta})}B(\boldsymbol{\theta})\boldsymbol{z}(s)\,ds + \int_{t-\delta}^{t} e^{(t-s)A(\boldsymbol{\theta})}\zeta(ds). \tag{5}$$

Assume that the continuous variables are observed every δ time units (so that δ is the length of the observation interval in terms of the basic time unit of the model) and let $x_\tau = x(\tau\delta)$ be the discrete observation of the continuous variable at time t. The exact discrete model is then

$$y_\tau = e^{\delta A(\boldsymbol{\theta})}y_{\tau-1} + \int_{(\tau-1)\delta}^{\tau\delta} e^{(\tau\delta-s)A(\boldsymbol{\theta})}B(\boldsymbol{\theta})\boldsymbol{z}(s)\,ds + \omega_\tau,$$

where

$$\omega_\tau = \int_{(\tau-1)\delta}^{\tau\delta} e^{(\tau\delta-s)A(\boldsymbol{\theta})}\zeta(ds). \tag{6}$$

If the $\boldsymbol{z}(t)$ are analytic functions of time, the integral of the term involving exogenous variables can be evaluated exactly, but otherwise, approximating the $\boldsymbol{z}(t)$ by a second-order polynomial in the neighborhood of each observation point allows the integral to be approximated by a linear function of $\boldsymbol{z}_t$, $\boldsymbol{z}_{t-1}$, and $\boldsymbol{z}_{t-2}$ with coefficients that are explicit functions of $\boldsymbol{\theta}$. The error in this approximation is of $\mathcal{O}(\delta^4)$ as δ tends to zero and under suitable regularity and smoothness conditions leads to an asymptotic bias in the parameter estimates obtained by a maximum likelihood procedure of $\mathcal{O}(\delta^3)$.

Given the properties of the disturbances $\zeta(dt)$, it can be shown that

$$E[\omega_t] = 0, \qquad E[\omega_t\ \ \omega_t'] = \int_0^\delta e^{sA(\boldsymbol{\theta})}\Omega e^{sA'(\boldsymbol{\theta})}\,ds,$$
$$E[\omega_t\quad \omega_s'] = 0 \quad \text{for all } t, s, t \neq s; \tag{7}$$

thus even if Ω is a diagonal matrix, this will not be true of the variance matrix of errors of the exact discrete model $E[\omega_t\ \ \omega_t']$ so that the ω_t are not independent but, given the properties of the innovation process $\zeta(dt)$, they will be serially uncorrelated. For this reason, and because of the cross-equation restrictions on the coefficients of the system, it is necessary to use a simultaneous equation estimator.

Observations generated by the continuous model (4) will satisfy the exact discrete model (6) irrespective of the observation interval δ, so the sampling properties of the parameters $\boldsymbol{\theta}$ can be derived from the sampling properties of (6). A full-information maximum-likelihood estimator of (6) allows all of the restrictions inherent in the underlying continuous model, and in any linearization of that model, to be imposed, and thus provides consistent and asymptotically efficient estimates. Moreover, as shown by Bergstrom (1983), Gaussian estimators can be obtained under the assumption that the integral of the white-noise innovation process is a Gaussian process without the need to assume that the innovations themselves have that property. The generality of these assumptions allows the innovations to be a mixture of Brownian motion and Poisson processes, thus allowing for a more plausible representation of economic behavior in that the innovations may come as discrete jumps at random intervals. Thus the use of a continuous system, and corresponding estimator, does not preclude discontinuities in the paths of the variables.

Assume that the model is to be estimated using a sample of $T+1$ equispaced observations of the variables $\boldsymbol{y}(t)$ and $\boldsymbol{z}(t)$ with an observation length of δ; for estimation purposes, and without loss of generality, the basic time unit is usually chosen such that $\delta = 1$. Although many parameters will be independent of the basic time unit, some parameters, such as rates of adjustment or their reciprocal, may vary in accordance with the basic time unit and must be defined in terms of this unit. In deriving some of the properties of these estimators the basic time unit will be fixed so that the limit as δ tends to zero can be considered. For simulation and prediction, δ may take on any value irrespective of the observation length used in estimation.

Exact estimators of mixed-order mixed stock–flow models also can be derived but are much more complex because flow variables, and derivatives of other variables, cannot be observed at a point but only as an integral over the observation period, that is

$$\boldsymbol{y}_\tau^0 = \int_{(\tau-1)\delta}^{\tau\delta} \boldsymbol{y}(s)\, ds.$$

As an illustration, consider a second-order differential-equation system in variables that can be measured instantaneously at a point in time, such as stocks and prices,

$$\begin{aligned} dD\boldsymbol{y}(t) &= A_1(\boldsymbol{\theta})D\boldsymbol{y}(t)\,dt + A_2(\boldsymbol{\theta})\boldsymbol{y}(t)\,dt + B_1(\boldsymbol{\theta})D\boldsymbol{z}(t)\,dt \\ &\quad + B_2(\boldsymbol{\theta})\boldsymbol{z}(t)\,dt + \zeta(dt), \end{aligned} \tag{8}$$

where $\boldsymbol{y}(t)$, $\boldsymbol{z}(t)$, $\boldsymbol{\theta}$, and $\zeta(dt)$ are as in (3). Let $E[\zeta(dt)\zeta'(dt)] = |dt|\Omega(\mu)$, with μ being the set of parameters that define Ω. This parameter set would have $m(m+1)/2$ elements if Ω were unrestricted but would have only m elements if Ω were diagonal. If some equations in (8) are identities, the corresponding rows and columns of Ω would be zero and the number of elements in μ reduced accordingly.

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

The solution to (8) can be written

$$\boldsymbol{y}(t) = F_1(\boldsymbol{\theta}, \delta)\boldsymbol{y}(t-\delta) + F_2(\boldsymbol{\theta}, \delta)\boldsymbol{y}(t-2\delta) + \Psi(t) + \omega(t), \tag{9}$$

where

$$\Psi(t) = \int_{t-\delta}^{t} P_1(t-s)[B_1 D\boldsymbol{z}(s) + B_2\boldsymbol{z}(s)]\,ds + \int_{t-2\delta}^{t-\delta} P_2(t-\delta-s)[B_1 D\boldsymbol{z}(s) + B_2\boldsymbol{z}(s)]\,ds,$$

and

$$\omega(t) = \int_{t-\delta}^{t} P_1(t-s)\zeta(ds) + \int_{t-2\delta}^{t-\delta} P_2(t-\delta-s)\zeta(ds);$$

the F_i and P_i are functions of the elements of the matrix $e^{A(\boldsymbol{\theta})}$ where

$$A(\boldsymbol{\theta}) = \begin{bmatrix} A_1(\boldsymbol{\theta}) & A_2(\boldsymbol{\theta}) \\ I & 0 \end{bmatrix},$$

$$F_1 = [e^{\delta A}]_{21}[e^{\delta A}]_{11}[e^{\delta A}]_{21}^{-1} + [e^{\delta A}]_{22},$$

$$F_2 = [e^{\delta A}]_{21}[e^{\delta A}]_{12} - [e^{\delta A}]_{21}[e^{\delta A}]_{11}[e^{\delta A}]_{21}^{-1}[e^{\delta A}]_{22},$$

$$P_1(s) = [e^{sA}]_{21}, \quad \text{and} \quad P_2(s) = [e^{\delta A}]_{21}[e^{sA}]_{11} - [e^{\delta A}]_{21}[e^{\delta A}]_{11}[e^{\delta A}]_{21}^{-1}[e^{sA}]_{21},$$

where the $[e^{sA}]_{ij}$ are the i, j partitions of order m of the matrix $e^{sA(\boldsymbol{\theta})}$;

$$E[\omega(t)\omega'(t)] = \delta\int_0^{\delta} P_1(s)\Omega P_1'(s)\,ds + \delta\int_0^{\delta} P_2(s)\Omega P_2'(s)\,ds,$$

$$E[\omega(t)\omega'(t-\delta)] = \delta\int_0^{\delta} P_2(s)\Omega P_1'(s)\,ds, \quad \text{and} \quad E[\omega(t)\omega'(t-k\delta)] = 0 \quad \text{for all } |k| > 1.$$

Again, by taking discrete observations of the continuous variable $x_\tau = x(\tau\delta)$ and choosing the basic time unit such that $\delta = 1$, the exact discrete model becomes

$$\boldsymbol{y}_t = F_1(\boldsymbol{\theta})\boldsymbol{y}_{t-1} + F_2(\boldsymbol{\theta})\boldsymbol{y}_{t-2} + \Psi_t + \omega_t \quad \text{for integer } t, \tag{10}$$

where Ψ_t and ω_t are defined as in (9). If the exogenous variables $\boldsymbol{z}(t)$ are not analytic functions of time, the integral Ψ_t may be approximated in a way similar to the corresponding term in (6) and this approximation will be exact if the $\boldsymbol{z}(t)$ are polynomials of order no more than two. The error of approximation is again of $\mathcal{O}(\delta^4)$.

The conditions on ω_t imply that there exists a set of random vectors ε_t such that ω_t can be written as the moving-average process

$$\omega_t = \varepsilon_t + G\varepsilon_{t-1}, \tag{11}$$

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

where G is a matrix of order m, the exact discrete model can be represented by the open autoregressive moving-average (VARMAX) process

$$\boldsymbol{y}_t = F_1(\boldsymbol{\theta})\boldsymbol{y}_{t-1} + F_2(\boldsymbol{\theta})\boldsymbol{y}_{t-2} + E_1(\boldsymbol{\theta})\boldsymbol{z}_t + E_2(\boldsymbol{\theta})\boldsymbol{z}_{t-1} + E_3(\boldsymbol{\theta})\boldsymbol{z}_{t-2} + \varepsilon_t + G(\boldsymbol{\theta}, \mu)\varepsilon_{t-1}, \tag{12}$$

where $E(\varepsilon_t) = 0$, $E(\varepsilon_t \ \varepsilon_t') = \Xi(\boldsymbol{\theta}, \mu)$ and $E(\varepsilon_s \ \varepsilon_t') = 0$, for all t and s, $s \neq t$.

More generally, a mixed rth-order differential-equation system, in which all variables can be observed instantaneously, that is, at a point in time, has an rth-order autoregressive, $(r-1)$th-order moving-average discrete representation that is stochastically equivalent to the continuous model. The coefficients of this model, corresponding to the F_i, G_i, in (12) will be even more complicated functions of the coefficient matrices of the differential model.

The restrictions inherent in the VARMAX process equivalent to a mixed stock–flow model are even more complex because flow variables are only observable as an integral over the observation interval; the exact discrete models equivalent to stock–flow systems are discussed by Bergstrom (1986) and reviewed by Wymer (1996). In this case, a mixed rth-order, mixed stock–flow differential-equation system in which a flow variable is defined by an rth-order equation has an rth-order autoregressive, rth-order moving-average discrete representation that is stochastically equivalent to the continuous model but with coefficients of the moving-average process that are not constant; if point observations are available for all variables defined by rth-order equations, the moving-average process is of order $(r-1)$. Thus, observations of continuous variables generated by a differential-equation system will be serially correlated and incorporate a moving-average error process even though the disturbance process of the continuous model has uncorrelated increments, except in the case of a first-order system in which all variables can be observed at a point in time.

The serial correlation in the observations of continuous variables arising from the moving-average process (11) or its equivalent in more general systems must be taken into account during estimation in order to prevent asymptotic bias in the estimators. Exact estimators of mixed-order, mixed stock–flow models have been derived and used as by Bergstrom et al. (1992) but these are extremely complex. To avoid the complications that arise from (11) or its generalizations, an approximation to this moving-average process that is independent of the parameters of the system may be derived for rth-order models as by Wymer (1972). This approximation, which has fixed coefficients, is of the same order as the moving average in the exact model. This approximate moving average can be inverted and truncated after a few terms and used to transform all of the series in the sample to eliminate the serial correlation inherent in the observations given by (10), at least to an approximation. This provides a system similar to the exact discrete model (12) but where the disturbances can be treated as serially uncorrelated.

3.2. Nonlinear Estimators

Although most of the theoretical development of continuous-time estimators has been within the context of linear models, in practice most of the models being estimated have been linearizations of some underlying nonlinear system. This approach is justified in that the estimated parameters are those of the nonlinear system and the linear model is estimated subject to all of the restrictions inherent in the theoretical structure and in the linearization. This does, however, introduce an approximation into the models being estimated and may cause serial correlation in the disturbances of the linearized model. These issues are discussed by Wymer (1993b, 1995).

Even when the disturbances of the nonlinear model are Gaussian, this need not be true of the disturbances of the linearized model. In addition, linearization may prevent some parameters from being identified, although they are identified in the nonlinear model, so that estimates of those parameter must be obtained in some other way and these estimates usually will be inconsistent. Even where parameters are formally identified in the linear model, however, the way in which they enter the linearized model may mean that they are poorly determined and the nonlinear model may provide more robust, and more precise, estimates of these parameters.

For these reasons, full-information maximum likelihood estimators analogous to the estimators of linear models have been developed to estimate the parameters of nonlinear systems directly as discussed by Wymer (1993b, 1995). Both a nonlinear exact discrete estimator and approximations to it are available. Although some asymptotic properties of these estimators are known, others have to be inferred heuristically from the properties of the linear estimators. Although the exact discrete estimator for linear models is derived using the analytical solution to the differential-equation system, for nonlinear models the system must be solved by numerical integration, but the principle is the same in both cases. In the pure case, for instance, where all variables are observed at a point in time, the estimator will be consistent and efficient and is analogous to the exact discrete estimator for linear models that provides estimates that are superefficient. If this estimator were applied to a linear model, the estimates would be the same as with the exact linear discrete estimator.

The derivation of the exact estimator of the nonlinear model (3) is analogous to that of the linear estimator. Let $\omega(t)$ be the vector of errors in the trajectories of the system such that $\omega(t) = \boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)$ where $\boldsymbol{y}(t)$ is the solution to (3) given initial conditions $\boldsymbol{y}(t-\delta)$ and $\hat{\boldsymbol{y}}(t)$ satisfies $D\hat{\boldsymbol{y}}(t) = \psi\{\hat{\boldsymbol{y}}(t), \boldsymbol{z}(t), \boldsymbol{\theta}\}$ given the same initial conditions. $E[\omega(t)\ \ \omega'(t)]$ is then the integral of an (unknown) function of the $\zeta(dt)$ over the interval $(0, \delta)$ and it follows from the properties of the error process $\zeta(dt)$ that the errors $\omega(t)$ are interdependent but serially uncorrelated. In the pure case, the endogenous variables $\boldsymbol{y}(t)$ are assumed to be observable at a point in time, such as stocks, prices, interest, and exchange rates, and the exogenous variables to be given analytic functions of time. For a given set of initial values of the parameters $\boldsymbol{\theta}$ and a set of initial values for the variables at the point $t-1$, the solution trajectories of (3) can be found by a numerical integration procedure. Let the vector of solution trajectories at t given the observed $\boldsymbol{y}(t-1)$ as initial values and the given set of $\boldsymbol{\theta}$ be $\hat{\boldsymbol{y}}(t;\boldsymbol{\theta})$. The residuals corresponding to $\omega(t)$ then can be calculated and used to form the likelihood function that then can be maximized to give full-information maximum likelihood estimates of the parameters.

This pure exact nonlinear discrete estimator requires the exogenous variables (or forcing functions) to be analytic functions of time so that the integration procedure is exact. Under those conditions the exact linear and nonlinear discrete estimators are the same for linear models but this will not be true for more general exogenous variables. In most econometric models [Bergstrom and Wymer (1976), for example, is an exception] the exogenous variables are defined only by a series of discrete observations. As in the linear estimators, the continuous path of exogenous variables may be approximated using a polynomial fitted to nearby observations. Although this introduces an asymptotic bias into the estimator, this bias is known to be small in the linear case and a similar result can be expected in nonlinear systems.

The underlying economic model may include flows and higher-order variables. Thus the functions ψ^* in the first-order system (2) or (3) will contain variables for which point observations are unavailable. The exact nonlinear estimator can be extended to such stock–flow and higher-order models but, as in the linear stock–flow model, the covariance function $E[\omega(s)\ \ \omega'(t)]$, $s \neq t$, will contain the product of terms in the disturbances $\zeta(ds)$ where the $\zeta(ds)$ are not disjoint, and thus the $\omega(t)$ will be serially correlated. Although serial correlation in the linear estimators may be eliminated, at least to an approximation, by transforming the data with the inverse of the moving-average process, such a transformation is not strictly applicable in a nonlinear system. Heuristically, however, it would seem that such a transformation would be useful, and this question is being examined. Without such a transformation, however, the estimator is no longer exact, but the analogy with the corresponding estimator of linear stock–flow models suggests that the asymptotic bias of the nonlinear stock–flow estimator will be small.

The exact nonlinear estimator has been used by to estimate macroeconomic models of the United Kingdom, the United States [see Donaghy (1993)], and Italy [see Gandolfo et al. (1996)]. Donaghy and Richard (1995) also have used the same estimator to estimate and test highly constrained, and highly nonlinear, currency substitution models involving homogeneity, symmetry, and curvature conditions.

A comparison of the linear and nonlinear estimators shows that, although many parameter estimates are not significantly different, some estimates are quite different; this occurs particularly with parameters that appear to be poorly identified in the linearized model. Although the costs can be high in terms of computing time, especially if the model is defined in the implicit form rather than the recursive form (3), the estimates suggest that there are substantial benefits in using the nonlinear estimator in that the asymptotic standard errors are very small relative to those of the estimators of the linearized model and to the approximate discrete estimator of the nonlinear model. The mean-square residuals are also smaller. It is considered that these results are due partly to the use of an exact rather than approximate estimator, which would be consistent with the estimators of linear models, but also because the specified relationships, and especially identities, are estimated directly without introducing errors of linearization and, perhaps more importantly, eliminating and serial correlation arising from linearization.

The use of full-information maximum-likelihood or Gaussian estimators allows the estimates to be used both for testing the overall structure of the continuous model and specific hypotheses. The stochastic equivalence between differential-equation systems and a VARMAX process makes a general VARMAX model a useful basis for such a test.

The exact discrete model (11) corresponding to the second-order differential-equation system (8), can be compared to the unrestricted second-order discrete VARMAX process

$$\boldsymbol{y}_t = \bar{F}_1(\boldsymbol{\theta})\boldsymbol{y}_{t-1} + \bar{F}_2(\boldsymbol{\theta})\boldsymbol{y}_{t-2} + \bar{E}_1(\boldsymbol{\theta})\boldsymbol{z}_t + \bar{E}_2(\boldsymbol{\theta})\boldsymbol{z}_{t-1} + \bar{E}_3(\boldsymbol{\theta})\boldsymbol{z}_{t-2} + \varepsilon_t + \bar{G}_1\varepsilon_{t-1}. \tag{13}$$

Although it would be useful to be able to test this VARMAX process against a higher-order one, the size of samples commonly available prevents this except in studies of some markets, especially financial markets, in which high-frequency data are available. Such a test would indicate whether the order of the continuous model was consistent with the data. Other tests that can be used to indicate whether the equations of the system are of the correct order are discussed by Wymer (1993a, 1996).

Under the assumption that the order of the continuous model, and the corresponding VARMAX process, is correct, the exact discrete model then can be tested against an unrestricted VARMAX process of the same order. This test, or sometimes variants of it with a partially restricted VARMAX process in which the $\bar{F}_i$ and $\bar{E}_i$ are unrestricted but where the $\bar{\boldsymbol{G}}_i$ are restricted in some general way, are used routinely in continuous-time studies.

The issues of cointegration and its relevance for continuous-time models are discussed by Wymer (1996). In such models, integration in the series is often automatically taken into account owing to the recursive nature of continuous systems in which the rate of change of a variable is a function of levels of itself and other variables and the tight specification of the structural model. The effects on estimation or testing of a model that contains integrated series is largely eliminated if the model is heavily overidentified as in the models considered in this paper. If testing shows that the residuals in the model are serially correlated, the approach suggested here is to look at this from the point of view of economic theory, to find whether there is a deficiency or misspecification in the structural model and to modify it accordingly, rather than to consider it as a statistical problem that can be eliminated by using a more sophisticated estimator.

3.3. Comparison with Ordinary Discrete Models

The fact that a linear continuous model has an exact VARMAX representation provides a link between the estimation of continuous-time models, ordinary discrete models, and time-series analysis. It is here that the distinction between the continuous-time approach to econometrics becomes apparent. For the purposes of this section, the exact discrete model (12) is compared with the unrestricted VARMAX process of the same order (13).

First, even if the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\mu}$ are distinct in the continuous model, the coefficients of the vector moving-average error process in the stochastically equivalent discrete model are highly nonlinear functions of all elements of $\boldsymbol{\theta}$ and $\boldsymbol{\mu}$. Moreover, even if Ω were a diagonal matrix, so that $\boldsymbol{\mu}$ contained only m elements, the error covariance matrix Ξ in (12) still would be a full matrix. Thus, simultaneous equation estimators are necessary to estimate the discrete model. Because such estimators are seldom used to estimate ordinary discrete models, the estimates of those models will be asymptotically biased if the data are generated by a continuous model.

Second, a comparison of the exact model (12) and the unrestricted VARMAX process (13) throws serious doubt on the validity, or even feasibility, of the general-to-specific methodology that has developed in econometrics over the past 15 years and which postulates that the starting point of econometric analysis should be the unrestricted process (13). Such a comparison makes it clear that there is no possibility of deriving the parameters, and the restrictions, inherent in the continuous model (8) from the coefficients of (13) even in simple systems, let alone in more complex models. The problem that arises with attempting to derive a sequence of successively more parsimonious models is that, if the underlying system is continuous, all coefficients of the VARMAX representation are functions of all parameters $\boldsymbol{\theta}$ of the underlying model owing to the way $e^{A(\boldsymbol{\theta})}$ enters the discrete form. Thus, eliminating some coefficients of the unrestricted model (13) immediately leads to bias in subsequent estimates of more parsimonious systems and this problem will be cumulative.

All elements of the matrices F_i, E_i, and G_i are functions of the elements of the matrix $e^{A(\boldsymbol{\theta})}$ in the exact model. Even if the continuous system (8) were linear in parameters and no parameter occurred in more than one equation, the discrete models, whether exact or approximate, would still involve highly nonlinear and cross-equation restrictions, thus requiring the use of full-information techniques. In fact, the coefficients of the differential-equation systems being estimated are generally heavily restricted, and highly nonlinear, functions of the set of parameters $\boldsymbol{\theta}$, with restrictions applying both within and across equations.

In contrast, the elements of $\overline{F}_i$ and $\overline{E}_i$ in an ordinary discrete model often are unrestricted, or the restrictions are of a simple and usually within-equation nature. An indication of the scale of this difference can be seen by comparing the estimators of the Bergstrom and Wymer (1976) model of the United Kingdom, which is a relatively small macroeconomic system. The model is a nonlinear mixed

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

second-order system with 2 second-order and 11 first-order equations, one of which is an identity, and in which the only exogenous variables are time and a constant. The parameter vector $\boldsymbol{\theta}$ has 31 elements and the parameter vector $\boldsymbol{\mu}$ defining the matrix Ω has 55 elements; $\Omega(\boldsymbol{\mu})$ is assumed to be a full matrix of order 10, allowing for the identity. As the identity is linearized, however, the error covariance matrix Ξ is assumed to be unrestricted and thus has 66 elements to be estimated. The unrestricted VARMAX model (13) has 264 elements in the matrices F_i and E_i and 121 elements in G whereas the error covariance matrix again has 66 elements to be estimated. Thus, even assuming that the economic behavior is linear, it is simply not conceivable that, beginning with the unrestricted model, 233 overidentifying restrictions in $\bar{F}_i$ and $\bar{E}_i$ and 121 in $\bar{G}_i$ could be found that would allow all coefficients to be defined in terms of 31 parameters. The scale of this problem increases exponentially with larger models.

If economic behavior is nonlinear, the situation becomes even worse. There is no way that estimation of an unrestricted linear model could lead to the overidentifying restrictions inherent in a nonlinear system and the linearization of that system. An alternative of using a power-series approximation to the economic system, say to the second order, would not help. Using the same example, there would be 120 terms in $\boldsymbol{y}_i$, $\boldsymbol{z}_j$, and $\boldsymbol{y}_i\boldsymbol{z}_j$ alone in each equation of the continuous model plus the error process; thus, the unrestricted form of this expansion would have over 3,000 coefficients to be estimated even leaving aside the error process. In the continuous-time approach, these coefficients are functions of only the 31 parameters that are to be estimated.

If the economic system is, in fact, continuous, a continuous model will provide not only a consistent, but far more efficient estimator of the parameters of the system relative to an unrestricted VARMAX process. This allows more powerful tests of the theory inherent in the model. The exact estimators provide asymptotically unbiased and efficient estimates of the parameters and predictors of postsample data so that a large gain can be expected in the efficiency of the exact continuous estimates compared with unrestricted VARMAX estimates, and the use of a parsimonious VARMAX model as an approximation for prediction purposes would lead to a biased predictor.

Continuous-time models have serious implications for research on discrete structural VAR processes, independently of the critiques by Cooley and LeRoy (1985) and Pagan (1987) of the structural VAR methodology.

A general autoregressive process is defined such that

$$x_t = \sum_{i=1}^{q} A_i x_{t-i} + \boldsymbol{u}_i, \qquad \textbf{(14)}$$

where x_t is a vector of variables and $\boldsymbol{u}_t$ is a vector of disturbances such that $E[\boldsymbol{u}_t] = 0$, $E[\boldsymbol{u}_t \ \boldsymbol{u}_t'] = V$, and $E[\boldsymbol{u}_t \ \boldsymbol{u}_s'] = 0$ for all t and $s \neq t$. Owing to the interpretation of this process as structural, the VAR is defined to be stationary, but as Pagan (1987) notes, this is unnecessary for the purpose of estimation. In

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

applications such as those of Sims (1980) or Blanchard and Quah (1989), either a time trend is included in each equation, or deviations about trends are used, or the x_t are defined as first differences. The value of q is chosen, perhaps by a sequence of likelihood ratio tests, and the model is estimated either unrestrictedly or to achieve some smoothness criterion. The model then is transformed to provide a process with orthogonal innovations e_t by premultiplying by a matrix K such that $KVK' = I$. Thus $e_t = Ku_t$ and (14) may be written

$$\sum_{i=0}^{q} B_i x_{t-i} = e_t, \tag{15}$$

where $B_0 = K$ and $B_i = KA_i$ for $i = 1 \cdots q$. The process (15) then is inverted to give the innovations representation

$$x_t = \sum_{i=0}^{\infty} C_i e_{t-i}. \tag{16}$$

In general, the matrix K is not unique and requires at least one assumption to make it just-identified. Various identifying assumptions have been made: for example, Sims (1980) assumed K was lower triangular with positive diagonal elements, giving results that depend crucially on the order of variables in x_t, whereas Blanchard and Quah (1989) assumed that monetary innovations have no permanent effect on real output, so that if the first elements of x_t and e_t refer to the output and real innovations and the second elements to the monetary variable and nominal innovations, K is chosen such that $\sum_{i=0}^{\infty}[C_i]_{12} = 0$. The coefficients of the C_i provide an impulse response function but, as pointed out by Cooley and LeRoy (1985) and emphasized by Pagan (1987), this requires prior assumptions on the causal structure of the economic model.

If the underlying economic system is continuous, the use of structural VAR processes for determining the properties of the system becomes invalid. The differential-equation models discussed here have an immediate causal interpretation but, as shown in (7) or (9), even if innovations in the continuous model are orthogonal so that the matrix Ω is diagonal, this will not be true of the errors of the equivalent discrete model even in a first-order system, whether or not restrictions are placed on the structural matrices $A_i(\boldsymbol{\theta})$. The reason for this can be seen from the definition of $\omega(t)$ in (6) or (9). The problem becomes even more complex in second- or higher-order systems, or in flow or mixed stock–flow models. If impulse functions are required, it is necessary to estimate an appropriately restricted structural model (4) and to use that to derive the impulse functions. The question of whether an assumption of orthogonal innovations can be justified economically, and the interpretation of such innovations, remains.

Another problem arises with discrete models if the data are generated by a continuous process. If the sample observations on the variables in an ordinary discrete model are found to be $I(1)$, for example, it is now common for the equations

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

to be estimated with the variables first-differenced and an error correction term added to satisfy long-run equilibrium. If the economic system generating the data is continuous, however, it can be seen from the exact discrete model (11) that differencing discrete observations of the path of the continuous variables will not remove the moving average inherent in those observations and any estimates will be biased.

Thus an atheoretical approach is not a satisfactory alternative to rigorous specification of a structural model on the basis of economic theory and the estimation of the parameters of that model using the exact discrete model, or some approximation to it, subject to all of the restrictions imposed by the theory. This allows hypotheses incorporated in the specification to be tested and modified or extended accordingly.

3.4. Models with Boundary Conditions

The exact nonlinear estimator is being extended to allow the estimation of nonlinear two-point boundary problems (or even multipoint), which would cover Hamiltonian systems, rational expectations, differential games, and other problems with forward-looking variables. The numerical solution of such models, some of which are highly nonlinear, with given parameters is now well developed and is being used routinely. Such a procedure could be embedded in the nonlinear estimator discussed above. The question of whether the parameters are identified would remain, especially in the case of data generated by an optimally controlled differential-equation system.

Assume that the basic time unit in the system is chosen such that $\delta = 1$, so that the sample consists of a set of observations $\boldsymbol{y}(t) = \boldsymbol{y}_t$, $t = 1 \cdots T$. An essential part of the nonlinear estimation procedure above is the integration of system (3) to find the solution $\hat{\boldsymbol{y}}(t)$ that satisfies $D\hat{\boldsymbol{y}}(s) = \psi\{\hat{\boldsymbol{y}}(s), \boldsymbol{z}(s), \boldsymbol{\theta}\}$ given initial conditions $\boldsymbol{y}(t-1) = \boldsymbol{y}_{t-1}$. The vector of errors $\omega(t) = \boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)$ in the estimated trajectories of the system then can be used to derive the error covariance matrix of the system and to calculate Gaussian or quasi-maximum likelihood estimates of the parameters.

Instead of solving the system over each observation interval given initial conditions only, the solution could be obtained subject to more general boundary conditions defined at (t_a, t_b); let these points be $(t-1, t+T_b)$ where T_b is fixed and often will be large. Exogenous variables must be defined over the interval $(0, T + T_b)$, perhaps as analytical functions of time. Assume that the first-order system (3) consists of m_a endogenous variables $\boldsymbol{y}_a(t)$ for which initial values are assumed to be given and m_b endogenous variables $\boldsymbol{y}_b(t)$ defined by the endpoint conditions. Thus, the first-order system becomes

$$\begin{aligned} d\boldsymbol{y}_a(s) &= \psi_a\{\boldsymbol{y}_a(s), \boldsymbol{y}_b(s), \boldsymbol{z}(s), \boldsymbol{\theta}\}\, ds + \zeta_a(ds), \\ d\boldsymbol{y}_b(s) &= \psi_b\{\boldsymbol{y}_a(s), \boldsymbol{y}_b(s), \boldsymbol{z}(s), \boldsymbol{\theta}\}\, ds + \zeta_b(ds), \end{aligned} \tag{17}$$

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

which can be solved numerically for the vector $\{\boldsymbol{y}_a(t)\ \ \boldsymbol{y}_b(t)\}$ for each $t = 1 \cdots T$ given $\boldsymbol{y}_a(t-1)$ and an appropriate set of boundary-point conditions defined at $t + T_b$. For simplicity, it is assumed that any zero-order equations, such as those that arise in optimal control problems, have been eliminated but the estimation and solution procedures being used do not require this. The errors in the trajectories can be calculated and used to give a quasi-maximum likelihood estimator of the parameters of the system. It must be emphasized that the properties of the errors, particularly their serial correlation properties, and the properties of the estimator have not been investigated at this stage. It also must be emphasized that to obtain consistent estimates it is necessary to reinitialize the solution procedure for each observation and that the horizon T_b should be constant; if appropriate, estimates with different values of T_b may be used to test whether T_b is sufficiently large.

4. DYNAMIC BEHAVIOR AND STRUCTURAL STABILITY

A study of the properties of the system includes not only determining its stability for the estimated or given parameter values, but also the question of whether it is structurally stable over some relevant range of parameter values such as the confidence intervals of the parameter estimates. Essentially, a model is structurally stable if small changes in parameters do not produce a qualitative change in its dynamic properties; points at which parameter changes do have a qualitative effect are bifurcation points. In very simple models with few parameters, bifurcation points can be found by an increasingly fine grid search of the relevant intervals for the parameter set. In larger models, sensitivity analysis can be used to determine the parameters that crucially affect the dynamics of the system and that are likely to be relevant. This may have policy implications. For example, if a model is structurally unstable for some parameter values, and if this can lead to some behavior of the system that is undesirable, it may be possible to introduce policies that will shift the relevant parameters away from critical values.

In previous work the asymptotic properties of the nonlinear system (1) have been studied by deriving some transformation of the variables, such as deviations about the steady state, which allows the model (2) to be written as an autonomous system as in (18). Thus if (1) has a steady state such that $\boldsymbol{y}(t) = \boldsymbol{y}^* e^{\rho t}$, assuming that the exogenous variables are analytical functions of time, the system may be solved for $\boldsymbol{y}^*$ and ρ as functions of the parameters $\boldsymbol{\theta}$. It then may be possible to define a set of variables [say $x(t)$, perhaps as deviations of $\boldsymbol{y}(t)$ about its steady state] to produce an autonomous system. Differentiation of the equilibrium point $\boldsymbol{y}^*(\boldsymbol{\theta})$ with respect to the parameters $\boldsymbol{\theta}$ allows an analysis of the steady state itself, and dynamical behavior in the neighborhood of the steady state may be investigated using the eigenvalues and eigenvectors of a linearization about the steady state. The structural stability of the model, that is, the effect on the dynamic properties of changes in the parameter values, may be studied using sensitivity analysis by differentiating the eigenvalues (and eigenvectors if required) with respect to the parameters. These techniques can be extended directly to a study of more general dynamical behavior in nonlinear systems.

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

4.1. Attractors

More complex properties of nonlinear systems may be considered by using phase space to define an attractor; a stable equilibrium solution or fixed point is a special case. For these purposes, assume that some transformation of system (2) can be written as the set of autonomous, first-order, nonlinear differential equations,

$$\dot{\boldsymbol{x}} = f\{x(t), \boldsymbol{\theta}\}, \qquad \textbf{(18)}$$

where $x(t)$ is a vector of n variables, f is a vector function, and $\boldsymbol{\theta}$ is a vector of p parameters. In the following, the solution path or orbit of the system with initial point $x(t_0)$ is denoted $\phi\{x(t_0), t\}$ or, where necessary, $\phi\{x(t_0), t, \boldsymbol{\theta}\}$; usually, $t_0 = 0$. The parameters $\boldsymbol{\theta}$ are omitted where they are fixed.[6]

System (18) is assumed to be coupled in that it is not separable into two independent systems. Because the system is autonomous, given an initial point $x(t_0)$, the system determines a set of solution curves $x(t) = \phi\{x(t_0), t\}$ in phase space such that $x(t_0) = \phi\{x(t_0), t_0\}$; in effect, this is a mapping of the trajectories in time-state space onto phase space, where t can be considered a parameter in phase space. Since the vector field is invariant with respect to t, solutions based at $t_0 \neq 0$ can be translated to $t_0 = 0$.

An attracting set A exists if there is an invariant n-dimensional neighborhood of A such that, if $x(0)$ is an initial point in the neighborhood of the attractor, the trajectory $\phi\{x(0), t\}$ will remain in that neighborhood and will converge on the attractor A as $t \to \infty$. An attractor then can be defined as an attracting set with a dense orbit; the requirement of a dense orbit, that is, an orbit that covers the whole of the attractor, ensures that the attractor is not the union of two or more smaller attractors. Because an attractor is a surface, it has zero volume in phase space.

Classical attractors, such as fixed points, limit cycles, and tori, lie on manifolds that are the analog of a surface. The dimension of these attractors can be seen by a suitable mapping of the attractor, so that a fixed point is of dimension 0, whereas a limit cycle, which can be mapped onto a line, has dimension 1 and is thus a T^1 torus, and a T^2 torus has dimension 2 because it can be mapped onto a plane. These attractors have integer dimensions.

A strange attractor belongs to a class of attractors that do not lie on manifolds. It is defined as an attractor that has a sensitive dependence on initial conditions $x(0)$ in a neighborhood of the attractor, and which is indecomposable in that it does not degenerate into two (or more) distinct attractors. Thus, for an initial point in some neighborhood of the attractor, the trajectory $\phi\{x(0), t\}$ approaches and remains arbitrarily close to the attractor for sufficiently large t, whereas small variations in the initial value of $x(0)$ lead to essentially different time paths of the system after some time interval. Although a strange attractor is strictly not a surface, it can be visualized as a surface consisting of a folded structure with an infinite number of very close layers with a finite volume of phase space between the layers. The attractors are not a closed curve (even of a complicated form) but consist of an aperiodic trajectory. Strange attractors have noninteger or fractal dimensions; it can be shown that the dimension of such attractors must be less than that of the phase space but greater than two. For given parameter values, a dynamical system will exhibit aperiodic or chaotic behavior if it possesses a strange attractor, providing that the trajectory passes through the neighborhood of the attractor as defined above. Although the system is deterministic, and hence given initial conditions lead to uniquely defined trajectories, the sensitivity of the system to those initial conditions means that the behavior of the system is apparently random.

Analysis of the stability of fixed-point attractors, and conditions for the existence and stability of other attractors, is based on a Taylor-series expansion of the system about an appropriate point or path. Assume that the system has an equilibrium or fixed point x^*, so that $\dot{\boldsymbol{x}} = f\{\phi(x^*, t), \boldsymbol{\theta}\} = 0$. Let $\tilde{\boldsymbol{x}} = x - x^*$, and expand (18) to give

$$D\tilde{x} = J(x, \boldsymbol{\theta})|_{x=x^*}\tilde{x} + h(\tilde{x}, x^*, \boldsymbol{\theta}) \quad \text{where } J(x, \boldsymbol{\theta}) = \left(\frac{\partial f_i}{\partial x_j}\right) \tag{19}$$

and h contains higher-order terms in $\tilde{\boldsymbol{x}}$. The nonlinear system will be asymptotically stable in the neighborhood of x^* if

$$\lim_{|\tilde{x}|\to 0} \frac{h(\tilde{x}, x^*, \boldsymbol{\theta})}{|\tilde{x}|}$$

is uniformly convergent in t and the eigenvalues of the linear term in the expansion (19), that is, the Jacobian $J(x, \boldsymbol{\theta})|_{x=x^*}$, have negative real parts. Because the system is assumed to be autonomous, h is independent of t, so the question of uniformity of convergence does not arise.

Let the model linearized about the fixed point or steady state be $A(\boldsymbol{\theta}) = J(x^*, \boldsymbol{\theta})$. If the eigenvalues of $A(\boldsymbol{\theta})$ are λ_i, some of which may be complex conjugate, the partial derivatives $\partial\lambda_i/\partial\boldsymbol{\theta}_\ell$ and $\partial\lambda_i/\partial a_{jk}$, where a_{jk} are elements of A, provide compact information on the dynamics of the model. First, these partial derivatives give an indication of which parameters or coefficients in the model are critical to its stability; often the number of these parameters is quite small. This has policy implications if those parameters are policy parameters or are such that they could be affected by policy. Second, if $|\partial\lambda_i/\partial\boldsymbol{\theta}_\ell|$ is large, it may be useful to examine the effects on the dynamics of the model of allowing $\boldsymbol{\theta}_\ell$ to vary within its asymptotic confidence interval; this gives an indication of whether effort should be made to get more precise estimates of that parameter. Third, knowledge of these partial derivatives often allows certain parameters and variables, or certain feedbacks, to be associated with particular cyclical behavior within the system, thus giving information on the overall dynamic properties of the system.

4.2. Lyapunov Exponents and the Dimension and Nature of Attractors

The analysis of models with fixed-point attractors can be extended to more complex systems. In doing so, a distinction must be made between conservative and dissipative systems; the latter always possesses attractors or repellers, such as fixed

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

points, tori, or strange attractors, whereas conservative or Hamiltonian systems do not but have an infinity of closed orbits so that any initial point always will lie on one of these orbits. The two types of systems can be distinguished using the generalized divergence of f. Let a set of initial conditions be contained in a vanishingly small hyperellipsoid V in n-dimensional space. This volume will change as a function of t as $x(t)$ changes, so that

$$\frac{dV}{dt} = \int\int_V \cdots \int \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right) dx_1 \cdots dx_n. \qquad \textbf{(20)}$$

The summation term is the generalized divergence or Lie derivative of f; dissipative systems are characterized by contracting volumes, that is, $dV/dt < 0$, whereas, in conservative or Hamiltonian systems, V is constant. This term, which is the trace of the Jacobian, is thus equal to the sum of the eigenvalues of $J(x, \boldsymbol{\theta})$. The Jacobian of a Hamiltonian system can be written as a block diagonal matrix with one block being the negative of another.

Any trajectory of a dissipative system will approach an attractor as $t \to \infty$. For systems exhibiting chaos in the neighborhood of a strange attractor, the trajectories have a sensitive dependence on the initial conditions, so that the separation of two nearby trajectories increases exponentially with time. This means that there is a stretching of the hyperellipsoid V in one direction, which is more than compensated by a contraction in other directions, so that the volume defined by the arbitrary initial conditions decreases with time. The hyperellipsoid cannot always be stretched in the same direction but must be folded such that it is located in the specified neighborhood of the initial conditions.

The Lyapunov exponents provide information on the relative rates of expansion or contraction of this hyperellipsoid in each dimension and hence on the asymptotic properties of a system. Let an initial point by x_0. Consider the solution of the system

$$\dot{\phi}(x_0, t) = f\{\phi(x_0, t), t\}, \qquad \phi(x_0, t_0) = x_0 \qquad \textbf{(21)}$$

and differentiate with respect to x_0 to give the variational (matrix) equation

$$\dot{\boldsymbol{\Phi}}(x_0, t) = J\{\phi(x_0, t)\}\Phi(x_0, t) \quad \text{where } \Phi(x_0, t) = \frac{\partial \phi(x_0, t)}{\partial x_0} \text{ and } \Phi(x_0, t_0) = I, \qquad \textbf{(22)}$$

which is the linearization of the vector field along the orbit $\phi(x_0, t)$. Thus if δx_0 is a small perturbation about x_0, then $\delta x(t) = \Phi(x_0, t)\delta x_0$. Let the eigenvalues of $\Phi(x_0, t)$ be $\mu_1(t), \ldots, \mu_n(t)$. The Lyapunov exponents then can be defined as

$$\lambda_i = \limsup_{t \to \infty} (1/t) \log|\mu_i|, \quad \text{for } i = 1, \ldots, n.$$

For simplicity, it is assumed that the exponents are arranged in descending order such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let the corresponding generalized eigenvectors be $\eta_1 \cdots \eta_n$.

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

The Lyapunov exponents are a generalization of the eigenvalues at an equilibrium point. Let x_0 be an equilibrium point so that $J\{\phi(x_0, t)\}$ is time-invariant and (22) becomes $\Phi(x_0, t) = e^{J\{\Phi(x_0,t)\}t}$. If λ_i^* are the eigenvalues of $J(x^*)$, then the Lyapunov exponents of the equilibrium point are given by the real part of λ_i^*, that is, $\lambda_i = \boldsymbol{R}(\lambda_i^*)$ because $(1/t)\log|\mu_i| = \boldsymbol{R}(\lambda_i^*)$. Thus the Lyapunov exponents indicate the rate of expansion or contraction of the hyperellipsoid in the neighborhood of the equilibrium point, or the average rate at which $x_0 \neq x^*$, and the subspace within which this occurs is defined by the corresponding eigenvectors.

Calculation of the Lyapunov exponents is by no means trivial owing to the numerical properties of the solution $\phi(x, t)$ in (21) and the variational equation (22). Because at least one Lyapunov exponent is positive in chaotic systems, the solutions $\Phi(x_0, t)$ are unbounded as $t \to \infty$ and the matrix is ill-conditioned. Thus, some form of orthonormalization of these solutions is necessary to maintain precision for calculating all Lyapunov exponents.

The Lyapunov exponents λ_i may be used to classify and help determine the form of the attractor.[7] For an attractor to exist, the volume of the hyperellipsoid defined by the generalized divergence must contract so that $\sum_{i=1}^{n} \lambda_i < 0$. It can be shown that for attractors other than an equilibrium point, at least one Lyapunov exponent must be zero. Thus, for nonchaotic attractors, for an asymptotically stable equilibrium point $\lambda_i < 0$ for all i, while for an asymptotically stable k-torus, k exponents must be zero and the remainder negative; thus, a limit cycle or T^1 torus has only one zero exponent and the rest negative. A nonchaotic attractor is nondegenerate or hyperbolic if its dimension equals the number of zero Lyapunov exponents. For chaotic attractors, at least one of the Lyapunov exponents must be positive and at least one zero. In addition, the dimension of a chaotic attractor must be a noninteger.

Lyapunov exponents and other measures such as the Hausdorff dimension and the correlation dimension may be used to determine the dimension of the attractor and hence the degree of order in the system. Let the set of points defining a surface in n-dimensional space be covered with n-dimensional hypercubes of length ε. The Kolmogorov capacity dimension of the object then can be defined as

$$D_K = \liminf_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},$$

where $N(\varepsilon)$ is the minimum number of hypercubes necessary to cover all points; this is closely related to the Hausdorff dimension D_H as defined by Guckenheimer and Holmes (1986, p. 285), with $D_H \leq D_K$ and, for the purposes of this paper, no distinction is made between these dimensions. The Hausdorff dimension is identical to the Euclidean dimension for surfaces in Euclidean space and defines the noninteger or fractal dimension of objects associated with Cantor sets as discussed by Lorenz (1989). Kolmogorov entropy K, which measures the degree of disorder or randomness in the system, can be defined for the purposes of this paper as $D_K \log(1/\varepsilon)$ for small ε; a precise definition is given by Eckmann and Ruelle

(1985). The extreme cases are limit cycles for which K is zero, whereas if the system is purely random, K is infinite. As noted by Lorenz (1989), a chaotic system has finite entropy so that $0 < K < \infty$.

The correlation dimension is defined for a single time series. Let x_t, $t = 1 \cdots T$ be an observed time series of a variable generated by the dynamical system but possibly observed with measurement error, and define its m history to be $x(m, i) = (x_i x_{i-1} \cdots x_{i-m+1})$ for $i = m \cdots T$. Because each m history is a point in m-dimensional space defining the delayed observed values of x_t in $x(m, i)$, m is called the embedding dimension. Takens (1981) showed that (i) if the variables of the true dynamical system are located on an attractor, that is, there are no transients, (ii) if the dynamical system and measurement function are smooth, and (iii) if $m > 2n - 1$, where n is the dimension of the system, then the behavior of the dynamical system can be identified from the series.

Let $C^m(\varepsilon)$ be the correlation integral (or correlation function) of the time series with embedding dimension m defined such that for a small positive number ε,

$$C^m(\varepsilon) = \lim_{T_m \to \infty} \frac{1}{T_m^2} \sum_{i,j=1}^{T_m} H(\varepsilon - \|x(m,i) - x(m,j)\|),$$

where $T_m = T - m + 1$ is the number of m histories, $\| \, \|$ is the Euclidean norm, and H is the Heaviside function $H(y) = 1$ if $y > 0$, $H(0) = \frac{1}{2}$, and 0 otherwise. The correlation dimension in embedding dimension m then is defined as

$$D_C(m) = \lim_{\varepsilon \to 0} \frac{\log C^m(\varepsilon)}{\log \varepsilon} \quad \text{and the correlation dimension as } D_C = \lim_{m \to \infty} D_C(m).$$

If required, the structural model can be used to generate long series of observations, which then can be used to calculate the correlation integral and correlation dimension. Following Eckmann and Ruelle (1985), the correlation dimension can be estimated by finding a constant ratio of

$$\frac{\log C^m(\varepsilon)}{\log \varepsilon},$$

that is, independent of m when m is large. The correlation integral and correlation dimension form a link between time-series analysis and the structural approach discussed in this paper.

The Lyapunov dimension is defined as

$$D_L = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|},$$

https://doi.org/10.1017/S1365100597003106 Published online by Cambridge University Press

where k is the largest number of eigenvalues such that the sum is positive, that is, k is such that $\sum_{i=1}^{k} \lambda_i > 0$ and $\sum_{i=1}^{k+1} \lambda_i < 0$. Hence,

$$\frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|} < 1$$

and so, if an attractor exists and at least λ_1 is positive, the Lyapunov dimension is a fractal.

The Lyapunov dimension and the correlation dimension provide upper and lower bounds on the Hausdorff dimension. Teman (1988) proved that $D_H \leq D_L$, whereas Grassberger and Procaccia (1983) showed that $D_C \leq D_H$; Kaplan and Yorke (1979) conjectured that $D_H = D_L$. Because positive Lyapunov exponents indicate the expansion of an initial hyperellipse in one or more directions and Kolmogorov entropy measures the average expansion in all directions, Kolmogorov entropy $K \leq \sum_i \lambda_i$ for $\lambda_i > 0$. An approximation to Kolmogorov entropy proposed by Grassberger and Procaccia (1983) and discussed by Eckmann and Ruelle (1985) is

$$K_2 = \lim_{m \to \infty} \lim_{\varepsilon \to 0} \frac{1}{\delta} \log \frac{C^m(\varepsilon)}{C^{m+1}(\varepsilon)},$$

where δ is the observation period of a continuous series $x(t)$. Because $K_2 \leq K$, these two measures provide upper and lower bounds for Kolmogorov entropy corresponding to the bounds on the Hausdorff dimension. For a limit cycle $C^m(\varepsilon) = C^{m+1}(\varepsilon)$ and hence $K_2 = 0$, whereas if the series is purely random, K is infinite; thus in terms of the correlation integral, K_2 approaches a finite, positive value from above as the embedding dimension is increased.

The approach of this paper provides more information about the asymptotic behavior of an economic system and the nature of its attractor than measures such as the correlation integral, correlation dimension, and the largest Lyapunov exponent that can be estimated from a single time series. These measures are discussed, for example, by Brock and Dechert (1988) and Barnett and Chen (1988). The amount of noise in some economic time series makes some of these tests quite poor at distinguishing between different types of behavior as shown in Barnett et al. (1994). One problem that may arise here is that if data are generated by a continuous stochastic system, the observations will be serially correlated for the reasons discussed above and this needs to be taken into account in calculating the correlation integral and associated functions. This problem does not arise, of course, with data generated by deterministic systems. Moreover, whereas the largest Lyapunov exponent being negative shows that the system will converge on some attractor, being positive is only a necessary condition for a strange attractor. Estimation of all, or at least several, of the largest exponents is required to find whether the sum of all exponents is negative and whether D_L is noninteger and to find an upper bound to Kolmogorov entropy. Essentially, the structural model, providing that it can be verified, imposes a degree of order on the data. Thus, rather than an atheoretical single time-series approach, the approach described here harnesses economic theory and time series for all variables to investigate the properties of the system.

4.3. Structural Stability

The second question of interest is the structural stability of the solution of the differential system (18), that is, whether the asymptotic properties of the system change qualitatively for different values of the parameters. For example, for some parameter values, the system may have a stable fixed point, whereas for others it may have some form of torus as an attractor or perhaps a strange attractor, whereas for still others it may be unstable. Such structural stability is of crucial importance when considering the policy implications of a system. A system is structurally stable when small perturbations produce topologically equivalent systems. Heuristically, the system will be structurally stable if there is a one-to-one mapping of the flow of the system for small changes in parameter values; if not, the system is structurally unstable.

In the dynamical system $\dot{\boldsymbol{x}} = f\{x(t), \boldsymbol{\theta}\}$, as the set of parameters $\boldsymbol{\theta}$ varies, the phase portrait of the system usually changes gradually so that the system is topologically unchanged, but at some values of $\boldsymbol{\theta}$, called bifurcation points, the topology may change, perhaps with a change in the number of fixed points or periodic orbits, or with a change in the nature of a strange attractor; thus the system becomes structurally unstable at that point. Specifically, a point $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ is a bifurcation point if there exists a $\boldsymbol{\theta}_1$ arbitrarily close to $\boldsymbol{\theta}_0$ such that $f(x, \boldsymbol{\theta}_0)$ and $f(x, \boldsymbol{\theta}_1)$ are topologically different. Such bifurcation points are characterized by a change in the number of eigenvalues of the system with zero real parts at that point so that the conditions for a point to be a bifurcation point can be defined in terms of the Jacobian of $f(x, \boldsymbol{\theta})$ or its eigenvalues.

The study of bifurcations involves an analysis of the successive terms in the Taylor-series expansion of (18) or (19), and particularly any degeneracy of those terms. The behavior of the system in the neighborhood of a bifurcation point can become even more complex when more than one parameter can lead to bifurcations or if other secondary parameters take on certain values affecting the nature of the bifurcation. A succession of bifurcations can lead to a strange attractor and aperiodic or chaotic dynamical behavior. Although successive bifurcations may arise as the parameter set $\boldsymbol{\theta}$ is slowly changed and chaotic behavior may occur within some subintervals, yet within these subintervals there may be embedded a secondary set of subintervals that produce windows of regular behavior. References to this literature are given by Wymer (1995).

Calculation of the partial derivatives of the Lyapunov exponents with respect to the parameters of the system using (22) will help to provide information on whether the system has an attractor in the neighborhood of the parameter values and the nature and stability of that attractor. This will suggest which parameters may usefully be investigated to find bifurcations and where such bifurcations might occur. The nature of any attractors, and especially strange attractors, in the neighborhood of the parameter estimates, perhaps defined as the confidence ellipsoid of these estimates, would be of particular interest. This is analogous to the use of sensitivity analysis in the study of the policy implications of linear models that have a steady state, but the asymptotic behavior of nonlinear models is likely to be far more complex.

The dynamic behavior and asymptotic properties of (2) may be considered from two points of view: first, the properties of the solutions for a given set of parameter values $\boldsymbol{\theta}$; and second, the structural stability of the model as one or more of these parameters varies. Although such analyses have been used routinely with many of the continuous-time models developed in the past, relatively recent developments in the study of aperiodic dynamical systems and strange attractors has allowed this work to be extended to an investigation of more complex dynamical behavior. Thus, the research discussed in this paper arises as a direct extension of previous work. In particular, some of the macromodels developed over the past 20 years, such as the Bergstrom and Wymer (1976) model of the United Kingdom, have eigenvalues that either are close to zero or are positive. Although this may show that the economic system is simply unstable, it may be an indication of much more complex behavior, particularly given the nature of the nonlinear structure these models.

5. CONCLUSION

Differential-equation systems in mathematics and the physical sciences have a very long history as has the theory of continuous stochastic processes. More importantly, a great deal of economic theory is continuous. The continuous-time approach in econometrics binds these together by allowing the theoretical model to be specified independently of the observation period of the data that are to be used for estimation and to provide full-information maximum likelihood estimates of the parameters of that model. The estimates then can be used in deriving the properties of the model and in forecasting. This approach fits well with the arguments of Frisch (1933) in an editorial in *Econometrica*: "each of these three view points, that of statistics, economic theory, and mathematics, is a necessary, but not by itself sufficient, condition for a real understanding of the quantitative relations in modern economic life. It is the *unification* of all three that is powerful. And it is that unification that constitutes econometrics."

The results of applied research in this field support this view and the expectations of Koopmans (1950) and Marschak (1950). Extensive references to applied work are given by Wymer (1993, 1995, 1996). The asymptotic properties of the continuous estimators appear to be a good indicator of the small-sample properties. The parameter estimates and forecast errors being obtained with these models appear to be more precise than those of ordinary discrete models. In particular, the approximate discrete estimators give estimates quite close to the exact estimators but the latter are much more precise in that the asymptotic standard errors are much smaller. This allows extensive hypothesis testing of the theoretical basis of the model from an economic point of view. The dynamic behavior of the model and the structural stability of any attractor then can be investigated.

NOTES

1. References to other work in this area are given by Wymer (1993a, 1996). Other estimators of continuous systems, often in the frequency domain, have been developed by A.W. Phillips (1959), Robinson (1976), and Harvey and Stock (1985). Although these estimators often allow more general disturbance processes than the estimators discussed in this paper, it is often more difficult or not possible to estimate overidentified systems. The question of identification has been discussed, for example, by P.C.B. Phillips (1973) and Hansen and Sargent (1983). Estimators of other forms of structural models, such as the asset pricing model of Lo (1988, 1991) and long-memory models defined by fractional differential-equation systems as in Chambers (1992), also are not discussed here.

2. It is assumed throughout this paper that the economic system being studied is dynamic and only dynamic models are considered. The term ordinary discrete model refers to models that are specified as a set of difference equations without being derived from a differential-equation system, although some of those models, may, in fact, be consistent with a differential-equation system.

3. Throughout this paper, a nonlinear model means one that is nonlinear in variables; all of the models discussed here may be nonlinear in parameters because even models linear in variables may have coefficients that are nonlinear functions of the parameters of the system.

4. For example, a class of models of dynamic demand for world monies has been developed and estimated by Donaghy and Richard (1993, 1995) to test various functional forms such as the Almost Ideal Demand System (AIDS) of Deaton and Meulbauer (1980), Modified AIDS (MAIDS), Generalized Modified AIDS (GMAIDS) and Generalized Regular AIDS (GRAIDS) as well as extensions of these systems incorporating the symmetric generalized McFadden or Barnett aggregator functions of Diewert and Wales (1987). It is intended that these dynamic functions form part of a more sophisticated multicountry dynamic model of interest and exchange-rate determination similar to that of Richard (1980). As one instance of this class, a model has been derived from a modified price-independent generalized log-linear (MPIGLOG) indirect utility function as in Cooper and McLaren (1992) incorporating the Symmetric Generalized Barnett (SGB) aggregator function, which provides global regularity. Use of the latter results in virtually global regularity as concavity can be imposed over the entire economic region. This is embedded in a dynamic system. The model was estimated subject to the homogeneity, symmetry, and regularity conditions using the exact nonlinear estimator with a sample of Divisia indices of volumes of money for five major countries.

5. In practice, there is no need to eliminate such variables explicitly, and the computer programs that have been developed to support the work in continuous-time systems usually allow the specification of models that are more general than those discussed here and that include zero-order equations if desired.

6. Although this paper considers only autonomous systems specifically, a nonautonomous system can be written as an autonomous system with little loss in generality, but with some change in interpretation. Specifically, if an $(n-1)$ dimensional nonautonomous system is $\dot{x} = g\{x(t), \theta, t\}$, defining $\dot{x}_n = 1$ allows this to be written as $\dot{x} = g\{x(t), \theta, x_n(t)\}$, $\dot{x}_n = 1$, which has the same form as the n-dimensional autonomous system (18).

7. Only dissipative systems are considered here. (Undiscounted) Hamiltonian or conservative systems with m state variables (and hence m degrees of freedom or costate variables, so that the system as defined above has dimension 2 m) have Lyapunov exponents such that $\sum_{i=1}^{2m} \lambda_i = 0$, at least two exponents are zero, and $\lambda_i = -\lambda_{2m-i+1}$ for $i = 1, \ldots, m$. All conservative systems are structurally unstable and do not possess attractors, but chaotic behavior can arise as local turbulence in the neighborhood of the Hamiltonian orbit when the system is perturbed.

Although the orbit of a discounted Hamiltonian system collapses onto a (saddle-path stable) fixed point, that point does not satisfy the conditions for an attractor. In particular, the Hamiltonian orbit is a function of the initial values of the known (backward-looking) variables and the trajectory does not converge on that point for arbitrary initial values of the unknown (forward-looking) variables in the neighborhood of that point. The Lyapunov exponents of these systems will be zero; the Ramsey model, for example, is a special case in that it has only two differential equations and hence two zero Lyapunov exponents.

The question of whether Hamiltonian systems in macroeconomics should be discounted or not is debatable and depends on the particular application. Although discounting greatly simplifies analysis of these systems, it precludes long-run behavior that may be more interesting and more relevant to an understanding of the economic system and macroeconomic policy than a fixed point.

REFERENCES

Barnett, W.A. (1990) Developments in monetary aggregation theory. *Journal of Policy Modelling* 12, 205–257.

Barnett, W.A. & P. Chen (1988) The aggregation-theoretic monetary aggregates are chaotic and have strange attractors: an econometric application of mathematical chaos. In W.A. Barnett, E.R. Berndt & H. White (eds.), *Dynamic Econometric Modelling*. Cambridge: Cambridge University Press.

Barnett, W.A., A.R. Gallant, M.J. Hinich, J.A. Jungeilges, D.T. Kaplan & M.J. Jensen. A single-blind controlled competition among tests for non-linearity and chaos. *Journal of Econometrics* (forthcoming).

Bergstrom, A.R. (ed.) (1976), *Statistical Inference in Continuous Time Economic Models*. Amsterdam: North-Holland.

Bergstrom, A.R. (1983) Gaussian estimation of structural parameters in higher order continuous-time dynamic models. *Econometrica* 51, 117–152.

Bergstrom, A.R. (1986) The estimation of open higher-order continuous time dynamic models with mixed stock and flow data. *Econometric Theory* 2, 350–373.

Bergstrom, A.R. (1988) The history of continuous time econometric models. *Econometric Theory* 4, 365–383.

Bergstrom, A.R. (1990) *Continuous Time Econometric Modelling*. Oxford: Oxford University Press.

Bergstrom, A.R., K.B. Nowman & C.R. Wymer (1992) Gaussian estimation of a second order continuous time macroeconometric model of the United Kingdom. *Economic Modelling*, 313–351.

Bergstrom, A.R. & C.R. Wymer (1976) A model of disequilibrium neoclassical growth and its application to the United Kingdom. In A.R. Bergstrom (ed.), *Statistical Inference in Continuous-Time Economic Models*, pp. 267–328. Amsterdam: North-Holland.

Blanchard, O.J. & D. Quah (1989) The dynamic effects of aggregate demand and supply disturbances. *American Economic Review* 79, 655–673.

Brock, W. & W.D. Dechert (1988) Theorems on distinguishing deterministic and random systems. In W.A. Barnett, E.R. Berndt & H. White (eds.), *Dynamic Econometric Modelling*. Cambridge: Cambridge University Press.

Chambers, M.J. (1992) The estimation of continuous time long-memory time series models. University of Essex Discussion Paper Series No. 410.

Cooley, T.F. & S.F. LeRoy (1985) Atheoretical macroeconometrics: a critique. *Journal of Monetary Economics* 16, 283–308.

Cooper, R.J. & K.R. McLaren (1992) An empirically oriented demand system with improved regularity properties. *Canadian Journal of Economics* 25, 652–668.

Deaton, A. & J. Muellbauer (1980) An almost ideal demand system. *American Economic Review* 70, 312–326.

Diewert, W.E. & T.J. Wales (1987) Flexible functional forms and global curvature conditions. *Econometrica* 55, 43–68.

Donaghy, K.P. (1993) A continuous time model of the United States economy. In G. Gandolfo (ed.), *Continuous Time Econometrics: Theory and Applications*, pp. 151–193. London: Chapman-Hall.

Donaghy, K.P. & D.M. Richard (1993) Flexible functional forms and generalized dynamic adjustment in the specification of the demand for money. In G. Gandolfo (ed.), *Continuous Time Econometrics: Theory and Applications*, pp. 229–259. London: Chapman-Hall.

Donaghy, K.P. & D.M. Richard (1995) Exact quasi-FIML estimation of dynamic systems of demand for World monies. Presented at International Symposium of Economic Modelling, Bologna, Italy, July 14, 1995.

Eckmann, J-P. & D. Ruelle (1985) Ergodic theory of chaos and strange attractors. *Reviews of Modern Physics* 57, 617–656.

Frisch, R. (1933) Editorial, *Econometrica* 1, 2.

Gandolfo, G. (1993) Continuous time econometrics has come of age. In G. Gandolfo (ed.), *Continuous Time Econometrics: Theory and Applications*, pp. 1–11. London: Chapman-Hall.

Gandolfo, G., P.C. Padoan, G. de Arcangelis & C.R. Wymer (1996) Non-linear estimation of a non-linear continuous time model. In W.A. Barnett, G. Gandolfo & C. Hillinger (eds.), *Dynamic Disequilibrium Modelling*, pp. 127–150. Cambridge: Cambridge University Press.

Grassberger, P. & I. Procaccia (1983) Estimation of Kolmogorov entropy from a chaotic signal. *Physical Review A* 28, 2591–2593.

Guckenheimer, J. & P. Holmes (1986) *Non-Linear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Berlin: Springer-Verlag.

Hansen, L.P. & T.J. Sargent (1983) The dimensionality of the aliasing problem in models with rational spectral densities. *Econometrica* 51, 377–388.

Harvey, A.C. & J.H. Stock (1985) The estimation of higher order continuous-time autoregressive models. *Econometric Theory* 1, 97–117.

Kaplan, J.L. & J.A. Yorke (1979). Functional differential equations and approximations at fixed points. In H.-O. Pietgen & H.-O. Walther (eds.), *Lecture Notes in Mathematics*, vol. 730. Berlin: Springer–Verlag.

Koopmans, T.C. (1950) Models involving a continuous-time variable. In T.C. Koopmans (ed.), *Statistical Inference in Dynamic Economic Models*, pp. 384–392. New York: Wiley.

Lo, A.W. (1988) Maximum likelihood estimation of generalized Ito processes with discretely sampled data. *Econometric Theory* 4, 231–247.

Lo, A.W. (1991) Long term memory in stock market prices. *Econometrica* 59, 1279–1313.

Lorenz, H-W. (1989) *Non-Linear Dynamical Economics and Chaotic Motion*. Berlin: Springer-Verlag.

Marschak, J. (1950) Statistical inference in economics: An introduction. In T.C. Koopmans (ed.), *Statistical Inference in Dynamic Economic Models*, pp. 1–50. New York: Wiley.

Pagan, A.R. (1987) Three econometric methodologies: a critical appraisal. *Journal of Economic Surveys* 1, 3–24; reprinted in Granger, C.W.J. (ed.) (1990), *Modelling Economic Series*, pp. 97–120. Oxford: Oxford University Press.

Phillips, A.W. (1959) The estimation of parameters in systems of stochastic differential equations. *Biometrika* 46, 67–76.

Phillips, P.C.B. (1973) The problem of identification in finite parameter continuous-time models. *Journal of Econometrics* 1, 351–362; reprinted in Bergstrom, A.R. (ed.) (1976), *Statistical Inference in Continuous-Time Economic Models*, pp. 123–134. Amsterdam: North-Holland.

Richard, D. (1980) A global adjustment model of exchange and interest rates. In D. Bigman & T. Taya (eds.), *The Functioning of Floating Exchange Rates: Theory, Evidence and Policy Implications*, pp. 243–276. Cambridge, MA: Ballinger.

Robinson, P.M. (1976) Fourier estimation of continuous-time models. In A.R. Bergstrom (ed.), *Statistical Inference in Continuous-Time Economic Models*, pp. 215–226. Amsterdam: North-Holland.

Sims, C.A. (1980) Macroeconomics and reality. *Econometrica* 48, 1–48.

Takens, F. (1981) Detecting strange attractors in turbulence. In D. Rand & L. Young (eds.), *Dynamical Systems and Turbulence*. Berlin: Springer-Verlag.

Teman, R. (1988) *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Berlin: Springer-Verlag.

Wymer, C.R. (1972) Econometric estimation of stochastic differential equation systems. *Econometrica*, 40, 565–577; reprinted in Bergstrom, A.R. (ed.) (1976), *Statistical Inference in Continuous Economic Models*, pp. 81–95. Amsterdam: North-Holland.

Wymer, C.R. (1993a) Continuous time models in macro-economics: specification and estimation. In G. Gandolfo (ed.), *Continuous Time Econometrics*: *Theory and Applications*, pp. 35–79. London: Chapman and Hall.

Wymer, C.R. (1993b) Estimation of non-linear differential equation systems. In P.C.B. Phillips (ed.), *Models, Methods and Applications of Econometrics*, pp. 91–114. Oxford: Blackwell.

Wymer, C.R. (1995) Advances in the estimation and analysis of non-linear differential equation models in economics. In L. Schoonbeek, E. Sterken & S.K. Kuipers (eds.), *Methods and Applications of Economic Dynamics*, pp. 41–76. Amsterdam: North-Holland.

Wymer, C.R. (1996) The role of continuous time disequilibrium models in macro-economics. In W.A. Barnett, G. Gandolfo & C. Hillinger (eds.), *Dynamic Disequilibrium Modelling*, pp. 67–123. Cambridge, England: Cambridge University Press.