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Generalized monotonicity in terms of differential inequalities

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Dedicated to the 80th birthday of professor Zoltán Daróczy

The classical notions of monotonicity and convexity can be characterized via the nonnegativity of the first and the second derivative, respectively. These notions can be extended applying Chebyshev systems. The aim of this note is to characterize generalized monotonicity in terms of differential inequalities, yielding analogous results to the classical derivative tests. Applications in the fields of convexity and differential inequalities are also discussed.

Keywords: Chebyshev system; generalized monotonicity; mean value theorem; differential inequality

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1. Introduction

The well-known characterization theorems of monotonic or convex functions via derivatives motivates the next problem: Find a suitable, and in some sense, important class of functions and then an adequate differential operator which is nonnegative exactly on this class. A natural candidate for such a suitable class can be obtained via Chebyshev systems:

Definition. Let I be a real interval. A continuous mapping $\boldsymbol{\omega}\colon I \to \mathbb{R}^n$ is an n-dimensional Chebyshev system if $\det\big(\boldsymbol{\omega}(t_1)\ldots\boldsymbol{\omega}(t_n)\big) > 0$ holds for any elements $t_1 < \cdots < t_n$ of the domain. We say that a function $f\colon I \to \mathbb{R}$ is monotone with respect to $\boldsymbol{\omega}$ (or briefly: $\boldsymbol{\omega}$-monotone), if, for all elements $t_0 \leqslant \cdots \leqslant t_n$ of I, the next inequality holds:

$$\det\begin{pmatrix} \boldsymbol{\omega}(t_0) & \ldots & \boldsymbol{\omega}(t_n) \\ f(t_0) & \ldots & f(t_n) \end{pmatrix} \geqslant 0. \tag{1.1}$$

Obviously, a Chebyshev system generates a base if it is evaluated at pairwise distinct points (the number of points coincides to the dimension of the range). Moreover, the base is positive for increasingly ordered points. This property implies that the linear hull of the components of a Chebyshev system consists of continuous functions having a unique interpolation property. Therefore the functions in the

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linear hulls behave like polynomials. Among Chebyshev systems, *extended complete* ones play a distinguished role. Such a system can be equivalently described via the positivity of its Wronskians' minors. Instead of the rigorous definition, we shall use this characteristic property (for precise details, see lemma 2.2 below).

The $\boldsymbol{\omega}$-monotonicity of a function has a geometrical meaning: Any element of the linear hull of $\boldsymbol{\omega}$ which agrees the function at n points, intersects the graph alternately in such a way, that it leaves below the function's graph. If the defining inequality is valid with equality, then the function is said to be *affine with respect to* $\boldsymbol{\omega}$ (briefly: $\boldsymbol{\omega}$-*affine*). Clearly, a function is $\boldsymbol{\omega}$-affine exactly if it belongs to the linear hull of the components of $\boldsymbol{\omega}$.

One of the most important example of a Chebyshev system is the so-called *polynomial system*, which is defined by $\boldsymbol{\omega}(t) = (1, \ldots, t^{n-1})$. The induced monotonicity notion is quoted as *higher-order monotonicity*, and was studied intensively first by Hopf [**5**] and by Popoviciu [**10**].

It is immediate to see that 1-monotonicity corresponds to the classical monotonicity, while 2-monotonicity reduces to the classical convexity. An excellent and detailed study on Chebyshev systems and generalized monotonicity is presented in the book of Karlin and Studden [**6**].

The aim of this note is to construct a differential operator $d_{\boldsymbol{\omega}}$ corresponding to a given Chebyshev system $\boldsymbol{\omega}$, so that, for smooth enough functions, $d_{\boldsymbol{\omega}}(f) \geqslant 0$ be valid exactly when f is monotone with respect to $\boldsymbol{\omega}$. Applications of the main result are also presented.

2. The main results

Throughout this note, we use the notation $\mathscr{C}^n(I, X)$ for the vector space of n times continuously differentiable functions acting on an interval I and mapping to a normed space X. As usual, the operator of the kth order differentiation is denoted by $d^{(k)}$. For $\boldsymbol{\omega} \in \mathscr{C}^n(I, \mathbb{R}^n)$, define the nth order linear differential operator

$$d_{\boldsymbol{\omega}} := \det \begin{pmatrix} \boldsymbol{\omega}^{(0)} & \ldots & \boldsymbol{\omega}^{(n)} \\ d^{(0)} & \ldots & d^{(n)} \end{pmatrix}.$$

This formal Wronskian is the key tool in characterizing both $\boldsymbol{\omega}$-affinity and $\boldsymbol{\omega}$-monotonicity. Among smooth enough functions, the first property is equivalent to belonging to the kernel of $d_{\boldsymbol{\omega}}$.

THEOREM 2.1. *Let I be an interval, and let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \mathscr{C}^n(I, \mathbb{R}^n)$ be a Chebyshev system. Then a function $f \in \mathscr{C}^n(I, \mathbb{R})$ is $\boldsymbol{\omega}$-affine if and only if $d_{\boldsymbol{\omega}}(f) = 0$ holds.*

Proof. Assume first that f is affine with respect $\boldsymbol{\omega}$, that is, $f = \alpha_1\omega_1 + \cdots + \alpha_n\omega_n$ holds. Then for all $k = 1, \ldots, n$, we have

$$\alpha_1\omega_1^{(k)} + \cdots + \alpha_n\omega_n^{(k)} - f^{(k)} = 0.$$

These equations can be considered as a homogeneous system of equations having a nontrivial solution $(\alpha_1, \ldots, \alpha_n, -1)$. Hence the base determinant, which is exactly $d_{\boldsymbol{\omega}}(f)$, must be singular.

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Conversely, if $d_{\boldsymbol{\omega}}(f) = 0$, then the rows of the determinant are linearly dependent. In particular, there exists coefficients $\beta_0, \beta_1, \ldots, \beta_n$ such that $0 = \beta_0 f + \beta_1\omega_1 + \cdots + \beta_n\omega_n$. The Chebyshev property ensures that $\beta_0 \neq 0$. This yields the $\boldsymbol{\omega}$-affinity of f. □

Moreover, under some additional conditions, $\boldsymbol{\omega}$-monotonicity is equivalent to the nonnegativity of $d_{\boldsymbol{\omega}}$. To prove this fact, two lemmas are needed. The first one characterizes extended and complete Chebyshev systems (consult [**6**, theorem 1.1, p. 376] and [**6**, theorem 1.2, p. 379]).

LEMMA 2.2. *A mapping* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \mathscr{C}^n([a, b], \mathbb{R}^n)$ *is an extended complete Chebyshev system if and only if the Wronskians* $W(\boldsymbol{\omega}_1), \ldots, W(\boldsymbol{\omega}_n)$ *are positive, where* $\boldsymbol{\omega}_k := (\omega_1, \ldots, \omega_k)$.

The second lemma is a consequence of the mean value theorem obtained by Páles [**7**]. This result subsumes a wide range of classical mean value theorems and can even be extended [**8**]. In what follows, we recall it here in a slightly rewritten form.

LEMMA 2.3. *Let* $f, g \in \mathscr{C}^n([a, b], \mathbb{R})$, *and let* $\boldsymbol{\omega} \in \mathscr{C}^n([a, b], \mathbb{R}^n)$ *be an extended complete Chebyshev system. Then, for all elements* $t_0 \leqslant \cdots \leqslant t_n$ *of* $[a, b]$, *there exists* $\xi \in [a, b]$ *such that*

$$\det\begin{pmatrix} \boldsymbol{\omega}(t_0) & \ldots & \boldsymbol{\omega}(t_n) \\ f(t_0) & \ldots & f(t_n) \end{pmatrix} \cdot d_{\boldsymbol{\omega}}(g)(\xi) = \det\begin{pmatrix} \boldsymbol{\omega}(t_0) & \ldots & \boldsymbol{\omega}(t_n) \\ g(t_0) & \ldots & g(t_n) \end{pmatrix} \cdot d_{\boldsymbol{\omega}}(f)(\xi). \tag{2.1}$$

Now we are in the position to formulate and prove our main result.

THEOREM 2.4. *Let* I *be an open interval, and let* $\boldsymbol{\omega} \in \mathscr{C}^n(I, \mathbb{R}^n)$ *be an extended complete Chebyshev system. Then a function* $f \in \mathscr{C}^n(I, \mathbb{R})$ *is* $\boldsymbol{\omega}$*-monotone if and only if* $d_{\boldsymbol{\omega}}(f) \geqslant 0$ *holds.*

Proof. Assume first that f is monotone with respect to $\boldsymbol{\omega}$, and fix $t_0 \in I$ arbitrarily. Then (1.1) holds for all points $t_0 < \cdots < t_n$ belonging to I. Assume that, for some $k \in \{1, \ldots, n\}$, we have already proved the inequality

$$0 \leqslant \det\begin{pmatrix} \boldsymbol{\omega}(t_0) & \ldots & \boldsymbol{\omega}^{(k-1)}(t_0) & \boldsymbol{\omega}(t_k) & \ldots & \boldsymbol{\omega}(t_n) \\ f(t_0) & \ldots & f^{(k-1)}(t_0) & f(t_k) & \ldots & f(t_n) \end{pmatrix}. \tag{2.2}$$

Taylor's Theorem guarantees the existence of a vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in]t_0, t_k[^n$ satisfying

$$\boldsymbol{\omega}(t_k) = \sum_{l=0}^{k-1} \frac{(t_k - t_0)^l}{l!}\boldsymbol{\omega}^{(l)}(t_0) + \frac{(t_k - t_0)^k}{k!}\langle \boldsymbol{\omega}^{(k)}, \boldsymbol{\xi}\rangle \tag{2.3}$$

under the convention $\langle \boldsymbol{\omega}^{(k)}, \boldsymbol{\xi}\rangle := \big(\omega_1^{(k)}(\xi_1), \ldots, \omega_n^{(k)}(\xi_n)\big)$. Substitute now the formula (2.3) into the inequality (2.2). Then, applying Taylor expansion also in the last

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row, the $(k+1)$st column can be written as the linear combination of the previous columns and a remainder. Thus,

$$0 \leqslant \det \begin{pmatrix} \dots & \boldsymbol{\omega}^{(k-1)}(t_0) & \sum_{l=0}^{k-1} \frac{(t_k-t_0)^l}{l!}\boldsymbol{\omega}^{(l)}(t_0) + \frac{(t_k-t_0)^k}{k!}\langle \boldsymbol{\omega}^{(k)}, \boldsymbol{\xi}\rangle & \boldsymbol{\omega}(t_{k+1}) & \dots \\ \dots & f^{(k-1)}(t_0) & \sum_{l=0}^{k-1} \frac{(t_k-t_0)^l}{l!} f^{(l)}(t_0) + \frac{(t_k-t_0)^k}{k!} f^{(k)}(\xi) & f(t_{k+1}) & \dots \end{pmatrix}$$

$$= \det \begin{pmatrix} \dots & \boldsymbol{\omega}^{(k-1)}(t_0) & \langle \boldsymbol{\omega}^{(k)}, \boldsymbol{\xi}\rangle & \boldsymbol{\omega}(t_{k+1}) & \dots \\ \dots & f^{(k-1)}(t_0) & f^{(k)}(\xi) & f(t_{k+1}) & \dots \end{pmatrix} \cdot \frac{(t_k - t_0)^k}{k!}.$$

Clearly, $\langle \boldsymbol{\omega}^{(k)}, \boldsymbol{\xi}\rangle \to \boldsymbol{\omega}^{(k)}(t_0)$ as $t_k \to t_0$. Therefore, eliminating the positive term $(t_k - t_0)^k/k!$ and then passing the limit $t_k \to t_0$, we arrive at the inequality (2.2) with $(k+1)$ instead of k. Repeating this process, finally, we arrive at $d_{\boldsymbol{\omega}}(f)(t_0) \geqslant 0$.

To verify the converse implication, take a function $g \in \mathscr{C}^n(I, \mathbb{R})$ fulfilling $d_{\boldsymbol{\omega}}(g) > 0$. Such a function does exist: Let $\tau \in I$ be fixed and define the space

$$\mathscr{C}_0^n(I, \mathbb{R}) := \{g \in \mathscr{C}^n(I, \mathbb{R}) \mid g^{(0)}(\tau) = 0, \dots, g^{(n-1)}(\tau) = 0\}.$$

By the Global Existence and Uniqueness Theorem, $d_{\boldsymbol{\omega}} \colon \mathscr{C}_0^n(I, \mathbb{R}) \to \mathscr{C}(I, \mathbb{R})$ is a bijection. In particular, if $\varphi \in \mathscr{C}(I, \mathbb{R})$ is positive, then there exists $g \in \mathscr{C}_0^n(I, \mathbb{R})$ such that $d_{\boldsymbol{\omega}}(g) = \varphi$, and hence the required property holds, indeed. Moreover, in this case $(\omega_1, \dots, \omega_n, g)$ is an extended $(n+1)$-dimensional Chebyshev system on any compact subinterval of I by lemma 2.2.

Let $t_0 < \dots < t_n$ be fixed elements of I. Then the right-hand side of (2.1) in lemma 2.3 is nonnegative by the above facts and by the assumption. Therefore the left-hand side of (2.1) is nonnegative, yielding the $\boldsymbol{\omega}$-monotonicity of f. □

The book of Karlin and Studden also presents a result linking generalized monotonicity with differential operators [**6**, theorem 3.2, p. 395]. Their operator is defined via a recursive process, and relays on the integral representation of extended complete Chebyshev systems [**6**, theorem 1.2, p. 379]. We believe, that our method is more direct, the connection between $\boldsymbol{\omega}$ and $d_{\boldsymbol{\omega}}$ is more concrete, and hence theorem 2.4 may give a new insight on the topic.

Observe that the necessity part of theorem 2.4 holds in case of *arbitrary* Chebyshev systems. Let us also mention that the main result of [**7**, **8**] involves the Taylor Theorem, as well. In this point of view, we might have applied the result of Páles in itself during the entire proof.

3. Applications

Two dimensional extended Chebyshev systems, quoted also as *positive regular pairs*, play a distinguished role in the field of Inequalities [**2**]. For positive regular pairs, theorem 2.4 reduces to the next statement:

Corollary 3.1. *If I is an open interval and $\boldsymbol{\omega} \in \mathscr{C}^2(I, \mathbb{R})$ is an extended complete Chebyshev system, then $f \in \mathscr{C}^2(I, \mathbb{R})$ is monotone with respect to $\boldsymbol{\omega}$ if and only if*

$$(\omega_1\omega_2' - \omega_1'\omega_2)f'' - (\omega_1\omega_2'' - \omega_1''\omega_2)f' + (\omega_1'\omega_2'' - \omega_1''\omega_2')f \geqslant 0.$$

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Not claiming completeness, let us list here some direct consequences of this Corollary. In each case, $d_{\boldsymbol{\omega}}$ can be written into a simpler form. The first case refers to *relative convexity*, of which classical convexity is a special case. The second and the third applications are about the hyperbolic and trigonometric pairs. The induced convexity notion of the trigonometric one was studied by Pólya [**4**, theorem 123, p. 98], who first noticed the corresponding differential inequality.

- If $\boldsymbol{\omega} = (1, \varphi)$ provided that $\varphi' > 0$, then $d_{\boldsymbol{\omega}} = \varphi' d^{(2)} - \varphi'' d^{(1)}$.
- If $\boldsymbol{\omega} = (\cosh, \sinh)$, then $d_{\boldsymbol{\omega}} = d^{(2)} - d^{(0)}$.
- If $\boldsymbol{\omega} = (\cos, \sin)$ and $I =]-\pi/2, \pi/2[$, then $d_{\boldsymbol{\omega}} = d^{(2)} + d^{(0)}$.

In fact, convexity induced by regular pairs can be studied in the more general framework of convexity induced by Beckenbach families [**1**]. According to a result of Peixoto [**9**], if the kernel of a second order nonlinear differential operator forms a Beckenbach family, then the solution set of the corresponding differential inequality can completely be described: A twice differentiable function is a solution of the differential inequality if and only if it is convex with respect to the Beckenbach family. Furthermore, by a theorem of Bonsall [**3**], Beckenbach convex functions are always twice differentiable in almost everywhere provided that the differential operator is linear.

However, the results of Peixoto and Bonsall cannot be applied in the case of higher-order monotonicity. On the other hand, theorem 2.4 still applies: One can easily see that the polynomial system $\boldsymbol{\omega}(t) = (1, \dots, t^{n-1})$ induces the differential operator

$$d_{\boldsymbol{\omega}} = (n-1)!(n-2)! \dots 0! d^{(n)}.$$

This easy observation immediately implies the theorem of Popoviciu [**10**] on the characterization of smooth enough higher-order monotone functions:

Corollary 3.2. *Let I be an open interval. Then, a function $f \in \mathscr{C}^n(I, \mathbb{R})$ is n-monotone if and only if $f^{(n)} \geqslant 0$ holds.*

The final applications of theorem 2.4 are about higher-order linear differential inequalities. In a sense, they are counterparts of the result of Peixoto [**9**]: Under suitable conditions, the solution of such an inequality are monotone functions with respect to a suitable Chebyshev system.

Corollary 3.3. *Let $\alpha_0, \dots, \alpha_{n-1} \colon I \to \mathbb{R}$ be continuous functions on an open interval I, and assume that the fundamental system of the linear differential equation*

$$d(\omega) := \omega^{(n)} + \alpha_{n-1}\omega^{(n-1)} + \dots + \alpha_0 \omega^{(0)} = 0$$

forms an extended complete Chebyshev system $\boldsymbol{\omega} \colon I \to \mathbb{R}^n$. Then a function $f \in \mathscr{C}^n(I, \mathbb{R})$ is a solution of the linear differential inequality $d(f) \geqslant 0$ if and only if it is $\boldsymbol{\omega}$-monotone.

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Proof. Let $W(\boldsymbol{\omega})$ be the Wronskian of $\boldsymbol{\omega}$, and let $d = d^{(n)} + \alpha_{n-1}d^{(n-1)} + \cdots + \alpha_0 d^{(0)}$. Furthermore, use the notation

$$\det_k\big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n)}\big)$$

for the determinant of that matrix which is obtained by cancelling the kth column of the matrix $\big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n)}\big)$. Obviously,

$$W(\boldsymbol{\omega}) = \det\big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n-1)}\big) = \det_n\big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n)}\big).$$

Since $d(\boldsymbol{\omega}) = 0$, we have $\boldsymbol{\omega}^{(n)} = -(\alpha_{n-1}\boldsymbol{\omega}^{(n-1)} + \cdots + \alpha_0 d^{(0)})$. Applying the expansion rule of determinants and these facts, one can arrive at

$$\begin{aligned}
\det\begin{pmatrix} \boldsymbol{\omega}^{(0)} & \ldots & \boldsymbol{\omega}^{(n)} \\ d^{(0)} & \ldots & d^{(n)} \end{pmatrix} &= \sum_{k=0}^{n}(-1)^{n+k}\det_k\big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n)}\big)d^{(k)} \\
&= W(\boldsymbol{\omega})d^{(n)} + \sum_{k=0}^{n-1}(-1)^{n+k}\det_k\Big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n-1)}, \boldsymbol{\omega}^{(n)}\Big)\,d^{(k)} \\
&= W(\boldsymbol{\omega})d^{(n)} + \sum_{k=0}^{n-1}(-1)^{n+k}\det_k\left(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n-1)}, -\sum_{l=0}^{n-1}\alpha_l\boldsymbol{\omega}^{(l)}\right)d^{(k)} \\
&= W(\boldsymbol{\omega})d^{(n)} + \sum_{k=0}^{n-1}(-1)^{n+k}\det_k\big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n-1)}, -\alpha_k\boldsymbol{\omega}^{(k)}\big)d^{(k)} \\
&= W(\boldsymbol{\omega})d^{(n)} + \sum_{k=0}^{n-1}\det\big(\boldsymbol{\omega}^{(0)}, \ldots, \boldsymbol{\omega}^{(n-1)}\big)\alpha_k d^{(k)} \\
&= W(\boldsymbol{\omega})d^{(n)} + \sum_{k=0}^{n-1}W(\boldsymbol{\omega})\alpha_k d^{(k)}.
\end{aligned}$$

This proves $d_{\boldsymbol{\omega}} = W(\boldsymbol{\omega})d$, and hence $d_{\boldsymbol{\omega}}$ and d are nonnegative simultaneously. Therefore, the statement of the Corollary follows from theorem 2.1. □

COROLLARY 3.4. *Assume that the polynomial* $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$ *of real coefficients has real zeros* $\lambda_1, \ldots, \lambda_k$ *with multiplicities* $m_1, \ldots, m_k$*. Then a function* $f \in \mathscr{C}^n(I, \mathbb{R})$ *is a solution of the differential inequality*

$$f^{(n)} + \alpha_{n-1}f^{(n-1)} + \cdots + \alpha_0 f^{(0)} \geqslant 0$$

if and only if, it is monotone with respect to $\boldsymbol{\omega}(t) = \big(t^{j_l}\exp(\lambda_l t)\big)_{j_l=0,\, l=1}^{m_l-1,\, k}$.

Proof. Clearly, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ is a fundamental system of the corresponding linear differential equation. Moreover, $(\omega_1, \ldots, \omega_k)$ is a fundamental system of some linear differential equation since the characteristic zeros are reals. Therefore the Wronskian of $(\omega_1, \ldots, \omega_k)$ is positive. Thus $\boldsymbol{\omega}$ is an extended complete Chebyshev system by lemma 2.2, and hence corollary 3.3 completes the proof. □

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Continuous interpolation families of arbitrary parameter can also be applied to establish even more general monotonicity notions. Some basic results for this kind of monotonicity were obtained by Tornheim [**11**]. These extensions involve both the cases of Beckenbach families and Chebyshev systems. As we have seen, in these particular settings, the induced monotonicity property can always be characterized via the nonnegativity of an adequate differential operator. However, the analogous problem for interpolation families of arbitrary parameters is still open and may be the topic of further research.

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