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EXTRAGRADIENT METHODS FOR QUASI-EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract

We study the extragradient method for solving quasi-equilibrium problems in Banach spaces, which generalizes the extragradient method for equilibrium problems and quasi-variational inequalities. We propose a regularization procedure which ensures strong convergence of the generated sequence to a solution of the quasi-equilibrium problem, under standard assumptions on the problem assuming neither any monotonicity assumption on the bifunction nor any weak continuity assumption of f in its arguments that in the many well-known methods have been used. Also, we give a necessary and sufficient condition for the solution set of the quasi-equilibrium problem to be nonempty and we show that, in this case, this iterative sequence converges strongly to a solution of the quasi-equilibrium problem. In other words, we prove strong convergence of the generated sequence to a solution of the quasi-equilibrium problem without assuming existence of a solution of the problem. Finally, we give an application of our main result to a generalized Nash equilibrium problem.

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1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$; E^* will denote the topological dual of E. The duality mapping $J : E \to \mathcal{P}(E^*)$ is defined by

$$J(x) = \{v \in E^* : \langle x, v\rangle = \|x\|^2 = \|v\|^2\}.$$

Let $C \subseteq E$ be a nonempty closed and convex set and $K(\cdot)$ be a multivalued mapping from C into itself such that for all $x \in C$, $K(x)$ is a nonempty closed and convex subset of C, and let $f : E \times E \to \mathbb{R}$ be a bifunction. The quasi-equilibrium problem QEP (f, K) consists of finding $x^* \in K(x^*)$, that is, a fixed point x^* of $K(\cdot)$, such that

$$f(x^*, y) \geq 0, \quad \forall y \in K(x^*). \tag{1-1}$$

The set of solutions of QEP (f, K) will be denoted as $S(f, K)$. We also denote the set of fixed points of the multivalued mapping $K(\cdot)$ by Fix(K). The associated Minty quasi-equilibrium problem, denoted by MQEP (f, K), can be expressed as finding $x^* \in K(x^*)$ such that $f(y, x^*) \leq 0$ for all $y \in K(x^*)$. The set of solutions of MQEP (f, K) will be denoted by $MS(f, K)$. When the constraint set $K(x)$ is equal to C for every $x \in C$, the quasi-equilibrium problem QEP (f, K) becomes a classical equilibrium problem EP (f, C), and the associated Minty quasi-equilibrium problem becomes a classical Minty equilibrium problem.

An example of a quasi-equilibrium problem is a quasi-variational inequality problem. Let $K(\cdot)$ be a multivalued mapping from C into itself such that for all $x \in C$, $K(x)$ is a nonempty closed and convex subset of C, consider a map $T : E \to E^*$ and define $f(x, y) = \langle T(x), y - x\rangle$, where $\langle\cdot, \cdot\rangle : E^* \times E \to \mathbb{R}$ denotes the duality pair, that is, $\langle z, x\rangle = z(x)$. Then QEP (f, K) is equivalent to the quasi-variational inequality problem QVIP (T, K), consisting of finding a point $x^* \in K(x^*)$ such that $\langle T(x^*), x - x^*\rangle \geq 0$ for all $x \in K(x^*)$. Usually for approximating solutions of the equilibrium problems, some monotonicity assumptions on the bifunction f are needed. We recall next two such properties for future reference: the bifunction f is said to be *monotone* if $f(x, y) + f(y, x) \leq 0$ for all $x, y \in E$ and *pseudo-monotone* if for any pair $x, y \in E$, $f(x, y) \geq 0$ implies that $f(y, x) \leq 0$.

The equilibrium problem encompasses, among its particular cases, convex optimization problems, variational inequalities (monotone or otherwise), Nash equilibrium problems and other problems of interest in many applications. The study of equilibrium problems goes back to Fan [11]. Subsequently, Brézis *et al.* [6] studied the problem with a coercivity assumption on f. Blum and Oettli [5] proved the existence of solutions to EP (f, C) with a monotonicity condition on f. Later, the equilibrium problems were studied extensively for existence of solutions (see, for example, [13] and the references therein). Recently, much research was devoted to the approximation of solutions to equilibrium problems (see, for example, [18, 19, 26] and the references therein). Equilibrium problems with monotone and pseudo-monotone bifunctions were studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors (for example, [4, 7, 8, 13, 15, 19]). Also, the quasi-equilibrium problems were studied in [3] and [33]. Recently, the second author and Iusem have studied the extragradient method with linesearch for solving nonsmooth equilibrium problems in Banach spaces. They proved weak and strong convergence of the generated sequence to a solution of the equilibrium problem, under standard assumptions on the bifunction (see [15, 16]). Other variants of the extragradient method can be found in [10, 12, 14, 17, 22, 24, 25, 31]. In this paper, we perform some modifications on the extragradient method in order to introduce and analyze the extragradient method with linesearch for solving quasi-equilibrium problems in Banach spaces, and we prove the strong convergence of the generated sequence to a solution of the quasi-equilibrium problem, under rather mild assumptions on the bifunction. Our convergence results hold without any monotonicity, smoothness and weak continuity assumptions on f. Also, we give a necessary and sufficient condition for the solution set of the quasi-equilibrium problem to be nonempty, and we show that in this case, this iterative sequence converges strongly to a solution of the quasi-equilibrium problem. In particular, our results contain the case where the bifunction is the sum of two (or finitely many) pseudo-monotone bifunctions or even the sum of a pseudo-monotone bifunction and a monotone bifunction. As we know, in this case, the sum need not be any more pseudo-monotone, where the pseudo-monotone case was studied by previous authors. One of the main reasons for studying quasi-equilibrium problems lies in the relation between them and quasi-variational inequalities, which mirrors the well-known relation between equilibrium problems and variational inequalities. Quasi-variational inequalities are themselves relevant because they encompass certain problems of interest in applications, which do not fall within the scope of variational inequalities. Perhaps one important instance of such applications is the generalized Nash equilibrium problem, which models a large number of real-life problems in economics and other areas (see [27, 28, 30]). In order to illustrate an application of our main result in this paper, we will give a concrete example of a real-life problem in Section 4.

In this paper, we study an extragradient method which improves upon [15] and [33] in the following ways.

(a) We deal with an extragradient method for solving quasi-equilibrium problems in Banach spaces, while [15] considered classical equilibrium problems. Also, [33] is restricted to finite-dimensional Euclidean spaces.

(b) We prove the strong convergence of the generated sequence to a solution of the quasi-equilibrium problem, without assuming the existence of a solution to the problem, while in [15] and [33] the authors assumed the existence of a solution for their equilibrium problems.

(c) The convergence analysis of the method in [15] requires the weak upper semicontinuity of $f(\cdot, y)$ for all $y \in E$ and the Lipschitz continuity of the bifunction f on bounded sets, which are quite restrictive conditions. Here we use neither Lipschitz continuity nor weak upper semicontinuity assumptions on the bifunction. Moreover, since the algorithm in [33] works only in finite-dimensional spaces, it is not even possible to get the weak convergence of the generated sequence in a Banach space by their algorithm.

(d) We deal with a rather general class of bifunctions, while [15] only considered pseudo-monotone bifunctions. On the other hand, [33] considered a multivalued mapping $K(\cdot) : C \to \mathcal{P}(C)$ which is $\bigstar$-nonexpansive and bounded valued, whereas we assume that $K(\cdot)$ is quasi-ϕ-nonexpansive, which is a weaker assumption, and we do not use the boundedness of $K(\cdot)$.

We also mention another difference between our algorithm and those of [15] and [33], regarding the step required for getting the strong convergence. Note that [33] considered only finite-dimensional Euclidean spaces with a different extragradient algorithm, and [15] dealt with Halpern's algorithm, which consists of a convex combination in E^* of the current iterate with a given point in E^*. Our method, instead, takes the projection of the initial iterate onto the intersection of three half spaces (see (3-10)). Although the projections may present some computational complexity in Banach spaces, they are needed for the convergence analysis of the generated sequence, especially when $K(\cdot)$ is quasi-ϕ-nonexpansive. In fact, the half space M_k defined in (3-12) helps us to prove the convergence of the generated sequence to a fixed point of $K(\cdot)$, and the half spaces L_k and N_k defined in (3-11) and (3-13) help us to prove the strong convergence of the sequence.

The paper is organized as follows. In Section 2, we introduce some preliminary material related to the geometry of Banach spaces. In Section 3, we introduce and analyze the extragradient method with linesearch for solving nonsmooth quasi-equilibrium problems in Banach spaces, and we prove the strong convergence of the generated sequence to a solution of the quasi-equilibrium problem. Also, we give a necessary and sufficient condition for the solution set of the quasi-equilibrium problem to be nonempty. Finally, in Section 4, we give an application of our main result to a generalized Nash equilibrium problem, which is formulated as a quasi-equilibrium problem.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, we have $\|(x + y)/2\| < 1 - \delta$. It is known that uniformly convex Banach spaces are reflexive and strictly convex.

A Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2-1}$$

exists for all $x, y \in S = \{z \in E : \|z\| = 1\}$. It is said to be *uniformly smooth* if the limit in (2-1) is attained uniformly for $x, y \in S$. It is well known that the spaces $L^p (1 < p < \infty)$ and the Sobolev spaces $W^{k,p}$ $(1 < p < \infty)$ are both uniformly convex and uniformly smooth.

It is well known that when E is smooth the duality operator J is single valued. Let E be a smooth Banach space. We define $\phi : E \times E \to \mathbb{R}$ as

$$\phi(x, y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2. \tag{2-2}$$

This function can be seen as a 'distance-like' function, better conditioned than the square of the metric distance, namely $\|x - y\|^2$; see, for example, [1, 21, 22].

It is easy to see that

$$0 \leq (\|x\| - \|y\|)^2 \leq \phi(x, y) \tag{2-3}$$

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for all $x, y \in E$. In Hilbert spaces, where the duality mapping J is the identity operator, we have $\phi(x, y) = \|x - y\|^2$. In the following, we will need the following three properties of ϕ, proved in [21].

PROPOSITION 2.1. *Let E be a smooth and uniformly convex Banach space. Take two sequences $\{x^k\}, \{y^k\} \subset E$. If $\lim_{k\to\infty} \phi(x^k, y^k) = 0$ and either $\{x^k\}$ or $\{y^k\}$ is bounded, then $\lim_{k\to\infty} \|x^k - y^k\| = 0$.*

PROPOSITION 2.2. *Let E be a reflexive, strictly convex and smooth Banach space. Take a nonempty, closed and convex set $C \subset E$. Then, for all $x \in E$, there exists a unique $x_0 \in C$ such that*

$$\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\}.$$

We define $P_C : E \to C$ by taking $P_C(x)$ as the unique element $x_0 \in C$ given by Proposition 2.2. The projection P_C is called the *generalized projection onto* C. When E is a Hilbert space, P_C is just the metric projection onto C.

The third result taken from [21] is the following proposition.

PROPOSITION 2.3. *Consider a smooth Banach space E and a nonempty, closed and convex set $C \subset E$. Let $x \in E$ and $x_0 \in C$. Then $x_0 = P_C(x)$ if and only if*

$$\langle z - x_0, J(x) - J(x_0)\rangle \le 0$$

for all $z \in C$.

We also need the following result from [29].

PROPOSITION 2.4 [29]. *Suppose that f and g are convex, proper and lower semicontinuous functions on the Banach space E and that there is a point in $D(f) \cap D(g)$ where one of them is continuous. Then*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x), \quad x \in D(\partial f) \cap D(\partial g).$$

We continue this section with a notational comment: when $\{x^k\}$ is a sequence in E, we denote strong convergence of $\{x^k\}$ to $x \in E$ by $x^k \to x$ and weak convergence by $x^k \rightharpoonup x$. In the following definitions, suppose that $C \subset E$ is a nonempty, closed and convex set.

DEFINITION 2.5. We say that $T : C \to C$ is a quasi-ϕ-nonexpansive mapping whenever $\mathrm{Fix}(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $(p, x) \in \mathrm{Fix}(T) \times C$.

DEFINITION 2.6. Let $K(\cdot)$ be a multivalued mapping from C into itself such that for all $x \in C$, $K(x)$ is a nonempty, closed and convex subset of C. We say that $K(\cdot)$ is quasi-ϕ-nonexpansive whenever the mapping $T(\cdot) = P_{K(\cdot)}(\cdot)$ is quasi-ϕ-nonexpansive, where P is the generalized projection.

DEFINITION 2.7. The multivalued mapping $K(\cdot)$ from C into itself is called lower semicontinuous at each $\bar{x} \in C$ if, whenever we have $\{x^k\} \subset C$ and $x^k \to \bar{x}$, then for any $\bar{y} \in K(\bar{x})$, there is a sequence $\{y^k\}$ with $y^k \in K(x^k)$ for all k such that $y^k \to \bar{y}$ as $k \to \infty$.

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In the following, we give an example of a multivalued mapping which is lower semicontinuous at each $\bar{x} \in C$.

EXAMPLE 2.8. Define $K(\cdot) : C \to \mathcal{P}(C)$ by $K(x) = B(0, \|x\|)$, where $B(0, \|x\|)$ denotes the closed ball of radius $\|x\|$ centered at 0. Suppose that $x^k \to \bar{x}$ and $\bar{y} \in K(\bar{x})$. Then, if $\bar{x} = 0$, we have $\bar{y} = 0$ and hence we can choose $y^k = 0$ for all k, where $y^k \in K(x^k)$. If $\bar{x} \neq 0$ and $\bar{y} \in K(\bar{x})$, we choose $y^k = (\langle x^k, J\bar{x}\rangle / \|\bar{x}\|^2)\bar{y}$. It is easy to see that $y^k \in K(x^k)$. Therefore, in both cases we have $y^k \in K(x^k)$ and $y^k \to \bar{y}$.

DEFINITION 2.9. The multivalued mapping $K(\cdot)$ from C into itself is said to be demiclosed if, whenever $x^k \rightharpoonup \bar{x}$ and $\lim_{k\to\infty} d(x^k, K(x^k)) = 0$, then $\bar{x} \in \mathrm{Fix}(K)$.

Note that when T is a quasi-ϕ-nonexpansive mapping, it is well known that $\mathrm{Fix}(T)$ is convex. Also, if T is demiclosed, then $\mathrm{Fix}(T)$ is closed (see Remark 3.2).

Now we introduce some conditions on the bifunction f and the multivalued mapping K that we will need for the convergence analysis.

B1: $f(x, x) = 0$ for all $x \in E$.
B2: f is continuous on $E \times E$ and uniformly on bounded subsets of E with respect to the second argument, and f is bounded on bounded subsets of $E \times E$.
B3: $f(x, \cdot) : E \to \mathbb{R}$ is convex for all $x \in E$.
B4: $K(\cdot) : C \to \mathcal{P}(C)$ is a multivalued mapping with nonempty, closed and convex values, quasi-ϕ-nonexpansive, demiclosed and lower semicontinuous at each $x \in C$.

Regarding B4, we mention that some demiclosedness-like property is needed for convergence of all variants of approximation methods of fixed point problems. Also, lower semicontinuity of $K(\cdot)$ is used to show that the sequence generated by the algorithm SEML in Section 3 converges strongly to a solution of QEP (f, K). We mention also that for the sequences generated by our algorithm in Section 3 to be well defined and bounded, we will assume that

$$DS(f, K) := \{x \in K(x) : f(y, x) \leq 0, \quad \forall y \in C\} \neq \emptyset.$$

However, if the sequence $\{v^k\}$ generated by the algorithm SEML is well defined and bounded, then the above condition is automatically satisfied. It is easy to see that $DS(f, K) \subseteq MS(f, K) \subseteq S(f, K)$ (see Remark 3.3 in Section 3).

3. Extragradient method with linesearch and strong convergence

In this section, we study the strong convergence of the sequence generated by a *strongly convergent variant of the extragradient method with linesearch (SEML)* to approximate a solution of the quasi-equilibrium problem. We also propose a regularization procedure on the extragradient method which ensures the strong convergence of the generated sequence to a solution of QEP (f, K). We will assume in the following that E is uniformly smooth and uniformly convex, that $C \subseteq E$ is nonempty closed

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and convex, that $f : E \times E \to \mathbb{R}$ is a bifunction, that $K(\cdot) : C \to \mathcal{P}(C)$ is a multivalued quasi-ϕ-nonexpansive mapping and that the assumptions B1–B4 are satisfied. For the sake of definiteness and boundedness of the iterative sequence $\{v^k\}$ generated by the following algorithm, we assume that $DS(f, K) \neq \emptyset$. However, we will show later that if the sequence $\{v^k\}$ generated by the algorithm is bounded, then $S(f, K) \neq \emptyset$. First we give the formal definition of the algorithm SEML.

1. Initialization: Fix $v^0 \in E$ and consider a sequence $\gamma_k \in [\varepsilon, 1/2]$ for some $\varepsilon \in (0, 1/2]$ and $k = 0, 1, 2, \ldots$. Take $\delta, \theta \in (0, 1), \hat{\beta}, \tilde{\beta}$ satisfying $0 < \hat{\beta} \leq \tilde{\beta}$ and a sequence $\{\beta_k\} \subseteq [\hat{\beta}, \tilde{\beta}]$.

2. Iterative step: Given v^k, define

$$x^k = P_{K(v^k)}(v^k), \tag{3-1}$$

$$z^k \in \mathrm{Argmin}_{y \in K(v^k)} \Big\{ f(x^k, y) + \frac{1}{2\beta_k} \|y\|^2 - \frac{1}{\beta_k} \langle y, Jx^k \rangle \Big\}. \tag{3-2}$$

If $z^k = v^k$, stop. Otherwise, let

$$\ell(k) = \min \Big\{ \ell \geq 0 : \beta_k f(y^\ell, x^k) - \beta_k f(y^\ell, z^k) \geq \frac{\delta}{2} \phi(z^k, x^k) \Big\}, \tag{3-3}$$

with

$$y^\ell = \theta^\ell z^k + (1 - \theta^\ell) x^k. \tag{3-4}$$

Set

$$\alpha_k := \theta^{\ell(k)}, \tag{3-5}$$

$$y^k := y^{\ell(k)} = \alpha_k z^k + (1 - \alpha_k) x^k. \tag{3-6}$$

Take $h^k \in \partial f(y^k, \cdot)(x^k)$ and define

$$H_k = \{y \in E : \langle y - x^k, h^k \rangle + f(y^k, x^k) \leq 0\}. \tag{3-7}$$

If $k = 0$, set $C_0 = C \cap H_0$. Otherwise, let

$$C_k = C_{k-1} \cap H_k, \tag{3-8}$$

$$w^k = P_{C_k}(x^k). \tag{3-9}$$

Determine the next approximation v^{k+1} as

$$v^{k+1} = P_{L_k \cap M_k \cap N_k}(v^0), \tag{3-10}$$

where

$$L_k = \{z \in E : \langle z - x^k, Jx^k - Jw^k \rangle \leq -\gamma_k \phi(x^k, w^k)\}, \tag{3-11}$$

$$M_k = \{z \in E : \langle z - v^k, Jv^k - Jx^k \rangle \leq -\gamma_k \phi(v^k, x^k)\}, \tag{3-12}$$

$$N_k = \{z \in E : \langle z - v^k, Jv^0 - Jv^k \rangle \leq 0\}. \tag{3-13}$$

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The above backtracking procedure for determining the right α_k is sometimes called an Armijo-type search (see [2]). It has been analyzed for VIP(T, C) in [23] and [20].

REMARK 3.1. In the above algorithm, it is worth mentioning that if we remove the steps (3-10)–(3-13) and replace C_k and w^k in (3-9), respectively, by $C_k \cap M_k$ and v^{k+1}, then with additional conditions such as the weak upper semicontinuity of f with respect to the first argument, the weak continuity of the duality mapping J and the weak lower semicontinuity of the multivalued mapping K at each $\bar{x} \in C$, we can get the weak convergence of the generated sequences $\{v^k\}$ and $\{x^k\}$ to a solution of the problem. However, since these are rather strong conditions to be imposed on the problem, we preferred to avoid this approach.

We proceed now to the convergence analysis of the algorithm SEML. We first give our strong convergence result for the algorithm SEML. The proof of the main theorem is divided to several lemmas and propositions. In order to establish the strong convergence of the sequence generated by the algorithm SEML, we need some intermediate results.

THEOREM 3.1. *Assume that E is a uniformly convex and uniformly smooth Banach space, f is a bifunction, $K(\cdot)$ is a multivalued quasi-ϕ-nonexpansive mapping and the assumptions B1–B4 are satisfied.*

(i) *If $DS(f, K) \neq \emptyset$, then the sequence $\{v^k\}$ generated by the algorithm SEML is well defined and bounded.*
(ii) *If the sequence $\{v^k\}$ is well defined and bounded, then the sequences $\{v^k\}$ and $\{x^k\}$ generated by the algorithm both converge strongly to an element of $S(f, K)$, which is therefore nonempty.*

We will give the proof of Theorem 3.1 at the end of this section, after proving the intermediary steps needed for the proof.

PROPOSITION 3.2. *Assume that f satisfies B1–B3. Take a closed and convex set $C \subseteq E$, $x \in E, \beta \in \mathbb{R}^+$. If*

$$z \in \operatorname{Argmin}_{y\in C}\Big\{f(x,y) + \frac{1}{2\beta}\|y\|^2 - \frac{1}{\beta}\langle y, Jx\rangle\Big\}, \tag{3-14}$$

then $\langle y - z, Jx - Jz\rangle \leq \beta[f(x,y) - f(x,z)]$ for all $y \in C$.

PROOF. Let $N_C(z)$ be the normal cone of C at $z \in C$, that is,

$$N_C(z) = \{v \in E^* : \langle y - z, v\rangle \leq 0, \quad \forall y \in C\}.$$

Since the minimand of (3-14) is convex by B3, and C is closed and convex, z satisfies the first-order optimality condition, given by

$$0 \in \partial\Big\{f(x,\cdot) + \frac{1}{2\beta}\|\cdot\|^2 - \frac{1}{\beta}\langle\cdot, Jx\rangle\Big\}(z) + N_C(z).$$

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Thus, in view of the definition of J and by Proposition 2.4, there exist $w \in \partial f(x,\cdot)(z)$ and $\bar{w} \in N_C(z)$ such that

$$0 = w + \frac{1}{\beta}Jz - \frac{1}{\beta}Jx + \bar{w}.$$

Therefore, since $\bar{w} \in N_C(z)$, we have $\langle y - z, -w - (1/\beta)Jz + (1/\beta)Jx\rangle \le 0$, so that, using the fact that $w \in \partial f(x,\cdot)(z)$,

$$\frac{1}{\beta}\langle y - z, Jx - Jz\rangle \le \langle y - z, w\rangle \le f(x, y) - f(x, z). \tag{3-15}$$

COROLLARY 3.3. *Assume that f satisfies B1–B3. Let $\{x^k\}$ and $\{z^k\}$ be the sequences generated by the algorithm SEML. Then $\langle y - z^k, Jx^k - Jz^k\rangle \le \beta_k f(x^k, y) - \beta_k f(x^k, z^k)$ for all $y \in K(v^k)$.*

PROOF. Follows from Proposition 3.2 and (3-2).

PROPOSITION 3.4. *Assume that f satisfies B1–B3. If the algorithm SEML stops at the kth iteration, then x^k is a solution of QEP(f, K).*

PROOF. If the algorithm stops at the kth iteration, then $z^k = v^k$ and hence $z^k = x^k$ by (3-1). Now the result follows from Corollary 3.3.

PROPOSITION 3.5. *Assume that f satisfies B1–B3. The following statements hold for the algorithm SEML.*

(i) *$\ell(k)$ is well defined, (that is, the linesearch for α_k is finite) and consequently the same holds for the sequence $\{y^k\}$.*
(ii) *If $x^k \ne z^k$, then $f(y^k, x^k) > 0$.*

PROOF. (i) If $x^k = z^k$, then $\ell(k) = 0$; therefore, let $x^k \ne z^k$. Assume by contradiction that

$$\beta_k[f(y^\ell, x^k) - f(y^\ell, z^k)] < \frac{\delta}{2}\phi(z^k, x^k) \tag{3-16}$$

for all ℓ. Note that the sequence $\{y^\ell\}$ is strongly convergent to x^k. In view of B2, taking limits in (3-16) as $\ell \to \infty$,

$$\beta_k[f(x^k, x^k) - f(x^k, z^k)] \le \frac{\delta}{2}\phi(z^k, x^k). \tag{3-17}$$

Since $x^k \in K(v^k)$ by (3-1), we apply Corollary 3.3 with $y = x^k$ in (3-17), obtaining

$$\langle x^k - z^k, Jx^k - Jz^k\rangle \le \frac{\delta}{2}\phi(z^k, x^k). \tag{3-18}$$

In view of the definition of ϕ, (3-18) implies that

$$\phi(z^k, x^k) + \phi(x^k, z^k) \le \delta\phi(z^k, x^k). \tag{3-19}$$

Since $\delta \in (0, 1)$, we get $\phi(x^k, z^k) < 0$, contradicting the nonnegativity of ϕ.

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(ii) Assume that $f(y^k, x^k) \leq 0$. Note that, using B1, B3 and (3-6),

$$0 = f(y^k, y^k) \leq \alpha_k f(y^k, z^k) + (1-\alpha_k) f(y^k, x^k).$$

Hence, $f(y^k, z^k) \geq 0$. On the other hand, by (3-3)–(3-6),

$$f(y^k, x^k) \geq f(y^k, z^k) + \frac{\delta}{2\beta_k}\phi(z^k, x^k) > f(y^k, z^k) \geq 0, \tag{3-20}$$

in contradiction with the assumption. Note that the strict inequality in (3-20) is due to the fact that $x^k \neq z^k$.

In order to prove the convergence theorem of the sequence generated by the algorithm, we need the following remarks and lemmas.

Remark 3.2. If $K : C \to \mathcal{P}(C)$ is a multivalued mapping satisfying B4, then $\mathrm{Fix}(K)$ is closed and convex.

Proof. Let $p_1, p_2 \in \mathrm{Fix}(K)$ and define $p_t = tp_1 + (1-t)p_2$, where $t \in [0,1]$. In order to prove the convexity of $\mathrm{Fix}(K)$, we must show that $p_t \in K(p_t)$. Let $Tp_t := P_{K(p_t)}(p_t)$, where P is the generalized projection. Note that by (2-2) and (2-3),

$$\begin{aligned}
0 \leq \phi(p_t, Tp_t) &= \|p_t\|^2 - 2\langle p_t, J(Tp_t)\rangle + \|Tp_t\|^2 \\
&= \|p_t\|^2 - t\|p_1\|^2 - (1-t)\|p_2\|^2 + t(\|p_1\|^2 - 2\langle p_1, J(Tp_t)\rangle + \|Tp_t\|^2) \\
&\quad + (1-t)(\|p_2\|^2 - 2\langle p_2, J(Tp_t)\rangle + \|Tp_t\|^2) \\
&= \|p_t\|^2 - t\|p_1\|^2 - (1-t)\|p_2\|^2 + t\phi(p_1, Tp_t) + (1-t)\phi(p_2, Tp_t) \\
&\leq \|p_t\|^2 - t\|p_1\|^2 - (1-t)\|p_2\|^2 + t\phi(p_1, p_t) + (1-t)\phi(p_2, p_t) \\
&= \|p_t\|^2 - 2t\langle p_1, Jp_t\rangle - 2(1-t)\langle p_2, Jp_t\rangle + \|p_t\|^2 = \phi(p_t, p_t) = 0.
\end{aligned}$$

Therefore, $\phi(p_t, Tp_t) = 0$. Now Proposition 2.1 shows that $Tp_t = p_t$. Since $Tp_t = P_{K(p_t)}(p_t)$, $p_t \in K(p_t)$, that is, $\mathrm{Fix}(K)$ is convex.

Now we show that $\mathrm{Fix}(K)$ is closed. Let $\{p^k\} \subset \mathrm{Fix}(K)$ be such that $p^k \to p$. Since we have $p^k \in K(p^k)$, then $\lim_{k\to\infty} d(p^k, K(p^k)) = 0$. Now the demiclosedness of K implies that $p \in \mathrm{Fix}(K)$, that is, $\mathrm{Fix}(K)$ is closed.

Remark 3.3. Assume that f and K satisfy B1–B4. Then $DS(f, K) \subseteq S(f, K)$. Also, $DS(f, K)$ is closed and convex.

Proof. Let $x^* \in DS(f, K)$ and suppose that $y \in K(x^*)$ is arbitrary. Define $p_t = tx^* + (1-t)y$, where $t \in [0, 1)$. Note that by B1 and B3,

$$0 = f(p_t, p_t) \leq tf(p_t, x^*) + (1-t)f(p_t, y). \tag{3-21}$$

Since $f(p_t, x^*) \leq 0$, (3-21) implies that $f(p_t, y) \geq 0$. Now, by using B2 and taking the limit as $t \to 1^-$, we get $f(x^*, y) \geq 0$. Since $y \in K(x^*)$ is arbitrary, then $x^* \in S(f, K)$. Finally, it follows from B2 and B3 that $DS(f, K)$ is closed and convex.

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LEMMA 3.6. *If $DS(f,K) \neq \emptyset$, then $DS(f,K) \subseteq L_k \cap M_k \cap N_k$. Therefore, the sequences $\{v^k\}$, $\{w^k\}$ and $\{x^k\}$ are well defined.*

PROOF. The proof is by induction. Note that $DS(f,K)$, L_k, M_k and N_k are closed and convex. We first show that $DS(f,K) \subseteq L_k \cap M_k \cap N_k$ for all $k \geq 0$. Let

$$D_k = \{z \in E : \phi(z, w^k) \leq \phi(z, x^k)\} = \{z \in E : \langle z - x^k, Jx^k - Jw^k\rangle \leq -\tfrac{1}{2}\phi(x^k, w^k)\}$$

and

$$F_k = \{z \in E : \phi(z, x^k) \leq \phi(z, v^k)\} = \{z \in E : \langle z - v^k, Jv^k - Jx^k\rangle \leq -\tfrac{1}{2}\phi(v^k, x^k)\}.$$

By $\gamma_k \in [\varepsilon, (1/2)]$, $D_k \subseteq L_k$ and $F_k \subseteq M_k$. Let $x^* \in DS(f,K)$ and note that $x^* \in H_k$ for all k; we also have $w^k = P_{C_k}(x^k)$ by (3-9). Now Proposition 2.3 implies that

$$\langle x^* - w^k, Jx^k - Jw^k\rangle \leq 0$$

or, equivalently,

$$\phi(w^k, x^k) + \phi(x^*, w^k) - \phi(x^*, x^k) \leq 0. \tag{3-22}$$

Therefore,

$$\phi(x^*, w^k) \leq \phi(x^*, x^k), \tag{3-23}$$

which implies that $DS(f,K) \subseteq D_k$ for all $k \geq 0$.

On the other hand, since $x^k = P_{K(v^k)}(v^k)$ and $P_{K(\cdot)}(\cdot)$ is a quasi-ϕ-nonexpansive mapping, that is, $K(\cdot)$ is a quasi-ϕ-nonexpansive mapping,

$$\phi(x^*, x^k) \leq \phi(x^*, v^k) \tag{3-24}$$

for all $x^* \in DS(f,K)$. Therefore, $DS(f,K) \subseteq D_k \cap F_k$ for all $k \geq 0$, which implies that $DS(f,K) \subseteq L_k \cap M_k$ for all $k \geq 0$. Next, by induction, we show that $DS(f,K) \subseteq L_k \cap M_k \cap N_k$ for all $k \geq 0$. Indeed, we have $DS(f,K) \subseteq L_0 \cap M_0 \cap N_0$, because $N_0 = E$. Assume that $DS(f,K) \subseteq L_k \cap M_k \cap N_k$ for some $k \geq 0$. Since $v^{k+1} = P_{L_k \cap M_k \cap N_k}(v^0)$, we have by Proposition 2.3 that

$$\langle z - v^{k+1}, Jv^0 - Jv^{k+1}\rangle \leq 0, \quad \forall z \in L_k \cap M_k \cap N_k.$$

Since $DS(f,K) \subseteq L_k \cap M_k \cap N_k$,

$$\langle z - v^{k+1}, Jv^0 - Jv^{k+1}\rangle \leq 0, \quad \forall z \in DS(f,K).$$

Now, since $\langle z - v^{k+1}, Jv^0 - Jv^{k+1}\rangle \leq 0, \ \forall z \in DS(f,K)$, the definition of N_{k+1} implies that $DS(f,K) \subseteq N_{k+1}$ and so $DS(f,K) \subseteq L_k \cap M_k \cap N_k$ for all $k \geq 0$. Finally, since $DS(f,K)$ is nonempty, $L_k \cap M_k \cap N_k$ is also nonempty and v^{k+1} is well defined. Now it is clear that the sequences $\{x^k\}$ and $\{w^k\}$ are well defined.

LEMMA 3.7. *If $DS(f,K) \neq \emptyset$, then the sequence $\{v^k\}$ generated by the algorithm SEML is bounded.*

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PROOF. From the definition of N_k, we have $v^k = P_{N_k}(v^0)$. For each $u \in DS(f, K) \subseteq N_k$, since P_{N_k} is the generalized projection onto N_k, we have $\langle u - v^k, Jv^0 - Jv^k \rangle \leq 0$ by Proposition 2.3; this implies that

$$\phi(v^k, v^0) \leq \phi(u, v^0). \tag{3-25}$$

Thus, the sequence $\{v^k\}$ is bounded by (2-3).

LEMMA 3.8. *Suppose that $\{v^k\}$, $\{w^k\}$ and $\{x^k\}$ are the sequences generated by the algorithm SEML. If $\{v^k\}$ is bounded, then the sequences $\{w^k\}$ and $\{x^k\}$ are bounded and*

$$\lim_{k\to\infty} \|v^{k+1} - v^k\| = \lim_{k\to\infty} \|v^k - x^k\| = \lim_{k\to\infty} \|x^k - w^k\| = 0.$$

PROOF. The definition of v^{k+1} implies that $v^{k+1} \in N_k$. Therefore, we have $\langle v^{k+1} - v^k, Jv^0 - Jv^k \rangle \leq 0$ by Proposition 2.3, which implies that

$$\phi(v^k, v^0) + \phi(v^{k+1}, v^k) - \phi(v^{k+1}, v^0) \leq 0.$$

Hence, $\phi(v^k, v^0) \leq \phi(v^{k+1}, v^0)$. So, the sequence $\{\phi(v^k, v^0)\}$ is nondecreasing. On the other hand, $\{v^k\}$ is bounded by the assumptions. Therefore, $\lim_{k\to\infty} \phi(v^k, v^0)$ exists. We also have

$$\phi(v^{k+1}, v^k) \leq \phi(v^{k+1}, v^0) - \phi(v^k, v^0).$$

Passing to the limit in the above inequality as $k \to \infty$,

$$\lim_{k\to\infty} \phi(v^{k+1}, v^k) = 0.$$

Now, by Proposition 2.1,

$$\lim_{k\to\infty} \|v^{k+1} - v^k\| = 0. \tag{3-26}$$

Since $v^{k+1} \in M_k$, from the definition of M_k,

$$\gamma_k \phi(v^k, x^k) \leq \langle v^k - v^{k+1}, Jv^k - Jx^k \rangle. \tag{3-27}$$

Therefore, by (2-3) and the Cauchy–Schwarz inequality,

$$\gamma_k (\|v^k\| - \|x^k\|)^2 \leq \|v^k - v^{k+1}\| \, \|Jv^k - Jx^k\| \leq \|v^k - v^{k+1}\| (\|v^k\| + \|x^k\|). \tag{3-28}$$

Now, regarding boundedness of $\{v^k\}$, $\gamma_k \geq \varepsilon > 0$ and $\lim_{k\to\infty} \|v^{k+1} - v^k\| = 0$ by (3-26), it is easy to see that $\{x^k\}$ is bounded. Also, by (3-27), we have

$$\gamma_k \phi(v^k, x^k) \leq \|v^k - v^{k+1}\| \, \|Jv^k - Jx^k\|.$$

Since $\{x^k\}$ and $\{v^k\}$ are bounded, $\lim_{k\to\infty} \|v^{k+1} - v^k\| = 0$ and $\gamma_k \geq \varepsilon > 0$, we have $\lim_{k\to\infty} \phi(v^k, x^k) = 0$. Therefore, Proposition 2.1 implies that

$$\lim_{k\to\infty} \|v^k - x^k\| = 0. \tag{3-29}$$

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In the following, note that

$$\|v^{k+1} - x^k\| \le \|v^{k+1} - v^k\| + \|v^k - x^k\|$$

and hence, by (3-26) and (3-29),

$$\lim_{k\to\infty} \|v^{k+1} - x^k\| = 0. \tag{3-30}$$

On the other hand, since $v^{k+1} \in L_k$, from the definition of L_k,

$$\gamma_k \phi(x^k, w^k) \le \langle x^k - v^{k+1}, Jx^k - Jw^k \rangle. \tag{3-31}$$

Again, by (2-3) and the Cauchy–Schwarz inequality,

$$\gamma_k(\|x^k\| - \|w^k\|)^2 \le \|x^k - v^{k+1}\| \, \|Jx^k - Jw^k\| \le \|x^k - v^{k+1}\|(\|x^k\| + \|w^k\|). \tag{3-32}$$

Once again, due to the boundedness of $\{x^k\}$, $\gamma_k \ge \varepsilon > 0$ and $\lim_{k\to\infty} \|v^{k+1} - x^k\| = 0$ by (3-30), we can obtain from (3-32) that $\{w^k\}$ is bounded. Also, by (3-31), we have $\gamma_k \phi(x^k, w^k) \le \|x^k - v^{k+1}\| \, \|Jx^k - Jw^k\|$. Since $\{x^k\}$ and $\{w^k\}$ are bounded,

$$\lim_{k\to\infty} \|v^{k+1} - x^k\| = 0 \quad \text{and} \quad \gamma_k \ge \varepsilon > 0,$$

we have $\lim_{k\to\infty} \phi(x^k, w^k) = 0$. Again, Proposition 2.1 implies that

$$\lim_{k\to\infty} \|x^k - w^k\| = 0.$$

Proposition 3.9. *Assume that E is uniformly convex and uniformly smooth, f is a bifunction, $K(\cdot)$ is a multivalued quasi-ϕ-nonexpansive mapping and the assumptions B1–B4 are satisfied.*

(i) *If there exists a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that $x^{k_n} \rightharpoonup p$, then $p \in C_\infty \cap \mathrm{Fix}(K)$, where $C_\infty = \cap_{k=0}^{\infty} C_k$.*

(ii) *$C_\infty \cap \mathrm{Fix}(K) \subseteq L_k \cap M_k \cap N_k$ for all k.*

Proof. (i) We first prove that $p \in \mathrm{Fix}(K)$. Note that we have $\lim_{n\to\infty} \|v^{k_n} - x^{k_n}\| = 0$ by Lemma 3.8, where, for each n, x^{k_n} is the generalized projection of v^{k_n} onto $K(v^{k_n})$. Therefore, we have $\lim_{n\to\infty} d(v^{k_n}, K(v^{k_n})) = 0$. Now, since K is demiclosed, we obtain $p \in K(p)$, that is, p is a fixed point of $K(\cdot)$. Now we prove that $p \in C_\infty$. Since $C_\infty = \cap_{k=0}^{\infty} C_k$, it is sufficient to prove that $p \in C_k$ for all integers k to obtain that $p \in C_\infty$. Note that the sequence $\{C_k\}$ is nonincreasing. Now let N be a fixed integer. Then there is $j > N$ such that for all $n \ge j$,

$$w^{k_n} \in C_{k_n} \subseteq C_N, \quad \forall n \ge j,$$

where $w^{k_n} = P_{C_{k_n}}(x^{k_n})$. Now, since $\lim_{n\to\infty} \|w^{k_n} - x^{k_n}\| = 0$ by Lemma 3.8, we have $w^{k_n} \rightharpoonup p$. Consequently, the set C_N is closed and $p \in C_N$. Since N is arbitrary,

$$p \in \cap_{k=0}^{\infty} C_k = C_\infty.$$

(ii) The proof is similar to the proof of Lemma 3.6. It suffices to replace $DS(f, K)$ by $C_\infty \cap \mathrm{Fix}(K)$.

REMARK 3.4. It is easy to see that $DS(f, K) \subseteq C_\infty \cap \mathrm{Fix}(K)$. Also, since K is demiclosed and quasi-ϕ-nonexpansive, the set $C_\infty \cap \mathrm{Fix}(K)$ is closed and convex.

REMARK 3.5. To the best of our knowledge, previous authors who investigated equilibrium problems first showed the weak convergence of the sequence generated by their algorithms. However, to show that this limit is a solution to the problem, they had to assume some strong conditions, such as, for example, the weak upper semicontinuity of the bifunction f with respect to its first argument. We note that in our algorithm, we will prove the strong convergence of our scheme to a solution of our quasi-equilibrium problem without assuming such strong conditions.

In the following proposition, we prove that the sequences $\{v^k\}$ and $\{x^k\}$ generated by the algorithm SEML converge strongly to an element of $C_\infty \cap \mathrm{Fix}(K)$.

PROPOSITION 3.10. *Assume that E is a uniformly convex and uniformly smooth Banach space, f is a bifunction, $K(\cdot)$ is a multivalued quasi-ϕ-nonexpansive mapping and the assumptions B1–B4 are satisfied. If the sequence $\{v^k\}$ generated by the algorithm SEML is bounded, then the sequences $\{v^k\}$ and $\{x^k\}$ are strongly convergent to an element of $C_\infty \cap \mathrm{Fix}(K)$.*

PROOF. If the sequence $\{v^k\}$ is bounded, then the sequence $\{x^k\}$ is bounded by Lemma 3.8. Assume that p is any weak limit point of the sequence $\{x^k\}$. Then there exists a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that $x^{k_n} \rightharpoonup p$ as $n \to \infty$. Note that Proposition 3.9 shows that $p \in C_\infty \cap \mathrm{Fix}(K)$ and hence $C_\infty \cap \mathrm{Fix}(K) \neq \emptyset$. It is clear that $C_\infty \cap \mathrm{Fix}(K)$ is closed and convex by Remark 3.4 and hence we can define $\bar{x} = P_{C_\infty \cap \mathrm{Fix}(K)}(v^0)$, where P is the generalized projection map onto $C_\infty \cap \mathrm{Fix}(K)$. In the following, we first prove the weak convergence of the sequence $\{x^k\}$. Finally, we show that $x^k \to \bar{x} = P_{C_\infty \cap \mathrm{Fix}(K)}(v^0)$. Note that $v^{k_n} \rightharpoonup p$ by Lemma 3.8. From the definition of N_k, we have $v^k = P_{N_k}(v^0)$. Since $C_\infty \cap \mathrm{Fix}(K) \subseteq N_k$ by Proposition 3.9(ii), and P_{N_k} is the generalized projection map onto N_k and hence for $\bar{x} \in C_\infty \cap \mathrm{Fix}(K) \subseteq N_k$, we have $\langle \bar{x} - v^k, Jv^0 - Jv^k \rangle \leq 0$ by Proposition 2.3. This implies that $\phi(v^k, v^0) \leq \phi(\bar{x}, v^0)$. Therefore,

$$\|v^k\|^2 - 2\langle v^k, Jv^0 \rangle + \|v^0\|^2 \leq \phi(\bar{x}, v^0). \tag{3-33}$$

Note that since $v^{k_n} \rightharpoonup p$, by the weak lower semicontinuity of the norm $\|\cdot\|$ and replacing k by k_n in (3-33),

$$\phi(p, v^0) = \|p\|^2 - 2\langle p, Jv^0 \rangle + \|v^0\|^2 \leq \liminf_{n\to\infty}(\|v^{k_n}\|^2 - 2\langle v^{k_n}, Jv^0 \rangle + \|v^0\|^2) \leq \phi(\bar{x}, v^0).$$

By the definition of $\bar{x}$ and $p \in C_\infty \cap \mathrm{Fix}(K)$, we have $\bar{x} = p$, that is, $x^{k_n} \rightharpoonup \bar{x}$. Hence, every weakly convergent subsequence of $\{x^k\}$ converges weakly to $\bar{x}$. This shows that $x^k \rightharpoonup \bar{x}$ and therefore $v^k \rightharpoonup \bar{x}$. Taking the liminf in (3-33), we get $\lim_{k\to\infty} \|v^k\| = \|\bar{x}\|$. Now note that

$$\lim_{n\to\infty} \phi(v^k, \bar{x}) = \lim_{n\to\infty} (\|v^k\|^2 - 2\langle v^k, J\bar{x} \rangle + \|\bar{x}\|^2) = 0.$$

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Therefore, by Proposition 2.1, we have $v^k \to \bar{x} = P_{C_\infty \cap \mathrm{Fix}(K)}(v^0)$. Now, since $\lim_{k\to\infty} \|v^k - x^k\| = 0$ by Lemma 3.8, we get $x^k \to \bar{x} = P_{C_\infty \cap \mathrm{Fix}(K)}(v^0)$.

PROPOSITION 3.11. *Assume that E is a uniformly convex and uniformly smooth Banach space, f is a bifunction, $K(\cdot)$ is a multivalued quasi-ϕ-nonexpansive mapping and the assumptions B1–B4 are satisfied. Let $\{x^k\}$, $\{y^k\}$, $\{z^k\}$ and $\{v^k\}$ be the sequences generated by the algorithm SEML. If the sequence $\{v^k\}$ is bounded and the algorithm does not have finite termination, then: (i) the sequence $\{z^k\}$ is bounded; (ii) $\lim_{k\to\infty} f(y^k, x^k) = 0$.*

PROOF. (i) We get from (3-2),

$$\beta_k f(x^k, z^k) + \tfrac{1}{2}\|z^k\|^2 - \langle z^k, Jx^k\rangle \le \beta_k f(x^k, x^k) + \tfrac{1}{2}\|x^k\|^2 - \langle x^k, Jx^k\rangle = \tfrac{-1}{2}\|x^k\|^2 \le 0, \tag{3-34}$$

using B1. From (3-34),

$$\|z^k\|^2 \le -2\beta_k f(x^k, z^k) + 2\langle z^k, Jx^k\rangle \le -2\beta_k f(x^k, z^k) + 2\|z^k\|\,\|x^k\|. \tag{3-35}$$

Take now $u^k \in \partial f(x^k, \cdot)(x^k)$. By definition of the subdifferential of $f(x^k, \cdot)$ evaluated at x^k,

$$\langle y - x^k, u^k\rangle \le f(x^k, y) - f(x^k, x^k) = f(x^k, y), \qquad \forall y \in E. \tag{3-36}$$

Let $B_1(x^k)$ be the closed ball of radius one centered at x^k. Since f is bounded on bounded sets and $x^k \to \bar{x}$ by Proposition 3.10, there is $M > 0$ such that $f(x^k, y) < M$ for all k and for all $y \in B_1(x^k)$. Then

$$\|u^k\| = \sup_{y\in B_1(x^k)} \langle y - x^k, u^k\rangle \le \sup_{y\in B_1(x^k)} f(x^k, y) \le M. \tag{3-37}$$

Therefore, $\{u^k\}$ is bounded. Now, from (3-36),

$$\langle z^k - x^k, u^k\rangle \le f(x^k, z^k) - f(x^k, x^k) = f(x^k, z^k). \tag{3-38}$$

Combining (3-35) and (3-38),

$$\|z^k\|^2 \le 2\beta_k\langle x^k - z^k, u^k\rangle + 2\|z^k\|\,\|x^k\| \le 2\tilde{\beta}\|u^k\|(\|z^k\| + \|x^k\|) + 2\|z^k\|\,\|x^k\|. \tag{3-39}$$

This implies the following inequality:

$$\|z^k\| \le 4\tilde{\beta}\|u^k\| + 2\|x^k\|. \tag{3-40}$$

In fact, (3-40) is obvious when $\|z^k\| \le \|x^k\|$ and it follows easily from (3-39) when $\|x^k\| \le \|z^k\|$.

Now, since the sequences $\{x^k\}$ and $\{u^k\}$ are bounded, the boundedness of $\{z^k\}$ follows from (3-40).

(ii) Since $h^k \in \partial f(y^k, \cdot)(x^k)$ by (3-7), we have by definition of the subdifferential of $f(y^k, \cdot)$ evaluated at x^k,

$$\langle y - x^k, h^k\rangle \le f(y^k, y) - f(y^k, x^k), \qquad \forall y \in E. \tag{3-41}$$

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Note that the sequences $\{x^k\}$ and $\{z^k\}$ are bounded by Proposition 3.10 and part (i), respectively; hence, the sequence $\{y^k\}$ is bounded too by (3-6). Let $B_1(x^k)$ be the closed ball of radius one centered at x^k. Now, since f is bounded on bounded sets, there is $M > 0$ such that $f(y^k, x^k) < M$ and $f(y^k, y) < M$ for all k and for all $y \in B_1(x^k)$. Then

$$\|h^k\| = \sup_{y \in B_1(x^k)} \langle y - x^k, h^k \rangle \leq \sup_{y \in B_1(x^k)} f(y^k, y) - f(y^k, x^k) \leq 2M. \tag{3-42}$$

Therefore, $\{h^k\}$ is bounded. On the other hand, since $w^k \in H_k$ by (3-7)–(3-9),

$$\langle w^k - x^k, h^k \rangle + f(y^k, x^k) \leq 0, \tag{3-43}$$

by (3-7). Also, by Lemma 3.8, we have $\lim_{k\to\infty} \|w^k - x^k\| = 0$. Using the Cauchy–Schwarz inequality,

$$-\|w^k - x^k\| \, \|h^k\| + f(y^k, x^k) \leq 0. \tag{3-44}$$

It follows from (3-44) that

$$\limsup_{k\to\infty} f(y^k, x^k) \leq 0. \tag{3-45}$$

Note that if the algorithm stops at iteration k, then x^k is a solution of QEP (f, K) by Proposition 3.4 and this case is ruled out in the current proposition. If the algorithm does not stop at any iteration, we consider separately the cases of $x^k = z^k$ for some k, and $x^k \neq z^k$ for all k. If $x^k = z^k$ for some k, then we have $x^k = y^k = z^k$ by (3-6); therefore, $f(y^k, x^k) = 0$ by B1. Now, if $x^k \neq z^k$ for all k, then Proposition 3.5(ii) implies that $f(y^k, x^k) > 0$. Therefore, in both cases, when the algorithm does not stop at any iteration,

$$f(y^k, x^k) \geq 0 \tag{3-46}$$

for all k. Now (3-45) and (3-46) imply that

$$\lim_{k\to\infty} f(y^k, x^k) = 0.$$

PROPOSITION 3.12. *Assume that E is uniformly convex and uniformly smooth, f is a bifunction, $K(\cdot)$ is a multivalued quasi-ϕ-nonexpansive mapping and the assumptions B1–B4 are satisfied. Let $\{x^k\}$ and $\{z^k\}$ be the sequences generated by the algorithm SEML. If $\{x^{k_i}\}$ is a subsequence of $\{x^k\}$ satisfying*

$$\lim_{i\to\infty} \phi(z^{k_i}, x^{k_i}) = 0, \tag{3-47}$$

then the sequence $\{x^k\}$ is strongly convergent to a solution of QEP (f, K).

PROOF. Since $x^k \to \bar{x} = P_{C_\infty \cap \mathrm{Fix}(K)}(v^0)$ by Proposition 3.10, it is enough to show that $\bar{x} \in S(f, K)$. From Proposition 2.1 and (3-47),

$$\lim_{i\to\infty} \|z^{k_i} - x^{k_i}\| = 0. \tag{3-48}$$

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Now continuity of f on $E \times E$ and (3-48) imply that

$$\lim_{k\to\infty} f(x^{k_i}, z^{k_i}) = 0. \tag{3-49}$$

Also, uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, we get from (3-48),

$$\lim_{i\to\infty} \|Jz^{k_i} - Jx^{k_i}\| = 0. \tag{3-50}$$

On the other hand, since $x^{k_i} \to \bar{x}$, we have $v^{k_i} \to \bar{x}$ by Lemma 3.8. Now take any $y \in K(\bar{x})$; since K is lower semicontinuous at $\bar{x} \in C$, there is a sequence $\{\tilde{y}^{k_i}\}$ such that $\tilde{y}^{k_i} \in K(v^{k_i})$ and $\tilde{y}^{k_i} \to y$. By Corollary 3.3,

$$\langle \tilde{y}^{k_i} - z^{k_i}, Jx^{k_i} - Jz^{k_i}\rangle \leq \beta_{k_i}[f(x^{k_i}, \tilde{y}^{k_i}) - f(x^{k_i}, z^{k_i})],$$

which implies that

$$-\|\tilde{y}^{k_i} - z^{k_i}\|\,\|Jx^{k_i} - Jz^{k_i}\| \leq \beta_{k_i}[f(x^{k_i}, \tilde{y}^{k_i}) - f(x^{k_i}, z^{k_i})]. \tag{3-51}$$

Now, taking the liminf in (3-51), we use (3-50) and (3-49), together with the boundedness of $\{z^k\}$ and $\{\tilde{y}^{k_i}\}$, in order to obtain that $\liminf_{i\to\infty} f(x^{k_i}, \tilde{y}^{k_i}) \geq 0$. Then B2 implies that $f(\bar{x}, y) \geq 0$. Since $y \in K(\bar{x})$ is arbitrary, we conclude that $\bar{x} \in S(f, K)$.

PROPOSITION 3.13. *Assume that E is a uniformly convex and uniformly smooth Banach space, f is a bifunction, $K(\cdot)$ is a multivalued quasi-ϕ-nonexpansive mapping and the assumptions B1–B4 are satisfied. If a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_k\}$ as defined in (3-5) converges to* 0, *then the sequence $\{x^k\}$ is strongly convergent to a solution of QEP (f, K).*

PROOF. For proving the result, we will use Proposition 3.12. Thus, we must show that

$$\lim_{i\to\infty} \phi(z^{k_i}, x^{k_i}) = 0.$$

For the sake of contradiction, and without loss of generality, let us assume that

$$\mathrm{liminf}_{i\to\infty}\phi(z^{k_i}, x^{k_i}) \geq \eta > 0, \tag{3-52}$$

taking into account the nonnegativity of $\phi(\cdot,\cdot)$. Define

$$\hat{y}^i = \frac{\alpha_{k_i}}{\theta} z^{k_i} + \left(1 - \frac{\alpha_{k_i}}{\theta}\right) x^{k_i}, \tag{3-53}$$

where $\alpha_{k_i} = \theta^{\ell(k_i)}$ by (3-5). Therefore,

$$\hat{y}^i - x^{k_i} = \frac{\alpha_{k_i}}{\theta}(z^{k_i} - x^{k_i}). \tag{3-54}$$

Note that, since $\lim_{i\to\infty} \alpha_{k_i} = 0$, $\ell(k_i) > 1$ for large enough i. Also, in view of (3-53), we have that $\hat{y}^i = y^{\ell(k_i)-1}$ in the inner loop of the linesearch for determining α_{k_i}, that is, in (3-4). Since $\ell(k_i)$ is the first integer for which the inequality in (3-3) holds, such

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inequality is reversed for $\ell(k_i) - 1$. That is,

$$\beta_{k_i}[f(\hat{y}^i, x^{k_i}) - f(\hat{y}^i, z^{k_i})] < \frac{\delta}{2}\phi(z^{k_i}, x^{k_i}) \tag{3-55}$$

for large enough i. On the other hand, since $\lim_{i\to\infty} \alpha_{k_i} = 0$ by hypothesis, and $\{z^{k_i} - x^{k_i}\}$ is bounded by Lemma 3.8 and Proposition 3.11i), it follows from (3-54) that

$$\lim_{i\to\infty} \|\hat{y}^i - x^{k_i}\| = 0. \tag{3-56}$$

Since by Proposition 3.10, $x^{k_i} \to \bar{x}$, and by B2, f is continuous with respect to the first argument, uniformly on bounded sets with respect to the second argument, and since δ belongs to $(0, 1)$, it follows from (3-56) that there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} \beta_{k_i}[f(x^{k_i}, x^{k_i}) - f(\hat{y}^i, x^{k_i})] &\leq \frac{\eta(1-\delta)}{8}, \\ \beta_{k_i}[f(\hat{y}^i, z^{k_i}) - f(x^{k_i}, z^{k_i})] &\leq \frac{\eta(1-\delta)}{8} \end{aligned} \tag{3-57}$$

for $i \geq m$, with η as in (3-52) and δ as in (3-55). Therefore,

$$\begin{aligned} \beta_{k_i}[f(x^{k_i}, x^{k_i}) - f(x^{k_i}, z^{k_i})] &\leq \beta_{k_i}[f(\hat{y}^i, x^{k_i}) - f(\hat{y}^i, z^{k_i})] + \frac{\eta(1-\delta)}{4} \\ &< \frac{\delta}{2}\phi(z^{k_i}, x^{k_i}) + \frac{(1-\delta)}{2}\phi(z^{k_i}, x^{k_i}) = \frac{1}{2}\phi(z^{k_i}, x^{k_i}) \end{aligned} \tag{3-58}$$

for all $i \geq m$, using (3-57) in the first inequality and (3-55) and (3-52) in the second one. Now we combine Corollary 3.3 and (3-58) in order to get

$$\langle x^{k_i} - z^{k_i}, Jx^{k_i} - Jz^{k_i}\rangle < \tfrac{1}{2}\phi(z^{k_i}, x^{k_i})$$

for all $i \geq m$ or, equivalently,

$$\phi(z^{k_i}, x^{k_i}) + \phi(x^{k_i}, z^{k_i}) < \phi(z^{k_i}, x^{k_i})$$

for all $i \geq m$, that is,

$$\phi(x^{k_i}, z^{k_i}) < 0$$

for all $i \geq m$, which contradicts the definition of ϕ, thus establishing the result.

PROOF OF THEOREM 3.1. We consider two cases related to the behavior of $\{\alpha_k\}$. First assume that there exists a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_k\}$ which converges to 0. In this case, the result is obtained by Proposition 3.13, that is, we get that $\{x^k\}$ strongly converges to $\bar{x} \in S(f, K)$.

Now we take a subsequence $\{\alpha_{k_i}\}$ of $\{\alpha_k\}$ bounded away from zero, say greater than or equal to η for large enough i. It follows from (3-3) and (3-6) that

$$\beta_{k_i}[f(y^{k_i}, x^{k_i}) - f(y^{k_i}, z^{k_i})] \geq \frac{\delta}{2}\phi(z^{k_i}, x^{k_i}). \tag{3-59}$$

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Note that

$$0 = f(y^{k_i}, y^{k_i}) \le \alpha_{k_i} f(y^{k_i}, z^{k_i}) + (1 - \alpha_{k_i}) f(y^{k_i}, x^{k_i}),$$

so that

$$\frac{1 - \alpha_{k_i}}{\alpha_{k_i}} f(y^{k_i}, x^{k_i}) \ge -f(y^{k_i}, z^{k_i}). \tag{3-60}$$

Multiplying (3-60) by β_{k_i} and adding (3-59), we easily get

$$\beta_{k_i} f(y^{k_i}, x^{k_i}) \ge \frac{\delta \alpha_{k_i}}{2} \phi(z^{k_i}, x^{k_i}). \tag{3-61}$$

Taking limits in (3-61) and using Proposition 3.11(ii), we obtain $\lim_{i\to\infty} \phi(z^{k_i}, x^{k_i}) = 0$.

Now we invoke Proposition 3.12 in order to get $\bar{x} \in S(f, K)$. We have shown that the limit $\bar{x}$ of $\{x^k\}$ belongs to $S(f, K)$ when the corresponding step sizes $\{\alpha_k\}$ either approach zero or remain bounded away from zero, establishing the claim.

4. Application to a generalized Nash equilibrium problem

In this section, we give an application of our main result to game theory. We first introduce constrained game problems. Let I be a finite index set and, for each $i \in I$, let X_i be a subset of a Banach space E_i. We use the notation

$$E = \prod_{i \in I} E_i, \quad X = \prod_{i \in I} X_i \quad \text{and} \quad X^i = \prod_{j \in I, j \ne i} X_j.$$

For each $x \in X$, x_i denotes its ith coordinate and x^i its projection on X^i. In the following, we occasionally write $x = (x_i, x^i)$. If I is the set of players, each player $i \in I$ has the strategy set X_i, a constraint correspondence $F_i : X^i \to 2^{X_i}$ and a loss function $f_i : X \to \mathbb{R}$. A constrained game $\Gamma = (X_i, F_i, f_i)_{i \in I}$ is defined as a family of ordered triples (X_i, F_i, f_i). A point $\hat{x} \in X$ is called an equilibrium point of Γ if, for each $i \in I$,

$$\hat{x}_i \in F_i(\hat{x}^i),$$

$$f_i(\hat{x}) \le f_i(y_i, \hat{x}^i), \quad \forall y_i \in F_i(\hat{x}^i). \tag{4-1}$$

If $F_i(x^i) = X_i$ for each $i \in I$, the constrained game $\Gamma = (X_i, F_i, f_i)_{i \in I}$ reduces to the conventional game $\Gamma = (X_i, f_i)_{i \in I}$ and an equilibrium point is said to be a Nash equilibrium point (see [9]).

In the following theorem, we use the above notation in order to approximate an equilibrium point of the problem.

THEOREM 4.1. *Assume that E is a uniformly convex and uniformly smooth Banach space. Consider the above constrained game $\Gamma = (X_i, F_i, f_i)_{i \in I}$, where I is the set of players, each player $i \in I$ has the strategy set X_i, which is a nonempty, closed and convex subset of E_i, a constraint correspondence $F_i : X^i \to 2^{X_i}$ and a loss function $f_i : X \to \mathbb{R}$, which is convex and uniformly continuous on bounded sets for all $i \in I$,*

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and the multivalued mapping $K(\cdot)$ from X into $\mathcal{P}(X)$ defined by

$$K(x) = \prod_{i\in I} F_i(x^i) \tag{4-2}$$

satisfies B4. If either there is $\hat{y} \in X$ such that $\hat{y}_i \in F_i(\hat{y}^i)$ and $f_i(\hat{y}_i, x^i) \leq f_i(x)$ for each $i \in I$ and all $x \in X$, or if the sequence $\{v^k\}$ generated by the algorithm SEML is bounded, then $\{v^k\}$ converges strongly to an equilibrium point of Γ.

PROOF. Define $f(x, y) = \sum_{i\in I}(f_i(y_i, x^i) - f_i(x_i, x^i))$. It is easy to see that f satisfies B1–B3. Then, by Theorem 3.1, the sequence $\{v^k\}$ converges strongly to some $\hat{x} \in K(\hat{x})$ such that $f(\hat{x}, y) \geq 0$ for all $y \in K(\hat{x})$. Now, choosing $y = (y_i, \hat{x}^i)$, we get $f(\hat{x}, y) = f_i(y_i, \hat{x}^i) - f_i(\hat{x}_i, \hat{x}^i) \geq 0$ for all $y_i \in F_i(\hat{x}^i)$. Since $i \in I$ is arbitrary, it follows that $\hat{x}$ is an equilibrium point of Γ. This completes the proof of the theorem.

Now we present a concrete example.

EXAMPLE 4.2. We consider an oil field with n oil wells. Some restrictions on the strategy set of the model are imposed on the amount of oil which is extracted. The oil is refined and then sold. We assume that there are n petroleum companies and each company i may extract from oil well i. Let x denote the vector in $\mathbb{R}^n$ with components x_i, where x_i is the amount of oil extracted from well i. We denote by $x_j^{\min}$ and $x_j^{\max}$ the lower and upper bounds for the amount of oil extracted from well j. Then the strategy set of the model takes the form

$$C := \{x = (x_1, \ldots, x_n); x_j^{\min} \leq x_j \leq x_j^{\max}, \quad \forall j = 1, \ldots, n\}.$$

Let $I = \{1, 2, \ldots, n\}$ denote the index set of all oil wells. We assume that the price p of the oil, which is a strictly decreasing function of the supply, is given by

$$p = a - b\sum_{i=1}^{n} x_i,$$

where a and b are two positive real numbers. Let c_j be the cost function for oil well j. We assume that for i different from j, c_i and c_j are independent of each other. However, the amount of oil x_i extracted from well i may depend on the other x_j. Now we specify our restrictions on the amount of oil x_i extracted from well i for all i. For each $i \in I$, suppose that

$$X_i = \{z : x_i^{\min} \leq z \leq x_i^{\max}\} \quad \text{and} \quad X^i = \prod_{j\in I, j\neq i} X_j.$$

Now we define $F_i : X^i \to 2^{X_i}$ by

$$F_i(x^i) = \begin{cases} \Big\{z_i \in X_i \mid \exists z^i \in X^i : z_i \leq \beta - \sum_{j\neq i} z_j\Big\} & \text{if } \sum_{i\in I} x_i > \beta, \\ \Big\{z_i \in X_i \mid \exists z^i \in X^i : z_i \leq \sum_{i\in I} x_i - \sum_{j\neq i} z_j\Big\} & \text{if } \sum_{i\in I} x_i \leq \beta \end{cases}$$

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for each $x_i \in X_i$, where x^i is the projection of x on X^i, β is the upper bound for the total amount of extracted oil and $\beta \geq \sum_{i=1}^{n} x_i^{\min}$. In fact, the total amount of extracted oil is constrained by the multivalued mapping $K(\cdot)$ from C into itself defined by

$$K(x) = F_1(x^1) \times F_2(x^2) \times \cdots \times F_n(x^n)$$

for all $x \in C$. It is easy to see that we can write the above definition of the mapping K at $x \in C$ as

$$K(x) = \begin{cases} \Big\{z \in C : \sum_{i\in I} z_i \leq \beta\Big\} & \text{if } \sum_{i\in I} x_i > \beta, \\ \Big\{z \in C : \sum_{i\in I} z_i \leq \sum_{i\in I} x_i\Big\} & \text{if } \sum_{i\in I} x_i \leq \beta. \end{cases} \tag{4-3}$$

Then, for each $x \in C$, $K(x)$ is a nonempty, closed and convex subset of C, and the total amount of extracted oil x must belong to $K(x)$. The profit made by the company i that owns oil well i is given by

$$h_i(x) = px_i - c_i(x_i) = \Big(a - b\sum_{k=1}^{n} x_k\Big)x_i - c_i(x_i)$$

for each $i = 1, \ldots, n$. Let us define

$$h_i(y_i, x^i) = \Big(a - b\Big(\sum_{j\neq i} x_j + y_i\Big)\Big)y_i - c_j(y_i).$$

Note that we write $x = (x_i, x^i)$. Clearly, if for all $x, y \in C$, we define $z = (z_1, \ldots, z_n)$ by

$$z_j = \begin{cases} x_j, & j \neq i, \\ y_j, & j = i \end{cases}$$

for all $j = 1, \ldots, n$, then we have $h_i(y_i, x^i) = h_i(z)$ for all $i = 1, \ldots, n$. If we define a loss function $f_i : C \to \mathbb{R}$ by $f_i(x) = -h_i(x)$ for all $i \in I$, then a point $\hat{x} \in C$ is an equilibrium point of $\Gamma = (X_i, F_i, f_i)_{i\in I}$ if, for each $i \in I$,

$$\begin{aligned} &\hat{x}_i \in F_i(\hat{x}^i), \\ &f_i(\hat{x}) \leq f_i(y_i, \hat{x}^i), \quad \forall y_i \in F_i(\hat{x}^i). \end{aligned} \tag{4-4}$$

Now, if we define

$$f(x, y) = \sum_{i=1}^{n} (h_i(x_i, x^i) - h_i(y_i, x^i)), \tag{4-5}$$

then the oligopolistic equilibrium problem model of oil markets is transformed to the following quasi-equilibrium problem: find $\hat{x} \in K(\hat{x})$ such that $f(\hat{x}, y) \geq 0$ for all $y \in K(\hat{x})$. Assume that $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ is a solution to the problem. By choosing $y_j = \hat{x}_j$ for all $j \neq i$, and the component y_i arbitrarily, since $f(\hat{x}, y) \geq 0$ for all $y \in K(\hat{x})$, we deduce that $h_i(\hat{x}_i, \hat{x}^i) - h_i(y_i, \hat{x}^i) \geq 0$ for all $y \in K(\hat{x})$. Since i is arbitrary, this implies that each

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equilibrium point corresponds to an optimal amount of oil to be extracted so that each company makes a profit. In other words,

$$\hat{x}_i \in F_i(\hat{x}^i),$$
$$f_i(\hat{x}) \leq f_i(y_i, \hat{x}^i), \quad \forall y_i \in F_i(\hat{x}^i). \tag{4-6}$$

However, unlike the case of an equilibrium problem where K does not depend on C, this may not correspond to the maximum profit for each company, unless the solution is unique.

Our main result provides an approximation scheme to the equilibrium solution of the above problem, with rather mild conditions on the data, as stated in the following theorem.

THEOREM 4.3. *Assume that the assumptions of the above example are satisfied, and that the cost function c_i is convex and uniformly continuous on bounded sets for all $i = 1, \ldots, n$. Let the bifunction f be defined as in (4-5), and the multivalued mapping $K(\cdot)$ from C into itself as in (4-3). Then, if the sequence $\{v^k\}$ is generated by the algorithm SEML, it converges to an element of $S(f, K)$.*

PROOF. It is obvious that f satisfies B1–B3. Now we show that the condition B4 is satisfied. By the definition of K,

$$K(x) = \Big\{ z = (z_1, \ldots, z_n) \in C : \sum_{i=1}^{n} z_i \leq \min\Big\{ \sum_{i=1}^{n} x_i, \beta \Big\}\Big\}. \tag{4-7}$$

Since $\beta \geq \sum_{i=1}^{n} x_i^{\min}$, it is obvious that K is a multivalued mapping with nonempty, closed and convex values, and $\mathrm{Fix}(K) \neq \emptyset$. We want to show that the condition B4 is satisfied. We first show that K is quasi-ϕ-nonexpansive. Let $p \in \mathrm{Fix}(K)$. For $x \in C$, we consider separately the cases where $x \in K(x)$ and $x \notin K(x)$. If $x \in K(x)$, we have $\|p - P_{K(x)}(x)\|^2 = \|p - x\|^2$, where P is the projection and, if $x \notin K(x)$, then we have $\|p - P_{K(x)}(x)\|^2 \leq \|p - x\|^2$. Therefore, we have $\|p - P_{K(x)}(x)\|^2 \leq \|p - x\|^2$ for all $x \in C$ and $p \in \mathrm{Fix}(K)$. This implies that K is quasi-ϕ-nonexpansive. Now we show that K is demiclosed. Let $x^k \to \bar{x}$ and $\lim_{k\to\infty} d(x^k, K(x^k)) = 0$. Then $\sum_{i=1}^{n} \bar{x}_i \leq \beta$. Now the definition of K shows that $\bar{x} \in K(\bar{x})$, that is, the mapping K is demiclosed. Finally, we prove that K is lower semicontinuous at each $\bar{x} \in C$. Suppose that $x^k \to \bar{x}$ and $\bar{y} \in K(\bar{x})$. For all $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$, we define

$$y^k = \begin{cases} \bar{y} & \text{if } \displaystyle\sum_{i=1}^{n} \bar{y}_i \leq \sum_{i=1}^{n} x_i^k, \\ x^k & \text{if } \displaystyle\sum_{i=1}^{n} \bar{y}_i > \sum_{i=1}^{n} x_i^k. \end{cases} \tag{4-8}$$

Then we have $y^k \in K(x^k)$ and $y^k \to \bar{y}$. This follows from the fact that $x^k \to \bar{x}$ and therefore $y^k = \bar{y}$, except maybe for finitely many values of k. This implies that K

is lower semicontinuous at each $\bar{x} \in C$. Now the result is a direct consequence of Theorem 3.1.

We note that a classical equilibrium problem arising from Nash–Cournot oligopolistic equilibrium models of electricity markets was discussed in [27], with a numerical example in [30].

The following is an infinite-dimensional example of application of our result.

EXAMPLE 4.4. Let $E = \ell^p = \{\xi = (\xi_1, \xi_2, \xi_3, \ldots) : (\sum_{i=1}^{\infty} |\xi_i|^p)^{1/p} < \infty\}$ for $1 < p < \infty$, and let $f : E \times E \to \mathbb{R}$ be a bifunction which is defined by

$$f(x, y) = \langle x - y, Jx\rangle \sum_{i=1}^{\infty} (x_i)^p.$$

Let $C = \{\xi = (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^p : \xi_i \geq 0, \forall i \in \mathbb{N}\}$, and $K(\cdot) : C \to \mathcal{P}(C)$ be defined by $K(x) = \{\frac{1}{2}x + tz | t \in \mathbb{R}^+, z \in B(x, \frac{1}{4}\|x\|)\}$ for each $x \in C$, where $B(x, \frac{1}{4}\|x\|)$ denotes the closed ball of radius $\frac{1}{4}\|x\|$ centered at x. It is easy to see that f satisfies B1–B3, and it is obvious that K is a multivalued mapping with nonempty, closed and convex values. We want to show that the condition B4 is satisfied. Note that for all $x \in C$, we have $x \in K(x)$ and therefore $\text{Fix}(K) \neq \emptyset$. Let $p \in \text{Fix}(K)$ and $x \in C$. Then we have $\|p - P_{K(x)}(x)\|^2 = \|p - x\|^2$, where P is the generalized projection. This implies that K is quasi-ϕ-nonexpansive. Also, by the definition of K, we have $x \in K(x)$ for all $x \in C$; therefore, the mapping K is demiclosed. Finally, we show that K is lower semicontinuous at each $\bar{x} \in C$. Suppose that $x^k \to \bar{x}$ and $\bar{y} \in K(\bar{x})$. Then there exist $t > 0$ and $\bar{z} \in B(\bar{x}, \frac{1}{4}\|\bar{x}\|)$ such that $\bar{y} = \frac{1}{2}\bar{x} + t\bar{z}$. It can be shown that there exists a sequence $z^k \in B(x^k, \frac{1}{4}\|x^k\|)$ such that $z^k \to \bar{z}$. Now we define $y^k = \frac{1}{2}x^k + tz^k$. Then we have $y^k \in K(x^k)$ and $y^k \to \bar{y}$. This implies that K is lower semicontinuous at each $\bar{x} \in C$. Hence, B4 is satisfied too. Now, if the sequence $\{v^k\}$ generated by the algorithm SEML is bounded, then, by Theorem 3.1, it converges strongly to an element of $S(f, K)$. In this example, we can see that $DS(f, K) = \emptyset$, while $S(f, K) \neq \emptyset$.

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