Euro. Jnl of Applied Mathematics (2021), *vol.* 32, *pp.* 337–369
doi:10.1017/S0956792520000170

Approximate solutions to one-phase Stefan-like problems with space-dependent latent heat†

J. BOLLATI and D. A. TARZIA

Departamento Matemática - FCE, Universidad Austral-CONICET, Paraguay 1950, S2000 FZF, Rosario, Argentina
emails: jbollati@austral.edu.ar; dtarzia@austral.edu.ar

(*Received 20 January 2020*; *revised 11 May 2020*; *accepted 19 May 2020*; *first published online 15 June 2020*)

The work in this paper concerns the study of different approximations for one-dimensional one-phase Stefan-like problems with a space-dependent latent heat. It is considered two different problems, which differ from each other in their boundary condition imposed at the fixed face: Dirichlet and Robin conditions. The approximate solutions are obtained by applying the heat balance integral method (HBIM), the modified HBIM and the refined integral method (RIM). Taking advantage of the exact analytical solutions, we compare and test the accuracy of the approximate solutions. The analysis is carried out using the dimensionless generalised Stefan number (Ste) and Biot number (Bi). It is also studied the case when Bi goes to infinity in the problem with a convective condition, recovering the approximate solutions when a temperature condition is imposed at the fixed face. Some numerical simulations are provided in order to assert which of the approximate integral methods turns out to be optimal. Moreover, we pose an approximate technique based on minimising the least-squares error, obtaining also approximate solutions for the classical Stefan problem.

Key words: Stefan problem, variable latent heat, heat balance integral method, refined heat balance integral method, exact solutions

2020 Mathematics Subject Classification: 80A22, 40C10, 35R35, 35K05, 35C05

1 Introduction

Stefan problems model heat transfer processes that involve a change of phase. They constitute a broad field of study since they appear in a great number of mathematical and industrial significance problems [1, 6, 10, 13]. A large bibliography on the subject is given in [25] and a review on analytical solutions in [26].

The Stefan problem with a space-dependent latent heat can be found in several physical processes. In [23], it was developed a mathematical model for the shoreline movement in a sedimentary basin using an analogy with the one-phase melting Stefan problem with a variable latent heat. Besides, in [31], it was introduced a two-phase Stefan problem with a general type of space-dependent latent heat from the background of the artificial ground-freezing technique.

† The present work has been partially sponsored by the Project PIP No. 0275 from CONICET-UA, Rosario, Argentina, by the Project ANPCyT PICTO Austral 2016 No. 0090 and by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement 823731 CONMECH.

The assumption of variable latent heat not only becomes meaningful in the study of the shoreline movement or in the soil freezing techniques but also in the nanoparticle melting [18] and in the one-dimensional consolidation with threshold gradient [29]. More references dealing with non-constant latent heat can be found in [3, 4, 7, 9, 14, 17, 21, 27, 30, 32, 33].

In this paper, we are going to consider two different Stefan-like problems (P) and (P_h) with space-dependent latent heat imposing different conditions at the fixed boundary. The first problem to consider can be stated as follows:

Problem (P). Find the location of the free boundary $x = s(t)$ and the temperature $T = T(x, t)$ at the liquid region $0 < x < s(t)$ such that

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < s(t), \quad t > 0, \tag{1.1a}$$

$$T(0, t) = \theta_\infty t^{\alpha/2}, \qquad t > 0, \tag{1.1b}$$

$$T(s(t), t) = 0, \qquad t > 0, \tag{1.1c}$$

$$k\frac{\partial T}{\partial x}(s(t), t) = -\gamma s(t)^\alpha \dot{s}(t), \qquad t > 0, \tag{1.1d}$$

$$s(0) = 0, \tag{1.1e}$$

Equation (1.1a) is the heat conduction equation in the liquid region where $a^2 = \frac{k}{\rho c}$ is the diffusion coefficient being k the thermal conductivity, ρ the density mass and c the specific heat capacity. At $x = 0$, a Dirichlet condition (1.1b) is imposed. It must be noticed that the temperature at the fixed boundary is time-dependent and it is characterised by a parameter $\theta_\infty > 0$. In addition, condition (1.1c) represents the fact that the phase change temperature is assumed to be 0 without loss of generality, condition (1.1d) is the corresponding Stefan condition and (1.1e) is the initial position of the free boundary.

The remarkable feature of the problem is related to the condition at the interface given by the Stefan condition (1.1d), where the latent heat by unit of volume is space-dependent defined by a power function of the position $\frac{\gamma}{\rho} x^\alpha(t)$ with γ a given positive constant and α an arbitrarily non-negative real value.

The second problem (P_h) arises by imposing a convective (Robin) condition at the fixed face $x = 0$ instead of a Dirichlet one. In mathematical terms, we can define (P_h) as follows:

Problem (P_h). Find the location of the free boundary $x = s_h(t)$ and the temperature $T_h = T_h(x, t)$ at the liquid region $0 < x < s_h(t)$ such that equations (1.1a) and (1.1c)–(1.1e) are satisfied, together with the Robin condition

$$k\frac{\partial T}{\partial x}(0, t) = \frac{h}{\sqrt{t}}\left[T(0, t) - \theta_\infty t^{\alpha/2}\right], \qquad t > 0. \tag{1.1b*}$$

Condition (1.1b*) states that the incoming heat flux at the fixed face is proportional to the difference between the material temperature and the ambient temperature. Here, $\theta_\infty t^{\alpha/2}$ characterises the bulk temperature at a large distance from the fixed face $x = 0$ and h represents the heat transfer at the fixed face. We will work under the assumption that $h > 0$ and $0 < T_h(0, t) < \theta_\infty t^{\alpha/2}$ in order to guarantee the melting process.

The exact solution to problem (P) was given in [32] for integer non-negative values of α and was generalised in [33] by taking α as a real non-negative constant. Besides, the exact solution of the problem (P_h) was provided in [3].

It is known that due to the non-linear nature of the Stefan problem, exact solutions are limited to a few cases and therefore it is necessary to solve them either numerically or approximately.

The idea in this paper is to take advantage of the exact solutions available in the literature testing the accuracy of different approximate integral methods.

The heat balance integral method (HBIM), introduced by Goodman [8], is an approximate technique which is usually employed for solving the location of the free boundary in phase-change problems. It consists in the transformation of the heat equation into an ordinary differential equation in time, assuming a quadratic profile in space for the temperature. For those profiles, several variants have been introduced in [28] and [20]. In addition, in [11, 12, 15, 16] this method has been applied defining new accurate temperature profiles. Moreover, for the case $\alpha = 0$, the explicit solution to the problem (P_h) for the two-phase process was given in [24] and this was useful to obtain the accuracy of different HBIMs to problem (P_h) in [2].

The paper will be structured as follows: in Section 2 we will give a brief introduction about the approximate methods to be implemented. Then, in Section 3, we will recall the exact solution to problem (P) that considers a Dirichlet condition at the fixed face and we will get some different approximate solutions that will be tested with the exact one. In Section 4, we will present the exact solution to the problem with a Robin condition at the fixed face, i.e. problem (P_h). We are going to implement the different approximate methods and we will test their accuracy. In all cases, we are going to provide numerical examples and comparisons. In addition, we will show that the approximate solutions to problem (P_h) converge to the approximate solutions to problem (P) when the heat transfer coefficient h goes to infinity. Finally, in Section 5, we will implement an approximate method that consists in minimising the least-squares error as in [19]. For the case $\alpha = 0$, we obtain different approximations for the problems (P) and (P_h) by using the least-squares approximate method.

2 Heat balance integral methods

The classical HBIM, described for first time in [8], was designed to approximate problems involving phase changes. This method consists in changing the heat equation (1.1a) by an ordinary differential equation in time that arises by assuming a suitable temperature profile consistent with the boundary conditions, integrating (1.1a) with respect to the spacial variable in an appropriate interval, and replacing the Stefan condition (1.1d) by a new equation obtained from the phase-change temperature (1.1c).

Therefore, if we derive condition (1.1c) with respect to time and take into account the heat equation (1.1a), we get

$$\frac{\partial T}{\partial x}(s(t), t)\dot{s}(t) + a^2 \frac{\partial^2 T}{\partial x^2}(s(t), t) = 0. \tag{2.1}$$

Clearing $\dot{s}$ and replacing it in the Stefan condition (1.1d) it gives

$$\frac{k}{\gamma s^{\alpha}(t)} \left[\frac{\partial T}{\partial x}(s(t), t) \right]^2 = a^2 \frac{\partial^2 T}{\partial x^2}(s(t), t). \tag{1.1d*}$$

This last condition is going to substitute the Stefan condition in the approximated problem obtained from the classical HBIM.

On the other hand, using equation (1.1a) and the condition (1.1c), we have

$$\frac{d}{dt}\int_0^{s(t)} T(x,t)dx = \int_0^{s(t)} \frac{\partial T}{\partial t}(x,t)dx + T(s(t),t)\dot{s}(t)$$
$$= \int_0^{s(t)} a^2 \frac{\partial^2 T}{\partial x^2}(x,t)dx = a^2\left[\frac{\partial T}{\partial x}(s(t),t) - \frac{\partial T}{\partial x}(0,t)\right].$$

Then, by applying the Stefan condition (1.1d) it results that

$$\frac{d}{dt}\int_0^{s(t)} T(x,t)dx = -a^2\left[\frac{\gamma}{k}s^{\alpha}(t)\dot{s}(t) + \frac{\partial T}{\partial x}(0,t)\right]. \qquad (1.1a^{\star})$$

The classical HBIM suggests to solve an approximate problem (P) through a new problem that arises from replacing the heat equation (1.1a) by (1.1a*) and the Stefan condition (1.1d) by (1.1d*) keeping the rest of the conditions of (P) the same. In short, the method consists in solving the problem governed by (1.1a*), (1.1b),(1.1c), (1.1d*) and (1.1e). *A priori*, this method will work better than the classical one due to the fact that it changes less conditions from the exact problem.

In [28], a modified integral balance method is presented. It postulates to change only the heat equation keeping the same the rest of conditions, even the Stefan condition. It means that it consists in solving an approximate problem given by (1.1a*), (1.1b), (1.1c), (1.1d) and (1.1e).

On the other hand, from the heat equation (1.1a), and the condition (1.1c) we have

$$\int_0^{s(t)}\int_0^{x} \frac{\partial T}{\partial t}(z,t)dzdx = \int_0^{s(t)}\int_0^{x} a^2\frac{\partial^2 T}{\partial z^2}(z,t)dz\,dx$$
$$= \int_0^{s(t)} a^2\left[\frac{\partial T}{\partial x}(x,t) - \frac{\partial T}{\partial x}(0,t)\right]dx$$
$$= a^2\left[T(s(t),t) - T(0,t) - \frac{\partial T}{\partial x}(0,t)s(t)\right],$$

that is to say

$$\int_0^{s(t)}\int_0^{x} \frac{\partial T}{\partial t}(z,t)dzdx = -a^2\left[T(0,t) + \frac{\partial T}{\partial x}(0,t)s(t)\right]. \qquad (1.1a^{\dagger})$$

The refined integral method (RIM) introduced in [20] suggests to solve an approximate problem given by (1.1a†), (1.1b), (1.1c), (1.1d) and (1.1e). That is to say, to replace the heat equation (1.1a) by (1.1a†).

In all cases, to solve the above approximated problems, it is necessary to adopt a suitable profile for the temperature. Throughout this paper, we will assume a quadratic profile in space

$$\widetilde{T}(x,t) = t^{\alpha/2}\theta_{\infty}\left[\widetilde{A}\left(1-\frac{x}{\widetilde{s}(t)}\right) + \widetilde{B}\left(1-\frac{x}{\widetilde{s}(t)}\right)^2\right], \qquad (2.2)$$

where $\widetilde{T}$ and $\widetilde{s}$ will be approximations of T and s, respectively. We can notice that in the chosen profile a power function of time arises in order to be compatible with the boundary conditions imposed in the exact problem.

It is worth to mention that for the approximations to the problem (P_h), it will be enough to consider the same approximate problems stated for (P), changing only the boundary condition (1.1b) by (1.1b*).

3 One-phase Stefan problem with Dirichlet condition

3.1 Exact solution

Before introducing the different approaching methods for problem (P), we present the exact solution, which was given in [32] and [33] for the cases when $\alpha \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$, respectively.

Let us define the following non-dimensional parameter:

$$\text{Ste} = \frac{k\theta_\infty}{\gamma a^{\alpha+2}} \tag{3.1}$$

which is called generalised Stefan number (Ste). We use the word 'generalised' since in case that the latent heat l is constant, i.e. $\alpha = 0$, we can recover the usual formula for the Ste, which assuming a zero phase-change temperature is given by $\text{Ste} = \frac{c\theta_\infty}{l}$. Notice that if we take $\alpha = 0$ then the Dirichlet condition at the fixed face is given by θ_∞ and from the Stefan condition (1.1d) the latent heat becomes $l = \gamma/\rho$.

Then, if we combine the results found in [32] and [33], we can rewrite the solution of the problem (P) (as it was done in the appendix of [5]), obtaining for each $\alpha \in \mathbb{R}_0^+$ that

$$T(x,t) = t^{\alpha/2}\left[AM\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + B\eta M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\eta^2\right)\right], \tag{3.2}$$

$$s(t) = 2a\nu\sqrt{t}, \tag{3.3}$$

where $\eta = \frac{x}{2a\sqrt{t}}$ is the similarity variable,

$$A = \theta_\infty, \qquad B = \frac{-\theta_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)}{\nu M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu^2\right)}, \tag{3.4}$$

and ν is the unique positive solution to the following equation:

$$\frac{\text{Ste}}{2^{\alpha+1}} f(z) = z^{\alpha+1}, \qquad z > 0, \tag{3.5}$$

where is defined by

$$f(z) = \frac{1}{zM\left(\frac{\alpha}{2}+1, \frac{3}{2}, z^2\right)} \tag{3.6}$$

and $M(a,b,z)$ is the Kummer function defined by

$$M(a,b,z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \qquad \text{(b cannot be a non-positive integer)} \tag{3.7}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

being $(a)_s$ the Pochhammer symbol:

$$(a)_s = a(a+1)(a+2)\dots(a+s-1), \qquad (a)_0 = 1. \tag{3.8}$$

Remark 3.1 *If $0 < Ste < 1$, the unique solution ν of equation (3.5) belongs to the interval* $(0, 1)$. *In fact, define $H(x) = \frac{Ste}{2^{\alpha+1}} f(z) - z^{\alpha+1}$. On the one hand, we have $H(0) = +\infty$ due to the fact that $M\left(\frac{\alpha}{2}+1, \frac{3}{2}, 0\right) = 1$. On the other hand, we obtain $H(1) < 0$ as $\frac{Ste}{2^{\alpha+1}} < 1 < M\left(\frac{\alpha}{2}+1, \frac{3}{2}, 1\right)$.*

3.2 Approximate solutions

We are going to implement the different approximate techniques for the problem (P) and test their accuracy taking advantage of the knowledge of the exact solution.

First of all, we introduce a problem (P_1) which arises when applying the classical heat balance integral problem to (P). According to the previous section, the problem (P_1) consists in finding the free boundary $s_1 = s_1(t)$ and the temperature $T_1 = T_1(x, t)$ in $0 < x < s_1(t)$ such that conditions (1.1a*), (1.1b),(1.1c), (1.1d*) and (1.1e) are verified.

Provided that T_1 assumes a quadratic profile in space like (2.2) we get the following result.

Theorem 3.2 *If $0 < Ste < 1$, there exists at least one solution to problem* (P_1)*, given by*

$$T_1(x,t) = t^{\alpha/2}\theta_\infty \left[A_1 \left(1 - \frac{x}{s_1(t)}\right) + B_1 \left(1 - \frac{x}{s_1(t)}\right)^2 \right], \tag{3.9}$$

$$s_1(t) = 2a\nu_1\sqrt{t}, \tag{3.10}$$

where the constants A_1, B_1 are defined as a function of ν_1 by

$$A_1 = \frac{-2\left[3\, 2^\alpha \nu_1^{\alpha+2} + \text{Ste}\left((-3 + (1+\alpha)\nu_1^2\right)\right]}{\text{Ste}\left(3 + (1+\alpha)\nu_1^2\right)}, \tag{3.11}$$

$$B_1 = \frac{3\left[2^{\alpha+1}\nu_1^{\alpha+2} + \text{Ste}\left(-1 + (1+\alpha)\nu_1^2\right)\right]}{\text{Ste}\left(3 + (1+\alpha)\nu_1^2\right)}, \tag{3.12}$$

and the coefficient ν_1 is a solution to the following equation:

$$\begin{aligned} &z^{2\alpha+4}(-3)\,2^{2\alpha+1}(\alpha-2) + z^{2\alpha+2}(-9)\,2^{2\alpha+1} + z^{4+\alpha}\,(-3)\;\; 2^\alpha(\alpha-3)(\alpha+1)\text{Ste} \\ &+ z^{\alpha+2}\,(-3)\;\; 2^{\alpha+1}(\alpha+7)\text{Ste} + z^\alpha 9\, 2^\alpha \text{Ste} + z^4 2(\alpha+1)^2\text{Ste}^2 \\ &+ z^2(-12)(\alpha+1)\text{Ste}^2 + 18\text{Ste}^2 = 0, \qquad z > 0. \end{aligned} \tag{3.13}$$

Proof First of all we shall notice that if T_1 adopts the profile (3.9), it is clear evident that the condition (1.1c) is automatically verified. From the imposed Dirichlet condition at the fixed boundary (1.1b) we get

$$A_1 + B_1 = 1. \tag{3.14}$$

In addition, we have that

$$\frac{\partial T_1}{\partial x}(x,t) = -t^{\alpha/2}\theta_\infty \left[\frac{A_1}{s_1(t)} + \frac{2B_1}{s_1(t)}\left(1 - \frac{x}{s_1(t)}\right)\right],$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

and

$$\frac{\partial^2 T_1}{\partial x^2}(x,t) = t^{\alpha/2}\theta_\infty \frac{2B_1}{s_1^2(t)}.$$

Therefore, from condition (1.1d*), we claim

$$\frac{k}{\gamma s_1^\alpha(t)} t^\alpha \theta_\infty^2 \frac{A_1^2}{s_1^2(t)} = a^2 t^{\alpha/2}\theta_\infty \frac{2B_1}{s_1^2(t)}.$$

Then, it follows that

$$s_1(t) = \left(\frac{A_1^2}{2B_1}\frac{k\theta_\infty}{\gamma a^2}\right)^{1/\alpha}\sqrt{t}.$$

Defining ν_1 such that $\nu_1 = \frac{1}{2a}\left(\frac{A_1^2}{2B_1}\frac{k\theta_\infty}{\gamma a^2}\right)^{1/\alpha}$, we deduce that

$$s_1(t) = 2a\nu_1\sqrt{t}, \tag{3.15}$$

where ν_1, A_1 and B_1 are related as

$$A_1^2 = \frac{2^{\alpha+1}\nu_1^\alpha}{\text{Ste}} B_1. \tag{3.16}$$

Condition (1.1a*) and

$$\begin{aligned}\frac{d}{dt}\int_0^{s_1(t)} T_1(x,t)dx &= \frac{d}{dt}\int_0^{s_1(t)} t^{\alpha/2}\theta_\infty\left[A_1\left(1-\frac{x}{s_1(t)}\right) + B_1\left(1-\frac{x}{s_1(t)}\right)^2\right]dx \\ &= \theta_\infty\left(\frac{A_1}{2}+\frac{B_1}{3}\right)\left(\frac{\alpha}{2}t^{\alpha/2-1}s_1(t) + t^{\alpha/2}\dot{s}_1(t)\right)\end{aligned}$$

give

$$\theta_\infty\left(\tfrac{A_1}{2}+\tfrac{B_1}{3}\right)\left(\tfrac{\alpha}{2}t^{\alpha/2-1}s_1(t)+t^{\alpha/2}\dot{s}_1(t)\right) = -a^2\left[\tfrac{\gamma}{k}s_1^\alpha(t)\dot{s}_1(t) + t^{\alpha/2}\theta_\infty\tfrac{(A_1+2B_1)}{s_1(t)}\right]. \tag{3.17}$$

According to (3.15), it results that

$$A_1\left((\alpha+1)\nu_1^2 - 1\right) + B_1\left(\tfrac{2}{3}(\alpha+1)\nu_1^2 - 2\right) = \tfrac{-2^{\alpha+1}\nu_1^{\alpha+2}}{\text{Ste}}. \tag{3.18}$$

Thus, we have obtained three equations (3.14), (3.16) and (3.18) for the unknown coefficients A_1, B_1 and ν_1.

From (3.14) and (3.18), it is obtained that A_1 and B_1 are given as a function of ν_1 by (3.11) and (3.12), respectively.

Then, equation (3.16) leads to the fact that ν_1 must be a positive solution to (3.13).

For the existence of solution to problem (P_1), it remains to prove that the function $w_1 = w_1(z)$, defined as the left-hand side of equation (3.13), has at least one positive root. This can be easily check by evaluating $w_1(0) = 18\text{Ste}^2 > 0$ and

$$w_1(1) = -\alpha^2(3\,2^\alpha - 2\text{Ste})\text{Ste} - 2\alpha(3\,4^\alpha + 4\text{Ste}^2) - 2(3\,4^\alpha + 3\,2^{\alpha+2}\text{Ste} - 4\text{Ste}^2)$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

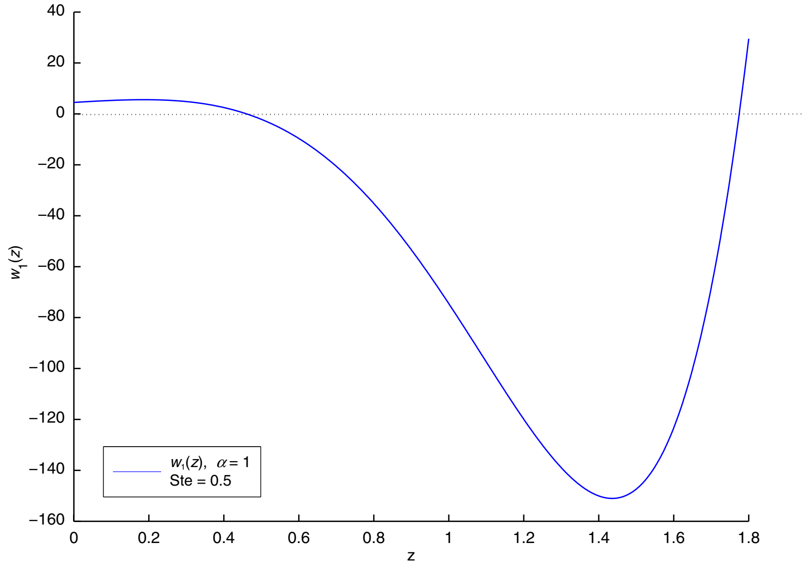

FIGURE 1. Plot of $w_1(z)$ for $\alpha = 1$ and Ste = 0.5.

From the assumption that $0 < \text{Ste} < 1$, we obtain $3\,2^\alpha - 2\text{Ste} > 0$, and

$$3\,4^\alpha + 3\,2^{\alpha+2}\text{Ste} - 4\text{Ste}^2 > 2^{\alpha+2}\text{Ste} - 4\text{Ste}^2 = 4\text{Ste}(3\,2^\alpha - \text{Ste}) > 0.$$

Therefore $w_1(1) < 0$. Consequently, we can assure that there exists at least one positive solution to equation (3.13) in the interval $(0, 1)$. □

Remark 3.3 *The approximated free boundary s_1 behaves as a square root of time just like the exact one s, it means that $s_1(t) = 2a\nu_1\sqrt{t}$ while $s(t) = 2a\nu\sqrt{t}$.*

Remark 3.4 *After Theorem 3.2 follows the question about uniqueness of solution. We found that there exists different values for α and $0 < \text{Ste} < 1$ that leads to multiple roots of equation (3.13), i.e. $w_1(z) = 0,\ z > 0$ (see Figure 1).*

However our study must be reduced to find the roots of $w_1(z)$ located in the interval $(0, 1)$ in view of the proof of Theorem 3.2 but also in view of Remark 3.1. For the particular case of $\alpha = 0$ the uniqueness analysis was given in [2].

Although we could not prove it analytically, by setting different values for α and Ste we can see that there exists just one root of the polynomial $w_1(z)$ located in the interval $(0, 1)$. In Figure 2, we illustrate this fact setting $\alpha = 0.5, 1, 1.5, 2, 3, 5, 10$ and Ste $= 0.5$. We have just plot between $0 \leqslant z \leqslant 0.5$ in order to appreciate better this fact.

With the purpose of testing the classical integral balance method and in view of the above remark we will only compare graphically the coefficient ν_1 that characterises the approximated free boundary problem s_1 with the coefficient ν that characterises the exact free boundary s. In Figure 3, we illustrate this comparisons for different values of $0 < \text{Ste} < 1$ and α.

For the comparisons we have assumed that $0 < \text{Ste} < 1$ not only due to the hypothesis in Theorem 3.2, but also because of the fact that in general, the majority of phase change materials under a realistic temperature present an Ste that does not exceed 1 (see [22]).

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

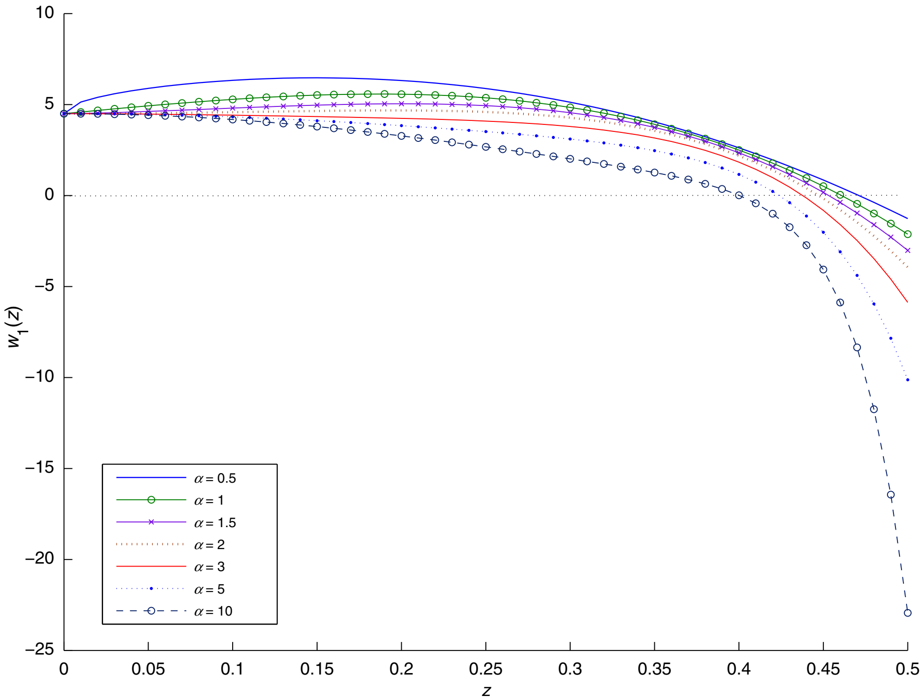

FIGURE 2. Plot of $w_1(z)$ for different values of α setting Ste $= 0.5$.

Now, we will turn to the modified integral balance method. In this case we state an approximated problem (P_2) for the problem (P) that is stated as follows: find the free boundary $s_2 = s_2(t)$ and the temperature $T_2 = T_2(x, t)$ in $0 < x < s_2(t)$ such that equation (1.1a*) and conditions (1.1b), (1.1c), (1.1d) and (1.1e) are satisfied.

Assuming a quadratic profile in space for T_2 we obtain the next theorem

Theorem 3.5 *The problem* (P_2) *has a unique solution given by*

$$T_2(x,t) = t^{\alpha/2}\theta_\infty \left[A_2\left(1 - \frac{x}{s_2(t)}\right) + B_2\left(1 - \frac{x}{s_2(t)}\right)^2 \right], \tag{3.19}$$

$$s_2(t) = 2a\nu_2\sqrt{t}, \tag{3.20}$$

where the constants A_2 and B_2 are given by

$$A_2 = \frac{6\text{Ste} - 2\text{Ste}\,\nu_2^2(\alpha+1) - 3\,2^{\alpha+1}\nu_2^{\alpha+2}}{\text{Ste}\,\left(\nu_2^2(\alpha+1)+3\right)}, \tag{3.21}$$

$$B_2 = \frac{-3\text{Ste} + 3\text{Ste}\,\nu_2^2(\alpha+1) + 3\,2^{\alpha+1}\nu_2^{\alpha+2}}{\text{Ste}\,\left(\nu_2^2(\alpha+1)+3\right)}, \tag{3.22}$$

and where ν_2 is the unique positive solution to the equation

$$z^{\alpha+4}2^\alpha(\alpha+1) + z^{\alpha+2}3\,2^{\alpha+1} + z^2\text{Ste}(\alpha+1) - 3\text{Ste} = 0, \qquad z > 0. \tag{3.23}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

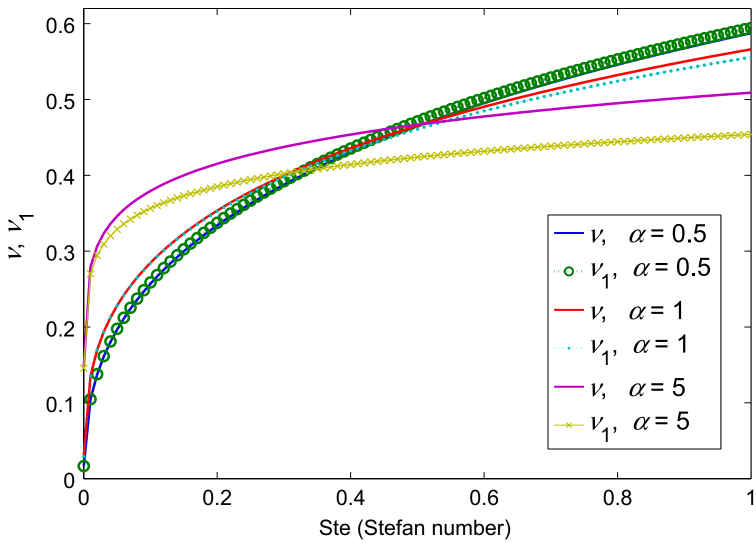

FIGURE 3. Plot of ν and ν_1 against Ste for different values of $\alpha = 0.5, 1, 5$.

Proof Condition (1.1c) is clearly checked from the chosen temperature profile.
From the Stefan condition (1.1d), we obtain

$$-kt^{\alpha/2}\theta_\infty \frac{A_2}{s_2(t)} = -\gamma s_2^\alpha(t)\dot{s}_2(t). \tag{3.24}$$

Therefore, it results that

$$s_2(t) = \left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)} \frac{k\theta_\infty}{\gamma} A_2 \right)^{1/(\alpha+2)} \sqrt{t}. \tag{3.25}$$

If we introduce the coefficient ν_2 such that $\nu_2 = \frac{1}{2a}\left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)} \frac{k\theta_\infty}{\gamma} A_2 \right)^{1/(\alpha+2)}$, the free boundary can be expressed as

$$s_2(t) = 2a\,\nu_2\sqrt{t}, \tag{3.26}$$

where the following relation holds:

$$A_2 = \frac{2^{\alpha+1}\nu_2^{\alpha+2}}{\text{Ste}}. \tag{3.27}$$

Taking into account the boundary condition at the fixed face (1.1b), we get

$$A_2 + B_2 = 1. \tag{3.28}$$

In addition, in virtue of equation (1.1a*), we get

$$A_2\left((\alpha+1)\nu_2^2 - 1\right) + B_2\left(\tfrac{2}{3}(\alpha+1)\nu_2^2 - 2\right) = \frac{-2^{\alpha+1}\nu_2^{\alpha+2}}{\text{Ste}}. \tag{3.29}$$

From equations (3.27), (3.28) and (3.29), we claim that A_2 and B_2 can be written in function of ν_2 through formulas (3.21) and (3.22), respectively. In addition, ν_2 must be a solution to equation (3.23). So that, to finish the proof, it remains to show that equation (3.23) has a unique positive

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

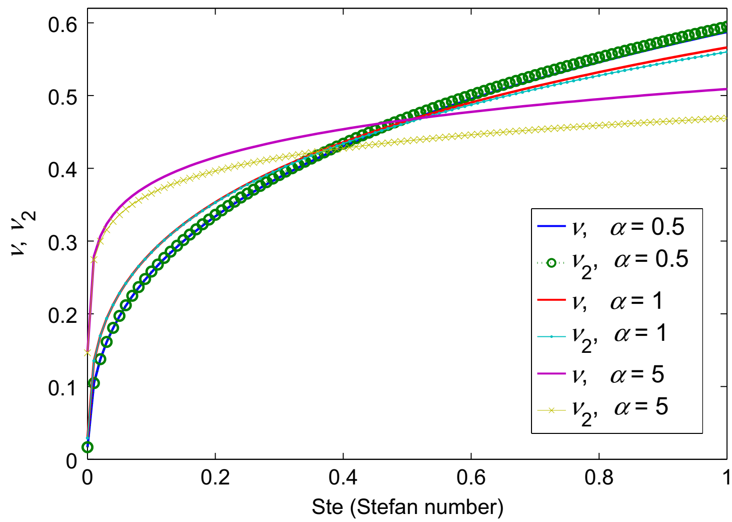

FIGURE 4. Plot of ν and ν_2 against Ste for different values of $\alpha = 0.5, 1, 5$.

solution, i.e. the function defined by the left-hand side of this equation $w_2 = w_2(z)$ has a unique positive root. This is easily checked by noting that

$$w_2(0) = -3\text{Ste} < 0, \qquad w_2(+\infty) = +\infty, \qquad \frac{dw_2}{dz}(z) > 0, \quad \forall z > 0. \qquad \square$$

In Figure 4, as we did for the classical HBIM, we compare the coefficients ν_2 (approximate) with ν (exact) for different values of $0 < \text{Ste} < 1$ and α.

The RIM intends to approximate the problem (P) through solving a problem (P_3) that consists in finding the free boundary $s_3 = s_3(t)$ and the temperature $T_3 = T_3(x, t)$ in $0 < x < s_3(t)$ such that equation (1.1a†) and conditions (1.1b), (1.1c), (1.1d) and (1.1e) are satisfied.

Under the assumption that T_3 adopts a quadratic profile in space like (2.2), we can state the following result.

Theorem 3.6 *The unique solution to problem* (P_3) *is given by*

$$T_3(x,t) = t^{\alpha/2}\left[A_3\theta_\infty\left(1-\frac{x}{s_3(t)}\right) + B_3\theta_\infty\left(1-\frac{x}{s_3(t)}\right)^2\right], \tag{3.30}$$

$$s_3(t) = 2a\nu_3\sqrt{t}, \tag{3.31}$$

where the constants A_3 and B_3 are given by

$$A_3 = \frac{6\text{Ste} - 2\text{Ste}\,\nu_3^2(\alpha+1) - 3\,2^{\alpha+1}\nu_3^{\alpha+2}}{\text{Ste}\,\left(\nu_3^2(\alpha+1)+3\right)}, \tag{3.32}$$

$$B_3 = \frac{-3\text{Ste} + 3\text{Ste}\,\nu_3^2(\alpha+1) + 3\,2^{\alpha+1}\nu_3^{\alpha+2}}{\text{Ste}\,\left(\nu_3^2(\alpha+1)+3\right)}, \tag{3.33}$$

and where ν_3 is the unique solution to equation

$$z^{\alpha+4}2^{\alpha+1}\alpha + z^{\alpha+2}3\,2^{\alpha+2} + z^2\text{Ste}(2+3\alpha) - 6\text{Ste} = 0, \qquad z > 0. \tag{3.34}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

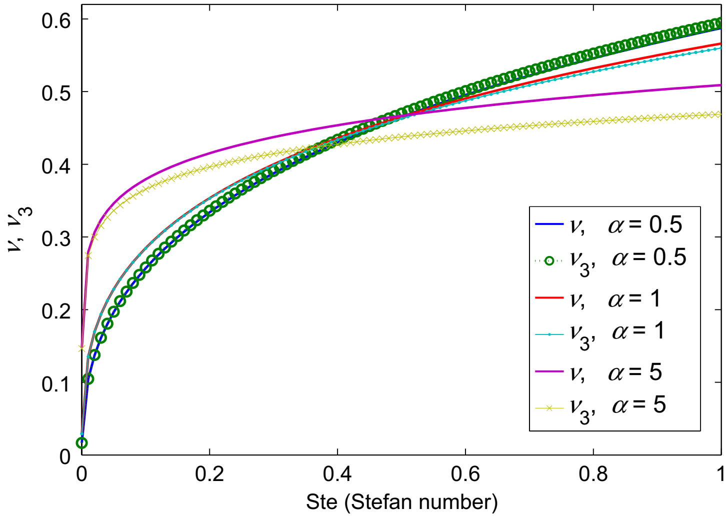

FIGURE 5. Plot of ν and ν_3 against Ste for different values of $\alpha = 0.5, 1, 5$.

Proof The proof is similar to the one of the Theorem 3.5. The only difference to take into account is the fact that equation(1.1a†) is equivalent to

$$\nu_3^2\left[A_3\left(\frac{1}{3}+\frac{2}{3}\alpha\right)+B_3\left(\frac{1}{3}+\frac{\alpha}{2}\right)\right]=B_3. \tag{3.35}$$

□

In Figure 5, we compare graphically the coefficient ν_3 that characterises the approximate free boundary s_3 with the coefficient ν that characterises the exact boundary s.

3.3 Comparisons between the approximate solutions and the exact one

In the previous section, we have applied three different methods to approximate the solution to the Stefan problem (P), with a Dirichlet condition at the fixed face and a variable latent heat.

For each method, we have stated a problem (P$_i$), $i = 1, 2, 3$ and we have compared graphically the dimensionless coefficients ν_i that characterises their free boundaries s_i, with the coefficient ν that characterises the exact free boundary s.

Then the goal will be to compare numerically, for different Ste, the coefficient ν given by (3.5) with the approximate coefficients ν_1, ν_2 and ν_3 defined by (3.13), (3.23) and (3.34), respectively.

In order that the comparisons be more representative, in Tables 1–3 we show the exact values obtained for ν, the approximate value ν_i and percentage error committed in each case $E(\nu_i) = 100\left|\frac{\nu-\nu_i}{\nu}\right|$, $i = 1, 2, 3$ for different values of Ste and α.

From the tables, we can notice that for $\alpha = 0.5$, the error committed by each method is lower than for $\alpha = 0$ or $\alpha = 5$. In all cases, the method which shows the greatest accuracy is the modified integral balance method. In other words, the best approximate problem to (P) is given by problem (P$_2$).

Besides, we can also provide an illustration at the exact temperature T with the approximate temperatures T_i, $i = 1, 2, 3$, given by (3.9), (3.19) and (3.30), respectively. If we consider $\alpha = 5$, Ste $= 0.5$, $\theta_\infty = 30$ and $a = 1$, we obtain Figures 6–9.

Table 1. *Dimensionless coefficients of the free boundaries and their percentage relative error for* $\alpha = 0$

Ste	ν	ν_1	$E_{rel}(\nu_1)$ (%)	ν_2	$E_{rel}(\nu_2)$ (%)	ν_3	$E_{rel}(\nu_3)$ (%)
0.1	0.2200	0.2232	1.4530	0.2209	0.3947	0.2218	0.7954
0.2	0.3064	0.3143	2.5729	0.3087	0.7499	0.3111	1.5213
0.3	0.3699	0.3827	3.4575	0.3738	1.0707	0.3780	2.1856
0.4	0.4212	0.4388	4.1687	0.4270	1.3618	0.4330	2.7953
0.5	0.4648	0.4869	4.7478	0.4723	1.6266	0.4804	3.3561
0.6	0.5028	0.5290	5.2236	0.5122	1.8683	0.5222	3.8729
0.7	0.5365	0.5666	5.6173	0.5477	2.0895	0.5599	4.3501
0.8	0.5669	0.6006	5.9443	0.5799	2.2923	0.5941	4.7913
0.9	0.5946	0.6316	6.2165	0.6094	2.4786	0.6255	5.1999
1.0	0.6201	0.6600	6.4432	0.6365	2.6500	0.6547	5.5786

Table 2. *Dimensionless coefficients of the free boundaries and their percentage relative error for* $\alpha = 0.5$

Ste	ν	ν_1	$E_{rel}(\nu_1)$ (%)	ν_2	$E_{rel}(\nu_2)$ (%)	ν_3	$E_{rel}(\nu_3)$ (%)
0.1	0.2569	0.2587	0.6956	0.2574	0.2001	0.2580	0.4012
0.2	0.3339	0.3372	0.9999	0.3349	0.3147	0.3360	0.6321
0.3	0.3876	0.3921	1.1718	0.3891	0.3974	0.3907	0.7995
0.4	0.4298	0.4353	1.2678	0.4318	0.4596	0.4338	0.9260
0.5	0.4650	0.4711	1.3143	0.4674	0.5067	0.4698	1.0225
0.6	0.4953	0.5018	1.3264	0.4980	0.5423	0.5007	1.0959
0.7	0.5220	0.5288	1.3133	0.5249	0.5684	0.5280	1.1508
0.8	0.5458	0.5528	1.2814	0.5491	0.5869	0.5523	1.1905
0.9	0.5675	0.5745	1.2352	0.5709	0.5989	0.5744	1.2173
1.0	0.5873	0.5943	1.1777	0.5909	0.6054	0.5946	1.2334

Table 3. *Dimensionless coefficients of the free boundaries and their percentage relative error for* $\alpha = 5$

Ste	ν	ν_1	$E_{rel}(\nu_1)$ (%)	ν_2	$E_{rel}(\nu_2)$ (%)	ν_3	$E_{rel}(\nu_3)$ (%)
0.1	0.3793	0.3563	6.0700	0.3723	1.8469	0.3656	3.6135
0.2	0.4151	0.3849	7.2853	0.4055	2.3333	0.3963	4.5496
0.3	0.4374	0.4020	8.0816	0.4256	2.6810	0.4145	5.2154
0.4	0.4537	0.4143	8.6859	0.4403	2.9615	0.4276	5.7505
0.5	0.4667	0.4239	9.1776	0.4518	3.2010	0.4377	6.2058
0.6	0.4775	0.4317	9.5943	0.4612	3.4122	0.4460	6.6060
0.7	0.4869	0.4384	9.9572	0.4693	3.6025	0.4529	6.9656
0.8	0.4950	0.4442	10.2795	0.4763	3.7766	0.4589	7.2936
0.9	0.5023	0.4492	10.5699	0.4826	3.9376	0.4642	7.5962
1.0	0.5090	0.4538	10.8345	0.4881	4.0880	0.4689	7.8780

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

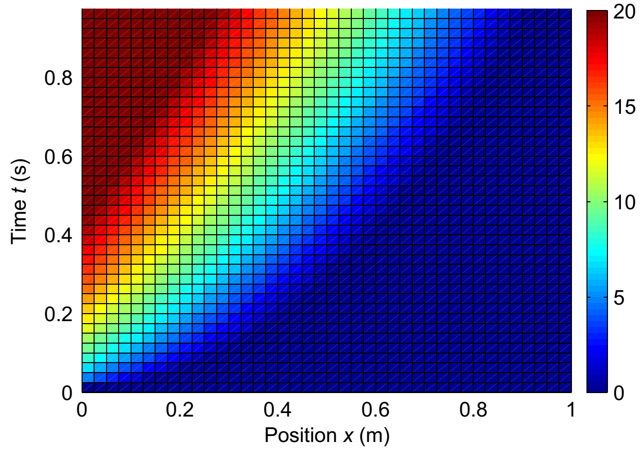

FIGURE 6. Colour map for T.

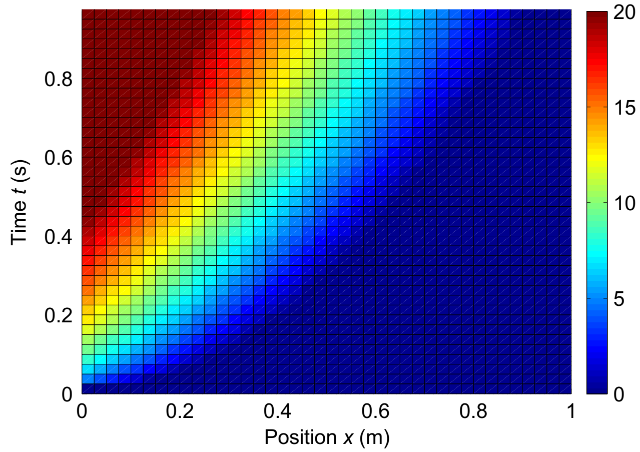

FIGURE 7. Colour map for T_1.

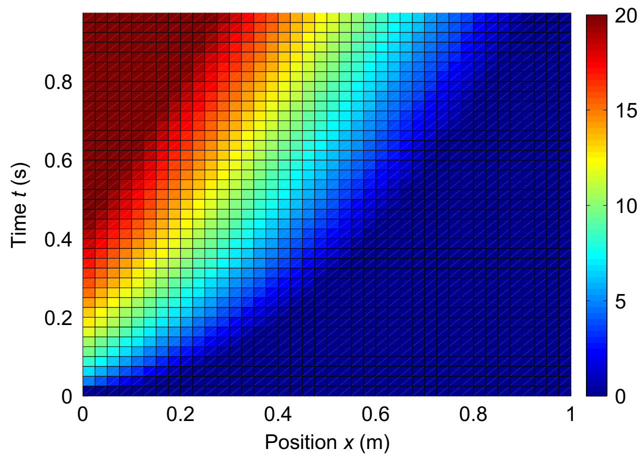

FIGURE 8. Colour map for T_2.

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

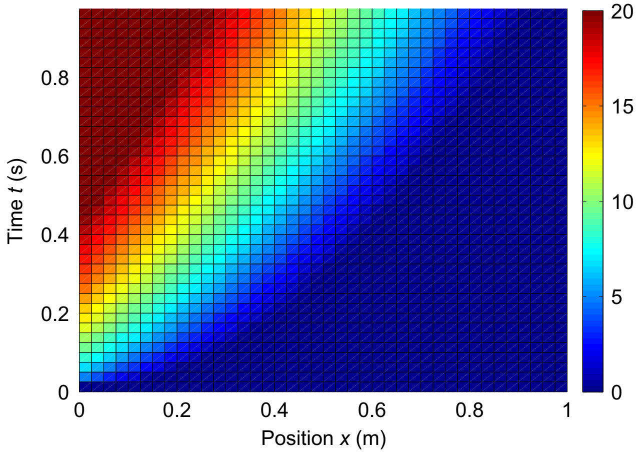

FIGURE 9. Colour map for T_3.

4 One-phase Stefan problem with Robin condition

In this section, we are going to present the exact solution of the problem with a Robin condition, then we will obtain different approximate solutions that will be compared and we will analyse their convergence when the coefficient that characterises the heat transfer at the fixed boundary goes to infinity.

4.1 Exact solution

We recall that the exact solution to problem (P_h) governed by equations (1.1a), (1.1b*) and (1.1c)–(1.1e) given in [3] can be written as

$$T_h(x,t) = t^{\alpha/2}\left[A_h M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + B_h \eta M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\eta^2\right)\right], \tag{4.1}$$

$$s_h(t) = 2a\nu_h\sqrt{t}, \tag{4.2}$$

where $\eta = \frac{x}{2a\sqrt{t}}$ is the similarity variable, the coefficients A_h and B_h are given by

$$A_h = \frac{-\nu_h M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu_h^2\right)}{M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu_h^2\right)} B_h, \tag{4.3}$$

$$B_h = \frac{-\theta_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu_h^2\right)}{\left[\frac{1}{2\text{Bi}} M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu_h^2\right) + \nu_h M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu_h^2\right)\right]}, \tag{4.4}$$

and with ν_h defined as the unique solution to the following equation:

$$\frac{\text{Ste}}{2^{\alpha+1}} \frac{1}{\left[\frac{1}{f(z)} + \frac{1}{2\text{Bi}} M\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{1}{2}, z^2\right)\right]} = z^{\alpha+1}, \qquad z > 0, \tag{4.5}$$

where Ste and f are given by (3.1) and (3.6), respectively, and where the Biot number (Bi) is defined by $\text{Bi} = \frac{ah}{k}$.

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

In [3], it was also proved that the unique solution to the exact problem with convective condition (P_h) converges pointwise to the unique solution to the problem with temperature condition (P) when the Bi goes to infinity (i.e. $h \to \infty$)

4.2 Approximate solutions and convergence

As it was done for the problem (P), we will now apply the classical integral balance method, the modified integral balance method and the RIM to the problem (P_h). For each method, we will state an approximate problem (P_{ih}), $i = 1, 2, 3$. Assuming a quadratic profile in space, we will obtain the solutions to the approximate problems. Finally, we will show that the solution of each problem (P_{ih}) converges to the solution of the problem (P_i) defined in the previous section, when $h \to \infty$. This fact is intuitively expected because the same happens to the exact problems (P_h) and (P).

We introduce an approximate problem (P_{1h}) that arises when applying the classical HBIM to the problem (P_h). It consists in finding the free boundary $s_{1h} = s_{1h}(t)$ and the temperature $T_{1h} = T_{1h}(x, t)$ in $0 < x < s_{1h}$ such that conditions: (1.1a⋆), (1.1b⋆),(1.1c), (1.1d⋆) and (1.1e) are satisfied.

Provided that T_{1h} adopts a quadratic profile in space, like (2.2) we can prove the next result.

Theorem 4.1 *If* $0 < \text{Ste} < 1$, $\alpha \geqslant 0$ *and* Bi *is large enough, there exists at least one solution to problem* (P_{1h})*, which is given by*

$$T_{1h}(x,t) = t^{\alpha/2}\theta_\infty \left[A_{1h}\left(1 - \frac{x}{s_{1h}(t)}\right) + B_{1h}\left(1 - \frac{x}{s_{1h}(t)}\right)^2 \right], \tag{4.6}$$

$$s_{1h}(t) = 2a\nu_{1h}\sqrt{t}, \tag{4.7}$$

where the constants A_{1h} *and* B_{1h} *are defined as a function of* ν_{1h}

$$A_{1h} = \frac{6\text{Ste} - 2\text{Ste}\,\nu_{1h}^2(\alpha+1) - \frac{3}{\text{Bi}}2^{\alpha+1}\nu_{1h}^{\alpha+1} - 3\,2^{\alpha+1}\nu_{1h}^{\alpha+2}}{\text{Ste}\left[\nu_{1h}^2(\alpha+1) + \frac{2}{\text{Bi}}\nu_{1h}(\alpha+1) + 3\right]}, \tag{4.8}$$

$$B_{1h} = \frac{-3\text{Ste} + 3\text{Ste}\,\nu_{1h}^2(\alpha+1) + \frac{3}{\text{Bi}}2^{\alpha}\nu_{1h}^{\alpha+1} + 3\,2^{\alpha+1}\nu_{1h}^{\alpha+2}}{\text{Ste}\left[\nu_{1h}^2(\alpha+1) + \frac{2}{\text{Bi}}\nu_{1h}(\alpha+1) + 3\right]}, \tag{4.9}$$

where ν_{1h} *is a solution to the following equation:*

$$\begin{aligned} &z^{2\alpha+4}(-3)2^{2\alpha+1}(\alpha-2) + z^{2\alpha+3}(-3)\frac{2^{2\alpha}}{\text{Bi}}(5\alpha-7) + z^{2\alpha+2}(-3)2^{2\alpha+1}\left(\frac{\alpha-2}{\text{Bi}^2}+3\right) \\ &+ z^{2\alpha+1}(-9)\frac{2^{2\alpha}}{\text{Bi}} + z^{\alpha+4}(-3)2^{\alpha}\text{Ste}(\alpha-3)(\alpha+1) + z^{\alpha+3}(-3)\frac{2^{\alpha+1}}{\text{Bi}}\text{Ste}(\alpha-1)(\alpha+1) \\ &+ z^{\alpha+2}(-3)2^{\alpha+1}\text{Ste}(\alpha+7) + z^{\alpha+1}3\frac{2^{\alpha+1}}{\text{Bi}}\text{Ste}(\alpha-5) + z^{\alpha}9\,2^{\alpha}\text{Ste} + z^4 2\text{Ste}^2(1+\alpha)^2 \\ &+ z^2(-12)\text{Ste}^2(\alpha+1) + 18\text{Ste}^2 = 0, \qquad z > 0. \end{aligned} \tag{4.10}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

Proof It can be easily checked that the chosen profile (4.6) verifies condition (1.1c). In addition, we have

$$\frac{\partial T_{1h}}{\partial x}(x,t) = -t^{\alpha/2}\theta_\infty \left[\frac{A_{1h}}{s_{1h}(t)} + \frac{2B_{1h}}{s_{1h}(t)}\left(1 - \frac{x}{s_{1h}(t)}\right)\right],$$

and

$$\frac{\partial^2 T_{1h}}{\partial x^2}(x,t) = t^{\alpha/2}\theta_\infty \frac{2B_{1h}}{s_{1h}^2(t)}.$$

In virtue of condition (1.1d*), the following equality holds:

$$\frac{k}{\gamma s_{1h}^{\alpha}(t)} t^{\alpha}\theta_\infty^2 \frac{A_{1h}^2}{s_{1h}^2(t)} = a^2 t^{\alpha/2}\theta_\infty \frac{2B_{1h}}{s_{1h}^2(t)}.$$

Consequently,

$$s_{1h}(t) = \left(\frac{A_{1h}^2}{2B_{1h}}\frac{k\theta_\infty}{\gamma a^2}\right)^{1/\alpha}\sqrt{t}.$$

Defining ν_{1h} such that $\nu_{1h} = \frac{1}{2a}\left(\frac{A_{1h}^2}{2B_{1h}}\frac{k\theta_\infty}{\gamma a^2}\right)^{1/\alpha}$, we conclude that

$$s_{1h}(t) = 2a\nu_{1h}\sqrt{t}, \tag{4.11}$$

where ν_{1h} is an unknown that is related with A_{1h} and B_{1h} in the following way:

$$A_{1h}^2 = \frac{2^{\alpha+1}\nu_{1h}^{\alpha}}{\text{Ste}} B_{1h}. \tag{4.12}$$

Then, condition (1.1a*) leads to

$$A_{1h}\left[(\alpha+1)\nu_{1h}^2 - 1\right] + B_{1h}\left[\frac{2}{3}(\alpha+1)\nu_{1h}^2 - 2\right] = -\frac{2^{\alpha+1}}{\text{Ste}}\nu_{1h}. \tag{4.13}$$

In addition, according to (1.1b*), we have

$$A_{1h}\left(1 + 2\text{Bi}\,\nu_{1h}\right) + 2B_{1h}\left(1 + \text{Bi}\,\nu_{1h}\right) = 2\text{Bi}\,\nu_{1h}. \tag{4.14}$$

Thus, we have obtained three equations (4.12), (4.13) and (4.14), for the three unknown coefficients A_{1h}, B_{1h} and ν_{1h}.

From (4.13) and (4.14), we obtain that A_{1h} and B_{1h} are given by (4.8) and (4.9), respectively.

Then, equation (4.12) leads to ν_{1h} as a positive solution to equation (4.10). If we denote by $\omega_{1h} = \omega_{1h}(z)$ the left-hand side of equation (4.10), we have

$$\omega_{1h}(0) = 18\,\text{Ste}^2 > 0 \tag{4.15}$$

and

$$\begin{aligned}\omega_{1h}(1) = &-\alpha^2\left(3\,2^{\alpha} - 2\text{Ste} + \tfrac{3}{\text{Bi}}2^{\alpha+1}\right)\text{Ste} - 2\alpha\left(3\,4^{\alpha} + 4\text{Ste}^2 + \tfrac{21}{\text{Bi}}2^{\alpha-1} - \tfrac{3}{\text{Bi}}2^{\alpha}\text{Ste}\right)\\ &- 2\left(3\,4^{\alpha} + 3\,2^{2+\alpha}\text{Ste} - 4\text{Ste}^2\right) + \tfrac{3}{\text{Bi}}\left(2^{2\alpha+3} - 2^{3+\alpha}\text{Ste}\right).\end{aligned} \tag{4.16}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

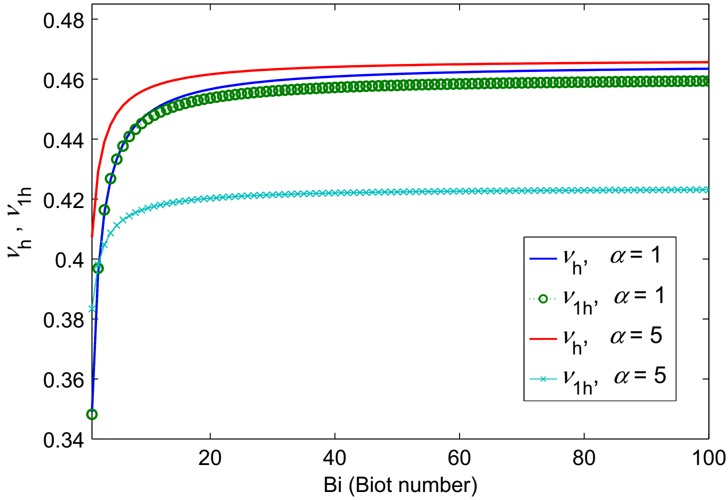

FIGURE 10. Plot of ν_h and ν_{1_h} against Bi for $\alpha = 1$ or 5 and $\mathrm{Ste} = 0.5$.

It can be noticed that if $0 < \mathrm{Ste} < 1$ and $\alpha \geqslant 0$, we have

$$3\,2^{\alpha} - 2\mathrm{Ste} + \frac{3}{\mathrm{Bi}}2^{\alpha+1} > 0,$$
$$3\,4^{\alpha} + 3\,2^{2+\alpha}\mathrm{Ste} - 4\mathrm{Ste}^2 > 0,$$

and

$$3\,4^{\alpha} + 4\mathrm{Ste}^2 + \frac{21}{\mathrm{Bi}}2^{\alpha-1} - \frac{3}{\mathrm{Bi}}2^{\alpha}\mathrm{Ste} = 3\,4^{\alpha} + 4\mathrm{Ste}^2 + \frac{3}{\mathrm{Bi}}2^{\alpha}\left(\frac{7}{2} - \mathrm{Ste}\right) > 0.$$

As $2^{2\alpha+3} - 2^{3+\alpha}\mathrm{Ste} = 2^{\alpha}2^{3}(2^{\alpha} - \mathrm{Ste}) > 0$, there exists a large enough Bi that makes $\omega_{1_h}(1) < 0$. In consequence, there will exists at least one solution to equation (4.10). □

With the aim of testing the accuracy of the classical HBIM and taking into account that the exact free boundary $s_h(t) = 2a\nu_h\sqrt{t}$ and the approximate one is given by $s_{1_h}(t) = 2a\nu_{1_h}\sqrt{t}$ we are going to compare graphically only the coefficients ν_h with ν_{1_h} for different values of Bi and α, fixing $\mathrm{Ste} = 0.5$ (see Figure 10).

The modified integral balance method defines a new approximated problem for (P_h) that will be called as problem (P_{2_h}) and which consists in finding the free boundary $s_{2_h} = s_{2_h}(t)$ and the temperature $T_{2_h} = T_{2_h}(x, t)$ in $0 < x < s_{2_h}(t)$ such that equations (1.1a⋆), (1.1b⋆) and (1.1c)–(1.1e) are satisfied.

Once again assuming a quadratic profile in space as (2.2) for the temperature T_{2_h}, we can state the following results.

Theorem 4.2 *Given* $\mathrm{Ste} > 0$ *and* $\alpha \geqslant 0$*, there exists a unique solution to the problem* (P_2) *which is given by*

$$T_{2_h}(x, t) = t^{\alpha/2}\left[A_{2_h}\theta_{\infty}\left(1 - \frac{x}{s_{2_h}(t)}\right) + B_{2_h}\theta_{\infty}\left(1 - \frac{x}{s_{2_h}(t)}\right)^2\right], \tag{4.17}$$

$$s_{2_h}(t) = 2a\nu_{2_h}\sqrt{t}, \tag{4.18}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

where the constants A_{2h} and B_{2h} are given by

$$A_{2h} = \frac{6\text{Ste} - 2\text{Ste}\, \nu_{2h}^2(\alpha+1) - \frac{3}{\text{Bi}} 2^{\alpha+1} \nu_{2h}^{\alpha+1} - 3\, 2^{\alpha+1} \nu_{2h}^{\alpha+2}}{\text{Ste}\left[\nu_{2h}^2(\alpha+1) + \frac{2}{\text{Bi}} \nu_{2h}(\alpha+1) + 3\right]}, \tag{4.19}$$

$$B_{2h} = \frac{-3\text{Ste} + 3\text{Ste}\, \nu_{2h}^2(\alpha+1) + \frac{3}{\text{Bi}} 2^{\alpha} \nu_{2h}^{\alpha+1} + 3\, 2^{\alpha+1} \nu_{2h}^{\alpha+2}}{\text{Ste}\left[\nu_{2h}^2(\alpha+1) + \frac{2}{\text{Bi}} \nu_{2h}(\alpha+1) + 3\right]}, \tag{4.20}$$

and where the coefficient ν_{2h} is the unique solution to the following equation:

$$\begin{aligned} &z^{\alpha+4} 2^{\alpha}(\alpha+1) + z^{\alpha+3} \tfrac{2^{\alpha+1}}{\text{Bi}}(\alpha+1) + z^{\alpha+2} 3\, 2^{\alpha+1} \\ &+ z^{\alpha+1} 3 \tfrac{2^{\alpha}}{\text{Bi}} + z^2 \text{Ste}(\alpha+1) - 3\text{Ste} = 0, \qquad z > 0. \end{aligned} \tag{4.21}$$

Proof It is clear immediate that the chosen profile temperature leads the condition (1.1c) to be automatically verified. From condition (1.1d), we obtain

$$-kt^{\alpha/2}\theta_{\infty} \frac{A_{2h}}{s_{2h}(t)} = -\gamma s_{2h}^{\alpha}(t) \dot{s}_{2h}(t). \tag{4.22}$$

Therefore,

$$s_{2h}(t) = \left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)} \frac{k\theta_{\infty}}{\gamma} A_{2h} \right)^{1/(\alpha+2)} \sqrt{t}. \tag{4.23}$$

Introducing the new coefficient ν_{2h} such that $\nu_{2h} = \frac{1}{2a}\left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)} \frac{k\theta_{\infty}}{\gamma} A_{2h} \right)^{1/(\alpha+2)}$, the free boundary can be expressed as

$$s_{2h}(t) = 2a\, \nu_{2h} \sqrt{t}, \tag{4.24}$$

where the following equality holds:

$$A_{2h} = \frac{2^{\alpha+1} \nu_{2h}^{\alpha+2}}{\text{Ste}}. \tag{4.25}$$

The convective boundary condition at $x = 0$, i.e. condition (1.1b*), leads to

$$A_{2h}(1 + 2\text{Bi}\, \nu_{2h}) + 2B_{2h}(1 + \text{Bi}\, \nu_{2h}) = 2\text{Bi}\, \nu_{2h}. \tag{4.26}$$

In addition, from (1.1a*) it results that

$$A_{2h}\left((\alpha+1)\nu_{2h}^2 - 1 \right) + B_{2h}\left(\tfrac{2}{3}(\alpha+1)\nu_{2h}^2 - 2 \right) = \frac{-2^{\alpha+1} \nu_{2h}^{\alpha+2}}{\text{Ste}}. \tag{4.27}$$

Taking into account equations (4.25)–(4.27), we obtain that A_{2h} and B_{2h} can be given as functions of ν_{2h} through formulas (4.19) and (4.20), respectively. Moreover, we get that ν_{2h} must be a solution to equation (4.21). To finish the proof, it remains to show that We shall notice first that

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

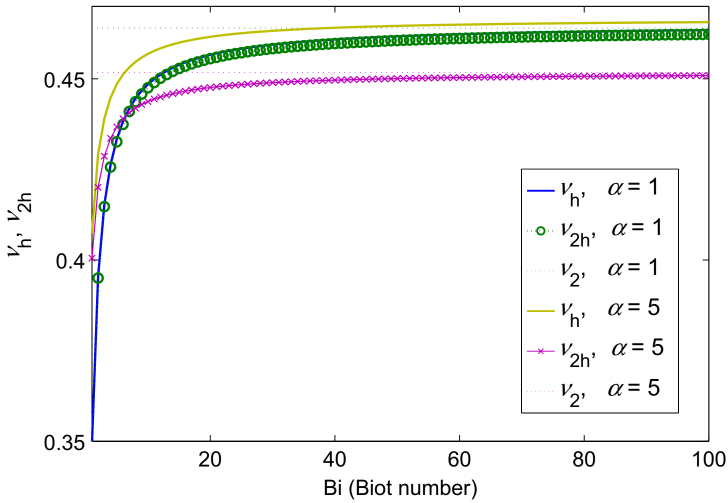

FIGURE 11. Plot of ν_h and ν_{2_h} against Bi for $\alpha = 1$ or 5 and Ste = 0.5.

(4.21) has a unique positive solution. If we define the function $w_{2_h} = w_{2_h}(z)$ as the left-hand side of equation (4.21), we have that

$$w_{2_h}(0) = -3\text{Ste} < 0, \qquad w_{2_h}(+\infty) = +\infty, \qquad \frac{dw_{2h}}{dz}(z) > 0, \quad \forall z > 0.$$

So we conclude that w_{2_h} has a unique positive root. □

In what follows, we will show that the unique solution to the problem (P_{2_h}) converges to the unique solution to the problem (P_2) when $h \to \infty$.

Theorem 4.3 *The solution to problem (P_{2_h}) given in Theorem 4.2 converges to the solution to problem (P_2) given by Theorem 3.5 when the coefficient h, which characterises the heat transfer in the fixed boundary, goes to infinity.*

Proof The free boundary of the problem (P_{2_h}) is characterised by a dimensionless coefficient ν_{2_h} which is the unique positive root of the function $\omega_{2_h} = \omega_{2_h}(z)$ defined as the left-hand side of equation (4.21). On the one hand, we can notice that if $h_1 < h_2$ then $\omega_{2h_1}(z) > \omega_{2h_2}(z)$ and consequently their unique positive root verify $\nu_{2h_1} < \nu_{2h_2}$.

On the other hand, if we define $\omega_2 = \omega_2(z)$ as the left-hand side of equation (3.23), we get

$$\omega_{2_h}(z) - \omega_2(z) = z^{\alpha+3}\frac{2^{\alpha+1}}{\text{Bi}}(\alpha+1) + z^{\alpha+1}3\frac{2^{\alpha}}{\text{Bi}} > 0, \quad \forall z > 0.$$

Therefore, $\{\nu_h\}_h$ is increasing and bounded from above by ν.

In addition, it is easily seen that when $h \to \infty$, or equivalently when $\text{Bi} \to \infty$, we obtain $\omega_{2h} \to \omega_2$ and so $\nu_{2_h} \to \nu_2$. Therefore, it is obtained that $s_{2_h}(t) \to s_2(t)$, for every $t > 0$. Showing that $A_{2_h} \to A_2$ and $B_{2_h} \to B_2$ we get $T_{2_h}(x,t) \to T_2(x,t)$ when $h \to \infty$ for every $t > 0$ and $0 < x < s_2(t)$. □

In Figure 11, we compare graphically, for different values of Bi > 1, the coefficient ν_{2_h} that characterises the free boundary s_{2_h} with the coefficient ν_h that characterises the exact free

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

boundary s_h, for different values of α, fixing Ste $= 0.5$. We shall notice that when the Bi increases then the value of ν_{2_h} gets closer to the value of ν_2.

Lastly, we will turn to the RIM applied to problem (P_h). We define a new approximate problem (P_{3_h}) which consists in finding the free boundary $s_{3_h} = s_{3_h}(t)$ and the temperature $T_{3_h} = T_{3_h}(x,t)$ in $0 < x < s_{3_h}(t)$ such that equations (1.1a†), (1.1b*) and (1.1c)–(1.1e) are verified.

Provided that T_{3_h} adopts a profile like (2.2), we state the following theorem.

Theorem 4.4 *Let* $0 < \mathrm{Ste} < 1$, $\alpha \geqslant 0$ *and* $\mathrm{Bi} \geqslant 0$, *then there exists a unique solution to problem* (P_{3h}) *which is given by*

$$T_{3_h}(x,t) = t^{\alpha/2}\left[A_{3_h}\theta_\infty\left(1-\frac{x}{s_{3_h}(t)}\right) + B_{3_h}\theta_\infty\left(1-\frac{x}{s_{3_h}(t)}\right)^2\right], \tag{4.28}$$

$$s_{3_h}(t) = 2a\nu_{3_h}\sqrt{t}, \tag{4.29}$$

where the constants A_{3_h} *and* B_{3_h} *are defined by*

$$A_{3_h} = \frac{12\nu_{3_h}\left(1-\nu_{3_h}^2\left(\frac{\alpha}{2}+\frac{1}{3}\right)\right)}{2\alpha\nu_{3_h}^3 + \left(\frac{5\alpha+2}{\mathrm{Bi}}\right)\nu_{3_h}^2 + \frac{6}{\mathrm{Bi}} + 12\nu_{3_h}}, \tag{4.30}$$

$$B_{3_h} = \frac{12\nu_{3_h}^3\left(\frac{2}{3}\alpha+\frac{1}{3}\right)}{2\alpha\nu_{3_h}^3 + \left(\frac{5\alpha+2}{\mathrm{Bi}}\right)\nu_{3_h}^2 + \frac{6}{\mathrm{Bi}} + 12\nu_{3_h}}, \tag{4.31}$$

and where ν_{3_h} *is the unique solution to the following equation:*

$$z^{\alpha+4}2^{\alpha+1}\alpha + z^{\alpha+3}\left(\frac{2^\alpha(2+5\alpha)}{\mathrm{Bi}}\right) + z^{\alpha+2}3\,2^{\alpha+2} + z^{\alpha+1}\frac{3\,2^{\alpha+1}}{\mathrm{Bi}}$$
$$+ z^2\mathrm{Ste}(2+3\alpha) - 6\mathrm{Ste} = 0, \qquad z > 0. \tag{4.32}$$

Proof The proof is similar to the one given in Theorem 4.2. The only difference lies in the fact that equation (1.1a†) is equivalent to

$$\nu_{3_h}^2\left[A_{3_h}\left(\tfrac{1}{3}+\tfrac{2}{3}\alpha\right) + B_{3_h}\left(\tfrac{1}{3}+\tfrac{\alpha}{2}\right)\right] = B_{3_h}. \tag{4.33}$$

□

The approximated problem (P_{3_h}) obtained when applying the RIM verifies the same convergence property than the exact problem (P_h).

Theorem 4.5 *The unique solution to problem (*P_{3_h}*) given by Theorem 4.4 converges to the unique solution to problem (*P_3*), given by Theorem 3.6, when the coefficient that characterises the heat transfer at the fixed face h goes to infinity.*

Proof The proof is analogous to the proof given in Theorem 4.3. □

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

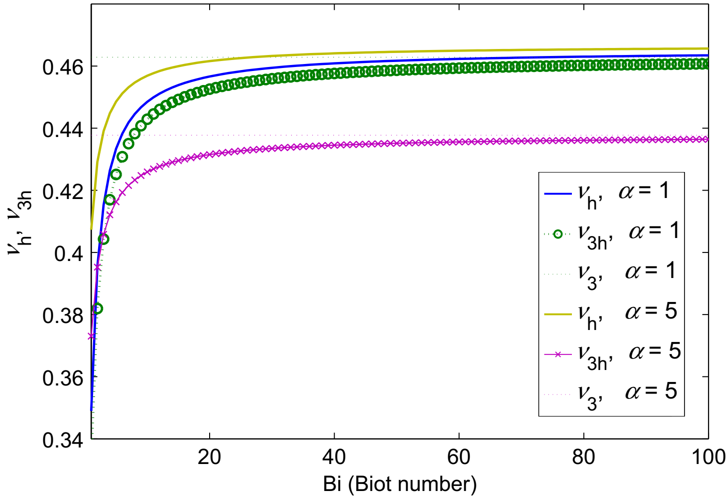

FIGURE 12. Plot of ν_h and ν_{3_h} against Bi for $\alpha = 1$ or 5 and Ste $= 0.5$.

In Figure 12, we compare graphically, for different values of $\mathrm{Bi} > 1$, the coefficient ν_{3h} that characterises the approximate free boundary s_{3h} with the coefficient ν_h corresponding to the exact free boundary s_h, for different values of α fixing Ste $= 0.5$. Once again, as Bi increases, the value ν_{3h} becomes closer to the value ν_3.

4.3 Comparisons between the approximate solutions and the exact one

In this section, we are going to compare the exact solution to the problem with a convective condition at the fixed face (P_h) with the approximate solutions obtained by applying the integral balance methods proposed in the previous sections.

For each method, we have defined a new problem (P_{ih}), $i = 1, 2, 3$ and we have compared graphically the coefficient ν_{ih} that characterises each free boundary s_{ih}, with the coefficient ν_h that corresponds to the exact free boundary s_h.

The goal is to compare numerically the coefficient ν_h given by (4.5) with the approximate coefficients ν_{1h}, ν_{2h} and ν_{3h} given by (4.10), (4.21) and (4.32), respectively.

In order that the comparisons be more representative, in Tables 4–6 we show the exact value ν_h, the approximate value ν_{ih} and the percentage error committed in each case $E(\nu_{ih}) = 100\left|\frac{\nu_h - \nu_{ih}}{\nu_h}\right|$, $i = 1, 2, 3$ for different values of Bi and α fixing Ste $= 0.5$.

From the above tables, we can deduce that for $\alpha = 0.5$, the percentage error committed is smaller than for the other cases. In all cases, as it happened with the problem (P), the method with best accuracy for approximating the problem (P_h) is the modified integral method, i.e. the best approximate problem is given by (P_{2h}).

We can also compare the exact temperature T_h with the approximate ones T_{ih}, $i = 1, 2, 3$, given by (4.6), (4.17) and (4.28), respectively. In Figures 13–16, we show a colour map for $\alpha = 5$, Ste $= 0.5$, $\theta_\infty = 30$, $a = 1$.

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

Table 4. *Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 0$ and $Ste = 0.5$*

Bi	ν_h	ν_{1h}	$E_{\text{rel}}(\nu_{1h})$ (%)	ν_{2h}	$E_{\text{rel}}(\nu_{2h})$ (%)	ν_{3h}	$E_{\text{rel}}(\nu_{3h})$ (%)
1	0.2926	0.2966	1.3828	0.2937	0.3939	0.2899	0.9103
10	0.4422	0.4681	5.8548	0.4484	1.4111	0.4545	2.7969
20	0.4533	0.4776	5.3525	0.4602	1.5151	0.4672	3.0744
30	0.4571	0.4807	5.1622	0.4642	1.5514	0.4716	3.1679
40	0.4590	0.4822	5.0628	0.4662	1.5699	0.4738	3.2148
50	0.4601	0.4832	5.0019	0.4674	1.5811	0.4751	3.2430
60	0.4609	0.4838	4.9606	0.4682	1.5886	0.4759	3.2618
70	0.4615	0.4842	4.9309	0.4688	1.5940	0.4766	3.2752
80	0.4619	0.4845	4.9085	0.4693	1.5980	0.4771	3.2853
90	0.4622	0.4848	4.8909	0.4696	1.6012	0.4774	3.2932
100	0.4625	0.4850	4.8768	0.4699	1.6037	0.4777	3.2994

Table 5. *Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 5$ and $Ste = 0.5$*

Bi	ν_h	ν_{1h}	$E_{\text{rel}}(\nu_{1h})$ (%)	ν_{2h}	$E_{\text{rel}}(\nu_{2h})$ (%)	ν_{3h}	$E_{\text{rel}}(\nu_{3h})$ (%)
1	0.3274	0.3293	0.5908	0.3280	0.1779	0.3160	3.4746
10	0.4459	0.4551	2.0484	0.4480	0.4543	0.4474	0.3370
20	0.4553	0.4631	1.7173	0.4574	0.4798	0.4583	0.6724
30	0.4585	0.4657	1.5912	0.4607	0.4886	0.4621	0.7874
40	0.4601	0.4671	1.5250	0.4623	0.4931	0.4640	0.8456
50	0.4610	0.4679	1.4844	0.4633	0.4958	0.4651	0.8807
60	0.4617	0.4684	1.4569	0.4640	0.4976	0.4659	0.9042
70	0.4622	0.4688	1.4370	0.4645	0.4989	0.4664	0.9210
80	0.4625	0.4691	1.4220	0.4648	0.4999	0.4668	0.9336
90	0.4628	0.4693	1.4103	0.4651	0.5006	0.4672	0.9434
100	0.4630	0.4695	1.4009	0.4653	0.5012	0.4674	0.9513

Table 6. *Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 0.5$ and $Ste = 0.5$*

Bi	ν_h	ν_{1h}	$E_{\text{rel}}(\nu_{1h})$ (%)	ν_{2h}	$E_{\text{rel}}(\nu_{2h})$ (%)	ν_{3h}	$E_{\text{rel}}(\nu_{3h})$ (%)
1	0.4073	0.3834	5.8702	0.4005	1.6647	0.3730	8.4069
10	0.4569	0.4170	8.7307	0.4437	2.8806	0.4259	6.7799
20	0.4616	0.4203	8.9507	0.4476	3.0301	0.4315	6.5196
30	0.4632	0.4214	9.0256	0.4489	3.0845	0.4335	6.4217
40	0.4641	0.4220	9.0633	0.4496	3.1126	0.4345	6.3703
50	0.4646	0.4224	9.0861	0.4501	3.1298	0.4351	6.3387
60	0.4649	0.4226	9.1012	0.4503	3.1414	0.4356	6.3173
70	0.4652	0.4228	9.1121	0.4505	3.1497	0.4359	6.3018
80	0.4654	0.4229	9.1203	0.4507	3.1560	0.4361	6.2901
90	0.4655	0.4230	9.1266	0.4508	3.1609	0.4363	6.2809
100	0.4656	0.4231	9.1317	0.4509	3.1649	0.4364	6.2736

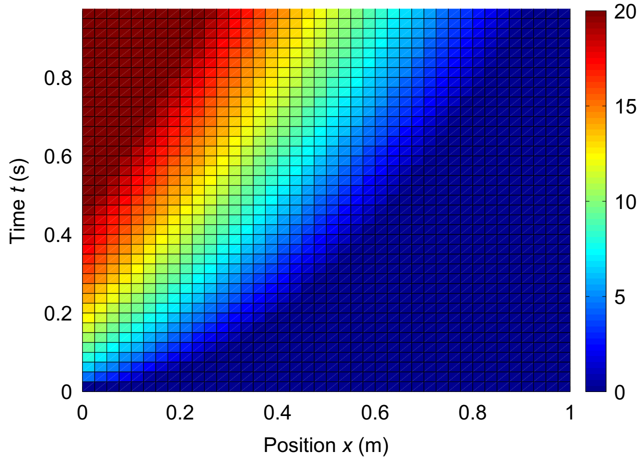

FIGURE 13. Colour map for T_h.

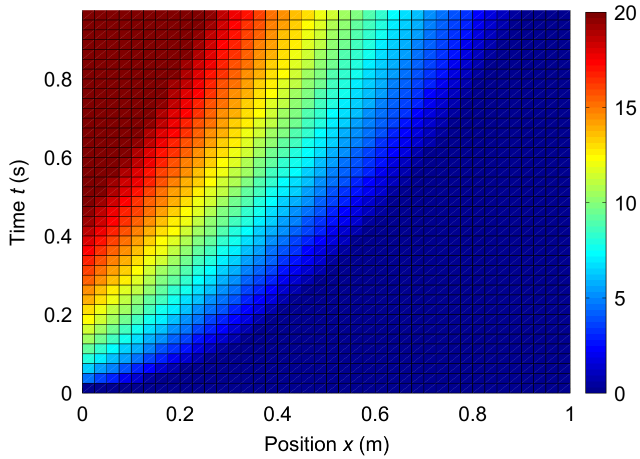

FIGURE 14. Colour map for T_{1h}.

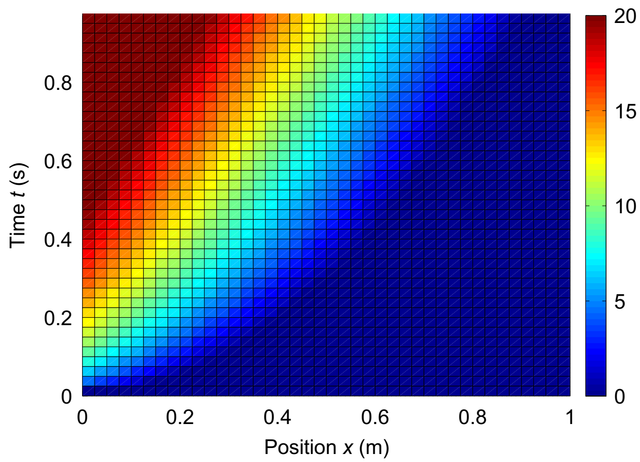

FIGURE 15. Colour map for T_{2h}.

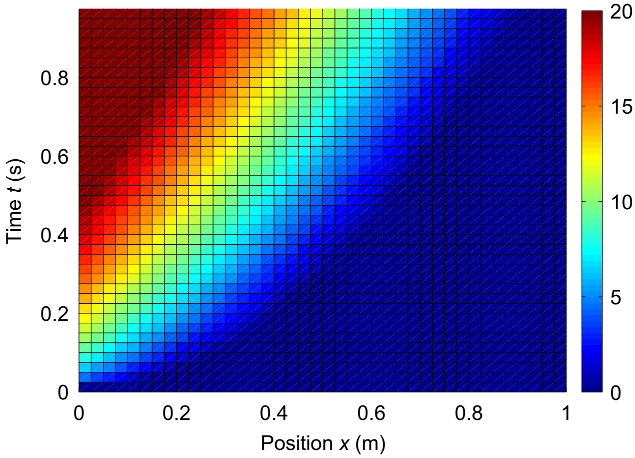

FIGURE 16. Colour map for T_{3_h}.

5 Minimising the least-squares error in the HBIM

In this section, we are going to analyse the least-squares error that we commit when assuming a quadratic profile in space. If we have an approximate solution for the heat equation given by $\hat{T}$, $\hat{s}$ such that

$$\hat{T}(x,t) = t^{\alpha/2}\theta_\infty \left[\hat{A}\left(1-\frac{x}{\hat{s}(t)}\right)+\hat{B}\left(1-\frac{x}{\hat{s}(t)}\right)^2\right], \tag{5.1}$$

with adequate coefficients $\hat{A}$, $\hat{B}$ and $\hat{s}$, then we can measure how far we are from the heat equation by computing the least-squares error (see [19]) given by

$$E = \int_0^{\hat{s}(t)} \left(\frac{\partial \hat{T}}{\partial t}(x,t) - a^2 \frac{\partial^2 \hat{T}}{\partial x^2}(x,t)\right)^2 dx \tag{5.2}$$

Taking into account that

$$\begin{aligned}\frac{\partial \hat{T}}{\partial t}(x,t) = {} & \frac{\alpha}{2} t^{\alpha/2-1}\theta_\infty \left[\hat{A}\left(1-\frac{x}{\hat{s}(t)}\right)+\hat{B}\left(1-\frac{x}{\hat{s}(t)}\right)^2\right] \\ & + t^{\alpha/2}\frac{\dot{\hat{s}}(t)}{\hat{s}^2(t)}x\theta_\infty\left[\hat{A}+2\hat{B}\left(1-\frac{x}{\hat{s}(t)}\right)\right]\end{aligned} \tag{5.3}$$

and

$$\frac{\partial^2 \hat{T}}{\partial x^2}(x,t) = t^{\alpha/2}\frac{2\hat{B}\theta_\infty}{\hat{s}^2(t)}, \tag{5.4}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

we get

$$E = \frac{\alpha^2}{4}\theta_\infty^2 t^{\alpha-2}\left(\frac{\hat{A}^2}{3}+\frac{\hat{A}\hat{B}}{2}+\frac{\hat{B}^2}{5}\right)+t^\alpha\theta_\infty^2\frac{\dot{\hat{s}}^2(t)}{\hat{s}^2(t)}\left(\frac{\hat{A}^2}{3}+\frac{\hat{A}\hat{B}}{3}+\frac{2\hat{B}^2}{15}\right)$$
$$+4t^\alpha a^4\theta_\infty^2\frac{\hat{B}^2}{\hat{s}^4(t)}+\alpha\theta_\infty^2 t^{\alpha-1}\frac{\dot{\hat{s}}(t)}{\hat{s}(t)}\left(\frac{\hat{A}^2}{6}+\frac{\hat{A}\hat{B}}{4}+\frac{\hat{B}^2}{10}\right)-2\alpha a^2\theta_\infty^2 t^{\alpha-1}\frac{\hat{B}}{\hat{s}^2(t)}\left(\frac{\hat{A}}{2}+\frac{\hat{B}}{3}\right)$$
$$-4a^2\theta_\infty^2 t^\alpha\frac{\dot{\hat{s}}(t)}{\hat{s}^3(t)}\hat{B}\left(\frac{\hat{A}}{2}+\frac{\hat{B}}{3}\right). \tag{5.5}$$

In case that the free boundary $\hat{s}(t)=2a\xi\sqrt{t}$ with $\xi>0$, by simple computations, the least-squares error becomes $E=E(\xi)$, given by the following expression:

$$E(\xi)=t^{\alpha-2}\frac{\theta_\infty^2}{\xi^4}\left[\frac{\xi^4}{4}\left(\alpha^2\left(\frac{\hat{A}^2}{3}+\frac{\hat{A}\hat{B}}{2}+\frac{\hat{B}^2}{5}\right)+2\alpha\left(\frac{\hat{A}^2}{6}+\frac{\hat{A}\hat{B}}{4}+\frac{\hat{B}^2}{10}\right)+\frac{\hat{A}^2}{3}+\frac{\hat{A}\hat{B}}{3}+\frac{2\hat{B}^2}{15}\right)\right.$$
$$\left.-\frac{\xi^2}{2}\hat{B}(\alpha+1)\left(\frac{\hat{A}}{2}+\frac{\hat{B}}{3}\right)+\frac{\hat{B}^2}{4}\right]. \tag{5.6}$$

Let us then define a new approximate problem (P_4) for the problem (P) that consists in finding the free boundary $s_4=s_4(t)$ and the temperature $T_4=T_4(x,t)$ in the domain $0<x<s_4(t)$ given by the profile (5.1) such that they minimise the least-squares error (5.5) subject to the conditions (1.1b), (1.1c), (1.1d) and (1.1e).

Theorem 5.1 *If a free boundary s_4 and a temperature T_4 constitute a solution to problem (P_4) then they are given by the expressions*

$$T_4(x,t)=t^{\alpha/2}\theta_\infty\left[A_4\left(1-\frac{x}{s_4(t)}\right)+B_4\left(1-\frac{x}{s_4(t)}\right)^2\right], \tag{5.7}$$
$$s_4(t)=2a\nu_4\sqrt{t}, \tag{5.8}$$

where the constants A_4 and B_4 are defined as a function of ν_4 as

$$A_4=\frac{2^{\alpha+1}\nu_4^{\alpha+2}}{\text{Ste}}, \qquad B_4=1-\frac{2^{\alpha+1}\nu_4^{\alpha+2}}{\text{Ste}}, \tag{5.9}$$

and where $\nu_4>0$ must minimise for every $t>0$, the function

$$E(\xi)=\frac{t^{\alpha-2}\theta_\infty^2}{60\text{Ste}^2}\frac{p(\xi)}{\xi^4}, \quad \forall t>0 \tag{5.10}$$

with

$$p(\xi)=\xi^{8+2\alpha}2^{2\alpha+1}(\alpha^2+\alpha+4)+5\,\xi^{2\alpha+6}2^{2\alpha+2}(1+\alpha)+15\,\xi^{2\alpha+4}2^{2\alpha+2}$$
$$+\xi^{\alpha+6}2^\alpha\text{Ste}(2+3\alpha+3\alpha^2)+5\,\xi^{\alpha+4}2^{\alpha+1}\text{Ste}(1+\alpha)$$
$$-15\,\xi^{2+\alpha}2^{\alpha+2}2^{\alpha+2}\text{Ste}+\xi^4\text{Ste}^2(2+3\alpha+3\alpha^2)$$
$$-10\xi^2\text{Ste}^2(1+\alpha)+15\text{Ste}^2. \tag{5.11}$$

Proof Provided that T_4 adopts a quadratic profile in space given by (5.7), then the condition (1.1c) holds immediately and the Stefan condition (1.1d) becomes equivalent to

$$-kt^{\alpha/2}\theta_\infty\frac{A_4}{s_4(t)}=-\gamma s_4^\alpha(t)\dot{s}_4(t). \tag{5.12}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

Then

$$s_4(t) = \left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)} \frac{k\theta_\infty}{\gamma} A_4 \right)^{1/(\alpha+2)} \sqrt{t}. \tag{5.13}$$

Introducing $\nu_4 = \frac{1}{2a}\left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)} \frac{k\theta_\infty}{\gamma} A_4 \right)^{1/(\alpha+2)}$, the free boundary becomes

$$s_4(t) = 2a\, \nu_4 \sqrt{t}, \tag{5.14}$$

and

$$A_4 = \frac{2^{\alpha+1} \nu_4^{\alpha+2}}{\text{Ste}}. \tag{5.15}$$

In addition, from the boundary condition at the fixed face (1.1b) we get

$$A_4 + B_4 = 1. \tag{5.16}$$

Then we obtain formulas (5.9) for the coefficients A_4 and B_4. Finally, as the free boundary s_4 is defined by (5.14), we have to minimise the least-squares error E given by (5.6). In addition, replacing A_4 and B_4 by the formulas given in (5.9), we get that ν_4 must minimise (5.10). □

Corollary 1 *For the classical Stefan problem, i.e. for the case $\alpha = 0$, we get that problem (P_4) has a unique solution given by*

$$T_4^{(0)}(x,t) = \theta_\infty \left[A_4^{(0)} \left(1 - \frac{x}{s_4^{(0)}(t)} \right) + B_4^{(0)} \left(1 - \frac{x}{s_4^{(0)}(t)} \right)^2 \right], \tag{5.17}$$

$$s_4^{(0)}(t) = 2a\nu_4^{(0)}\sqrt{t}, \tag{5.18}$$

where the superscript (0) *makes reference to the value of $\alpha = 0$ and the constants $A_4^{(0)}$ and $B_4^{(0)}$ are defined as a function of $\nu_4^{(0)}$ as*

$$A_4^{(0)} = \frac{2(\nu_4^{(0)})^2}{\text{Ste}}, \qquad B_4 = 1 - \frac{2(\nu_4^{(0)})^2}{\text{Ste}} \tag{5.19}$$

being $\nu_4^{(0)} > 0$ the value where the function $E^{(0)}$ attains its minimum

$$E^{(0)}(\xi) = \frac{t^{-2}\theta_\infty^2}{60\text{Ste}^2} \frac{p^{(0)}(\xi)}{\xi^4}, \quad \forall t > 0 \tag{5.20}$$

with

$$p^{(0)}(\xi) = 8\xi^8 + 2(10+\text{Ste})\xi^6 + 2(30 + 5\text{Ste} + \text{Ste}^2)\xi^4 \\ - 10\text{Ste}(6+\text{Ste})\xi^2 + 15\text{Ste}^2 \tag{5.21}$$

In addition, $\nu_4^{(0)}$ can be obtained as the unique positive root of the following real polynomial

$$r(\xi) = 32\xi^8 + 4(10+\text{Ste})\xi^6 + 20\text{Ste}(6+\text{Ste})\xi^2 - 60\text{Ste}^2. \tag{5.22}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

Table 7. *Dimensionless coefficients of the free boundaries and their percentage relative error for $\alpha = 0$*

Ste	ν	ν_2	$E_{\mathrm{rel}}(\nu_2)$ (%)	ν_4	$E_{\mathrm{rel}}(\nu_4)$ (%)
0.1	0.2200	0.2209	0.3947	0.2209	0.3855
0.2	0.3064	0.3087	0.7499	0.3086	0.7168
0.3	0.3699	0.3738	1.0707	0.3736	1.0040
0.4	0.4212	0.4270	1.3618	0.4265	1.2551
0.5	0.4648	0.4723	1.6266	0.4716	1.4762
0.6	0.5028	0.5122	1.8683	0.5112	1.6722
0.7	0.5365	0.5477	2.0895	0.5464	1.8470
0.8	0.5669	0.5799	2.2923	0.5783	2.0037
0.9	0.5946	0.6094	2.4786	0.6074	2.1449
1.0	0.6201	0.6365	2.6500	0.6342	2.2727

Remark 5.2 *Due to formula (5.20), we have that the error we commit when approximating with problem (P_4) for the case $\alpha = 0$ is inversely proportional to the square of time, i.e. $E^{(0)} \propto 1/t^2$.*

Proof From Theorem 5.1, we need to minimise the function $E(\xi)$ given by (5.10) for the case $\alpha = 0$. So, it is clear evident that we need to minimise the function $E^{(0)}(\xi)$ given by (5.20) which is equivalent to minimise the function $F^{(0)}(\xi) = \frac{p^{(0)}(\xi)}{\xi^4}$. Therefore, let us show that $F^{(0)}$ has a unique positive value where the minimum is attained. Observe that $F^{(0)}$ is a continuous function in $\mathbb{R}^+$. Moreover if we compute its derivative, we obtain

$$F'^{(0)}(\xi) = \frac{r(\xi)}{\xi^5}$$

with r given by (5.22). As r is a polynomial that verifies $r(0) = -60\mathrm{Ste}^2 < 0$, $r(+\infty) = +\infty$, and $r'(\xi) > 0$, $\forall \xi > 0$, we obtain that there exists a unique value $\xi_0 > 0$ such that $r(\xi_0) = 0$. In addition, we can assure that $r(\xi) < 0$, for every $\xi < \xi_0$ and $r(\xi) > 0$, for every $\xi > \xi_0$. Consequently, we have

$$F'^{(0)}(\xi) < 0, \ \forall \xi < \xi_0, \qquad F'^{(0)}(\xi_0) = 0, \qquad F'^{(0)}(\xi) > 0, \ \forall \xi > \xi_0.$$

We can conclude that $F^{(0)}$ decreases in $(0, \xi_0)$ and increases in $(\xi_0, +\infty)$. This means that $F^{(0)}$ has a unique minimum that is attained at ξ_0. Calling $\nu_4^{(0)} = \xi_0$, we get that $\nu_4^{(0)}$ is the unique positive root of r and minimises the error function $E^{(0)}$. □

Taking into account the last result we show in Table 7 the coefficient ν that characterises the exact free boundary of problem (P), the approximate coefficient ν_2 obtained by the modified integral balance method (which until now was the most accurate technique) and the coefficient ν_4 defined by the Corollary 1 for different values of Ste numbers. Computing also the percentage relative error committed in each case we assure that the approximate problem (P_4) is the best approximation we can obtain adopting a quadratic profile in space for the temperature.

In a similar way, we can define a new approximate problem (P_{4h}) for the problem (P_h) that consists in finding the free boundary $s_{4h} = s_{4h}(t)$ and the temperature $T_{4h} = T_{4h}(x,t)$ in $0 < x <$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

$s_{4h}(t)$ given by the profile (5.1) such that they minimise the least-squares error (5.5) subject to the conditions (1.1b⋆) and (1.1c)–(1.1e).

Theorem 5.3 *If a free boundary s_{4h} and a temperature T_{4h} constitute a solution to problem (P_{4h}) then they are given by the expressions:*

$$T_{4h}(x,t) = t^{\alpha/2}\theta_\infty \left[A_{4h}\left(1-\frac{x}{s_{4h}(t)}\right)+B_{4h}\left(1-\frac{x}{s_{4h}(t)}\right)^2\right], \tag{5.23}$$

$$s_{4h}(t) = 2a\nu_{4h}\sqrt{t}, \tag{5.24}$$

where the constants A_{4h} and B_{4h} are defined as a function of ν_{4h} as

$$A_{4h} = \frac{2^{\alpha+1}\nu_{4h}^{\alpha+2}}{\text{Ste}}, \qquad B_{4h} = \frac{2\text{Bi}\,\nu_{4h} - A_{4h}(1+2\text{Bi}\,\nu_{4h})}{2(1+\text{Bi}\,\nu_{4h})} \tag{5.25}$$

and where $\nu_{4h} > 0$ must minimise for every $t > 0$, the real function:

$$\begin{aligned} E_h(\xi) = {} & \frac{t^{\alpha-2}\theta_\infty^2}{60\,\text{Ste}^2(\frac{1}{\text{Bi}}+\xi)^2}\cdot \\ & \Big\{p(\xi)+\frac{1}{\text{Bi}}\left[2^{2\alpha}\left(7\alpha^2+7\alpha+18\right)x^{2\alpha+7}+25\;2^{2\alpha+1}(\alpha+1)x^{2\alpha+5},\right. \\ & +2^\alpha\left(9\alpha^2+9\alpha+6\right)\text{Ste}x^{\alpha+5}+15\;2^{2\alpha+2}x^{2\alpha+3} \\ & \left.-5\;2^{\alpha+1}(\alpha+1)\text{Ste}\,x^{\alpha+3}-15\;2^{\alpha+1}\text{Ste}\,x^{\alpha+1}\right] \\ & +\frac{1}{\text{Bi}^2}\left[4^{\alpha+1}\left(2\alpha^2+2\alpha+3\right)x^{2\alpha+6}+5\;4^{\alpha+1}(\alpha+1)x^{2\alpha+4}+15\;4^\alpha x^{2\alpha+2}\right]\Big\} \end{aligned} \tag{5.26}$$

with $p(\xi)$ given by formula (5.11).

Proof It is clear immediate that the chosen profile temperature leads the condition (1.1c) to be automatically verified. From condition (1.1d), we obtain

$$-kt^{\alpha/2}\theta_\infty\frac{A_{4h}}{s_{4h}(t)} = -\gamma s_{4h}^\alpha(t)\dot{s}_{4h}(t). \tag{5.27}$$

Therefore,

$$s_{4h}(t) = \left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)}\frac{k\theta_\infty}{\gamma}A_{4h}\right)^{1/(\alpha+2)}\sqrt{t}. \tag{5.28}$$

Introducing the new coefficient ν_{4h} such that $\nu_{4h} = \frac{1}{2a}\left(\frac{(\alpha+2)}{(\frac{\alpha}{2}+1)}\frac{k\theta_\infty}{\gamma}A_{4h}\right)^{1/(\alpha+2)}$, the free boundary can be expressed as

$$s_{4h}(t) = 2a\,\nu_{4h}\sqrt{t}, \tag{5.29}$$

where the following equality holds:

$$A_{4h} = \frac{2^{\alpha+1}\nu_{4h}^{\alpha+2}}{\text{Ste}}. \tag{5.30}$$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

The convective boundary condition at $x = 0$, i.e. condition (1.1b*), leads to

$$A_{4h}(1 + 2\text{Bi}\,\nu_{4h}) + 2B_{4h}(1 + \text{Bi}\,\nu_{4h}) = 2\text{Bi}\,\nu_{4h}. \tag{5.31}$$

Therefore, we obtain the formulas given in (5.25). Replacing A_{4h}, B_{4h} and s_{4h} for their expressions in function of ν_{4h}, minimising the least-squares error (5.5) is equivalent to minimising (5.26) (obtained by Mathematica software). □

Corollary 2 *For the classical Stefan problem, i.e. for the case $\alpha = 0$, we get that if $\text{Bi} > \frac{1}{\sqrt{12}}$ and $\text{Ste} < \frac{1}{2\text{Bi}^2}$, then (P_{4h}) has a unique solution given by*

$$T_{4h}^{(0)}(x,t) = \theta_\infty \left[A_{4h}^{(0)} \left(1 - \frac{x}{s_{4h}^{(0)}(t)}\right) + B_{4h}^{(0)} \left(1 - \frac{x}{s_{4h}^{(0)}(t)}\right)^2 \right], \tag{5.32}$$

$$s_{4h}^{(0)}(t) = 2a\nu_{4h}^{(0)}\sqrt{t}, \tag{5.33}$$

where the superscript (0) *makes reference to the value of $\alpha = 0$ and the constants $A_{4h}^{(0)}$ and $B_{4h}^{(0)}$ are defined as a function of $\nu_{4h}^{(0)}$ as*

$$A_{4h}^{(0)} = \frac{2(\nu_{4h}^{(0)})^2}{\text{Ste}}, \qquad B_{4h}^{(0)} = \frac{2\text{Bi}\,\nu_{4h}^{(0)} - A_{4h}^{(0)}(1 + 2\text{Bi}\nu_{4h}^{(0)})}{2(1 + \nu_{4h}^{(0)}\text{Bi})}, \tag{5.34}$$

being $\nu_{4h}^{(0)} > 0$ the value where the function $E_h^{(0)}$ attains its minimum

$$\begin{aligned} E_h^{(0)}(\xi) = \frac{t^{-2}\theta_\infty^2}{60\text{Ste}^2 x^2 (\frac{1}{\text{Bi}} + x)^2} \Big\{ & p^{(0)}(\xi) + \frac{1}{\text{Bi}} \big[2x(9x^6 + (3\text{Ste} + 25)x^4 \\ & + 5(6 - \text{Ste})x^2 - 15\text{Ste})\big] + \frac{1}{\text{Bi}^2} x^2 (12x^4 + 20x^2 + 15) \Big\}, \end{aligned} \tag{5.35}$$

where $p^{(0)}$ is given by (5.21). Moreover, $\nu_{4h}^{(0)}$ can be obtained as the unique positive root of the following polynomial:

$$\begin{aligned} r_h(\xi) = {} & 16\text{Bi}^3\xi^9 + 51\text{Bi}^2\xi^8 + \xi^7\left(2\text{Bi}^3\text{Ste} + 20\text{Bi}^3 + 57\text{Bi}\right) \\ & + \xi^6\left(7\text{Bi}^2\text{Ste} + 65\text{Bi}^2 + 24\right) \\ & + \text{Bi}(3\text{Ste} + 25)\xi^5 + \xi^4\left(\text{Bi}^2\left(2\text{Ste}^2 + 15\text{Ste} + 30\right) + 20\right) \\ & + 5\text{Bi}(3 + (-1 + 12\text{Bi}^2)\text{Ste} + 2\text{Bi}^2\text{Ste}^2)\xi^3 + 45\text{Bi}^2\text{Ste}\xi^2 \\ & + 15\text{BiSte}(1 - 2\text{Bi}^2\text{Ste})\xi - 15\text{Bi}^2\text{Ste}^2. \end{aligned} \tag{5.36}$$

Proof When we replace $\alpha = 0$ in Theorem 5.3, we immediately obtain the formulas (5.34) and (5.35). In order to prove that there exists a unique value that minimises the least-squares error, we compute $E_h'(\xi)$ and we get that $E_h'(\xi) = \frac{\theta_\infty}{30\text{Ste}^2 t^2 \xi^3 (\text{Bi}\xi + 1)^3} r_h(\xi)$ with r_h given by (5.36). We can observe that $r_h(0) < 0$, $r_h(+\infty) = +\infty$, $r_h' > 0$ under the hypothesis that $\text{Bi} > \frac{1}{\sqrt{12}}$ and $\text{Ste} < \frac{1}{2\text{Bi}^2}$. Therefore, we can assure that there exists a unique ξ_{h0} such that $r_h(\xi_{h0}) = 0$. In addition, we have that $r_h(\xi) < 0$, $\forall \xi < \xi_{h0}$ and $r_h(\xi) > 0$, $\forall \xi > \xi_{h0}$. Then we get that $E_h(\xi)$ decreases $\forall \xi < \xi_{h0}$

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

Table 8. *Coefficients of the free boundaries and their percentage relative error for $\alpha = 0$ and $Ste = 0.02$*

Bi	ν_h	ν_{2h}	$E_{\rm rel}(\nu_{2h})$ (%)	ν_{4h}	$E_{\rm rel}(\nu_{4h})$ (%)
1.0000	0.0193	0.0193	0.0002	0.0193	0.0002
2.0000	0.0350	0.0350	0.0023	0.0350	0.0022
3.0000	0.0468	0.0468	0.0066	0.0468	0.0064
4.0000	0.0553	0.0553	0.0120	0.0553	0.0117
5.0000	0.0617	0.0617	0.0175	0.0617	0.0172

Table 9. *Coefficients of the free boundaries and their percentage relative error for $\alpha = 0$ and $Ste = 0.02$*

Bi	ν_h	ν_{2h}	$E_{\rm rel}(\nu_{2h})$ (%)	ν_{4h}	$E_{\rm rel}(\nu_{4h})$ (%)
1	0.2926	0.2937	0.3939	0.2933	0.2600
10	0.4422	0.4484	1.4111	0.4477	1.2478
20	0.4533	0.4602	1.5151	0.4595	1.3576
30	0.4571	0.4642	1.5514	0.4635	1.3962
40	0.4590	0.4662	1.5699	0.4655	1.4158
50	0.4601	0.4674	1.5811	0.4667	1.4277
60	0.4609	0.4682	1.5886	0.4675	1.4357
70	0.4615	0.4688	1.5940	0.4681	1.4414
80	0.4619	0.4693	1.5980	0.4686	1.4457
90	0.4622	0.4696	1.6012	0.4689	1.4491
100	0.4625	0.4699	1.6037	0.4692	1.4518

and increases $\forall \xi > \xi_{h0}$. Consequently, we obtain that ξ_{h0} constitutes the unique minimum of the least-squares error. □

In view of the above result, in Table 8, for $\alpha = 0$ we compare the coefficient ν_h that characterises the exact free boundary problem with the coefficient ν_{2h} corresponding to the modified integral method, which was until now the most accurate, and we also compare with the coefficient ν_{2h} obtained when minimising the least-squares error. We fix Ste $= 0.02$ and vary Bi between 1 and 5. The value of this parameters are chosen in order to verify the hypothesis of Corollary 2. By computing the percentage relative error of each method, we conclude that the approximate problem (P_{4h}) gives us the best approximate solution to problem (P_h).

In case we decide to use the formula (5.36) to compute ν_{4h} without satisfying the hypothesis of the Corollary 2, fixing Ste $= 0.5$ and varying Bi from 1 to 100 we get the results shown in Table 9.

6 Conclusions

In this paper, we have studied different approximate methods for one-dimensional one-phase Stefan problems where the main feature consists in taking a space-dependent latent heat. We have considered two different problems that differ from each other in their boundary condition

https://doi.org/10.1017/S0956792520000170 Published online by Cambridge University Press

at the fixed face $x = 0$: Dirichlet or Robin condition. We have implemented the classical HBIM, a modified integral method and the RIM. Exploiting the knowledge of the exact solution of both problems (available in literature), we have studied the accuracy of the different approximations obtained. All the analysis have been carried out using dimensionless parameters like Ste and Bi. Furthermore, we have studied the case when Bi goes to infinity in the problem with a convective condition, recovering the approximate solutions when a temperature condition is imposed at the fixed face. We provided some numerical simulations and we have concluded that, in the majority of cases, the modified integral method is the most reliable in terms of accuracy. When approaching by the minimisation of the least-squares error, we get better approximations but only for the case $\alpha = 0$ (where we could prove existence and uniqueness of solution). The least accurate method was the classical HBIM, not only to the high percentage error committed but also because we could not obtain a result that assures uniqueness of the approximate solution.

Acknowledgement

The authors would like to thank the two anonymous referees for their helpful comments.

Conflict of interest

None.

References

[1] Alexiades, V. & Solomon, A. D. (1993) *Mathematical Modeling of Melting and Freezing Processes*, Hemisphere Publishing Corp., Washington.

[2] Bollati, J., Semitiel, J. & Tarzia, D. (2018) Heat balance integral methods applied to the one-phase Stefan problem with a convective boundary condition at the fixed face. *Appl. Math. Comput.* **331**, 1–19.

[3] Bollati, J. & Tarzia, D. (2018) Explicit solution for the one-phase Stefan problem with latent heat depending on the position and a convective boundary condition at the fixed face. *Commun. Appl. Anal.* **22**, 309–332.

[4] Bollati, J. & Tarzia, D. (2018) Exact solution for a two-phase Stefan problem with variable latent heat and a convective boundary condition at the fixed face. *Z. Angew. Math. Phys.* **69-38**, 1–15.

[5] Bollati, J. & Tarzia, D. (2018) One-phase Stefan problem with a latent heat depending on the position of the free boundary and its rate of change. *Electron. J. Diff. Equations* **2018-10**, 1–12.

[6] Cannon, J. R. (1984) *The One-Dimensional Heat Equation*, Addison-Wesley, Menlo Park, California.

[7] Dorian, T. (2014) *Spatial and Temporal Variability of Latent Heating in the Tropics Using TRMM Observations*. Master of science thesis, University of Misconsin-Madison.

[8] Goodman, T. (1958) The heat balance integral methods and its application to problems involving a change of phase. *Trans. ASME* **80**, 335–342.

[9] Gottlieb, H. (2002) Exact solution of a Stefan problem in a nonhomogeneous cylinder. *Appl. Math. Lett.* **15**, 167–172.

[10] Gupta, S. C. (2003) *The Classical Stefan Problem. Basic Concepts, Modelling and Analysis*, Elsevier, Amsterdam.

[11] Hristov, J. (2009) The heat-balance integral method by a parabolic profile with unspecified exponent: analysis and benchmark exercises. *Therm. Sci.* **13**, 27–48.

[12] Hristov, J. (2009) Research note on a parabolic heat-balance integral method with unspecified exponent: an entropy generation approach in optimal profile determination. *Therm. Sci.* **13**, 49–59.

[13] LUNARDINI, V. J. (1991) *Heat Transfer with Freezing and Thawing*, Elsevier Science Publishers B. V.

[14] MCCONNELL, T. (1991) The two-sided stefan problem with a spatially dependent latent heat. *Trans. Am. Math. Soc.* **326**, 669–699.

[15] MITCHELL, S. L. (2012) Applying the combined integral method to one-dimensional ablation. *Appl. Math. Modell.* **36**, 127–138.

[16] MITCHELL, S. L. & MYERS, T. (2010) Improving the accuracy of heat balance integral methods applied to thermal problems with time dependent boundary conditions. *Int. J. Heat Mass Transfer* **53**, 3540–3551.

[17] PRIMICERIO, M. (1970) Stefan-like problems with space-dependent latent heat. *Meccanica* **5**, 187–190.

[18] RIBERA, H. & MYERS, T. (2016) A mathematical model for nanoparticle melting with size dependent latent heat and melt temperature. *Microfluid Nanofluid* **20-147**, 1–13.

[19] RIBERA, H., MYERS, T. & MAC DAVETTE, M. (2019) Optimising the heat balance integral method in spherical and cilyndrical stefan problems. *Appl. Math. Comput.* **354**, 216–231.

[20] SADOUN, N., SI-AHMED, E. & COLINET, J. (2006) On the refined integral method for the one-phase stefan problem with time-dependent boundary conditions. *Appl. Math. Modell.* **30**, 531–544.

[21] SALVA, N. N. & TARZIA, D. A. (2011) Explicit solution for a Stefan problem with variable latent heat and constant heat flux boundary conditions. *J. Math. Anal. Appl.* **379**, 240–244.

[22] SOLOMON, A. D. (1979) An easily computable solution to a two-phase Stefan problem. Solar Energy **33**, 525–528.

[23] SWENSON, J., VOLLER, V., PAOLA, C., PARKER, G. & MARR, J. (2000) Fluvio-deltaic sedimentation: a generalized stefan problem. *Eur. J. Appl. Math.* **11**, 433–452.

[24] TARZIA, D. A. (2017) Relationship between Neumann solutions for two-phase Lamé-Clapeyron-Stefan problems with convective and temperature boundary conditions. *Therm. Sci.* **21-1**, 187–197.

[25] TARZIA, D. A. (2000) A bibliography on moving-free boundary problems for heat diffusion equation. The Stefan problem, MAT-Serie A 2, 1–297.

[26] TARZIA, D. A. (2011) Explicit and approximated solutions for heat and mass transfer problems with a moving interface. In: M. El-Amin (editor), *Advanced Topics in Mass Transfer*, Ch. 20, In Tech, Rijeka, pp. 439–484.

[27] VOLLER, V. R., SWENSON, J. & PAOLA, C. (2004) An analytical solution for a Stefan problem with variable latent heat. *Int. J. Heat Mass Transfer.* **47**, 5387–5390.

[28] WOOD, A. S. (2001) A new look at the heat balance integral method. *Appl. Math. Modell.* **25**, 815–824.

[29] ZHOU, Y., BU, W. & LU, M. (2013) One-dimensional consolidation with a threshold gradient: a Stefan problem with rate-dependent latent heat. *Int. J. Numer. Anal. Methods Geomechan.* **37**, 2825–2832.

[30] ZHOU, Y., HU, X., LI, T., ZHANG, D. & ZHOU, G. (2018) Similarity type of general solution for one-dimensional heat conduction in the cylindrical coordinate. *Int. J. Heat Mass Transfer* **119**, 542–550.

[31] ZHOU, Y., SHI, X. & ZHOU, G. (2018) Exact solution for a two-phase Stefan problem with power-type latent heat. *J. Eng. Math.* **110**, 1–13.

[32] ZHOU, Y., WANG, Y. & BU, W. (2014) Exact solution for a Stefan problem with a latent heat a power function of position. *Int. J. Heat Mass Transfer* **69**, 451–454.

[33] ZHOU, Y. & XIA, L. (2015) Exact solution for Stefan problem with general power-type latent heat using Kummer function. *Int. J. Heat Mass Transfer* **84**, 114–118.