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Infinitesimal elastic–plastic Cosserat micropolar theory. Modelling and global existence in the rate-independent case

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In this article we investigate the regularizing properties of generalized continua of Cosserat micropolar type in the elasto-plastic case. We propose an extension of classical infinitesimal elasto-plasticity to include consistently non-dissipative micropolar effects. It is shown that the new model is thermodynamically admissible and allows for a unique, global-in-time solution of the corresponding rate-independent initial–boundary-value problem. The method of choice is the Yosida approximation and a passage to the limit.

1. Introduction

This article addresses the modelling and mathematical analysis of *geometrically linear* generalized continua of *Cosserat micropolar* type in the elastic and elasto-plastic cases. General continuum models involving *independent rotations* were introduced by the Cosserat brothers [13]. In fact, their original motivation came from the theory of surfaces, where the moving three-frame (Gauss frame) had been used successfully.

Their development was largely forgotten until it was rediscovered at the beginning of the 1960s [1, 19, 22, 30, 31, 37, 42, 48, 53–55]. At that time theoretical investigations of non-classical continuum theories were the main motivation [34]. The Cosserat concept has been generalized in various directions; for an overview of these so-called *microcontinuum* theories see [8, 20, 21].

Among the first contributions extending the Cosserat framework to infinitesimal elasto-plasticity we have to mention [7, 36, 47]. More recent infinitesimal elastic–plastic formulations have been investigated in [15, 17, 33, 43]. These models directly comprise joint elastic and plastic Cosserat effects. Lately, the models have been extended to a finite elastic–plastic setting as well (see, for example, [24, 28, 29, 44–46, 49] and references therein). Most of these extensions also directly comprise joint elastic and plastic Cosserat effects but we pretend that their physical and mathematical significance is at present much more difficult to assess than models where Cosserat effects are restricted to the elastic response of the material (see [24] and references therein). Our own contribution will be of the second type.

Apart from the theoretical development, the Cosserat-type models are today increasingly advocated as a means of regularizing the pathological mesh size dependence of localization computations where shear failure mechanisms [5, 6, 12, 38, 40] play a dominant role; for applications in plasticity see the non-exhaustive list [14–17, 33, 43]. The mathematical difficulties that occur reflect the physical fact that upon localization the validity limit of the classical models is reached. In models without any internal length the deformation should be homogeneous on the scale of a representative volume element of the material [39].

The incorporation of a length-scale, which is natural in a Cosserat theory, has the potential to remove the mesh sensitivity. The presence of the internal length-scale causes the localization zones to have finite width. However, the actual length-scale of a material is difficult to establish experimentally and theoretically [35] and its determination remains basically an open question, as is the determination of other material constants that also appear in the Cosserat framework. It is also not entirely clear how the shear band width depends on the characteristic length.

The mathematical analysis of Cosserat micropolar models is at present restricted to the infinitesimal, linear elastic models (see, for example, [18, 25, 26, 32]). To the best of our knowledge, the elasto-plastic situation has not been dealt with mathematically.

As far as classical rate-independent elasto-plasticity is concerned we remark that global existence for the displacement has been shown only in a very weak, measure-valued sense, while the stresses could be shown to remain in $L^2(\Omega)$. For this result we refer, for example, to [3, 11, 52]. If hardening or viscosity is added, then global classical solutions are found (see, for example, [2, 9, 10]). A complete theory for the classical rate-independent case remains elusive (see the remarks in [11]).

While the infinitesimal Cosserat micropolar elasto-plasticity model in its various versions is interesting mathematically in its own right, we concentrate here on its possible regularizing properties. We emphasize that our non-dissipative formulation seems to provide just the correct amount of regularization missing in the classical elasto-plastic problem. This being our main thrust, we do not investigate Cosserat models in which additional Cosserat effects have been introduced for the plastic behaviour as well.

Our contribution is organized as follows. First, we review the basic concepts of the geometrically linear elastic Cosserat micropolar theories in a variational context and present various existence results.

The formulation is then consistently extended to infinitesimal elasto-plasticity with non-dissipative micropolar effects. The decisive stress tensor is none other than the linearized elastic Eshelby energy–momentum tensor.

Subsequently, we mathematically study the rate-independent case obtained and show, by means of the Yosida approximation and a passage to the limit, that the rate-independent problem admits a unique, global-in-time solution for displacements and microrotations in standard Sobolev spaces under fairly mild assumptions on the data. The relevant notation is found in the appendix.

2. The infinitesimal elastic Cosserat model

Let us start by recalling the infinitesimal Cosserat approach. First, in the purely elastic case, an infinitesimal Cosserat theory can be obtained by introducing the additive decomposition of the macroscopic displacement gradient ∇u into infinitesimal *microrotation* $\bar{A} \in \mathfrak{so}(3, \mathbb{R})$ (infinitesimal Cosserat rotation tensor) and an infinitesimal *micropolar stretch tensor* (or first Cosserat deformation tensor) $\bar{\varepsilon} \in \mathbb{M}^{3\times 3}$ with

$$\nabla u = \bar{\varepsilon} + \bar{A}, \tag{2.1}$$

where $\bar{\varepsilon}$ is not necessarily symmetric, such that (2.1) is in general not the decomposition of ∇u into infinitesimal continuum stretch $\operatorname{sym}(\nabla u)$ and infinitesimal continuum rotation $\operatorname{skew}(\nabla u)$.

In the quasi-static case, the Cosserat theory is then obtained from a variational principle [46, p. 51] or [50] for the infinitesimal displacement $u : [0, T] \times \bar{\Omega} \mapsto \mathbb{R}^3$ and the independent infinitesimal microrotation $\bar{A} : \bar{\Omega} \mapsto \mathfrak{so}(3, \mathbb{R})$:

$$\left.\begin{aligned} \mathcal{E}(u, \bar{A}) = \int_{\Omega} W(\nabla u, \bar{A}, \mathrm{D}_x \bar{A}) - \langle f, u\rangle - \langle M, \bar{A}\rangle \,\mathrm{d}V \qquad\qquad & \\ - \int_{\Gamma_{\mathrm{S}}} \langle N, u\rangle \,\mathrm{d}S - \int_{\Gamma_{\mathrm{C}}} \langle M_{\mathrm{c}}, \bar{A}\rangle \,\mathrm{d}S \mapsto \min \text{ w.r.t. } (u, \bar{A}), & \\ \bar{A}|_{\Gamma} = \bar{A}_{\mathrm{d}}, \quad u|_{\Gamma} = g_{\mathrm{d}}(t, x) - x. \qquad\qquad\qquad\qquad & \end{aligned}\right\} \tag{2.2}$$

Here W represents the elastic energy density and $\Omega \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ is that part of the boundary where Dirichlet conditions g_{d}, $\bar{A}_{\mathrm{d}}$ for infinitesimal displacements and rotations, respectively, are prescribed, while $\Gamma_{\mathrm{S}} \subset \partial\Omega$ is a part of the boundary where traction boundary conditions N are applied with $\Gamma \cap \Gamma_{\mathrm{S}} = \emptyset$. In addition, $\Gamma_{\mathrm{C}} \subset \partial\Omega$ is the part of the boundary where external surface couples M_{c} are applied with $\Gamma \cap \Gamma_{\mathrm{C}} = \emptyset$. The classical volume force is denoted by f and the additional volume couple by M. Variation of the action $\mathcal{E}$ with respect to u yields the equation for linearized balance of linear momentum and variation of $\mathcal{E}$ with respect to $\bar{A}$ yields the linearized version of balance of angular momentum.

2.1. Infinitesimal elastic Cosserat theory

It remains to specify the analytic form of the energy density W. A linearized version of material frame-indifference implies the reduction

$$W(\nabla u, \bar{A}, \mathrm{D}_x \bar{A}) = W(\bar{\varepsilon}, \mathrm{D}_x \bar{A}), \tag{2.3}$$

and for infinitesimal displacements u and small curvature $\mathrm{D}_x \bar{A}$ a quadratic ansatz is appropriate:

$$W(\bar{\varepsilon}, \mathrm{D}_x \bar{A}) = W_{\mathrm{mp}}^{\mathrm{infin}}(\bar{\varepsilon}) + W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D}_x \bar{A}), \tag{2.4}$$

with an additive decomposition of the energy density into microstretch $\bar{\varepsilon}$ and curvature parts.

In the isotropic case we assume for the stretch energy

$$\begin{aligned} W_{\text{mp}}^{\text{infin}}(\bar{\varepsilon}) &= \mu\|\text{sym}(\bar{\varepsilon})\|^2 + \mu_c\|\text{skew}(\bar{\varepsilon})\|^2 + \tfrac{1}{2}\lambda\,\text{tr}[\text{sym}(\bar{\varepsilon})]^2 \\ &= \mu\|\text{sym}\,\nabla u\|^2 + \mu_c\|\text{skew}(\nabla u) - \bar{A}\|^2 + \tfrac{1}{2}\lambda\,\text{tr}[\text{sym}(\nabla u)]^2, \end{aligned} \tag{2.5}$$

where the *Cosserat couple modulus* $\mu_c \geqslant 0$ is an additional parameter, complementing the two Lamé constants $\mu, \lambda > 0$.

For the curvature term we assume that

$$W_{\text{curv}}^{\text{small}}(\mathrm{D}_x\bar{A}) = \mu(\tfrac{1}{12}L_c^2)(\alpha_5\|\text{sym}\,\mathrm{D}_x\bar{A}\|^2 + \alpha_6\|\text{skew}\,\mathrm{D}_x\bar{A}\|^2 + \alpha_7\,\text{tr}[\mathrm{D}_x\bar{A}]^2). \tag{2.6}$$

Here, $L_c > 0$ with units of length introduces a specific *internal characteristic length* into the elastic formulation. In general, we assume $\alpha_5 > 0$, α_6, $\alpha_7 \geqslant 0$.

Two observations are essential. First, if $\mu_c = 0$, the infinitesimal problem completely decouples: the infinitesimal microrotations $\bar{A}$ have no influence at all on the macroscopic behaviour of the infinitesimal displacements and classical infinitesimal elasticity results.

Second, the choice $\alpha_6, \alpha_7 = 0$ is possible, since coercivity of the reduced curvature expression can still be concluded on account of the classical first Korn inequality applied to $\text{sym}\,\mathrm{D}_x\bar{A}$.[1]

In the limit of zero internal length-scale $L_c = 0$ and for $\mu_c > 0$,[2] the balance of angular momentum reads

$$D_{\bar{A}}W_{\text{mp}}(\nabla u, \bar{A}) \in \text{Sym} \quad \Leftrightarrow \quad D_{\bar{A}}W_{\text{mp}}(\nabla u, \bar{A}) = 0, \tag{2.7}$$

and implies already that infinitesimal continuum rotations and infinitesimal microrotations coincide: $\text{skew}(\nabla u) = \bar{A}$, and this in turn is equivalent to the symmetry of the infinitesimal Cauchy stress σ or the so-called *Boltzmann axiom*.

If we now consider $\mu_c > 0$, it is standard to prove that the corresponding minimization problem admits a unique minimizing pair $(u, \bar{A}) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3, \mathbb{R}))$. Existence results of this type have been obtained in, for example, [18, 25, 26, 32].

[1]For $\bar{A} \in \mathfrak{so}(3, \mathbb{R})$ we have

$$\bar{A} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}, \qquad \text{axl}(\bar{A}) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \qquad \nabla\,\text{axl}(\bar{A}) = \begin{pmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{pmatrix},$$

$$\text{sym}\,\nabla\,\text{axl}(\bar{A}) = \begin{pmatrix} \alpha_x & \frac{1}{2}(\alpha_y + \beta_x) & \frac{1}{2}(\alpha_z + \gamma_x) \\ \frac{1}{2}(\alpha_y + \beta_x) & \beta_y & \frac{1}{2}(\beta_z + \gamma_y) \\ \frac{1}{2}(\alpha_z + \gamma_x) & \frac{1}{2}(\beta_z + \gamma_y) & \gamma_z \end{pmatrix},$$

$$\|\text{sym}\,\nabla\,\text{axl}(\bar{A})\|^2 = \alpha_x^2 + \beta_y^2 + \gamma_z^2 + \tfrac{1}{2}(\alpha_y + \beta_x)^2 + \tfrac{1}{2}(\alpha_z + \gamma_x)^2 + \tfrac{1}{2}(\beta_z + \gamma_y)^2,$$

$$\|\text{sym}\,\mathrm{D}_x\bar{A}\|^2 = \|\text{sym}\,\nabla(\bar{A}\cdot e_1)\|^2 + \|\text{sym}\,\nabla(\bar{A}\cdot e_2)\|^2 + \|\text{sym}\,\nabla(\bar{A}\cdot e_3)\|^2.$$

Looking at the last line the standard Korn's inequality applied to columns of $\bar{A}$ yields coercivity of $\|\text{sym}\,\mathrm{D}_x\bar{A}\|^2$.

[2]Also corresponding to the limit of arbitrary large samples, which can be seen by a simple scaling argument.

THEOREM 2.1 (existence for infinitesimal elastic Cosserat model). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_{\mathrm{d}} \in H^1(\Omega, \mathbb{R}^3)$ and $\bar{A}_{\mathrm{d}} \in H^1(\Omega, \mathfrak{so}(3, \mathbb{R}))$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_{\mathrm{S}}, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathfrak{so}(3, \mathbb{R}))$ together with $M_{\mathrm{c}} \in L^2(\Gamma_{\mathrm{C}}, \mathfrak{so}(3, \mathbb{R}))$. Then models based on (2.2) with (2.5) and (2.6) admit a unique minimizing solution pair $(u, \bar{A}) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathfrak{so}(3, \mathbb{R}))$. The solution is smoother if the data are smoother.*

Proof. We apply the direct method of variations. For simplicity we only assume that $M, M_{\mathrm{c}} = 0$. First we observe that infimizing sequences $(u_k, \bar{A}_k)$ of (2.2) exist and

$$\begin{aligned}
\infty &> \int_\Omega W_{\mathrm{mp}}^{\mathrm{infin}}(\nabla u_k - \bar{A}_k) + W_{\mathrm{curv}}^{\mathrm{small}}(\mathrm{D}_x \bar{A}_k) - \langle f, u_k \rangle \,\mathrm{d}V \\
&\geqslant \int_\Omega \mu_{\mathrm{c}} \|\nabla u_k - \bar{A}_k\|^2 \,\mathrm{d}V - \|f\|_{L^2} \|u_k\|_{H^1(\Omega)} \\
&= \int_\Omega \mu_{\mathrm{c}} \|\mathrm{sym}(\nabla u_k - \bar{A}_k)\|^2 + \mu_{\mathrm{c}} \|\mathrm{skew}(\nabla u_k - \bar{A}_k)\|^2 \,\mathrm{d}V - \|f\|_{L^2} \|u_k\|_{H^1(\Omega)} \\
&\geqslant \int_\Omega \mu_{\mathrm{c}} \|\mathrm{sym}\, \nabla(u_k - g_{\mathrm{d}} + g_{\mathrm{d}})\|^2 \,\mathrm{d}V - \|f\|_{L^2} \|u_k\|_{H^1(\Omega)} \\
&\geqslant \tfrac{1}{2} \mu_{\mathrm{c}} c_K \|u_k - g_{\mathrm{d}}\|_{H^1(\Omega)}^2 - \mu_{\mathrm{c}} \|\mathrm{sym}\, \nabla g_{\mathrm{d}}\|_{L^2(\Omega)}^2 - \|f\|_{L^2} \|u_k\|_{H^1(\Omega)},
\end{aligned} \tag{2.8}$$

showing that u_k is bounded in $H^1(\Omega)$. We have used the fact that sym is orthogonal to skew and used the classical first Korn inequality together with the boundary conditions for u_k. Similarly, we obtain boundedness of $\bar{A}_k$ in $H^1(\Omega, \mathfrak{so}(3, \mathbb{R}))$. We can choose a subsequence of $(u_k, \bar{A}_k)$ converging strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. By overall convexity of the energy density in $(\nabla u, \mathrm{D}_x \bar{A})$, the limit pair is a minimizer.

For uniqueness, we consider the second derivative of the strains

$$\begin{aligned}
D^2_{(\nabla u, \bar{A})} & W(\nabla u - \bar{A}) \cdot ((\nabla \phi, \delta \bar{A}), (\nabla \phi, \delta \bar{A})) \\
&\geqslant \mu_{\mathrm{c}} \|\nabla \phi - \delta \bar{A}\|^2 \\
&= \mu_{\mathrm{c}} \|\mathrm{sym}\, \nabla \phi\|^2 + \mu_{\mathrm{c}} \|\mathrm{skew}(\nabla \phi - \delta \bar{A})\|^2 \\
&\geqslant \mu_{\mathrm{c}} \|\mathrm{sym}\, \nabla \phi\|^2.
\end{aligned} \tag{2.9}$$

By the classical first Korn inequality for $\phi \in H^{1,2}_{\circ}(\Omega)$ we obtain uniform positivity of the second derivative upon integration. The functional is strictly convex; the solution is unique.

Since the resulting field equations of force balance and balance of angular momentum are linear and uniformly elliptic with constant coefficients, the standard elliptic regularity theory applies so that, for pure Dirichlet boundary conditions, the smoother the data the smoother the solution. □

The corresponding *infinitesimal gradient-constrained Cosserat micropolar* model (or *indeterminate couple stress* model) is obtained formally by setting $\mu_{\mathrm{c}} = \infty$ and has the form (simplified curvature term: $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$)

$$\left.\begin{aligned} &\int_\Omega \mu\|\operatorname{sym}\nabla u\|^2 + \tfrac{1}{2}\lambda\operatorname{tr}[\operatorname{sym}\nabla u]^2 + \mu(\tfrac{1}{12}L_c^2)\|\mathrm{D}_x\operatorname{skew}(\nabla u)\|^2 - \langle f, u\rangle\,\mathrm{d}V \\ &\qquad - \int_{\Gamma_S}\langle N, u\rangle\,\mathrm{d}S - \int_{\Gamma_C}\langle M_c, \operatorname{skew}(\nabla u)\rangle\,\mathrm{d}S \mapsto \min \text{ w.r.t. } u, \\ &\sigma^{\mathrm{loc}} = 2\mu\operatorname{sym}(\nabla u) + \lambda\operatorname{tr}[\operatorname{sym}(\nabla u)]\cdot\mathbb{1} \in \mathrm{Sym}, \quad \text{constitutive stress}, \\ &\quad u|_{\partial\Omega}(x) = g_d(x) - x, \qquad \operatorname{skew}(\nabla u)|_{\partial\Omega} = \operatorname{skew}(\nabla g_d)|_{\partial\Omega}. \end{aligned}\right\} \tag{2.10}$$

Using the same methods as before we obtain the following theorem.

THEOREM 2.2 (existence for infinitesimal gradient case). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary of class C^1 and assume for the boundary data $g_d \in H^2(\Omega, \mathbb{R}^3)$. Moreover, let $f \in L^2(\Omega, \mathbb{R}^3)$ and suppose $N \in L^2(\Gamma_S, \mathbb{R}^3)$ together with $M_c \in L^2(\Gamma_C, \mathfrak{so}(3, \mathbb{R}))$. Then a model based on (2.10) admits a unique minimizing solution $u \in H^1(\Omega) \cap \{\nabla \operatorname{curl} u \in L^2(\Omega)\}$ (cf. [18]).*

3. Infinitesimal Cosserat micropolar elasto-plasticity

3.1. Non-dissipative extension to micropolar elasto-plasticity

Now we extend the formulation of micropolar elasticity to cover infinitesimal elasto-plasticity as well. It should be made clear that there exist various ways of obtaining such an extension; for an overview of the competing models we refer to the instructive survey article by Forest and Sievert [23]. Incidentally, the Cosserats themselves [13, p. 5] had already envisaged the application of their general theory to plasticity and fracture. Without restricting generality we base the following considerations on a simplified curvature expression by setting $\alpha_5 = \alpha_6 = 1$, $\alpha_7 = 0$.

The basic idea of a *non-dissipative* extension is quite simple. Consider the additive decomposition of the total micropolar stretch into elastic and plastic parts

$$\bar{\varepsilon} = \bar{\varepsilon}_e + \bar{\varepsilon}_p, \tag{3.1}$$

and assume that microrotational effects remain purely elastic: $\bar{A}_e := \bar{A}$. Now we replace formally $\bar{\varepsilon}$ in equation (2.5) with $\bar{\varepsilon}_e$, which yields (note that $\|\mathrm{D}_x\bar{A}_e\|^2 = 2\|\nabla\operatorname{axl}(\bar{A}_e)\|^2$)

$$\begin{aligned} \mathcal{E}(\bar{\varepsilon}_e, \bar{A}_e) &= \int_\Omega \mu\|\operatorname{sym}\bar{\varepsilon}_e\|^2 + \mu_c\|\operatorname{skew}(\bar{\varepsilon}_e)\|^2 + \tfrac{1}{2}\lambda\operatorname{tr}[\bar{\varepsilon}_e]^2 + 2\mu(\tfrac{1}{12}L_c^2)\|\nabla\operatorname{axl}(\bar{A}_e)\|^2\,\mathrm{d}V \\ &= \int_\Omega \mu\|\varepsilon - \operatorname{sym}\bar{\varepsilon}_p\|^2 + \mu_c\|\operatorname{skew}(\nabla u - \bar{A}_e - \bar{\varepsilon}_p)\|^2 \\ &\qquad + \tfrac{1}{2}\lambda\operatorname{tr}[\varepsilon - \bar{\varepsilon}_p]^2 + 2\mu(\tfrac{1}{12}L_c^2)\|\nabla\operatorname{axl}(\bar{A}_e)\|^2\,\mathrm{d}V \end{aligned} \tag{3.2}$$

as thermodynamic potential, where $\varepsilon = \operatorname{sym}\nabla u$ is the symmetric part of the displacement gradient. We need to supply a consistent flow rule for $\bar{\varepsilon}_p$ (note again that $\bar{A}_{\mathrm{e}}$ acts solely elastically). By choosing

$$\dot{\varepsilon}_{\mathrm{p}}(t) \in \mathfrak{f}(T_{\mathrm{E}}), \quad T_{\mathrm{E}} := -\partial_{\bar{\varepsilon}_{\mathrm{p}}} W_{\mathrm{mp}}^{\mathrm{infin}}(\bar{\varepsilon}_{\mathrm{e}}) = \partial_{\bar{\varepsilon}_{\mathrm{e}}} W_{\mathrm{mp}}^{\mathrm{infin}}(\bar{\varepsilon}_{\mathrm{e}}), \quad \bar{\varepsilon}_{\mathrm{e}} = \bar{\varepsilon} - \bar{\varepsilon}_{\mathrm{p}}, \qquad (3.3)$$
$$W_{\mathrm{mp}}^{\mathrm{infin}}(\bar{\varepsilon}_{\mathrm{e}}) = \mu\|\mathrm{sym}\,\bar{\varepsilon}_{\mathrm{e}}\|^2 + \mu_{\mathrm{c}}\|\mathrm{skew}(\bar{\varepsilon}_{\mathrm{e}})\|^2 + \tfrac{1}{2}\lambda\,\mathrm{tr}[\bar{\varepsilon}_{\mathrm{e}}]^2,$$

with a constitutive multifunction $\mathfrak{f}$ such that $\langle \mathfrak{f}(\Sigma), \Sigma\rangle \geqslant 0, \forall \Sigma \neq 0$ the *reduced dissipation inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varepsilon, \bar{A}_{\mathrm{e}}, \bar{\varepsilon}_{\mathrm{p}}) \leqslant 0 \qquad (3.4)$$

at fixed in time $(\nabla u, \bar{A}_{\mathrm{e}})$ is automatically satisfied, thus ensuring the second law of thermodynamics.

We assume that the multifunction $\mathfrak{f}$ takes trace free symmetric values only, i.e. $\mathfrak{f}(T_{\mathrm{E}}) \in \mathrm{Sym}(3) \cap \mathfrak{sl}(3,\mathbb{R})$. This sets the *infinitesimal plastic spin* $\mathrm{skew}(\bar{\varepsilon}_{\mathrm{p}})$ to *zero* and restricts attention to incompressible plasticity as in the classical formulation of ideal plasticity. Since then $\bar{\varepsilon}_{\mathrm{p}} \in \mathrm{Sym}(3)$, we may identify $\bar{\varepsilon}_{\mathrm{p}} = \mathrm{sym}(\bar{\varepsilon}_{\mathrm{p}}) = \varepsilon_{\mathrm{p}}$, formally as in ideal plasticity. We have thus obtained our infinitesimal model, as follows.

3.2. Infinitesimal elasto-plastic Cosserat model

The infinitesimal system in variational form with non-dissipative Cosserat effects reads

$$\left.\begin{aligned}
&\int_\Omega \mu\|\varepsilon - \varepsilon_{\mathrm{p}}\|^2 + \mu_{\mathrm{c}}\|\mathrm{skew}(\nabla u - \bar{A}_{\mathrm{e}})\|^2 + \tfrac{1}{2}\lambda\,\mathrm{tr}[\varepsilon]^2 + 2\mu(\tfrac{1}{12}L_{\mathrm{c}}^2)\|\nabla\,\mathrm{axl}(\bar{A}_{\mathrm{e}})\|^2\\
&\qquad -\langle f, u\rangle - \langle M, \bar{A}_{\mathrm{e}}\rangle\,\mathrm{d}V - \int_{\Gamma_{\mathrm{S}}}\langle N, u\rangle\,\mathrm{d}S - \int_{\Gamma_{\mathrm{C}}}\langle M_{\mathrm{c}}, \bar{A}_{\mathrm{e}}\rangle\,\mathrm{d}S\\
&\qquad\qquad \mapsto \min\ \text{w.r.t. } (u, \bar{A}_{\mathrm{e}}) \text{ at fixed } \varepsilon_{\mathrm{p}},\\
&\dot{\varepsilon}_{\mathrm{p}}(t) \in \mathfrak{f}(T_{\mathrm{E}}), \qquad T_{\mathrm{E}} = 2\mu(\varepsilon - \varepsilon_{\mathrm{p}}),\\
&u|_\Gamma = g_{\mathrm{d}}(t,x) - x, \quad \bar{A}_{\mathrm{e}}|_\Gamma = \mathrm{skew}(\nabla g_{\mathrm{d}}(t,x))|_\Gamma.
\end{aligned}\right\} \qquad (3.5)$$

The corresponding system of partial differential equations coupled with the flow rule is given by (note that $\|\bar{A}_{\mathrm{e}}\|^2 = 2\|\,\mathrm{axl}(\bar{A}_{\mathrm{e}})\|^2$ for $\bar{A}_{\mathrm{e}} \in \mathfrak{so}(3,\mathbb{R})$)

$$\left.\begin{aligned}
&\mathrm{div}\,\sigma = -f, \quad x \in \Omega,\\
&\sigma = 2\mu(\varepsilon - \varepsilon_{\mathrm{p}}) + 2\mu_{\mathrm{c}}(\mathrm{skew}(\nabla u) - \bar{A}_{\mathrm{e}}) + \lambda\,\mathrm{tr}[\varepsilon]\cdot \mathbb{1},\\
&-\mu(\tfrac{1}{12}L_{\mathrm{c}}^2)\Delta\,\mathrm{axl}(\bar{A}_{\mathrm{e}}) = \mu_{\mathrm{c}}\,\mathrm{axl}(\mathrm{skew}(\nabla u) - \bar{A}_{\mathrm{e}}) + \tfrac{1}{2}\,\mathrm{axl}(\mathrm{skew}(M)),\\
&\dot{\varepsilon}_{\mathrm{p}}(t) \in \mathfrak{f}(T_{\mathrm{E}}), \quad T_{\mathrm{E}} = 2\mu(\varepsilon - \varepsilon_{\mathrm{p}}),\\
&u|_\Gamma(t,x) = g_{\mathrm{d}}(t,x) - x, \quad \bar{A}_{\mathrm{e}}|_\Gamma = \mathrm{skew}(\nabla g_{\mathrm{d}}(t,x))|_\Gamma,\\
&\sigma\cdot \boldsymbol{n}|_{\Gamma_{\mathrm{S}}}(t,x) = N, \quad \sigma\cdot\boldsymbol{n}|_{\partial\Omega\setminus\{\Gamma\cup\Gamma_{\mathrm{S}}\}}(t,x) = 0,\\
&\mu(\tfrac{1}{12}L_{\mathrm{c}}^2)\nabla\,\mathrm{axl}(\bar{A}_{\mathrm{e}})\cdot\boldsymbol{n}|_{\Gamma_{\mathrm{C}}}(t,x) = \tfrac{1}{2}\,\mathrm{axl}(\mathrm{skew}(M_{\mathrm{c}})),\\
&\mu(\tfrac{1}{12}L_{\mathrm{c}}^2)\nabla\,\mathrm{axl}(\bar{A}_{\mathrm{e}})\cdot\boldsymbol{n}|_{\partial\Omega\setminus\{\Gamma\cup\Gamma_{\mathrm{C}}\}}(t,x) = 0,\\
&\mathrm{tr}[\varepsilon_{\mathrm{p}}(0)] = 0, \quad \varepsilon_{\mathrm{p}}(0) \in \mathrm{Sym}(3).
\end{aligned}\right\} \qquad (3.6)$$

We remark that the derivation of this model is intrinsically thermodynamically correct but that it can also be obtained as the linearized version of a corresponding geometrically exact model [41] based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts, which, in fact, was derived prior to this linearized model.

In [17, p. 815] an elasto-plastic model based on the infinitesimal theory with dissipative Cosserat effects has been investigated by means of localized considerations. They show that the Cosserat couple modulus $\mu_c > 0$ has a decisive influence on localization effects, essentially excluding mode II shear failure. In the light of our development with non-dissipative Cosserat effects, however, it is difficult to transfer this insight directly.

3.3. Mathematical analysis of the elasto-plastic model

For brevity of notation, in this section we write A instead of $\bar{A}_e$ and l_c instead of the positive constant $\mu(\frac{1}{12}L_c^2)$. Moreover, we study general Dirichlet boundary conditions, which means that the boundary data for the displacement and for the microrotation may be prescribed independently. For simplicity we only consider $M = 0$.

The goal of this subsection is to prove that the following initial–boundary-value problem,

$$\left.\begin{aligned} \operatorname{div}\sigma &= -f, \\ \sigma &= 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\operatorname{skew}(\nabla u) - A) + \lambda\,\operatorname{tr}[\varepsilon]\cdot \mathbb{1}, \\ -l_c\Delta\operatorname{axl}(A) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u) - A), \\ \dot{\varepsilon}_p &\in \mathfrak{f}(T_E), \quad T_E = 2\mu(\varepsilon - \varepsilon_p), \\ u|_{\partial\Omega} &= u_d, \quad A|_{\partial\Omega} = A_d, \quad \varepsilon_p(0) = \varepsilon_p^0, \end{aligned}\right\} \tag{3.7}$$

possesses global-in-time L^2-solutions, assuming that the given data f, u_d, A_d, ε_p^0 satisfy some natural restrictions and $\mathfrak{f} : D(\mathfrak{f}) \subset \operatorname{Sym}(3) \to \mathcal{P}(\operatorname{Sym}(3))$ is supposed to be a *maximal monotone* mapping [4, definition 1, p. 140] with $0 \in \mathfrak{f}(0)$. This is, for example, verified for the flow function corresponding to classical ideal plasticity.[3] Here, for any set X the symbol $\mathcal{P}(X)$ denotes the family of all subsets of X. This mapping defines the maximal monotone operator

$$\mathfrak{f} : L^2(\Omega, \operatorname{Sym}(3)) \to \mathcal{P}(L^2(\Omega, \operatorname{Sym}(3)))$$

with the domain

$$\begin{aligned} \mathcal{D}(\mathfrak{f}) = \{T \in L^2(\Omega, \operatorname{Sym}(3)) : T(x) \in D(\mathfrak{f}) \text{ a.e. in } \Omega \text{ and there exists} \\ S \in L^2(\Omega, \operatorname{Sym}(3)) \text{ with } S(x) \in \mathfrak{f}(T(x)) \text{ a.e. in } \Omega\}. \end{aligned} \tag{3.8}$$

System (3.7) contains only one physical nonlinearity, the constitutive multifunction $\mathfrak{f}$, which is assumed to be maximal monotone. Such a nonlinear mapping can be approximated by single-valued, global Lipschitz functions $\mathfrak{f}_\eta$, called in the literature the *Yosida approximation* (see, for example, [4, theorem 2, p. 144]). Hence, our idea is quite natural: we rewrite (3.7) with $\mathfrak{f}_\eta$ instead of $\mathfrak{f}$ and try to pass to the limit $\eta \to 0^+$.

[3] Maximal monotonicity shows $\langle \mathfrak{f}(T_E) - \mathfrak{f}(0), T_E - 0\rangle = \langle \mathfrak{f}(T_E), T_E\rangle \geqslant 0$, which implies the reduced dissipation inequality (3.4).

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Thus, for all $\eta > 0$ we study first the following *approximated* initial–boundary-value problem

$$\left.\begin{aligned} \operatorname{div}\sigma^\eta &= -f,\\ \sigma^\eta &= 2\mu(\varepsilon^\eta - \varepsilon_{\mathrm{p}}^\eta) + 2\mu_{\mathrm{c}}(\operatorname{skew}(\nabla u^\eta) - A^\eta) + \lambda\operatorname{tr}[\varepsilon^\eta]\cdot \mathbb{1},\\ -l_{\mathrm{c}}\Delta\operatorname{axl}(A^\eta) &= -\mu_{\mathrm{c}}\operatorname{axl}(A^\eta) + \mu_{\mathrm{c}}\operatorname{axl}(\operatorname{skew}(\nabla u^\eta)),\\ \dot\varepsilon_{\mathrm{p}}^\eta &= \mathfrak{f}_\eta(T_{\mathrm{E}}^\eta), \quad T_{\mathrm{E}}^\eta = 2\mu(\varepsilon^\eta - \varepsilon_{\mathrm{p}}^\eta),\\ u^\eta|_{\partial\Omega} &= u_{\mathrm{d}}, \quad A^\eta|_{\partial\Omega} = A_{\mathrm{d}}, \quad \varepsilon_{\mathrm{p}}^\eta(0) = \varepsilon_{\mathrm{p}}^0. \end{aligned}\right\} \tag{3.9}$$

Theorem 3.1 (global existence and uniqueness for approximated problem). *Let us assume that the given data possess the following regularity: for all times $T > 0$,*

$$\begin{aligned} f &\in C([0,T], L^2(\Omega,\mathbb{R}^3)),\\ u_{\mathrm{d}} &\in C([0,T], H^{1/2}(\partial\Omega,\mathbb{R}^3)),\\ A_{\mathrm{d}} &\in C([0,T], H^{3/2}(\partial\Omega,\mathfrak{so}(3,\mathbb{R}))) \end{aligned}$$

and the initial data $\varepsilon_{\mathrm{p}}^0$ belong to $L^2(\Omega,\mathrm{Sym}(3))$. Then the approximated problem has a global-in-time, unique solution $(u^\eta,\varepsilon^\eta,\varepsilon_{\mathrm{p}}^\eta,A^\eta)$ with the regularity

$$\begin{aligned} u^\eta &\in C([0,T], H^1(\Omega,\mathbb{R}^3)), & \varepsilon^\eta &\in C([0,T], L^2(\Omega,\mathrm{Sym}(3))),\\ \varepsilon_{\mathrm{p}}^\eta &\in C^1([0,T], L^2(\Omega,\mathrm{Sym}(3))), & A^\eta &\in C([0,T], H^2(\Omega,\mathfrak{so}(3,\mathbb{R}))). \end{aligned}$$

If the given data are more regular in time, or, more precisely, if

$$\left.\begin{aligned} \dot f &\in C([0,T], L^2(\Omega,\mathbb{R}^3)),\\ \dot u_{\mathrm{d}} &\in C([0,T], H^{1/2}(\partial\Omega,\mathbb{R}^3)),\\ \dot A_{\mathrm{d}} &\in C([0,T], H^{3/2}(\partial\Omega,\mathfrak{so}(3,\mathbb{R}))), \end{aligned}\right\} \tag{3.10}$$

then the unique solution is also C^1 in time.

Proof. We give a sketch of the proof, which is otherwise standard. Note that the approximated system of equations contains only global Lipschitz nonlinearities. Hence, we use Banach's Fix Point Theorem. For a fixed time $T > 0$ let us denote by X the Banach space $C([0,T], L^2(\Omega,\mathrm{Sym}(3)))$. We define an operator $P : X \to X$ as follows: for $\varepsilon \in X$ we solve the integral equation

$$\varepsilon_{\mathrm{p}}(t) = \int_0^t \mathfrak{f}_\eta(2\mu(\varepsilon(\tau) - \varepsilon_{\mathrm{p}}(\tau)))\,\mathrm{d}\tau + \varepsilon_{\mathrm{p}}^0. \tag{3.11}$$

By the regularity of $\mathfrak{f}_\eta$ it follows that this nonlinear integral equation is uniquely solvable in X. Then, for the solution ε_{p}, we study the elliptic boundary-value problem

$$\left.\begin{aligned} \operatorname{div}(2\mu(\varepsilon - \varepsilon_{\mathrm{p}}) + 2\mu_{\mathrm{c}}(\operatorname{skew}(\nabla u) - A) + \lambda\operatorname{tr}[\varepsilon]\cdot\mathbb{1}) &= -f,\\ -l_{\mathrm{c}}\Delta\operatorname{axl}(A) &= -\mu_{\mathrm{c}}\operatorname{axl}(A) + \mu_{\mathrm{c}}\operatorname{axl}(\operatorname{skew}(\nabla u)),\\ u|_{\partial\Omega} = u_{\mathrm{d}}, \quad A|_{\partial\Omega} &= A_{\mathrm{d}}, \end{aligned}\right\} \tag{3.12}$$

for the pair (u, A) of unknown functions. This problem has a unique solution u with the regularity $C([0,T], H^1(\Omega, \mathbb{R}^3))$ and $A \in C([0,T], H^2(\Omega, \mathfrak{so}(3,\mathbb{R})))$. Finally, we set $P(\varepsilon) := \frac{1}{2}(\nabla u + \nabla^{\mathrm{T}} u)$. By direct inspection of (3.12) it is not difficult to see that for two input functions $\varepsilon^1, \varepsilon^2 \in X$ we have

$$\|P(\varepsilon^1)(t) - P(\varepsilon^2)(t)\|_{L^2(\Omega)} \leqslant C\|\varepsilon_{\mathrm{p}}^1(t) - \varepsilon_{\mathrm{p}}^2(t)\|_{L^2(\Omega)},$$

where $\varepsilon_{\mathrm{p}}^1(t)$ and $\varepsilon_{\mathrm{p}}^2(t)$ are solutions of (3.11) with the input functions ε^1 and ε^2, respectively, and the positive constant C does not depend on these input functions and is independent of t. Hence, looking at (3.12) we see that for short times T the operator P is a contraction. Moreover, the contraction constant depends only on the Lipschitz constant of the function $\mathfrak{f}_\eta$ and on the time T. Hence, for small T the mapping P possesses a unique fixed point in X and this function defines a local-in-time solution of the approximated system. Next, using the fact that the length of the existence interval does not depend on the given data, we may extend the solution with the same time-step and obtain a global-in-time, unique solution. Finally, we see that the solution ε_{p} is even more regular in time; this means $\varepsilon_{\mathrm{p}} \in C^1([0,T], L^2(\Omega, \mathrm{Sym}(3)))$. Then for given data satisfying (3.10) we conclude that the solution is C^1 in time. □

The main idea of the last proof was based on the global Lipschitz property of the nonlinear function $\mathfrak{f}_\eta$. However, we have not yet used the physical structure of the problem. Next, we prove that the energy associated with the problem is bounded independently of the parameter η. The energy is defined by

$$\mathcal{E}(u, \varepsilon, \varepsilon_{\mathrm{p}}, A)(t) = \int_\Omega (\mu\|\varepsilon - \varepsilon_{\mathrm{p}}\|^2 + \tfrac{1}{2}\lambda \operatorname{tr}[\varepsilon]^2 + \mu_{\mathrm{c}}\|\operatorname{skew}(\nabla u) - A\|^2 + 2l_{\mathrm{c}}\|\nabla \operatorname{axl}(A)\|^2)\,\mathrm{d}x.$$

THEOREM 3.2 (coerciveness of the energy). *The energy function is elastically coercive with respect to* ∇u. *This means that* $\exists C_{\mathrm{E}} > 0$, $\forall u \in H_\circ^{1,2}(\Omega)$, $\forall A \in H_\circ^{1,2}(\Omega)$, $\forall \varepsilon_{\mathrm{p}} \in L^2(\Omega)$

$$\mathcal{E}(u, \varepsilon, \varepsilon_{\mathrm{p}}, A) \geqslant C_{\mathrm{E}}(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)}).$$

Moreover, $\exists C_{\mathrm{E}} > 0$, $\forall u_{\mathrm{d}}, A_{\mathrm{d}} \in H^{1/2}(\partial\Omega)$, $\exists C_{\mathrm{d}} > 0$, $\forall \varepsilon_{\mathrm{p}} \in L^2(\Omega)$, $\forall u \in H^{1,2}(\Omega)$, $\forall A \in H^{1,2}(\Omega)$ *with* $u|_{\partial\Omega} = u_{\mathrm{d}}$ *and* $A|_{\partial\Omega} = A_{\mathrm{d}}$ *it holds that*

$$\mathcal{E}(u, \varepsilon, \varepsilon_{\mathrm{p}}, A) + C_{\mathrm{d}} \geqslant C_{\mathrm{E}}(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)}).$$

Proof. We begin with the first statement. From the definition of the energy we see that

$$\begin{aligned}\mathcal{E}(u, \varepsilon, \varepsilon_{\mathrm{p}}, A) &\geqslant \int_\Omega (\tfrac{1}{2}\lambda \operatorname{tr}[\varepsilon]^2 + \mu_{\mathrm{c}}\|\operatorname{skew}(\nabla u) - A\|^2 + 2l_{\mathrm{c}}\|\nabla \operatorname{axl}(A)\|^2)\,\mathrm{d}x \\ &\geqslant \int_\Omega (\tfrac{1}{2}\lambda|\operatorname{div} u|^2 + \mu_{\mathrm{c}}\|\operatorname{skew}(\nabla u)\|^2 + \mu_{\mathrm{c}}\|A\|^2 \\ &\qquad - 2\mu_{\mathrm{c}}\langle\operatorname{skew}(\nabla u), A\rangle + 2l_{\mathrm{c}}\|\nabla \operatorname{axl}(A)\|^2)\,\mathrm{d}x \\ &\geqslant \int_\Omega (\tfrac{1}{2}\lambda|\operatorname{div} u|^2 + \tfrac{1}{2}\mu_{\mathrm{c}}\|\operatorname{skew}(\nabla u)\|^2 - \mu_{\mathrm{c}}\|A\|^2 + 2l_{\mathrm{c}}\|\nabla \operatorname{axl}(A)\|^2)\,\mathrm{d}x.\end{aligned} \tag{3.13}$$

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Next we estimate the negative term using the Poincaré inequality and the definition of the energy

$$\begin{aligned}\mu_c\|A\|^2_{L^2(\Omega)} &= 2\mu_c\|\operatorname{axl}(A)\|^2_{L^2(\Omega)}\\ &\leqslant 2\mu_c C_\Omega\|\nabla\operatorname{axl}(A)\|^2_{L^2(\Omega)}\\ &\leqslant \frac{\mu_c C_\Omega}{l_c}\mathcal{E}(u,\varepsilon,\varepsilon_p,A),\end{aligned}$$

where the constant $C_\Omega > 0$ depends on the domain Ω only. Moreover, we use the following well-known estimate [27, p. 36]:

$$\|\nabla u\|^2_{L^2(\Omega)} \leqslant C^{\mathrm{curl}}_{\mathrm{div}}(\|\operatorname{div} u\|^2_{L^2(\Omega)} + \|\operatorname{curl} u\|^2_{L^2(\Omega)}),$$

where the positive constant $C^{\mathrm{curl}}_{\mathrm{div}}$ does not depend on u. Inserting the latter two inequalities into the main estimate (3.13) we have

$$\left(1+\frac{\mu_c C_\Omega}{l_c}\right)\mathcal{E}(u,\varepsilon,\varepsilon_p,A) \geqslant \frac{1}{2C^{\mathrm{curl}}_{\mathrm{div}}}\min\{\lambda,\mu_c\}\|\nabla u\|^2_{L^2(\Omega)} + 2l_c\|\nabla\operatorname{axl}(A)\|^2_{L^2(\Omega)}.$$

From this the first statement follows immediately. It is important to see that the estimate follows from the positivity of μ_c, λ and $l_c = \frac{1}{12}\mu L_c^2$.

The second statement follows from the first one if we select fixed functions $\tilde{u}, \tilde{A} \in H^{1,2}(\Omega)$ such that $\tilde{u}|_{\partial\Omega} = u_d$, $\tilde{A}|_{\partial\Omega} = A_d$ and use the first inequality for the differences $u - \tilde{u}$ and $A - \tilde{A}$. □

The coerciveness of the energy is the crucial one in our existence theory. In classical rate-independent plasticity, $\operatorname{curl} u$ is not controlled. We also note here that the energy $\mathcal{E}$ already controls the $L^2(\Omega)$ norm of the inelastic strain ($\|\varepsilon_p\|_{L^2(\Omega)} \leqslant \|\varepsilon\|_{L^2(\Omega)} + \|\varepsilon - \varepsilon_p\|_{L^2(\Omega)}$).

Theorem 3.3 (energy estimate for the approximate sequence). *Let us assume the given data satisfy (3.10) and that* $\{(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)\}$ *is the solution of the approximate problem. Then for all times* $T > 0$ *there exists a positive constant* $C(T)$*, independent of* η*, such that*

$$\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(t) \leqslant C(T) \quad \textit{for all } t \in [0,T). \tag{3.14}$$

Proof. Calculating the time derivative of the energy, we obtain

$$\begin{aligned}&\dot{\mathcal{E}}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(t)\\ &\quad= \int_\Omega (2\mu\langle\varepsilon^\eta-\varepsilon_p^\eta,\dot\varepsilon^\eta-\dot\varepsilon_p^\eta\rangle + \lambda\operatorname{tr}[\varepsilon^\eta]\operatorname{tr}[\dot\varepsilon^\eta]\\ &\qquad\quad + 2\mu_c\langle\operatorname{skew}(\nabla u^\eta)-A^\eta,\operatorname{skew}(\nabla\dot u^\eta)-\dot A^\eta\rangle + 4l_c\langle\nabla\operatorname{axl}(A^\eta),\nabla\operatorname{axl}(\dot A^\eta)\rangle)\,\mathrm{d}x\\ &\quad= -\int_\Omega\langle T_E^\eta,\dot\varepsilon_p^\eta\rangle\,\mathrm{d}x + \int_\Omega\langle\sigma^\eta,\nabla\dot u^\eta\rangle\,\mathrm{d}x\\ &\qquad\quad - 2\mu_c\int_\Omega\langle\operatorname{skew}(\nabla u^\eta)-A^\eta,\dot A^\eta\rangle\,\mathrm{d}x + 4l_c\int_\Omega\langle\nabla\operatorname{axl}(A^\eta),\nabla\operatorname{axl}(\dot A^\eta)\rangle\,\mathrm{d}x.\end{aligned}$$

The first integral on the right-hand side of the last equality is non-negative. In the second and the fourth integrals we integrate by parts to obtain

$$\begin{aligned}\dot{\mathcal{E}}(u^\eta,\varepsilon^\eta,\varepsilon_{\mathrm p}^\eta,A^\eta)(t)&\\ \leqslant{}&\int_\Omega\langle f,\dot u^\eta\rangle\,\mathrm dx+\int_{\partial\Omega}\langle\sigma^\eta\cdot n,\dot u^\eta\rangle\,\mathrm ds\\ &-4\mu_{\mathrm c}\int_\Omega\langle\operatorname{axl}\operatorname{skew}(\nabla u^\eta)-\operatorname{axl}(A^\eta),\operatorname{axl}(\dot A^\eta)\rangle\,\mathrm dx\\ &-4l_{\mathrm c}\int_\Omega\langle\Delta\operatorname{axl}(A^\eta),\operatorname{axl}(\dot A^\eta)\rangle\,\mathrm dx+4l_{\mathrm c}\int_{\partial\Omega}\langle\nabla\operatorname{axl}(A^\eta)\cdot n,\operatorname{axl}(\dot A^\eta)\rangle\,\mathrm ds.\end{aligned}$$

Using the equation for the microrotations and the boundary conditions, we finally obtain

$$\begin{aligned}\dot{\mathcal{E}}(u^\eta,\varepsilon^\eta,\varepsilon_{\mathrm p}^\eta,A^\eta)(t)\leqslant{}&\int_\Omega\langle f,\dot u^\eta\rangle\,\mathrm dx+\int_{\partial\Omega}\langle\sigma^\eta\cdot n,\dot u_{\mathrm d}\rangle\,\mathrm ds\\ &+4l_{\mathrm c}\int_{\partial\Omega}\langle\nabla\operatorname{axl}(A^\eta)\cdot n,\operatorname{axl}(\dot A_{\mathrm d})\rangle\,\mathrm ds.\end{aligned}\tag{3.15}$$

Note that the boundary integrals are defined in the sense of the duality between the spaces

$$H^{1/2}(\partial\Omega,\mathbb R^3)\quad\text{and}\quad H^{-1/2}(\partial\Omega,\mathbb R^3).$$

Integrating (3.15) in time we arrive at the inequality

$$\begin{aligned}\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_{\mathrm p}^\eta,A^\eta)(t)\leqslant{}&\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_{\mathrm p}^\eta,A^\eta)(0)\\ &+\int_0^t\int_\Omega\langle f,\dot u^\eta\rangle\,\mathrm dx+\int_0^t\int_{\partial\Omega}\langle\sigma^\eta\cdot n,\dot u_{\mathrm d}\rangle\,\mathrm ds\\ &+4l_{\mathrm c}\int_0^t\int_{\partial\Omega}\langle\nabla\operatorname{axl}(A^\eta)\cdot n,\operatorname{axl}(\dot A_{\mathrm d})\rangle\,\mathrm ds.\end{aligned}\tag{3.16}$$

By the continuity with respect to time we conclude that the initial values $u^\eta(0)$, $\varepsilon^\eta(0)$, $A^\eta(0)$ are solutions of the following linear elliptic boundary-value problem

$$\left.\begin{gathered}\operatorname{div}\sigma^\eta(0)=-f,\\ \sigma^\eta(0)=2\mu(\varepsilon^\eta(0)-\varepsilon_{\mathrm p}^\eta(0))+2\mu_{\mathrm c}(\operatorname{skew}(\nabla u^\eta(0))-A^\eta(0))+\lambda\operatorname{tr}[\varepsilon^\eta(0)]\cdot\mathbb 1,\\ -l_{\mathrm c}\Delta\operatorname{axl}(A^\eta(0))=-\mu_{\mathrm c}\operatorname{axl}(A^\eta(0))+\mu_{\mathrm c}\operatorname{axl}(\operatorname{skew}(\nabla u^\eta(0))),\\ u^\eta(0)|_{\partial\Omega}=u_{\mathrm d},\quad A^\eta(0)|_{\partial\Omega}=A_{\mathrm d},\end{gathered}\right\}\tag{3.17}$$

where $\varepsilon^\eta(0)=\frac12(\nabla u^\eta(0)+\nabla^{\mathrm T}u^\eta(0))$. The unique solution of (3.17) satisfies

$$u^\eta(0)\in H^1(\Omega,\mathbb R^3),\quad\varepsilon^\eta(0)\in L^2(\Omega,\operatorname{Sym}(3)),\quad A^\eta(0)\in H^2(\Omega,\mathfrak{so}(3,\mathbb R))$$

and is independent of η. The initial energy value $\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_{\mathrm p}^\eta,A^\eta)(0)$ is a constant. Next, we analyse the first integral from the right-hand side of (3.16). Integrating partially in time, we obtain

$$\begin{aligned}
&\int_0^t \int_\Omega \langle f, \dot{u}^\eta \rangle \,\mathrm{d}x \,\mathrm{d}\tau \\
&\quad = -\int_0^t \int_\Omega \langle \dot{f}, u^\eta \rangle \,\mathrm{d}x \,\mathrm{d}\tau + \int_\Omega \langle f(t), u^\eta(t) \rangle \,\mathrm{d}x - \int_\Omega \langle f(0), u^\eta(0) \rangle \,\mathrm{d}x \\
&\quad \leqslant \frac{1}{2} \int_0^t \|\dot{f}\|_{L^2}^2 \,\mathrm{d}\tau + \frac{1}{2} \int_0^t \|u^\eta\|_{L^2}^2 \,\mathrm{d}\tau + \|f(0)\|_{L^2} \|u^\eta(0)\|_{L^2} + \|f(t)\|_{L^2} \|u^\eta(t)\|_{L^2}.
\end{aligned}$$

By Poincaré's inequality we conclude that

$$\begin{aligned}
\|u^\eta(t)\|_{L^2} &\leqslant \|u^\eta(t) - \tilde{u}_{\mathrm{d}}(t)\|_{L^2} + \|\tilde{u}_{\mathrm{d}}(t)\|_{L^2} \\
&\leqslant \operatorname{diam}(\Omega)(\|\nabla u^\eta(t)\|_{L^2} + \|\nabla \tilde{u}_{\mathrm{d}}(t)\|_{L^2}) + \|\tilde{u}_{\mathrm{d}}(t)\|_{L^2},
\end{aligned}$$

where $\tilde{u}_{\mathrm{d}}$ is a function from $H^1(\Omega, \mathbb{R}^3)$ with $\tilde{u}_{\mathrm{d}|\partial\Omega} = u_{\mathrm{d}}$. By the coercivity of the energy with respect to the gradient of u^η there exists a positive constant C_{E} and a function $C_{\mathrm{d}}(t)$ independent of η such that $\|\nabla u^\eta(t)\|_{L^2} \leqslant C_{\mathrm{E}} \mathcal{E}^{1/2}(u^\eta, \varepsilon^\eta, \varepsilon_{\mathrm{p}}^\eta, A^\eta)(t) + C_{\mathrm{d}}(t)$. Using the latter results we have

$$\begin{aligned}
\left| \int_0^t \int_\Omega \langle f, \dot{u}^\eta \rangle \,\mathrm{d}x \,\mathrm{d}\tau \right| &\leqslant C \int_0^t \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_{\mathrm{p}}^\eta, A^\eta)(\tau) \,\mathrm{d}\tau \\
&\quad + C \|f(t)\|_{L^2} \mathcal{E}^{1/2}(u^\eta, \varepsilon^\eta, \varepsilon_{\mathrm{p}}^\eta, A^\eta)(t) + C(t), \qquad (3.18)
\end{aligned}$$

where the constants C, $C(t)$ do not depend on η and $C(t)$ depends on the given data only. The second integral in (3.16) is estimated (by the trace theorem in the space $L^2(\mathrm{div})$ [51, ch. 1]) as follows:

$$\begin{aligned}
\left| \int_0^t \int_{\partial\Omega} \langle \sigma^\eta \cdot n, \dot{u}_{\mathrm{d}} \rangle \,\mathrm{d}s \right| &\leqslant \int_0^t \|\sigma^\eta \cdot n\|_{H^{-1/2}} \|\dot{u}_{\mathrm{d}}\|_{H^{1/2}} \,\mathrm{d}\tau \\
&\leqslant C \int_0^t (\|\sigma^\eta\|_{L^2} + \|\operatorname{div} \sigma^\eta\|_{L^2}) \|\dot{u}_{\mathrm{d}}\|_{H^{1/2}} \,\mathrm{d}\tau \\
&\leqslant C \int_0^t \|f\|_{L^2} \|\dot{u}_{\mathrm{d}}\|_{H^{1/2}} \,\mathrm{d}\tau \\
&\leqslant +C \int_0^t \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_{\mathrm{p}}^\eta, A^\eta)(\tau) \,\mathrm{d}\tau + C \int_0^t \|\dot{u}_{\mathrm{d}}\|_{H^{1/2}}^2 \,\mathrm{d}\tau, \qquad (3.19)
\end{aligned}$$

where $C > 0$ does not depend on η. To estimate the last integral in (3.16) we use the H^2-regularity of the microrotations

$$\begin{aligned}
&\left| \int_0^t \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta) \cdot n, \operatorname{axl}(\dot{A}_{\mathrm{d}}) \rangle \,\mathrm{d}s \right| \\
&\qquad \leqslant \int_0^t \|\nabla \operatorname{axl}(A^\eta) \cdot n\|_{H^{-1/2}} \|\operatorname{axl}(\dot{A}_{\mathrm{d}})\|_{H^{1/2}} \,\mathrm{d}\tau \\
&\qquad \leqslant C \int_0^t (\|\nabla \operatorname{axl}(A^\eta)\|_{L^2} + \|\Delta \operatorname{axl}(A^\eta)\|_{L^2}) \|\operatorname{axl}(\dot{A}_{\mathrm{d}})\|_{H^{1/2}} \,\mathrm{d}\tau
\end{aligned}$$

$$= C\int_0^t \left(\|\nabla \operatorname{axl}(A^\eta)\|_{L^2} + \frac{\mu_c}{l_c}\|\operatorname{skew}(\nabla u^\eta) - A^\eta\|_{L^2}\right)\|\operatorname{axl}(\dot A_d)\|_{H^{1/2}}\,d\tau$$
$$\leqslant \tilde C\int_0^t \mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(\tau)\,d\tau + \tilde C\int_0^t \|\operatorname{axl}(\dot A_d)\|^2_{H^{1/2}}\,d\tau, \tag{3.20}$$

where again the constants C, $\tilde C$ do not depend on η. On inserting (3.18)–(3.20) into (3.16) we obtain

$$\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(t) \leqslant C_1\|f(t)\|_{L^2}\mathcal{E}^{1/2}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(t) + C_2\int_0^t \mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta)(\tau)\,d\tau + C_3(t), \tag{3.21}$$

where C_1, C_2, $C_3(t)$ do not depend on η and $C_3(t)$ depends only on the given data. Next, using the inequality $ab \leqslant \delta a^2 + (1/4\delta)b^2$, we separate in the first term on the right-hand side the energy with a small factor and absorb this expression by the left-hand side. Finally, Gronwall's lemma completes the proof. □

The energy estimate proved in the last theorem yields boundedness of the stresses $\{\sigma^\eta\}$ in the space $L^\infty((0,T),L^2(\Omega,\operatorname{Sym}(3)))$ and of the microrotations $\{A^\eta\}$ in the space $L^\infty((0,T),H^1(\Omega,\mathfrak{so}(3,\mathbb{R})))$. Moreover, using the fact that the energy controls the gradient of the displacement, the sequence $\{u^\eta\}$ is bounded in the space $L^\infty((0,T),H^1(\Omega,\mathbb{R}^3))$ and consequently the sequence of strains $\{\varepsilon^\eta\}$ and the sequence of inelastic strains $\{\varepsilon_p^\eta\}$ are bounded in the space

$$L^\infty((0,T),L^2(\Omega,\operatorname{Sym}(3))).$$

Hence, for a subsequence (again denoted using the superscript η) we find that, for all $T > 0$,

$$\begin{aligned}
\sigma^\eta &\overset{*}{\rightharpoonup} \sigma \quad \text{in } L^\infty((0,T),L^2(\Omega,\operatorname{Sym}(3))),\\
A^\eta &\overset{*}{\rightharpoonup} A \quad \text{in } L^\infty((0,T),H^1(\Omega,\mathfrak{so}(3,\mathbb{R}))),\\
u^\eta &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty((0,T),H^1(\Omega,\mathbb{R}^3)),\\
\varepsilon^\eta &\overset{*}{\rightharpoonup} \varepsilon \quad \text{in } L^\infty((0,T),L^2(\Omega,\operatorname{Sym}(3))),\\
\varepsilon_p^\eta &\overset{*}{\rightharpoonup} \varepsilon_p \quad \text{in } L^\infty((0,T),L^2(\Omega,\operatorname{Sym}(3)))
\end{aligned}$$

and the limit functions satisfy

$$\varepsilon = \tfrac{1}{2}(\nabla u + \nabla^T u), \qquad \sigma = 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\operatorname{skew}(\nabla u) - A) + \lambda\operatorname{tr}[\varepsilon]\cdot\mathbb{1}.$$

Moreover, we see that the sequence $\{\operatorname{div}\sigma^\eta\}$ is constant with respect to η and consequently is bounded in the space $L^\infty((0,T),L^2(\Omega,\mathbb{R}^3))$, and the sequence $\{\Delta\operatorname{axl}(A^\eta)\}$ is bounded in the space $L^\infty((0,T),L^2(\Omega,\mathbb{R}^3))$. Using the closedness of the differential operators in Sobolev spaces, the limit functions satisfy the system

$$\begin{aligned}
\operatorname{div}\sigma &= -f,\\
-l_c\Delta\operatorname{axl}(A) &= -\mu_c\operatorname{axl}(A) + \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u)),\\
u|_{\partial\Omega} = u_d, &\qquad A|_{\partial\Omega} = A_d.
\end{aligned}$$

Thus, to complete the existence theory for the infinitesimal elasto-plastic Cosserat model we should prove only that the limit functions satisfy the differential inclusion $(3.7)_4$. The sequence $T_{\mathrm{E}}^{\eta} = 2\mu(\varepsilon^{\eta} - \varepsilon_{\mathrm{p}}^{\eta})$ converges weakly-$*$ to $T_{\mathrm{E}} = 2\mu(\varepsilon - \varepsilon_{\mathrm{p}})$ and the sequence

$$\int_0^t \mathfrak{f}_{\eta}(T_{\mathrm{E}}^{\eta})\,\mathrm{d}\tau = \varepsilon_{\mathrm{p}}^{\eta} - \varepsilon_{\mathrm{p}}^{0}$$

also converges weakly-$*$ to $\varepsilon_{\mathrm{p}} - \varepsilon_{\mathrm{p}}^{0}$. To conclude that the limit functions ε_{p} and T_{E} satisfy the differential inclusion, we need estimates for the sequence $\{\mathfrak{f}_{\eta}(T_{\mathrm{E}}^{\eta})\}$. Hence, the next step in our existence theory is an estimate for time derivatives of the approximate sequence.

THEOREM 3.4 (energy estimate for time derivatives). *Suppose that the given data possess more time regularity as in the last theorem and satisfy additionally: for all times $T > 0$*

$$\left.\begin{aligned} \ddot{f} &\in L^2((0,T)\times\Omega,\mathbb{R}^3),\\ \partial_t^3 u_{\mathrm{d}} &\in L^2((0,T),H^{1/2}(\partial\Omega,\mathbb{R}^3)),\\ \partial_t^3 A_{\mathrm{d}} &\in L^2((0,T),H^{1/2}(\partial\Omega,\mathfrak{so}(3,\mathbb{R}))).\end{aligned}\right\}\tag{3.22}$$

Moreover, assume that the initial data $\varepsilon_{\mathrm{p}}^{0} \in L^2(\Omega,\mathrm{Sym}(3))$ are chosen such that the initial value of the reduced Eshelby tensor $T_{\mathrm{E}}(0) = 2\mu(\varepsilon(0) - \varepsilon_{\mathrm{p}}^{0})$ defined by system (3.17) belongs to the domain of the maximal monotone operator $\mathfrak{f}$. Then there exists a positive constant $C(T)$, independent of the parameter η, such that

$$\mathcal{E}(\dot{u}^{\eta},\dot{\varepsilon}^{\eta},\dot{\varepsilon}_{\mathrm{p}}^{\eta},\dot{A}^{\eta})(t) \leqslant C(T) \quad \textit{for all } t\in[0,T).$$

Proof. For $h > 0$ let us denote by $(u_h^{\eta}(t),\varepsilon_h^{\eta}(t),\varepsilon_{\mathrm{p},h}^{\eta}(t),A_h^{\eta}(t))$ the shifted functions $(u^{\eta}(t+h),\varepsilon^{\eta}(t+h),\varepsilon_{\mathrm{p}}^{\eta}(t+h),A^{\eta}(t+h))$ and calculate the energy evaluated on the differences $(u_h^{\eta} - u^{\eta},\dots)$. Then for the time derivative we have

$$\begin{aligned}
&\dot{\mathcal{E}}(u_h^{\eta} - u^{\eta},\varepsilon_h^{\eta} - \varepsilon^{\eta},\varepsilon_{\mathrm{p},h}^{\eta} - \varepsilon_{\mathrm{p}}^{\eta},A_h^{\eta} - A^{\eta})(t)\\
&\quad = \int_{\Omega} 2\mu\langle \varepsilon_h^{\eta} - \varepsilon^{\eta} - \varepsilon_{\mathrm{p},h}^{\eta} + \varepsilon_{\mathrm{p}}^{\eta}, \dot{\varepsilon}_h^{\eta} - \dot{\varepsilon}^{\eta} - \dot{\varepsilon}_{\mathrm{p},h}^{\eta} + \dot{\varepsilon}_{\mathrm{p}}^{\eta}\rangle\,\mathrm{d}x\\
&\qquad + 2\mu_{\mathrm{c}}\int_{\Omega}\langle \mathrm{skew}(\nabla u_h^{\eta} - \nabla u^{\eta}) - A_h^{\eta} + A^{\eta}, \mathrm{skew}(\nabla\dot{u}_h^{\eta} - \nabla\dot{u}^{\eta}) - \dot{A}_h^{\eta} + \dot{A}^{\eta}\rangle\,\mathrm{d}x\\
&\qquad + \lambda\int_{\Omega}\mathrm{tr}[\varepsilon_h^{\eta} - \varepsilon^{\eta}]\,\mathrm{tr}[\dot{\varepsilon}_h^{\eta} - \dot{\varepsilon}^{\eta}]\,\mathrm{d}x + 4l_{\mathrm{c}}\int_{\Omega}\langle\nabla\,\mathrm{axl}(A_h^{\eta} - A^{\eta}),\nabla\,\mathrm{axl}(\dot{A}_h^{\eta} - \dot{A}^{\eta})\rangle\,\mathrm{d}x\\
&\quad = -\int_{\Omega}\langle T_{\mathrm{E},h}^{\eta} - T_{\mathrm{E}}^{\eta}, \dot{\varepsilon}_{\mathrm{p},h}^{\eta} - \dot{\varepsilon}_{\mathrm{p}}^{\eta}\rangle\,\mathrm{d}x + \int_{\Omega}\langle\sigma_h^{\eta} - \sigma^{\eta},\nabla\dot{u}_h^{\eta} - \nabla\dot{u}^{\eta}\rangle\,\mathrm{d}x\\
&\qquad + 4\mu_{\mathrm{c}}\int_{\Omega}\langle\mathrm{axl}\,\mathrm{skew}(\nabla u_h^{\eta} - \nabla u^{\eta}) - \mathrm{axl}(\dot{A}_h^{\eta} - \dot{A}^{\eta})\rangle\,\mathrm{d}x\\
&\qquad + 4l_{\mathrm{c}}\int_{\Omega}\langle\nabla\,\mathrm{axl}(A_h^{\eta} - A^{\eta}),\nabla\,\mathrm{axl}(\dot{A}_h^{\eta} - \dot{A}^{\eta})\rangle\,\mathrm{d}x,
\end{aligned}\tag{3.23}$$

where $T_{\mathrm{E},h}^{\eta}(t) = T_{\mathrm{E}}^{\eta}(t+h)$ and $\sigma_h^{\eta}(t) = \sigma^{\eta}(t+h)$. By the monotonicity of the Yosida approximation the first term on the right-hand side of (3.23) is non-positive. Similar to the energy estimate in theorem 3.3 we integrate by parts in the second and in

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the fourth integral and use the equation for microrotations. Hence, we arrive at the inequality

$$\begin{aligned}\dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{\mathrm{p},h}^\eta - \varepsilon_{\mathrm{p}}^\eta, A_h^\eta - A^\eta)(t)&\\ \leqslant \int_\Omega \langle f_h - f, \dot{u}_h^\eta - \dot{u}^\eta\rangle \,\mathrm{d}x + \int_{\partial\Omega} \langle (\sigma_h^\eta - \sigma^\eta)\cdot n, \dot{u}_{\mathrm{d},h} - \dot{u}_{\mathrm{d}}\rangle \,\mathrm{d}s&\\ + 4l_{\mathrm{c}} \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A_h^\eta - A^\eta)\cdot n, \operatorname{axl}(\dot{A}_{\mathrm{d},h} - \dot{A}_{\mathrm{d}})\rangle \,\mathrm{d}s,& \qquad (3.24)\end{aligned}$$

where $f_h(t) = f(t+h)$, $u_{\mathrm{d},h}(t) = u_{\mathrm{d}}(t+h)$ and $A_{\mathrm{d},h}(t) = A_{\mathrm{d}}(t+h)$. Next, we integrate (3.24) in time and obtain

$$\begin{aligned}\mathcal{E}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{\mathrm{p},h}^\eta - \varepsilon_{\mathrm{p}}^\eta, A_h^\eta - A^\eta)(t)&\\ \leqslant \mathcal{E}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{\mathrm{p},h}^\eta - \varepsilon_{\mathrm{p}}^\eta, A_h^\eta - A^\eta)(0)&\\ + \int_0^t \int_\Omega \langle f_h - f, \dot{u}_h^\eta - \dot{u}^\eta\rangle \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_{\partial\Omega} \langle (\sigma_h^\eta - \sigma^\eta)\cdot n, \dot{u}_{\mathrm{d},h} - \dot{u}_{\mathrm{d}}\rangle \,\mathrm{d}s\,\mathrm{d}\tau&\\ + 4l_{\mathrm{c}} \int_0^t \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A_h^\eta - A^\eta)\cdot n, \operatorname{axl}(\dot{A}_{\mathrm{d},h} - \dot{A}_{\mathrm{d}})\rangle \,\mathrm{d}s\,\mathrm{d}\tau.& \qquad (3.25)\end{aligned}$$

Before we divide (3.25) by h^2 we should shift in the integral terms the shift operator onto given data. We calculate this with details for the first integral only.

$$\begin{aligned}\int_0^t \int_\Omega \langle f_h - f, \dot{u}_h^\eta - \dot{u}^\eta\rangle \,\mathrm{d}x\,\mathrm{d}\tau&\\ = \int_\Omega \int_0^t \langle f(\tau+h) - f(\tau), \dot{u}^\eta(\tau+h)\rangle d\tau \,\mathrm{d}x&\\ - \int_\Omega \int_0^t \langle f(\tau+h) - f(\tau), \dot{u}^\eta(\tau)\rangle d\tau \,\mathrm{d}x&\\ = (\tau + h = s \text{ in the first integral})&\\ = \int_\Omega \int_h^{t+h} \langle f(s) - f(s-h), \dot{u}^\eta(s)\rangle \,\mathrm{d}s\,\mathrm{d}x&\\ - \int_\Omega \int_0^t \langle f(s+h) - f(s), \dot{u}^\eta(s)\rangle \,\mathrm{d}s\,\mathrm{d}x&\\ = -\int_\Omega \int_h^{t+h} \langle f(s+h) - 2f(s) + f(s-h), \dot{u}^\eta(s)\rangle \,\mathrm{d}s\,\mathrm{d}x&\\ - \int_\Omega \int_0^h \langle f(s+h) - f(s), \dot{u}^\eta(s)\rangle \,\mathrm{d}x\,\mathrm{d}s&\\ + \int_\Omega \int_t^{t+h} \langle f(s+h) - f(s), \dot{u}^\eta(s)\rangle \,\mathrm{d}s\,\mathrm{d}x.& \qquad (3.26)\end{aligned}$$

In the same manner we transform the second and the third integral terms from (3.25). Next, we insert (3.26) and the results for other terms into (3.25), divide by

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h^2 and pass to the limit $h \to 0^+$. Hence, we conclude with the following inequality:

$$\begin{aligned}\mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_{\mathrm{p}}, \dot A^\eta)(t) \leqslant \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_{\mathrm{p}}, \dot A^\eta)(0) &- \int_0^t \int_\Omega \langle \ddot f, \dot u^\eta\rangle \,\mathrm{d}x\,\mathrm{d}\tau \\ &- \int_\Omega \langle \dot f(0), \dot u^\eta(0)\rangle \,\mathrm{d}x + \int_\Omega \langle \dot f(t), \dot u^\eta(t)\rangle\,\mathrm{d}x \\ &- \int_0^t \int_{\partial\Omega} \langle \sigma^\eta \cdot n, \partial_t^3 u_{\mathrm{d}}\rangle \,\mathrm{d}s\,\mathrm{d}\tau - \int_{\partial\Omega} \langle \sigma^\eta(0)\cdot n, \partial_t^2 u_{\mathrm{d}}(0)\rangle\,\mathrm{d}s \\ &+ \int_{\partial\Omega} \langle \sigma^\eta(t)\cdot n, \partial_t^2 u_{\mathrm{d}}(t)\rangle\,\mathrm{d}s \\ &- 4l_{\mathrm{c}} \int_0^t \int_{\partial\Omega} \langle \nabla\,\mathrm{axl}(A^\eta)\cdot n, \mathrm{axl}(\partial_t^3 A_{\mathrm{d}})\rangle\,\mathrm{d}s\,\mathrm{d}\tau \\ &- 4l_{\mathrm{c}} \int_{\partial\Omega} \langle \nabla\,\mathrm{axl}(A^\eta)(0)\cdot n, \mathrm{axl}(\partial_t^2 A_{\mathrm{d}})(0)\rangle\,\mathrm{d}s \\ &+ 4l_{\mathrm{c}} \int_{\partial\Omega} \langle \nabla\,\mathrm{axl}(A^\eta)(t)\cdot n, \mathrm{axl}(\partial_t^2 A_{\mathrm{d}})(t)\rangle\,\mathrm{d}s. \end{aligned} \tag{3.27}$$

To obtain the initial energy for time derivatives we observe that $\dot\varepsilon^\eta_{\mathrm{p}}(0) = \mathfrak{f}_\eta(T^\eta_{\mathrm{E}}(0)) = \mathfrak{f}_\eta(T_{\mathrm{E}}(0))$. By using assumption $T_{\mathrm{E}}(0) \in \mathcal{D}(\mathfrak{f})$ we find that the sequence $\{\mathfrak{f}_\eta(T_{\mathrm{E}}(0))\}$ is bounded in $L^2(\Omega, \mathrm{Sym}(3))$. The other initial values $\dot u^\eta(0)$, $\dot\varepsilon^\eta(0)$ and $\dot A^\eta(0)$ are solutions of (3.17) with $\dot\varepsilon^\eta_{\mathrm{p}}(0)$ instead of $\varepsilon^0_{\mathrm{p}}$. Consequently, the initial energy for time derivatives is bounded. The integral term on the right-hand side of (3.27) can be estimated in the same manner as in the proof of theorem 3.3. Thus we arrive at the following inequality

$$\begin{aligned}\mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_{\mathrm{p}}, \dot A^\eta)(t) \leqslant C_1 \|\dot f(t)\|_{L^2} &\mathcal{E}^{1/2}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_{\mathrm{p}}, \dot A^\eta)(t) \\ &+ C_2 \int_0^t \mathcal{E}(\dot u^\eta, \dot\varepsilon^\eta, \dot\varepsilon^\eta_{\mathrm{p}}, \dot A^\eta)(\tau)\,\mathrm{d}\tau + C_3(t),\end{aligned}$$

where C_1, C_2, $C_3(t)$ do not depend on η and $C_3(t)$ depends only on the given data. Similarly to the proof of theorem 3.3, this concludes the statement. □

The energy estimate for time derivatives yields that the sequence $\{\mathfrak{f}_\eta(T^\eta_{\mathrm{E}})\}$ is bounded in $L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))$. Hence, $\varepsilon_{\mathrm{p}}(0) = \varepsilon^0_{\mathrm{p}}$ and we can select a subsequence (denoted again by the superscript η) with

$$\mathfrak{f}_\eta(T^\eta_{\mathrm{E}}) \overset{*}{\rightharpoonup} \mathfrak{f}_0 \quad \text{in } L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3))).$$

This shows that the limit function $T_{\mathrm{E}} = 2\mu(\varepsilon - \varepsilon_{\mathrm{p}})$ belongs to $\mathcal{D}(\mathfrak{f})$. To end our existence theory we need only to prove that

$$\mathfrak{f}_0(t,x) \in \mathfrak{f}(T_{\mathrm{E}}(t,x)) \quad \text{a.e. in } (0,T)\times\Omega. \tag{3.28}$$

From the definition of a maximal monotone operator it is easy to see that its graph is weakly–strongly closed. Thus we have to improve the weak convergence of the sequence $\{T^\eta_{\mathrm{E}}\}$.

Theorem 3.5 (strong convergence of the stresses). *Let us assume that the given data satisfy all requirements of theorem 3.4. Then*

$$\mathcal{E}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_{\mathrm{p}}^\eta - \varepsilon_{\mathrm{p}}^\nu, A^\eta - A^\nu)(t) \to 0$$

for $\eta, \nu \to 0^+$ *uniformly on bounded time intervals.*

Proof. Calculating the time derivative of the energy evaluated on the differences of two approximation steps we obtain

$$\begin{aligned}
&\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_{\mathrm{p}}^\eta - \varepsilon_{\mathrm{p}}^\nu, A^\eta - A^\nu)(t)\\
&\quad = 2\mu \int_\Omega \langle \varepsilon^\eta - \varepsilon^\nu - \varepsilon_{\mathrm{p}}^\eta + \varepsilon_{\mathrm{p}}^\nu, \dot\varepsilon^\eta - \dot\varepsilon^\nu - \dot\varepsilon_{\mathrm{p}}^\eta + \dot\varepsilon_{\mathrm{p}}^\nu \rangle \,\mathrm{d}x\\
&\qquad + \lambda \int_\Omega \operatorname{tr}[\varepsilon^\eta - \varepsilon^\nu] \operatorname{tr}[\dot\varepsilon^\eta - \dot\varepsilon^\nu] \,\mathrm{d}x\\
&\qquad + 4 l_{\mathrm{c}} \int_\Omega \langle \nabla \operatorname{axl}(A^\eta - A^\nu), \nabla \operatorname{axl}(\dot A^\eta - \dot A^\nu) \rangle \,\mathrm{d}x\\
&\qquad + 2\mu_{\mathrm{c}} \int_\Omega \langle \operatorname{skew}(\nabla u^\eta - \nabla u^\nu) - A^\eta + A^\nu, \operatorname{skew}(\nabla \dot u^\eta - \nabla \dot u^\nu) - \dot A^\eta + \dot A^\nu \rangle \,\mathrm{d}x.
\end{aligned}$$

Using that the given data for both approximation steps are the same we conclude that

$$\dot{\mathcal{E}}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_{\mathrm{p}}^\eta - \varepsilon_{\mathrm{p}}^\nu, A^\eta - A^\nu)(t) = -\int_\Omega \langle T_{\mathrm{E}}^\eta - T_{\mathrm{E}}^\nu, \mathfrak{f}_\eta(T_{\mathrm{E}}^\eta) - \mathfrak{f}_\nu(T_{\mathrm{E}}^\nu) \rangle \,\mathrm{d}x. \tag{3.29}$$

Next, we estimate the right-hand side of (3.29). This estimation is a standard one in the theory of maximal monotone operators (cf. the proof of [4, theorem 1, p. 147]). Nevertheless, for completeness of the proof we insert it here. By definition of the Yosida approximation we have

$$\mathfrak{f}_\ell(T_{\mathrm{E}}^\ell) \in \mathfrak{f}(J_\ell(T_{\mathrm{E}}^\ell)) \quad \text{where } J_\ell(T_{\mathrm{E}}^\ell) = T_{\mathrm{E}}^\ell - \ell \mathfrak{f}_\ell(T_{\mathrm{E}}^\ell) \text{ and } \ell = \eta, \nu \tag{3.30}$$

is the resolvent of the operator $\mathfrak{f}$. Hence, by (3.30) we have

$$\begin{aligned}
&-\int_\Omega \langle T_{\mathrm{E}}^\eta - T_{\mathrm{E}}^\nu, \mathfrak{f}_\eta(T_{\mathrm{E}}^\eta) - \mathfrak{f}_\nu(T_{\mathrm{E}}^\nu) \rangle \,\mathrm{d}x\\
&\qquad = -\int_\Omega \langle J_\eta(T_{\mathrm{E}}^\eta) - J_\nu(T_{\mathrm{E}}^\nu), \mathfrak{f}_\eta(T_{\mathrm{E}}^\eta) - \mathfrak{f}_\nu(T_{\mathrm{E}}^\nu) \rangle \,\mathrm{d}x\\
&\qquad\quad - \int_\Omega \langle \eta \mathfrak{f}_\eta(T_{\mathrm{E}}^\eta) - \nu \mathfrak{f}_\nu(T_{\mathrm{E}}^\nu), \mathfrak{f}_\eta(T_{\mathrm{E}}^\eta) - \mathfrak{f}_\nu(T_{\mathrm{E}}^\nu) \rangle \,\mathrm{d}x\\
&\qquad \leqslant \tfrac{1}{4}\eta \|\mathfrak{f}_\nu(T_{\mathrm{E}}^\nu)\|_{L^2}^2 + \frac{\nu}{4} \|\mathfrak{f}_\eta(T_{\mathrm{E}}^\eta)\|_{L^2}^2\\
&\qquad = \tfrac{1}{4}\eta \|\dot\varepsilon_{\mathrm{p}}^\nu\|_{L^2}^2 + \tfrac{1}{4}\nu \|\dot\varepsilon_{\mathrm{p}}^\eta\|_{L^2}^2.
\end{aligned}$$

Inserting the last result into (3.29) and integrating in time we finally obtain

$$\mathcal{E}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_{\mathrm{p}}^\eta - \varepsilon_{\mathrm{p}}^\nu, A^\eta - A^\nu)(t) \leqslant \tfrac{1}{4} t(\eta + \nu) C(T) \quad \text{for all } t \in [0, T),$$

where the constant $C(T)$ is from the statement of theorem 3.4. The last inequality immediately completes the proof. □

Using (3.30) and the fact that the resolvent J_η is a global Lipschitz operator with the Lipschitz constant less than or equal to 1, we see that the sequence $\{J_\eta(T_{\mathrm{E}}^\eta)\}$ converges strongly to the function T_{E} (note that the sequence $\{\mathfrak{f}_\eta(T_{\mathrm{E}}^\eta)\}$ is bounded). Thus, the weak limit $\mathfrak{f}_\eta(T_{\mathrm{E}}^\eta) \overset{*}{\rightharpoonup} \mathfrak{f}_0$ belongs to the set $\mathfrak{f}(T_{\mathrm{E}})$ and the limit functions $(u, \varepsilon, \varepsilon_{\mathrm{p}}, A)$ satisfy (3.7).

THEOREM 3.6 (uniqueness of solutions). *Let us assume that the given data f, u_{d}, A_{d}, $\varepsilon_{\mathrm{p}}^0$ satisfy all requirements of theorem 3.4 Then system (3.7) possesses a unique, global-in-time solution $(u, \varepsilon, \varepsilon_{\mathrm{p}}, A)$.*

Proof. Assume that $(u^1, \varepsilon^1, \varepsilon_{\mathrm{p}}^1, A^1)$ and $(u^2, \varepsilon^2, \varepsilon_{\mathrm{p}}^2, A^2)$ are two solutions of (3.7) for the same given data. Then for the energy function evaluated on differences of these solutions we have

$$
\begin{aligned}
&\dot{\mathcal{E}}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_{\mathrm{p}}^1 - \varepsilon_{\mathrm{p}}^2, A^1 - A^2)(t) \\
&\quad = 2\mu \int_\Omega \langle \varepsilon^1 - \varepsilon^2 - \varepsilon_{\mathrm{p}}^1 + \varepsilon_{\mathrm{p}}^2, \dot\varepsilon^1 - \dot\varepsilon^2 - \dot\varepsilon_{\mathrm{p}}^1 + \dot\varepsilon_{\mathrm{p}}^2 \rangle \, \mathrm{d}x \\
&\qquad + \lambda \int_\Omega \operatorname{tr}[\varepsilon^\eta - \varepsilon^\nu] \operatorname{tr}[\dot\varepsilon^\eta - \dot\varepsilon^\nu] \, \mathrm{d}x + 4 l_{\mathrm{c}} \int_\Omega \langle \nabla \operatorname{axl}(A^1 - A^2), \nabla \operatorname{axl}(\dot A^1 - \dot A^2) \rangle \, \mathrm{d}x \\
&\qquad + 2\mu_{\mathrm{c}} \int_\Omega \langle \operatorname{skew}(\nabla u^1 - \nabla u^2) - A^1 + A^2, \operatorname{skew}(\nabla \dot u^1 - \nabla \dot u^2) - \dot A^1 + \dot A^2 \rangle \, \mathrm{d}x \\
&\quad = - \int_\Omega \langle T_{\mathrm{E}}^1 - T_{\mathrm{E}}^2, \dot\varepsilon_{\mathrm{p}}^1 - \dot\varepsilon_{\mathrm{p}}^2 \rangle \, \mathrm{d}x \\
&\quad \leqslant 0.
\end{aligned}
$$

Immediately, this yields that

$$
\begin{aligned}
\mathcal{E}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_{\mathrm{p}}^1 - \varepsilon_{\mathrm{p}}^2, A^1 - A^2)(t)& \\
\leqslant \mathcal{E}(u^1 - u^2, \varepsilon^1 - \varepsilon^2, \varepsilon_{\mathrm{p}}^1 - \varepsilon_{\mathrm{p}}^2, A^1 - A^2)(0) &= 0
\end{aligned}
$$

and the statement follows from coerciveness of the energy function (see the first statement of theorem 3.2). □

Finally, we formulate the following existence theorem, which we have proved.

THEOREM 3.7 (existence for the infinitesimal elasto-plastic Cosserat model). *Suppose that the given data f, u_{d}, A_{d} satisfy the following conditions. For all times $T > 0$,*

$$
\begin{aligned}
f &\in C^1([0,T], L^2(\Omega, \mathbb{R}^3)), & \ddot f &\in L^2((0,T) \times \Omega, \mathbb{R}^3), \\
u_{\mathrm{d}} &\in C^2([0,T], H^{1/2}(\partial\Omega, \mathbb{R}^3)), & \partial_t^3 u_{\mathrm{d}} &\in L^2((0,T) H^{1/2}(\partial\Omega, \mathbb{R}^3)), \\
A_{\mathrm{d}} &\in C^2([0,T], H^{3/2}(\partial\Omega, \mathfrak{so}(3,\mathbb{R}))), & \partial_t^3 A_{\mathrm{d}} &\in L^2((0,T) H^{1/2}(\partial\Omega, \mathfrak{so}(3,\mathbb{R}))).
\end{aligned}
$$

Moreover, assume that the initial data $\varepsilon_{\mathrm{p}}^0 \in L^2(\Omega, \operatorname{Sym}(3))$ are chosen such that the initial value of the reduced Eshelby tensor $T_{\mathrm{E}}(0) = 2\mu(\varepsilon(0) - \varepsilon_{\mathrm{p}}^0)$ defined by system (3.17) belongs to the domain of the maximal monotone operator $\mathfrak{f}$. Then *system (3.7) possesses a global-in-time, unique solution* $(u, \varepsilon, \varepsilon_{\mathrm{p}}, A)$ *with the following regularity. For all times* $T > 0$,

$$\begin{aligned} u &\in H^{1,\infty}((0,T), H^1(\Omega, \mathbb{R}^3)), \\ \varepsilon, \varepsilon_{\mathrm{p}} &\in H^{1,\infty}((0,T), L^2(\Omega, \mathrm{Sym}(3))), \\ A &\in H^{1,\infty}((0,T), H^2(\Omega, \mathfrak{so}(3,\mathbb{R}))). \end{aligned}$$

Remark 3.8. In the analysis part we have assumed that the values of the constitutive multifunction $\mathfrak{f}$ are trace free. This means that the existence theory developed works well for trace-free flow rules, corresponding to incompressible plasticity only. For constitutive multifunctions possessing general values we are not able to show the coerciveness of the energy.

Note that, for the model to be well posed in the rate-independent case, we did not need the so-called *safe load* condition, which is otherwise unavoidable.

4. Discussion

The infinitesimal Cosserat model has been extended to elasto-plasticity where Cosserat effects remain, in contrast to others proposals in the literature, non-dissipative. As only difference from classical rate-independent infinitesimal plasticity we have introduced an additional infinitesimal microrotation $\bar{A}_{\mathrm{e}}$, influencing only the elastic behaviour of the model. This minor change is shown to completely regularize the pathological behaviour of rate-independent classical plasticity theory. Decisive in our analysis is the observation that the infinitesimal microrotations provide an independent control of the rotation $\operatorname{curl} u$, otherwise not present in the theory. This extra resistance against elastic shear is also a welcome feature from a modelling and numerical point of view.

Since this modification of classical rate-independent plasticity is not operative in uniaxial tension/compression we may arguably say that the provided regularization is optimal. Numerical calculations based on this modification are ‘cheap’, in the sense that the resulting system remains of second order.

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Appendix A. Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing two-dimensional Hausdorff measure. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second-order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by

$$\langle X, Y \rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}[XY^{\mathrm{T}}],$$

and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$ (we use these symbols indifferently for tensors and vectors). The identity tensor on $\mathbb{M}^{3\times3}$ is denoted by $\mathbb{1}$, so that $\mathrm{tr}[X] = \langle X, \mathbb{1}\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-algebra theory, i.e. $\mathfrak{so}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times3} | X^{\mathrm{T}} = -X\}$ are skew symmetric second-order tensors and $\mathfrak{sl}(3,\mathbb{R}) := \{X \in \mathbb{M}^{3\times3} | \,\mathrm{tr}[X] = 0\}$ are traceless tensors. We set

$$\mathrm{sym}(X) = \tfrac{1}{2}(X^{\mathrm{T}} + X) \quad \text{and} \quad \mathrm{skew}(X) = \tfrac{1}{2}(X - X^{\mathrm{T}})$$

such that $X = \mathrm{sym}(X) + \mathrm{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\mathrm{dev}\, X = X - \frac{1}{3}\,\mathrm{tr}[X]\mathbb{1} \in \mathfrak{sl}(3,\mathbb{R})$.

For a second-order tensor X we let $X \cdot e_i$ be the application of the tensor X to the column vector e_i and we define the third-order tensor

$$\mathfrak{h} = \mathrm{D}_x X(x) = (\nabla(X(x)\cdot e_1), \nabla(X(x)\cdot e_2), \nabla(X(x)\cdot e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in (\mathbb{M}^{3\times3})^3.$$

For $\mathfrak{h}$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^{3} \|\mathfrak{h}^i\|^2$ together with $\mathrm{sym}(\mathfrak{h}) := (\mathrm{sym}\,\mathfrak{h}^1, \mathrm{sym}\,\mathfrak{h}^2, \mathrm{sym}\,\mathfrak{h}^3)$ and $\mathrm{tr}[\mathfrak{h}] := (\mathrm{tr}[\mathfrak{h}^1], \mathrm{tr}[\mathfrak{h}^2], \mathrm{tr}[\mathfrak{h}^3]) \in \mathbb{R}^3$. The first and second differential of a scalar-valued function $W(F)$ are written $D_F W(F)\cdot H$ and $D_F^2 W(F)\cdot(H,H)$, respectively. Sometimes we also use $\partial_X W(X)$ to denote the first derivative of W with respect to X. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega)$, $H^{1,2}(\Omega)$, $H_\circ^{1,2}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions.

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