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POLYGONAL QUASICONFORMAL MAPPINGS AND CHORD-ARC CURVES

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Abstract

In this paper we show that a polygonal quasiconformal mapping always corresponds to a chord-arc curve. Furthermore, we find that the set of curves corresponding to polygonal quasiconformal mappings is path connected in the set of all bounded chord-arc curves.

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1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be its boundary. A Jordan curve Γ is said to be a chord-arc curve if there exists a constant C such that for every ξ_1, $\xi_2 \in \partial\mathbb{D}$,

$$\mathcal{L}(\gamma) \le C|\xi_1 - \xi_2|,$$

where γ is the 'shorter' arc of Γ joining ξ_1 and ξ_2 and $\mathcal{L}(\gamma)$ denotes its arc length. A domain Ω in the plane is said to be a chord-arc domain if its boundary is a chord-arc curve. A weaker condition than chord-arc is Ahlfors' three-point condition: a Jordan curve γ satisfies the three-point condition if there is a constant C such that for any three points z_1, z_2 and z_3 on the curve γ with $z_3 \in (z_1, z_2)$, $|z_1 - z_3| \le C|z_1 - z_2|$.

Suppose that Γ is an oriented Jordan curve in the plane which separates the plane into two complementary regions Ω_+ and Ω_-. Let f and g be conformal mappings of $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ onto Ω_+ and Ω_-, respectively. These two mappings extend homeomorphically to the boundary and hence $f^{-1} \circ g$ determines an oriented homeomorphism h of the unit circle to itself. Furthermore, if Γ is a chord-arc curve, then the welding $h = f^{-1} \circ g$ belongs to the group $SQ(\partial\mathbb{D})$ of strongly quasisymmetric homeomorphisms of the unit circle, that is, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|E| \le \delta|I| \Rightarrow |h(E)| \le \varepsilon|h(I)|$$

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whenever $I \subset \partial\mathbb{D}$ is an interval and $E \subset I$ is a measurable subset. From [5] or [2], we know that $SQ(\partial\mathbb{D})$ is the group of all homeomorphisms of the unit circle for which the associated measure $dh = h'ds$ is absolutely continuous with density h' belonging to the class of weights A_∞ introduced by Muckenhoupt. We can define a distance in $SQ(\partial\mathbb{D})$ by $d(h,k) = \|\log(h') - \log(k')\|_{BMO}$ to make $SQ(\partial\mathbb{D})$ a topological space, since $\log(h')$ is in $BMO(\partial\mathbb{D})$, the space of functions of bounded mean oscillation. The important problem of the connectivity of the manifold of chord-arc domains remains open. See [1] for more results on this topic.

Let $M(\mathbb{D})$ denote the unit sphere of all essentially bounded measurable functions in $\mathbb{D}$. For a given $\mu \in M(\mathbb{D})$, there exists a unique quasiconformal self-mapping f^μ of $\mathbb{D}$ fixing $1, -1$ and i and satisfying

$$\frac{\partial f^\mu}{\bar{\partial} z} = \mu \frac{\partial f^\mu}{\partial z} \quad \text{a.e. } z \in \mathbb{D}.$$

The measurable function μ is called the Beltrami coefficient of f^μ. Similarly, there exists a unique quasiconformal homeomorphism of the plane f_μ which is conformal outside of the unit disc $\mathbb{D}$ with the normalisation

$$f_\mu(1) = 1, \quad f_\mu(i) = i \quad \text{and} \quad f_\mu(-1) = -1.$$

In the unit disc $\mathbb{D}$, we have again

$$\frac{\partial f^\mu}{\bar{\partial} z} = \mu \frac{\partial f^\mu}{\partial z} \quad \text{a.e. } z \in \mathbb{D}.$$

If $\mu \in M(\mathbb{D})$, then f^μ has well-defined boundary values giving a quasisymmetric homeomorphism of the unit circle. We define an equivalence relation on $M(\mathbb{D})$ by $\mu \sim \nu$ if $f^\mu|_{\partial\mathbb{D}} = f^\nu|_{\partial\mathbb{D}}$. The equivalence class which contains μ is denoted by $[\mu]$ or $[f^\mu]$ and the set of all the equivalence classes is the universal Teichmüller space $T(\mathbb{D})$.

For any $\mu \in M(\mathbb{D})$, let $f^\mu \in [\mu]$ be a quasiconformal self-homeomorphism of the unit disc $\mathbb{D}$. Define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \sim \mu\}.$$

A quasiconformal mapping f^μ is extremal if $\|\mu\|_\infty = k_0(\mu)$. It is well known that there always exists at least one extremal mapping in each point of $T(\mathbb{D})$. If the complex Beltrami coefficient of f^μ is of the form $k\bar{\varphi}/|\varphi|$, where $k = \|\mu\|_\infty$ and φ is a holomorphic quadratic differential with finite norm

$$\|\varphi\| = \int_{\mathbb{D}} |\varphi(z)|\, dx\, dy,$$

where $z = x + iy$, then we call f^μ a Teichmüller mapping and φ the associated quadratic differential for f^μ. It is well known that the Teichmüller mapping f^μ is the unique extremal mapping in $[\mu]$. A well-known criterion for $f^\mu \in [\mu]$ to be extremal is the following theorem due to Hamilton–Krushkal–Reich–Strebel.

Theorem 1.1 [4]. *Let $[\mu] \in T(\mathbb{D})$. Then $f^\mu \in [\mu]$ is extremal if and only if*

$$\sup\Big\{\Re \iint_{\mathbb{D}} \mu(z)\varphi(z)\, dx\, dy\Big\} = \|\mu\|_\infty,$$

where the sup *is taken over all holomorphic quadratic differentials with norm one.*

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Polygonal quasiconformal mappings were introduced by Strebel (see [10, 12]) and they play a fundamental role in the theory of extremal quasiconformal mappings. They are defined as follows.

Let $\mathbb{D}(z_1, z_2, \ldots, z_n)$ denote the unit disc $\mathbb{D}$ with $n \geq 4$ anticlockwise ordered distinguished points $z_1, z_2, \ldots, z_n$ fixed on $\partial\mathbb{D}$; $\mathbb{D}(z_1, z_2, \ldots, z_n)$ is called an n-polygon. For a pair of n-polygons, $\mathbb{D}(z_1, z_2, \ldots, z_n)$ and $\mathbb{D}(w_1, w_2, \ldots, w_n)$, with vertices corresponding to each other in the same order, there always exists a Teichmüller mapping $f : \mathbb{D}(z_1, z_2, \ldots, z_n) \to \mathbb{D}(w_1, w_2, \ldots, w_n)$ such that $f(z_j) = w_j$ for $j = 1, 2, \ldots, n$ (see [10, 12]). The Teichmüller mapping f will be called an n-polygonal quasiconformal mapping or polygonal quasiconformal mapping. Every polygonal quasiconformal mapping determines a pair of quadratic differentials φ_n on $\mathbb{D}(z_1, \ldots, z_n)$ and ψ_n on $\mathbb{D}(w_1, \ldots, w_n)$ with $\|\varphi_n\| = \|\psi_n\| = 1$, which are called polygonal differentials. The two differentials are real along the sides of the boundary of the polygons and have at most simple poles at the vertices. The quadratic differentials φ_n and ψ_n can be analytically continued outside of the unit disc and consequently they are rational. The critical points of the differentials are the poles and zeros of the differentials. The set of all critical points of a polygonal differential is a finite set.

With the foregoing background we now present the results in this paper. In [7], the authors proved that if $\mu \in M(\mathbb{D})$ is the Beltrami coefficient of a polygonal quasiconformal mapping f, then the Hausdorff dimension of $f_\mu(\partial\mathbb{D})$ is one. In this paper, we will give a stronger result that the curve $f_\mu(\partial\mathbb{D})$ is not only rectifiable, but also a chord-arc curve.

Theorem 1.2. *Let $\mu \in M(\mathbb{D})$ be the Beltrami coefficient of a polygonal quasiconformal mapping f^μ. Then the curve $f_\mu(\mathbb{D})$ is a chord-arc curve.*

Let Q be the set of all the curves γ, where $\gamma = f_\mu(\partial\mathbb{D})$ and $\mu \in M(\mathbb{D})$ is almost everywhere equal to the Beltrami coefficient of a polygonal quasiconformal mapping. From Theorem 1.2, we easily deduce the following corollary.

Corollary 1.3. *The set Q is a path-connected subset of the manifold of bounded chord-arc curves.*

2. Proofs of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. Let $\mathbb{D}(z_1, z_2, \ldots, z_n)$ and $\mathbb{D}(w_1, w_2, \ldots, w_n)$ be a pair of n-polygons and f be the polygonal quasiconformal mapping between them. By the extremal Teichmüller theory, we know that the Beltrami coefficient of f has the form $\mu(z) = k|\varphi(z)|/\varphi(z)$, where k is the essential norm of the Beltrami coefficient and $\varphi(z)$ is a holomorphic quadratic differential. For $\mu(z)$, there exists a conformal mapping g such that $g \circ f = f_\mu$ on the unit disc. It is easy to see that $f(z)$ restricted to the unit circle is equal to the conformal welding $g^{-1} \circ f_\mu$, since f_μ is conformal outside the unit disc; denote its extension by $h = g^{-1} \circ f_\mu$.

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Let $z_0 \in \partial\mathbb{D}$ and $\xi_0 = f_\mu(z_0)$.

Claim 1: The smoothness or nonsmoothness of the welding h at z_0 is dependent only on the local nature of the curve $f_\mu(\partial\mathbb{D})$ around the point ξ_0.

The proof of the claim is similar to the proof of [8, Lemma I.1]. Let D_r be a disc with centre z_0 and radius r. Suppose that g_1 is a conformal mapping from the half-disc $D_r \cap \mathbb{D}$ onto a topological half-disc Ω bounded by a portion of the curve $f_\mu(\partial\mathbb{D})$ around the relevant point $\xi_0 \in f_\mu(\partial\mathbb{D})$. Let $h_1 = g_1^{-1} \circ f_\mu$ be the welding on the arc $\partial\mathbb{D} \cap D_r$. Then

$$h = g^{-1} \circ f_\mu = g^{-1} \circ g_1 \circ g_1^{-1} \circ f_\mu = g^{-1} \circ g_1 \circ h_1.$$

Furthermore, the mapping $g^{-1} \circ g_1$ maps the interval $\partial\mathbb{D} \cap D_r$ onto an interval of the unit circle. By the Schwarz reflection principle, the mapping $g^{-1} \circ g_1$ on the half-disc $D_r \cap \mathbb{D}$ extends conformally throughout a full disc containing the arc $\partial\mathbb{D} \cap D_r$. So $g^{-1} \circ g_1$ is real analytic on the arc $\partial\mathbb{D} \cap D_r$. Hence, h_1 and h have the same smoothness or nonsmoothness properties.

In the following we give another representation of h. Let φ on $\mathbb{D}(z_1, \ldots, z_n)$ and ψ on $\mathbb{D}(w_1, \ldots, w_n)$ be the polygonal differentials associated with f. From the introduction, we know that φ and ψ are real on the unit circle. Furthermore, they have rational extensions to the whole plane. The trajectory structures of φ and ψ partition the unit disc into finitely many horizontal strips R_j $(j = 1, 2, \ldots, J)$ in $\mathbb{D}(z_1, \ldots, z_n)$ and R'_j $(j = 1, 2, \ldots, J)$ in $\mathbb{D}(w_1, \ldots, w_n)$. Let $\Phi(z) = \int \sqrt{\varphi(z)}\, dz$ and $\Psi(w) = \int \sqrt{\psi(w)}\, dw$, where $\sqrt{\varphi(z)}$ and $\sqrt{\psi(w)}$ denote the principal values of the square roots. These horizontal strips R_j and R'_j $(j = 1, 2, \ldots, J)$ are mapped by the conformal mappings $\Phi(z)$ and $\Psi(z)$ onto the Euclidean horizontal rectangles

$$\Phi(R_j) = \{\zeta = \xi + i\eta : 0 < \xi < a_j, 0 < \eta < b_j\};$$
$$\Psi(R'_j) = \{\zeta' = \xi' + i\eta' : 0 < \xi' < Ka_j, 0 < \eta' < b_j\}.$$

The polygonal mapping f in R_j $(j = 1, 2, \ldots, J)$ satisfies the relation

$$\Psi \circ f \circ \Phi^{-1}(\zeta) = K\xi + i\eta, \quad \zeta = \xi + i\eta.$$

Set

$$F = \Psi \circ f \circ \Phi^{-1}$$

and

$$h = f|_{\partial\mathbb{D}} = \Psi^{-1} \circ F \circ \Phi|_{\partial\mathbb{D}}.$$

Claim 2: The curve $f_\mu(\partial\mathbb{D})$ is rectifiable.

We first show that the curve $f_\mu(\partial\mathbb{D})$, except for a finite number of points, is locally rectifiable. By [8], if the curve $f_\mu(\partial\mathbb{D})$ has a 'corner' of positive angle at some point, then the welding for $f_\mu(\partial\mathbb{D})$ will have a 'power law' behaviour at the corresponding point. Thus, the welding $h = g \circ f_\mu$ will have vanishing or infinite derivative there. Furthermore, smooth curves always correspond to C^∞ welding.

The sets of critical points of φ and ψ in the closure of the unit disc, denoted by E_1 and E_2, respectively, are finite sets. For any $e^{i\theta} \in \partial\mathbb{D} \setminus (E_1 \cup \Phi \circ f^{-1}(E_2))$, by the

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representation of h and the trajectory structures of φ and ψ, there exists $r > 0$ such that φ is real in $(e^{i(\theta-r)}, e^{i(\theta+r)}) \subset \partial\mathbb{D} \setminus E$ and ψ is real in $(h(e^{i(\theta-r)}), h(e^{i(\theta+r)}))$. It is easy to see that h is a smooth map from $(e^{i(\theta-r)}, e^{i(\theta+r)})$ to $(h(e^{i(\theta-r)}), h(e^{i(\theta+r)}))$. By Claim 1 and [6, Theorem 4.2, page 60], $f_\mu((e^{i(\theta-r)}, e^{i(\theta+r)}))$ is rectifiable. So, except for a finite number of points, the curve $f_\mu(\partial\mathbb{D})$ is locally rectifiable.

Now we discuss the local properties of h at a critical point of φ. Without loss of generality, we suppose that $p_0 = 1$ is a zero of order n and the representation near p_0 is

$$\varphi(z) = (z-1)^n(a_n + a_{n+1}(z-1) + \cdots), \quad a_n \neq 0.$$

In a sufficiently small neighbourhood of p_0, we can select a single-valued branch of the square root, say $(a_n + a_{n+1}(z-1) + \cdots)^{1/2} = b_0 + b_1(z-1) + \cdots$. Then

$$\sqrt{\varphi(z)} = (z-1)^{n/2}(b_0 + b_1(z-1) + \cdots)$$

and, by integrating term by term,

$$\Phi(z) = (z-1)^{(n+2)/2}(c_0 + c_1(z-1) + \cdots)$$

with

$$c_k = \frac{2b_k}{n+2(k+1)}.$$

Similarly, when $p_0 = 1$ is a pole of order one, the representation of $\Phi(z)$ near p_0 is

$$\Phi(z) = (z-1)^{1/2}(c_0 + c_1(z-1) + \cdots).$$

Let

$$\zeta(z) = (c_0 + c_1(z-1) + \cdots)^{2/(n+2)}, \quad n \geq -1$$

be a single-valued branch of the right-hand side in some sufficiently small neighbourhood of p_0. Then

$$\Phi(z) = ((z-1)\zeta)^{(n+2)/2}.$$

From the introduction, $\varphi(z)$ is real on the unit circle $\partial\mathbb{D}$. For odd $n \geq -1$, $\Phi(p_0)$ is the intersection of a horizontal trajectory and a vertical trajectory of φ. Hence, $\Phi(z)$, restricted to the unit circle, is real or pure imaginary on the different sides of p_0. So there exists a subarc γ_0 of $f_\mu(\partial\mathbb{D})$ which contains p_0 as an interior point such that

$$h = \Psi^{-1} \circ F \circ \Phi|_{\gamma_0} = \Psi^{-1}(K\Phi)|_{\gamma_0}$$

or

$$h = \Psi^{-1} \circ F \circ \Phi|_{\gamma_0} = \Psi^{-1} \circ \Phi|_{\gamma_0}$$

on different sides of p_0. As in [8, pages 299–301], $f_\mu(p_0)$ is the common eye of two logarithmic spirals. When n is even, $\Phi|\gamma_0$ is real, so

$$h = \Psi^{-1} \circ F \circ \Phi|_{\gamma_0} = \Psi^{-1}(K\Phi)|_{\gamma_0}$$

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and $f_\mu(\partial\mathbb{D})$ has a tangent at $f_\mu(p_0)$. Hence, whether p_0 is a pole of order one or a zero point, $f_\mu(\partial\mathbb{D})$ is locally rectifiable near $f_\mu(p_0)$. By the compactness of $f_\mu(\partial\mathbb{D})$, the claim follows.

Now, to prove the theorem, we only need to show that there exists a bi-Lipschitz mapping between the unit circle and the curve $f_\mu(\partial\mathbb{D})$. Let $m = \mathcal{L}(f_\mu(\partial\mathbb{D}))$ denote the length of the curve $f_\mu(\partial\mathbb{D})$ so that $0 < m < \infty$. We identify the unit circle with the interval $[0, 2\pi)$ (identifying $0, 2\pi$ in the usual way). Fix a point $p \in f_\mu(\partial\mathbb{D})$ and choose an orientation of $f_\mu(\partial\mathbb{D})$. Define $T : f_\mu(\partial\mathbb{D}) \to [0, 2\pi)$ by $T(\xi) = (2\pi/m)\mathcal{L}(I(p, \xi))$, where $\mathcal{L}(I(p, \xi))$ is the length of the subarc of $f_\mu(\partial\mathbb{D})$ with end points p and ξ.

It is easy to see that T is bijective and continuous on the curve $f_\mu(\partial\mathbb{D})$. Since f_μ is a quasiconformal homeomorphism of the plane, $f_\mu(\partial\mathbb{D})$ is a quasicircle. So $f_\mu(\partial\mathbb{D})$ satisfies Ahlfors' three-point condition. Hence, $f_\mu(\partial\mathbb{D})$ does not contain a closed angle. By Ahlfors' three-point condition and the compactness of $f_\mu(\partial\mathbb{D})$,

$$0 < c_1 \leq \frac{\mathcal{L}(I(\xi_1, \xi_2))}{d(f_\mu^{-1}(\xi_1), f_\mu^{-1}(\xi_2))} \leq c_2,$$

where $I(\xi_1, \xi_2)$ is the 'shorter' closed subarc of $f_\mu(\partial\mathbb{D})$ with end points ξ_1, ξ_2 and c_1, c_2 are two constants. Thus, the mapping T is bi-Lipschitz and the theorem follows. □

Remark 2.1. For more detail of the method that we use to prove that $f_\mu(\partial\mathbb{D})$ is bi-Lipschitz equivalent to the unit circle, see [3]. Schechter [11] asserts that f_μ is of class $\mathbb{C}^{1+\varepsilon}$ provided that μ is a compactly supported function in $\mathrm{Lip}(\varepsilon, \mathbb{C})$. The main result of [9] identifies a class of nonsmooth functions μ which determine bi-Lipschitz quasiconformal mappings f_μ. It is easy to see that the polygonal quasiconformal mappings do not belong to the above two cases.

Proof of Corollary 1.3. Choose a curve γ from Q. By Theorem 1.2, γ is a chord-arc curve. By the definition of Q, there exists a bounded measure function $\mu \in M(\mathbb{D})$ of the form $k|\varphi|/\varphi$, where φ is a polygonal quadratic differential associated with a polygonal quasiconformal mapping f^μ and k is the essential norm of μ. Let $\mu_t = t\mu$. By Theorem 1.1, f^{μ_t} is a polygonal quasiconformal mapping. So γ connects with the unit circle (the case for $t = 0$) by the path $f_{t\mu}(\partial\mathbb{D})$. The corollary follows. □

Remark 2.2. For any quasiconformal homeomorphism f of the unit disc, we can choose a sequence of polygonal quasiconformal mappings $\{f_n\}$ such that $\{f_n\} \to f$ pointwise almost everywhere on the unit circle. But this does not mean that the polygonal mapping is dense in the set of all quasiconformal homeomorphisms of the unit disc. In [7], the authors gave some quasiconformal homeomorphisms that cannot be approached by polygonal mappings. However, the examples given in [7] correspond to curves with Hausdorff dimensions bigger than one. So they are not chord-arc curves. Let MCD denote the manifold of all chord-arc curves. We ask whether or not the set Q is dense in MCD under the BMO metric defined in the introduction.

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