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Transversal local rigidity of discrete abelian actions on Heisenberg nilmanifolds

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Abstract. In this paper we prove a perturbative result for a class of $\mathbb{Z}^2$ actions on Heisenberg nilmanifolds that have Diophantine properties. Along the way we prove cohomological rigidity and obtain a tame splitting for the cohomology with coefficients in smooth vector fields for such actions.

Key words: local rigity, abelian actions, Heisenberg nilmanifolds
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Contents

1. *Introduction*

Starting with the seminal work of Katok and Spatzier on Anosov actions [11], smooth local classification of abelian actions with hyperbolic features has deserved a lot of attention. Hyperbolicity implies existence of invariant geometric structures whose properties were exploited in obtaining very strong local classification results [6, 17]. The main goal of local classification is completely understanding the dynamics of smooth actions, which are small perturbations of the given action.

For actions with no hyperbolicity, such as parabolic and elliptic actions, there are no convenient invariant geometric structures and the methods from the hyperbolic theory are not applicable [5]. Also, for parabolic and elliptic actions the local classification results are weaker than for hyperbolic actions and the methods used are more analytical. For elliptic abelian actions the main feature allowing local classification has been the Diophantine property [12, 13] for torus translations, while the main strategy for proving local classification results has been the method of successive iterations labeled in the 1960s by the KAM method after Kolmogorov, Arnold and Moser, who devised it for the purpose of showing persistence of Diophantine tori in Hamiltonian dynamics. The method has been more recently adapted to certain kinds of parabolic *continuous*-time actions in [7] and later used in [3, 18]. This adapted method is described for general Lie group actions in [2].

In this paper we apply this adapted KAM method of successive iterations to a class of *discrete*-time abelian actions that are parabolic, meaning that the derivative of the action has polynomial growth. We describe a class of discrete abelian actions on a $(2n+1)$-dimensional Heisenberg nilmanifold, which on the induced torus have certain Diophantine properties. For the purpose of this introduction we call these actions 'Diophantine'. We show that these Diophantine actions belong to a *finite-dimensional* $(4g-1)$-dimensional family of algebraic actions for which we prove a local classification result. Namely we show that a small perturbation of the family around the Diophantine member contains a smooth conjugate of that Diophantine action. This implies that every perturbed family contains an element which is dynamically the same as the Diophantine action. This phenomenon has been previously labeled *transversal local rigidity* and has been studied for classes of *continuous-time* actions [3, 7]. For *discrete* abelian actions, we are not aware of any results in the literature where *transversal* local rigidity is proved and where it does not follow from a stronger local (or global) rigidity result for actions of $\mathbb{Z}$ or $\mathbb{R}$.

The analytic method of obtaining local classification results interprets the local conjugation problem as a nonlinear operator, which after linearization describes the *cohomology* over the unperturbed actions. The linearized version of the local classification problem is precisely *the first cohomology group* with coefficients in smooth vector fields. If the

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first cohomology is finite dimensional and both first and second coboundary operators have inverses with sufficiently nice *tame* norm estimates, then one can reasonably hope to employ the KAM iterative method. Tameness means that the C^r norm of the solution can be bounded by the $C^{r+\sigma}$ norm of the given data, where r is arbitrarily large while σ is a constant. In short, the analytic method has two major ingredients: a detailed analysis of the first cohomology and coboundary operators, and an application of the KAM iteration. Such detailed analysis of cohomology is usually hard to perform, and usually needs to use the full machinery of the representation theory, which is why results are often restricted to actions on manifolds of smaller dimension and simpler structure of representation spaces. This is the main reason that there is a lack of local rigidity results for parabolic actions on higher step nilmanifolds.

We remark that even when careful analysis of first cohomology is possible, the inverses of coboundary operators may lack tameness, in which case the KAM method may not work. Namely, in [**8**], we carried out analysis of the first cohomology for the discrete parabolic homogeneous action on $SL(2,\mathbb{R})\times SL(2,\mathbb{R})/\Gamma$. However, the inverse of the second coboundary operator turned out not to be tame; in fact, Tanis and Wang [**14**] proved that there can be no tame inverse (see also [**15**, Theorem 2.2]). No local classification results have been obtained for this example.

In this paper we perform detailed analysis of cohomology for a class of discrete-time actions with Diophantine properties on $(2n+1)$-dimensional Heisenberg nilmanifolds. It turns out that their cohomology is finite dimensional and we can obtain tame estimates for solutions of coboundary operators. Once we get complete cohomological information, we use the KAM method to prove transversal local rigidity. This is similar to the proof of the main results in [**2**, **7**], except that in the case of discrete actions we have somewhat more complicated (linear and nonlinear) operators to work with. As far as we know this is the first example of a *discrete parabolic* (but not elliptic) abelian action for which some kind of local rigidity property holds.

The analysis of first cohomology for the corresponding *continuous-time* group actions on Heisenberg nilmanifolds has been carried out in [**1**]. In the continuation of the work presented in this paper, we intend to address local classification of the $\mathbb{R}^k$ actions described in [**1**] as well as their discrete subactions.

1.1. *Setting.* Let $n\geq 2$ be an integer. The Heisenberg group over $\mathbb{R}^n$ is the set $\mathsf{H}:=\mathsf{H}(n)=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, and it is equipped with the group multiplication

$$(\mathbf{x},\boldsymbol{\xi},t)\cdot(\mathbf{x}',\boldsymbol{\xi}',t')=(\mathbf{x}+\mathbf{x}',\boldsymbol{\xi}+\boldsymbol{\xi}',t+t'+\tfrac{1}{2}(\mathbf{x}'-\mathbf{x}\cdot\boldsymbol{\xi}')).$$

The Lie algebra of H is the vector space $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, which is generated by the vector fields

$$(X_i)_{i=1}^n,\quad (\Lambda_i)_{i=1}^n,\quad Z$$

that satisfy the commutation relations

$$[X_i,X_j]=0,\quad [\Lambda_i,\Lambda_j]=0,\quad [X_i,\Lambda_j]=\delta_{ij}Z,\quad i,j\in\{1,2,\ldots,n\}.$$

The set $\Gamma:=\mathbb{Z}^n\times\mathbb{Z}^n\times\frac{1}{2}\mathbb{Z}\subset\mathsf{H}$ is the *standard lattice* of H. The lattice is cocompact and the compact quotient manifold $M:=\Gamma\backslash\mathsf{H}$ is called the *standard Heisenberg nilmanifold*.

Even though our proofs are written for the case of the standard lattice Γ, this not a restriction; the results in fact automatically hold for general lattices of H due to the complete description of all lattices in H and the corresponding representation of $\Gamma\backslash\mathsf{H}$ by Tolimieri in [16].

Let $L^2(M)$ be the space of complex-valued square-integrable functions on M. As in [1], we define the Laplacian on $L^2(M)$ by

$$\triangle := -Z^2 - \sum_{i=1}^{n} X_i^2 + \Lambda_i^2. \tag{1}$$

Then $\triangle$ is an essentially self-adjoint, non-negative operator, and $(I+\triangle)^s$ is defined by the spectral theorem for all $s>0$. The space $W^s(M)$ is the Sobolev space of s-differentiable functions defined to be the maximal domain of $(I+\triangle)^s$, and it is equipped with the inner product

$$\langle f, g\rangle_s := \langle (I+\triangle)^s f, g\rangle. \tag{2}$$

The norm of a function $f \in W^s(M)$ is denoted $\|f\|_s$. Because M is compact, we have

$$C^\infty(M) := \bigcap_{s\geq 0} W^s(M).$$

For $\mathbf{m} \in \mathbb{Z}^{2n}$, let

$$\begin{aligned} &\mathbf{m} := (m_1, m_2, \ldots, m_{2n}),\\ &\mathbf{m}_1 := (m_1, m_2, \ldots, m_n), \quad \mathbf{m}_2 := (m_{n+1}, m_{n+2}, \ldots, m_{2n}). \end{aligned} \tag{3}$$

Then let

$$\boldsymbol{\tau} := (\tau_1, \tau_2, \ldots, \tau_n, \mathbf{0}), \quad \boldsymbol{\eta} := (\mathbf{0}, \eta_1, \eta_2, \ldots, \eta_n)$$

be Diophantine over $\mathbb{Z}^n$ in $\mathbb{R}^n$ and satisfy

$$\sum_{j=1}^{n} \tau_j \eta_j = 0. \tag{4}$$

By Diophantine, we mean that there are constants $c := c_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ and $\gamma := \gamma_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ such that for any $\mathbf{m} \in \mathbb{Z}^{2n}$ and $p \in \mathbb{Z}$, we have

$$\begin{aligned} |\boldsymbol{\tau}\cdot\mathbf{m} - p| &> c|\mathbf{m}_1\cdot\mathbf{m}_1|^{-\gamma} \quad \text{if } \mathbf{m}_1 \neq 0,\\ |\boldsymbol{\eta}\cdot\mathbf{m} - p| &> c|\mathbf{m}_2\cdot\mathbf{m}_2|^{-\gamma} \quad \text{if } \mathbf{m}_2 \neq 0. \end{aligned} \tag{5}$$

The above condition is saying that the vectors of the form $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ are simultaneously Diophantine, which is the natural condition that has appeared in previous works on classification of perturbations of actions by translations on the torus (for example in [19] or [13]).

Next let

$$Y_{\boldsymbol{\tau}} := \sum_{i=1}^{n} \tau_i X_i, \quad Y_{\boldsymbol{\eta}} := \sum_{i=1}^{n} \eta_i \Lambda_i$$

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and notice that these vector fields commute because

$$[Y_\tau, Y_\eta] = 0 \tag{6}$$

is equivalent to (4).

We consider the $\mathbb{Z}^2$ right action on M given by

$$\rho(m_1, m_2)(x) := x \exp(m_1 Y_\tau + m_2 Y_\eta). \tag{7}$$

The action ρ induces a $\mathbb{Z}^2$ action on $L^2(M)$ (which we also denote by ρ), defined by

$$\rho(m_1, m_2)(f) := f \circ \rho(m_1, m_2).$$

1.2. *Results on cohomological rigidity.* Let $\rho : \mathbb{Z}^k \to \mathrm{Diff}^\infty(M)$ be a smooth $\mathbb{Z}^k$ action on a compact manifold M. Let V be a ρ-module, by which we mean that there is a $\mathbb{Z}^k$ action on V, which we label by ρ_*. Let $C^l(\mathbb{Z}^k, V)$ denote the space of multilinear maps from $\mathbb{Z}^k \times \cdots \times \mathbb{Z}^k$ to V.

Then we have the cohomology sequence

$$C(\mathbb{Z}^k, V) \xrightarrow{\mathbf{d}_1} C^1(\mathbb{Z}^k, V) \xrightarrow{\mathbf{d}_2} C^2(\mathbb{Z}^k, V), \tag{8}$$

where the operators $\mathbf{d}_1$ and $\mathbf{d}_2$ are defined as follows. For $H \in C(\mathbb{Z}^k, V) = V$ and $\beta \in C^1(\mathbb{Z}^k, V)$, define

$$\begin{aligned} \mathbf{d}_1 H(g) &= \rho_*(g)H - H, \\ (\mathbf{d}_2\beta)(g_1, g_2) &= (\rho_*(g_2)\beta(g_1) - \beta(g_1)) - (\rho_*(g_1)\beta(g_2) - \beta(g_2)). \end{aligned} \tag{9}$$

The first cohomology $H^1_\rho(V)$ over the action ρ with coefficients in the module V is defined to be $\mathrm{Ker}(\mathbf{d}_2)/\mathrm{Im}(\mathbf{d}_1)$. Elements of $\mathrm{Ker}(\mathbf{d}_2)$ are called *cocycles* over ρ with coefficients in V, and elements of $\mathrm{Im}(\mathbf{d}_1)$ are called *coboundaries* over ρ with coefficients in V.

We consider here two situations:

(1) $V = C^\infty(M)$ and $\rho_*(g)f = f \circ \rho(g)$ for any $g \in \mathbb{Z}^k$ and any $f \in C^\infty(M)$; and
(2) $V = \mathrm{Vect}^\infty M$ and $\rho_*(g)X = D\rho(g)X \circ \rho(g)^{-1}$ for any $g \in \mathbb{Z}^k$ and any $X \in \mathrm{Vect}^\infty M$.

We say that $H^1_\rho(C^\infty(M))$ is *constant* if up to a modification by a constant cocycle, every cocycle is a coboundary. This means that $H^1_\rho(C^\infty(M))$ is isomorphic to $\mathbb{R}^k$.

Now let M be the homogeneous space $\Gamma \setminus G$, where G is a Lie group with Lie algebra $\mathfrak{g}$ and Γ is a lattice in G. Let ρ be a $\mathbb{Z}^k$ action on M by right multiplication. Then ρ induces an action ρ_* on $\mathfrak{g}$ via the adjoint operator ad. This action makes $\mathfrak{g}$ into a module, so one can consider the cohomology $H^1_\rho(\mathfrak{g})$, which is of course finite dimensional. If $H^1_\rho(\mathrm{Vect}^\infty M) = H^1_\rho(\mathfrak{g})$, that is, if the cohomology with coefficients in vector fields is the same as the cohomology over ρ with coefficients in *constant* vector fields, then we say that $H^1_\rho(\mathrm{Vect}^\infty M)$ is *constant*. In particular, the constant $H^1_\rho(\mathrm{Vect}^\infty M)$ is *exceptionally small*: it is finite dimensional.

THEOREM 1.1. *For the action ρ defined in §1.1, both $H^1_\rho(C^\infty(M))$ and $H^1_\rho(\mathrm{Vect}^\infty M)$ are constant. Moreover, in both cases, the operators $\mathbf{d}_1$ and $\mathbf{d}_2$ have tame inverses. Namely, there exist positive constants σ and s_0, and there exists a left inverse $\mathbf{d}_i^*$ of $\mathbf{d}_i^*$, for $i = 1, 2$,*

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such that for all $s \geq s_0$ *there is a constant* $C_s > 0$ *such that* $\|\mathbf{d}_i^* \gamma_i\|_s \leq C_s \|\gamma_i\|_{s+\sigma}$, *where* γ_i *is a cochain in* $\mathrm{Im}(\mathbf{d}_i)$.

The above theorem for $H^1_\rho(C^\infty(M))$ is a direct consequence of the following two results which contain precise information on estimates for the norms of solutions to cohomological equations, which is essential for application of the KAM method. The precise formulation of Theorem 1.1 for $H^1_\rho(\mathrm{Vect}^\infty M)$, with concrete estimates, is Proposition 3.7 in §3.5. The property of a cohomology group described in Theorem 1.1 is usually in the literature referred to as *tame splitting in cohomology*. For more details on cohomology operators and splitting in cohomology, we refer to [3], [8] or [13].

We define the first coboundary operators associated to the generators of ρ. These are operators $L_{\boldsymbol{\tau}}$ and $L_{\boldsymbol{\eta}}$ on $L^2(M)$ given by

$$\begin{aligned} L_{\boldsymbol{\tau}} f &:= f \circ \rho(1, 0) - f, \\ L_{\boldsymbol{\eta}} f &:= f \circ \rho(0, 1) - f. \end{aligned} \tag{10}$$

THEOREM 1.2. *For any* $s \geq 0$ *and for any* $\epsilon > 0$, *there is a constant* $C_{s,\epsilon} := C_{s,\epsilon,\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ *such that for any* $f, g \in C^\infty(M)$ *of zero average with respect to the Haar measure, and that satisfy* $L_{\boldsymbol{\tau}} g = L_{\boldsymbol{\eta}} f$, *there is a solution* $P \in C^\infty(M)$ *such that*

$$L_{\boldsymbol{\tau}} P = f \quad \text{and} \quad L_{\boldsymbol{\eta}} P = g$$

and

$$\|P\|_s \leq C_{s,\epsilon}(\|f\|_{s+\max\{2\gamma,\, 3n/2+1+\epsilon\}} + \|g\|_{s+2\gamma}),$$

where γ *is the Diophantine exponent in (5).*

THEOREM 1.3. *For any* $s \geq 0$ *and for any* $\epsilon > 0$, *there is a constant* $C_{s,\epsilon} := C_{s,\epsilon,\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ *such that for any* $f, g, \phi \in C^\infty(M)$ *of zero average, and that satisfy* $L_{\boldsymbol{\eta}} f - L_{\boldsymbol{\tau}} g = \phi$, *there exists a non-constant function* $P \in C^\infty(M)$ *such that*

$$\begin{aligned} &\|g - L_{\boldsymbol{\eta}} P\|_s \leq C_{s,\epsilon} \|\phi\|_{s+\sigma(n,\gamma,\epsilon)}, \\ &\|f - L_{\boldsymbol{\tau}} P\|_s \leq C_{s,\epsilon} \|\phi\|_{s+\sigma(n,\gamma,\epsilon)}, \\ &\|P\|_s \leq C_{s,\epsilon}(\|f\|_{s+\sigma(n,\gamma,\epsilon)} + \|g\|_{s+\sigma(n,\gamma,\epsilon)}), \end{aligned}$$

where $\sigma(n, \gamma, \epsilon) := \max\{2\gamma, 3n + 1 + \epsilon\}$.

Remark 1.4. Results of this section can be viewed as the first step of obtaining discrete counterparts of the results of Cosentino and Flaminio on Lie group actions on Heisenberg nilmanifolds [1]. An additional difficulty in the discrete case is that the space of obstructions to solutions of the cohomological equation is infinite dimensional in each irreducible, infinite-dimensional representation. We trust that the following general result holds: for actions of Lie groups P considered in [1], every non-degenerate lattice subaction of P satisfies the statement of Theorem 1.1.

Remark 1.5. The Diophantine constants in (5) could have different values for $\boldsymbol{\tau}$ and for $\boldsymbol{\eta}$. It would not affect results, only the values of the constants in the estimates. For simplicity we used the same γ throughout.

Remark 1.6. We note that for a typical element of the action ρ, the first cohomology is infinite dimensional as a consequence of the results of Flaminio and Forni in [9]. The results in [9] hold for nilmanifolds of any step, and it is an interesting open problem to construct $\mathbb{R}^k$ and $\mathbb{Z}^k$ homogeneous actions satisfying Theorem 1.1 on nilmanifolds of step greater than two.

1.3. *Transversal local rigidity result.* Let ρ be a smooth action of a discrete group G by diffeomorphisms of a smooth compact manifold M. Suppose that there exists a *finite-dimensional family* $\{\rho^\lambda\}_{\lambda\in\mathbb{R}^d}$ of smooth G actions on M such that $\rho^0 = \rho$, and the family is C^1 transversally, that is, it is C^1 in the parameter λ.

The action ρ is *transversally locally rigid with respect to the family* $\{\rho^\lambda\}$ if every *sufficiently small* perturbation of the family ρ^λ in a neighborhood of $\lambda = 0$ intersects the smooth conjugacy class of ρ, where the smooth conjugacy class of ρ consists of all actions $\{h \circ \rho \circ h^{-1} : h \in \mathrm{Diff}^\infty(M)\}$. By *sufficiently small* we mean that the perturbed family consists of sufficiently small perturbations of the elements in the initial family, in a fixed C^l norm determined by the given initial family data, and that transversally in the direction of the parameter λ the perturbed family is close to the initial one in the C^1 topology. The precise smallness conditions we need are given in Theorem 3.10, which is the more precise formulation of our main local rigidity result.

THEOREM 1.7. *Let ρ be the $\mathbb{Z}^2$ action defined in* (7)*, where $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ are Diophantine as in* (5)*. Then ρ is transversally locally rigid with respect to an explicit* $(4n-1)$*-dimensional family of homogeneous $\mathbb{Z}^2$ actions.*

The explicit family of actions is defined in §3.1.

1.4. *Structure of the paper.* The paper has two parts with analysis of different flavor. In §2 we prove the cohomological results in Theorems 1.2 and 1.3. These results are further used in §3.5 to prove Proposition 3.7. All these results together imply directly Theorem 1.1. The main analytic tool for the proof of cohomological results is representation theory on the Heisenberg nilmanifold. The calculation in finite-dimensional representation is significantly simpler and is written in the appendix. The main calculation in infinite-dimensional representation is done in §2.3. In the second part of the paper we apply cohomological results to prove Theorem 1.7. We describe the finite-dimensional family relative to which transversal rigidity holds in §3.2 and we prove the main iterative step needed for Theorem 1.7 in §3.6.

2. *Proofs of Theorems 1.2 and 1.3*

2.1. *Representation spaces.* Let $L^2(M)$ be the Hilbert space of complex-valued square-integrable functions with respect to the H-invariant volume form for M. By the Stone–von Neumann theorem, the space $L^2(M)$ decomposes into an orthogonal sum of irreducible, unitary representations that are unitarily equivalent to certain one-dimensional or infinite-dimensional models that we describe at the top of §§2.2 and 2.3. Moreover, by irreducibility, Sobolev spaces $W^s(M)$ are also decomposable in the above sense, because vector fields in $\mathfrak{h}$ split into irreducible, unitary representation spaces, and the infinitesimal representations of $\mathfrak{h}$ extend to representations of the enveloping algebra. For this reason, we may prove our Sobolev estimates concerning coboundary operators (Theorems 1.2 and 1.3) in simpler, orthogonal components of $W^s(M)$, and then we glue the estimates together at the end (see (50)).

2.2. *Finite-dimensional representations.* The one-dimensional representations are unitarily equivalent to characters $\rho_{\mathbf{m}}$ of $\mathbb{R}^{2n}$ in $L^2(\mathbb{T}^{2n})$, for $\mathbf{m} \in \mathbb{Z}^{2n}$, and are given by

$$\rho_{\mathbf{m}}(\mathbf{x}, \boldsymbol{\xi}, t) f = e^{2\pi i \mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})} f. \tag{11}$$

For each integer $1 \leq j \leq n$, the derived representations of $\rho_{\mathbf{m}}$ are

$$X_j = 2\pi i m_j, \quad \Lambda_j = 2\pi i m_{n+j}, \quad Z = 0.$$

Write

$$\rho := \bigoplus_{\mathbf{m} \in \mathbb{Z}^4} \rho_{\mathbf{m}}.$$

So, given $f \in L^2(\mathbb{T}^{2n})$, we have the orthogonal decomposition

$$f(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\mathbf{m} \in \mathbb{Z}^4} f_{\mathbf{m}} e^{2\pi i \mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})},$$

where $\triangle$ acts on irreducible, unitary representations of $L^2(\mathbb{T}^{2n})$ by

$$\rho_{\mathbf{m}}(\triangle) = 4\pi^2 \mathbf{m} \cdot \mathbf{m}.$$

For $s > 0$, the subspace of s-differentiable functions is $W^s(\mathbb{T}^{2n}) \subset L^2(\mathbb{T}^{2n})$, defined to be the maximal domain of the operator $(I + \rho(\triangle))^{s/2}$ on $L^2(\mathbb{T}^{2n})$ with inner product and norm given by (2). In particular,

$$\|f\|_s^2 = \sum_{\mathbf{m} \in \mathbb{Z}^{2n}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^s |f_{\mathbf{n}}|^2. \tag{12}$$

We denote the space of smooth functions in $L^2(\mathbb{T}^{2n})$ by

$$W^\infty(\mathbb{T}^{2n}) := \bigcap_{s \geq 0} W^s(\mathbb{T}^{2n}).$$

Furthermore, for every s, we have $W^s(\mathbb{T}^{2n}) = \mathbb{C}\langle 1 \rangle \oplus W_0^s(\mathbb{T}^{2n})$, where $W_0^s(\mathbb{T}^{2n})$ is the Sobolev space of s-differentiable, zero-average functions on $\mathbb{T}^{2n}$. So, it follows that

$$W^\infty(\mathbb{T}^{2n}) = \mathbb{C}\langle 1 \rangle \oplus W_0^\infty(\mathbb{T}^{2n}),$$

where $W_0^\infty(\mathbb{T}^{2n}) = \bigcap_{s \geq 0} W_0^s(\mathbb{T}^{2n})$.

The two propositions below establish Theorems 1.2 and 1.3 in the case of finite-dimensional representations. The proofs are straightforward and deferred to the appendix.

PROPOSITION 2.1. *There is a constant $C_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ such that for any zero-average $f, g \in W_0^\infty(\mathbb{T}^{2n})$ that satisfy $L_{\boldsymbol{\tau}} g = L_{\boldsymbol{\eta}} f$, there is a solution $P \in W^\infty(\mathbb{T}^{2n})$ such that*

$$L_{\boldsymbol{\tau}} P = f \quad \text{and} \quad L_{\boldsymbol{\eta}} P = g$$

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and, for any $s \geq 0$*,*

$$\|P\|_s \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}}(\|f\|_{s+2\gamma} + \|g\|_{s+2\gamma}).$$

PROPOSITION 2.2. *There is a constant* $C_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ *such that for any* $\phi \in W^\infty(\mathbb{T}^{2n})$ *and any non-constant zero-average functions* $f, g \in W_0^\infty(\mathbb{T}^{2n})$ *that satisfy* $L_{\boldsymbol{\eta}} f - L_{\boldsymbol{\tau}} g = \phi$*, there is a non-constant function* $P \in W^\infty(\mathbb{T}^{2n})$ *such that for any* $s \geq 0$*,*

$$\begin{aligned}
&\|g - L_{\boldsymbol{\eta}} P\|_s \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} \|\phi\|_{s+2\gamma},\\
&\|f - L_{\boldsymbol{\tau}} P\|_s \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} \|\phi\|_{s+2\gamma},\\
&\|P\|_s \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}}(\|f\|_{s+2\gamma} + \|g\|_{s+2\gamma}).
\end{aligned}$$

2.3. *Schrödinger representations.* Next we consider the irreducible, infinite-dimensional representations. By the Stone–von Neumann theorem, any infinite-dimensional representation is unitarily equivalent to a Schrödinger representation of H on $L^2(\mathbb{R}^n)$ with a parameter $h \neq 0$. When acting on the right, this is

$$(\mu_h(\mathbf{x}, \boldsymbol{\xi}, t)\phi)(y) = e^{-iht + i\epsilon|h|^{1/2}\boldsymbol{\xi}\cdot y - (1/2)ih\boldsymbol{\xi}\cdot\mathbf{x}}\phi(y - |h|^{1/2}\mathbf{x}), \tag{13}$$

where $\epsilon = \text{sign}(h) = \pm 1$. For integers $1 \leq j \leq n$, we have

$$\mu_h(X_j) = -|h|^{1/2}\frac{\partial}{\partial y_j}, \quad \mu_h(\Lambda_j) = i\epsilon|h|^{1/2} y_j, \quad \mu_h(Z) = -ih.$$

The derived representation extends to the enveloping algebra of the Lie algebra of H.

In §1.1, we noted that we will work with the standard lattice and, in this case, the Schrödinger representations are parameterized by $h \in 2\pi\mathbb{Z} \setminus \{0\}$ (see [**1**, §3.2]). Then observe that

$$|Z| = |h|$$

and define the operator $\Box$ in the model μ_h to be

$$\begin{aligned}
\mu_h(\Box) &:= \frac{1}{2\pi}\bigg(|\mu_h(Z)| - \sum_{i=1}^n \mu_h(X_i^2) + \mu_h(\Lambda_i^2)\bigg)\\
&= \frac{|h|}{2\pi}\bigg(1 + \sum_{i=1}^n y_i^2 - \frac{\partial^2}{\partial y_i^2}\bigg),
\end{aligned}$$

which is homogeneous in $|h|$. The operator $\mu_h(\Box)$ is related to $\mu_h(\triangle)$ by

$$\mu_h(\Box) = \frac{1}{2\pi}(\mu_h(\triangle) + |\mu_h(Z)| + \mu_h(Z)^2). \tag{14}$$

Define $W^s(\mu_h, \mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ to be Hilbert Sobolev space of s-differentiable functions, that is, the maximal domain of the operator $\mu_h(\Box)^s$ on $L^2(\mathbb{R}^n)$ with inner product

$$\langle \mu_h(\Box)^s f, g\rangle_{L^2(\mathbb{R}^n)} = \bigg(\frac{|h|}{2\pi}\bigg)^s \bigg\langle\bigg(I + \sum_{i=1}^n y_i^2 - \frac{\partial^2}{\partial y_i^2}\bigg)^s f, g\bigg\rangle_{L^2(\mathbb{R}^n)}.$$

Denote the Sobolev norm of this operator by

$$|||f|||_s := \langle \mu_h(\Box)^s f, f\rangle_{L^2(\mathbb{R}^n)}. \tag{15}$$

Clearly, the space of smooth functions in $L^2(\mathbb{R}^n)$ with respect to $\mu_h(\Box)$ is the Schwartz space

$$\mathscr{S}(\mathbb{R}^n) = \bigcap_{s\geq 0} W^s(\mu_h, \mathbb{R}^n).$$

Following [1], estimates of linear operators with respect to the Laplacian (1) are a consequence of such estimates in the homogeneous norm. This estimate differs from Lemma 3.15 of [1] in that our homogeneous operator $\mu_h(\Box)$ includes the term $|\mu_h(Z)|$, whereas that operator in [1] does not. The argument below is the same as in [1, Lemma 3.15].

LEMMA 2.3. *Let $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ be a linear map for the representation μ_h such that for every $s \geq 0$, there are a constant $C_s > 0$ and some $t \geq 0$ satisfying*

$$|||Tf|||_s \leq C_s |||f|||_{s+t}.$$

Then, for every $s \geq 0$, there is a constant $C_{s,t} = 2^{(s+t)/2} \max_{0\leq k\leq s}\{C_k\} > 0$ such that

$$\|Tf\|_s \leq C_{s,t}\|f\|_{s+t}.$$

Proof. First let $s \geq 0$ be an integer. Then

$$\begin{aligned}\|Tf\|_s^2 &= \bigg\langle \bigg(I - \mu_h(Z)^2 - \sum_{i=1}^n \mu_h(X_i)^2 + \mu_h(\Lambda_i)^2 \bigg)^s Tf, Tf \bigg\rangle \\ &\leq \sum_{k=0}^s \binom{s}{k} (1+h^2)^{s-k} \langle \mu_h(\Box)^k Tf, Tf\rangle. \end{aligned} \tag{16}$$

By the definition of $|||Tf|||_k$, by (14) and because all terms are positive, we have

$$\begin{aligned}(16) &\leq \sum_{k=0}^s \binom{s}{k} (1+h^2)^{s-k} C_k^2 \langle \mu_h(\Box)^{k+t} f, f\rangle \\ &= \max_{0\leq k\leq s}\{C_k^2\}\langle (I - \mu_h(Z)^2 + \mu_h(\Box))^s \mu_h(\Box)^{t/2} f, \mu_h(\Box)^{t/2} f\rangle. \end{aligned} \tag{17}$$

Because $\mu_h(Z)^2$ is a constant, $(I - \mu_h(Z)^2 + \mu_h(\Box))$ and $\mu_h(\Box)$ commute. Then, using that $-\mu_h(Z)^2$ is positive again, we have

$$\begin{aligned}(17) &\leq \max_{0\leq k\leq s}\{C_k^2\}\langle (I - \mu_h(Z)^2 + 2\pi\mu_h(\Box))^{s+t} f, f\rangle \\ &= \max_{0\leq k\leq s}\{C_k^2\}\langle (I + \mu_h(\triangle) + 2\pi|\mu_h(Z)|)^{s+t} f, f\rangle \\ &\leq 2^{s+t}\max_{0\leq k\leq s}\{C_k^2\}\|f\|_{s+t}^2.\end{aligned}$$

The estimate for $s \geq 0$ follows by interpolation. □

We will use the above lemma to reduce our estimates to the case $h = 1$. Because the norm (15) is homogeneous in h, by rescaling by the factor $|h|^{s/2}$ from $|||f|||_s$, we can restrict ourselves to the case $|h| = 2\pi$, as in [1]. In what follows, we set $h = 2\pi$, as the argument for $h = -2\pi$ is analogous.

Then, to simplify notation, we write

$$X_j = -\frac{\partial}{\partial y_j}, \quad \Lambda_j = iy_j, \quad Z = -i$$

and we refer to the Schrödinger representation on $L^2(\mathbb{R}^n)$ as

$$\mu := \mu_{2\pi}.$$

For $s > 0$, we denote $W^s(\mathbb{R}^n) := W^s(\mu, \mathbb{R}^n)$.

It will be convenient to define the Sobolev space $W^s(\mathbb{R}^{n-1})$ that is the maximal domain of the operator $I + \sum_{i=2}^n y_i^2 - (\partial^2/\partial y_i^2)$ on $L^2(\mathbb{R}^{n-1})$. We use the same notation for the inner product, where in this setting

$$\langle f, g\rangle_s := \bigg\langle \bigg(I + \sum_{i=2}^n y_i^2 - \frac{\partial^2}{\partial y_i^2} \bigg)^s f, g \bigg\rangle_{L^2(\mathbb{R}^{n-1})}.$$

Definition 2.4. Denote the norm of $W^s(\mathbb{R}^{n-1})$ by $|f|_s$.

2.3.1. *Change of variable.* Define

$$\tau = \sqrt{\sum_{j=1}^n \tau_j^2}.$$

Let $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in O(n)$ be an $n \times n$ matrix with orthonormal rows $\mathbf{a}_i$ such that

$$\mathbf{a}_1 = \frac{1}{\tau}(\tau_1, \tau_2, \ldots, \tau_n).$$

Observe that (τ_j) and (η_j) span a two-dimensional subspace of $\mathbb{R}^n$, so we can choose $\mathbf{a}_2$ to be such that

$$(\eta_1, \eta_2, \ldots, \eta_n) \in \operatorname{span}(\{\mathbf{a}_1, \mathbf{a}_2\}).$$

Further, choose the signs of the vectors $\mathbf{a}_j$, for $2 \le j \le n$, so that $A \in \mathrm{SO}(n)$. Then A is the determinant one rotation of $\mathbb{R}^n$ such that

$$\begin{aligned} A(\tau_1, \tau_2, \ldots, \tau_n) &= (\tau, 0, \ldots, 0), \\ A(\eta_1, \eta_2, \ldots, \eta_n) &= (\nu_1, \nu_2, 0, \ldots, 0) \end{aligned} \tag{18}$$

for some $(\nu_1, \nu_2) \in \mathbb{R}^2$.

For $y = (y_1, y_2, \ldots, y_n)$, define $z = (z_1, z_2, \ldots, z_n)$ via matrix–vector multiplication by

$$z = Ay.$$

Therefore,

$$f(y) := f \circ A^{-1}(z). \tag{19}$$

Clearly, because A is an orthogonal matrix, the operator $U_A : L^2(\mathbb{R}^n, dy) \to L^2(\mathbb{R}^n, dz)$ given by

$$U_A f = f \circ A^{-1}$$

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is unitary. Let $\tilde{\mu}$ be the representation on H such that for any $g \in \mathsf{H}$, $\tilde{\mu}(g) : L^2(\mathbb{R}^n, dz) \to L^2(\mathbb{R}^n, dz)$ is given by

$$\tilde{\mu}(g) := U_A \mu(g) U_A^{-1}.$$

So, $\tilde{\mu}$ is unitarily equivalent to μ.

Now we compute a basis for $\mathfrak{h}$ in terms of the derived representations of $\tilde{\mu}$. For each $j \in \{1, 2, \ldots, n\}$, let $(x_{j,t}, \lambda_{j,t}, z_{j,t})_{t\in[-1,1]}$ be smooth curves in H such that

$$X_j = \frac{d}{dt}\mu(x_{j,t})|_{t=0}, \quad \Lambda_j = \frac{d}{dt}\mu(\lambda_{j,t})|_{t=0}, \quad Z_j = \frac{d}{dt}\mu(z_{j,t})|_{t=0}.$$

Then set

$$\tilde{X}_j = \frac{d}{dt}\tilde{\mu}(x_{j,t})|_{t=0}, \quad \tilde{\Lambda}_j = \frac{d}{dt}\tilde{\mu}(\lambda_{j,t})|_{t=0}, \quad \tilde{Z}_j = \frac{d}{dt}\tilde{\mu}(z_{j,t})|_{t=0}.$$

Let A^{-1} be the matrix

$$A^{-1} = (b_{ij})$$

for some coefficients b_{ij}. A calculation shows that for $j \in \{1, 2, \ldots, n\}$,

$$\tilde{X}_j = -\sum_{k=1}^{n} b_{jk}\frac{\partial}{\partial z_k}, \quad \tilde{\Lambda}_j = i\sum_{j=k}^{n} b_{jk}z_k, \quad \tilde{Z} = -i.$$

One can check that these operators satisfy the commutation relations

$$[\tilde{X}_i, \tilde{X}_j] = 0, \quad [\tilde{\Lambda}_i, \tilde{\Lambda}_j] = 0, \quad [\tilde{X}_i, \tilde{\Lambda}_j] = \delta_{ij}\tilde{Z}$$

for $i, j \in \{1, 2, \ldots, n\}$.

LEMMA 2.5. *We have*

$$\tilde{\mu}(\square) = I + \sum_{i=1}^{n} z_i^2 - \frac{\partial^2}{\partial z_i^2}.$$

Proof. By definition,

$$\tilde{\mu}(\square) = I + \sum_{i=1}^{n} -\tilde{X}_i^2 - \tilde{\Lambda}_i^2. \tag{20}$$

Notice that

$$\begin{aligned}\tilde{X}_i^2 &= \bigg(-\sum_{j=1}^{n} b_{ij}\frac{\partial}{\partial z_j}\bigg)^2\\ &= \sum_{j,m=1}^{n} b_{ij}b_{im}\frac{\partial^2}{\partial z_j \partial z_m}.\end{aligned}$$

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Because the columns of A^{-1} are orthonormal, we get

$$\begin{aligned}\sum_{i=1}^{n} \tilde{X}_i^2 &= \sum_{i=1}^{n} \sum_{j,m=1}^{n} b_{ij} b_{im} \frac{\partial^2}{\partial z_j \partial z_m} \\ &= \sum_{j,m=1}^{n} \sum_{i=1}^{n} b_{ij} b_{im} \frac{\partial^2}{\partial z_j \partial z_m} \\ &= \sum_{1\le j\neq m\le n} \frac{\partial}{\partial z_j \partial z_m} \sum_{i=1}^{n} b_{ij} b_{im} + \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} \sum_{i=1}^{n} b_{ij}^2 \\ &= \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{i=1}^{n} \tilde{\Lambda}_i^2 &= - \sum_{j,m=1}^{n} \sum_{i=1}^{n} b_{ij} b_{im} z_j z_m \\ &= - \sum_{1\le j\neq m\le n} z_j z_m \sum_{i=1}^{n} b_{ij} b_{im} - \sum_{j=1}^{n} z_j^2 \sum_{i=1}^{n} b_{ij}^2 \\ &= - \sum_{j=1}^{n} z_j^2.\end{aligned}$$

Hence,

$$(20) = I + \sum_{i=1}^{n} z_i^2 - \frac{\partial^2}{\partial z_i^2}. \qquad \square$$

Finally, we compute the operator $\tilde{\mu}(\exp(Y_{\boldsymbol{\kappa}}))$ for $\boldsymbol{\kappa} \in \{\boldsymbol{\tau}, \boldsymbol{\eta}\}$.

LEMMA 2.6. *For any $f \in L^2(\mathbb{R}^n)$ and $z \in \mathbb{R}^n$, we have*

$$\begin{aligned}\tilde{\mu}(\exp(Y_{\boldsymbol{\tau}})) f(z) &= f(z - (\tau, 0, \ldots, 0)), \\ \tilde{\mu}(\exp(Y_{\boldsymbol{\eta}})) f(z) &= \exp(i\nu_2 z_2) f(z)\end{aligned}$$

for some $\nu_2 \in \mathbb{R}^*$.

Proof. To help keep track of which coordinate system we are working in, note that $U_A f = f \circ A^{-1}$, where $z = Ay$. So,

$$U_A : L^2(\mathbb{R}^n, dy) \to L^2(\mathbb{R}^n, dz), \quad U_A^{-1} : L^2(\mathbb{R}^n, dz) \to L^2(\mathbb{R}^n, dy)$$

and of course the Schrödinger representation μ satisfies

$$\mu(g) : L^2(\mathbb{R}^n, dy) \to L^2(\mathbb{R}^n, dy)$$

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for any $g \in \mathrm{H}$. Then

$$\begin{aligned}\tilde{\mu}(\exp(Y_{\boldsymbol{\tau}}))f(z) &:= U_A\mu(\exp(Y_{\boldsymbol{\tau}}))U_A^{-1}f(z)\\ &= \mu(\exp(Y_{\boldsymbol{\tau}}))U_A^{-1}f(A^{-1}z)\\ &= \mu(\exp(Y_{\boldsymbol{\tau}}))U_A^{-1}f(y)\\ &= U_A^{-1}f(y_1-\tau_1,\ldots,y_n-\tau_n)\\ &= f(A(y_1-\tau_1,\ldots,y_n-\tau_n))\\ &= f(Ay-A(\tau_1,\ldots,\tau_n))\\ &= f(z-(\tau,0,\ldots,0)).\end{aligned}$$

Next, recall that $\boldsymbol{\eta} = (\mathbf{0}, \eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^{2n}$ and define $\underline{\boldsymbol{\eta}} := (\eta_1, \eta_2, \ldots, \eta_n)$. Then $\mu(\exp(Y_{\boldsymbol{\eta}}))$ is the multiplication operator

$$\mu(\exp(Y_{\boldsymbol{\eta}}))f(y) = e^{i\underline{\boldsymbol{\eta}}\cdot y}\cdot f(y).$$

So,

$$\begin{aligned}\tilde{\mu}(Y_{\boldsymbol{\eta}})f(z) &:= U_A\mu(\exp(Y_{\boldsymbol{\eta}}))U_A^{-1}f(z)\\ &= \mu(\exp(Y_{\boldsymbol{\eta}}))U_A^{-1}f(A^{-1}z)\\ &= \mu(\exp(Y_{\boldsymbol{\eta}}))U_A^{-1}f(y)\\ &= e^{i\underline{\boldsymbol{\eta}}\cdot y}U_A^{-1}f(y)\\ &= e^{i\underline{\boldsymbol{\eta}}\cdot y}f(Ay)\\ &= e^{i\underline{\boldsymbol{\eta}}\cdot A^{-1}z}f(z)\\ &= e^{iA\underline{\boldsymbol{\eta}}\cdot z}f(z).\end{aligned} \tag{21}$$

Now recall from (18) that $A\underline{\boldsymbol{\eta}} = (\nu_1, \nu_2, 0, \ldots, 0)$ for some $(\nu_1, \nu_2) \in \mathbb{R}^2$, so

$$(21) = \exp(i(\nu_1 z_1 + \nu_2 z_2))f(z). \tag{22}$$

Furthermore, observe that the assumption $[Y_{\boldsymbol{\tau}}, Y_{\boldsymbol{\eta}}] = 0$ from (6) is equivalent to the condition

$$\sum_{j=1}^{n}\tau_j\eta_j = 0.$$

We also have $A^{-1}(\tau, 0, \ldots, 0) = (\tau_j)$, where $A^{-1} = (b_{ij})$. Then, for all $1 \le j \le n$,

$$b_{j1} = \frac{\tau_j}{\tau}.$$

Because $A \in \mathrm{SO}(n)$, we have

$$a_{1j} = b_{j1} = \frac{\tau_j}{\tau}.$$

Hence,

$$\nu_1 = (A\boldsymbol{\eta})_1 = \sum_{j=1}^{n} a_{1j}\eta_j = \frac{1}{\tau}\sum_{j=1}^{n} \tau_j\eta_j = 0. \tag{23}$$

Because A is a rotation and $\nu_1 = 0$, we get that $|\nu_2| = |\boldsymbol{\eta}| > 0$. Finally, because A is a real matrix and $\boldsymbol{\eta} \in \mathbb{R}^n$, it follows that $\nu_2 \in \mathbb{R}^*$. The lemma now follows from (22) and (23). □

For $\boldsymbol{\kappa} \in \{\boldsymbol{\tau}, \boldsymbol{\eta}\}$, the operator $L_{\boldsymbol{\kappa}}$ is defined on functions of the $\mathbf{z}$-variable by

$$L_{\boldsymbol{\kappa}} := \tilde{\mu}(\exp(Y_{\boldsymbol{\kappa}})) - I,$$

so, by the above lemma,

$$L_{\boldsymbol{\kappa}} f(z) = \begin{cases} f(z - (\tau, 0, \ldots, 0)) - f(z) & \text{if } \boldsymbol{\kappa} = \boldsymbol{\tau}, \\ [\exp(i\nu_2 z_2) - 1] f(z) & \text{if } \boldsymbol{\kappa} = \boldsymbol{\eta}. \end{cases} \tag{24}$$

The coordinates $(z_3, z_4, \ldots, z_n)$ will not play a central role, so for any $f \in L^2(\mathbb{R}^n)$ and for any $z \in \mathbb{R}^n$, define

$$\begin{aligned} \mathbf{z}_3 &:= (z_3, z_4, \ldots, z_n) \in \mathbb{R}^{n-2}, \\ f_{\mathbf{z}_3}(z_1, z_2) &:= f(z). \end{aligned}$$

For $j = 1, 2$, let $\mathcal{F}_j$ be the Fourier transform in the z_j-variable, so

$$\begin{aligned} \mathcal{F}_1 f_{\mathbf{z}_3}(\omega_1, z_2) &:= \int_{\mathbb{R}} f_{\mathbf{z}_3}(z_1, z_2) e^{-2\pi i \omega_1 z_1}\, dz_1, \\ \mathcal{F}_2 f_{\mathbf{z}_3}(z_1, \omega_2) &:= \int_{\mathbb{R}} f_{\mathbf{z}_3}(z_1, z_2) e^{-2\pi i \omega_2 z_2}\, dz_2. \end{aligned}$$

We begin with a short lemma.

LEMMA 2.7. *For any $s \geq 0$ and for any $\epsilon \in (0, 1)$, there is a constant $C_\epsilon > 0$ such that for any $\mathbf{z} \in \mathbb{R}^n$ and for any $f \in W^{s+n/2+\epsilon}(\mathbb{R}^n)$, the functions f, $\mathcal{F}_1 f$ and $\mathcal{F}_2 f$ are continuous on $\mathbb{R}^n$, and*

$$|f_{\mathbf{z}_3}(z_1, z_2)| \leq \frac{C_\epsilon}{(1 + \sum_{i=1}^{n} z_i^2)^{s/2}} |||f|||_{s+n/2+\epsilon}, \tag{25}$$

$$|\mathcal{F}_2 f_{\mathbf{z}_3}(z_1, \omega_2)| \leq \frac{C_\epsilon}{(1 + \omega_2^2 + \sum_{\substack{1 \leq i \leq n \\ i \neq 2}} z_i^2)^{s/2}} |||f|||_{s+n/2+\epsilon}. \tag{26}$$

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Similarly, for any $(\omega, z_2) \in \mathbb{R}^2$*, for any* $r \geq 0$ *and for any* $f \in W^{s+r+n/2+\epsilon}(\mathbb{R}^n)$*,*

$$|f_{\mathbf{z}_3}(z_1, z_2)| \leq \frac{C_\epsilon}{(1+z_1^2)^{r/2}\big(1+\sum_{i=2}^n z_i^2\big)^{s/2}} |||f|||_{s+r+n/2+\epsilon}, \tag{27}$$

$$|\mathcal{F}_1 f_{\mathbf{z}_3}(\omega_1, z_2)| \leq \frac{C_\epsilon}{(1+\omega_1^2)^{r/2}\big(1+\sum_{i=2}^n z_i^2\big)^{s/2}} |||f|||_{r+s+n/2+\epsilon}. \tag{28}$$

Proof. The Sobolev embedding theorem implies that there is a constant $C_\epsilon > 0$ such that

$$\begin{aligned}
\bigg\| \bigg(I + \sum_{i=1}^n z_i^2\bigg)^{s/2} f \bigg\|_{C^{0,\epsilon}(\mathbb{R}^n)} &\leq C_\epsilon \bigg|\bigg|\bigg| \bigg(I + \sum_{i=1}^n z_i^2\bigg)^{s/2} f \bigg|\bigg|\bigg|_{n/2+\epsilon} \\
&\leq C_\epsilon \bigg|\bigg|\bigg| \bigg(I + \sum_{i=1}^n z_i^2 - \frac{\partial}{\partial z_i^2}\bigg)^{s/2} f \bigg|\bigg|\bigg|_{n/2+\epsilon} \\
&= C_\epsilon |||f|||_{s+n/2+\epsilon}.
\end{aligned}$$

This implies the first inequality and that f is continuous. The inequality (26) follows in the same way by applying the inverse Fourier transform $\mathcal{F}_2^{-1}$. Then $\mathcal{F}_2 f$ is also continuous.

For (28), the Sobolev embedding theorem again gives a constant $C_\epsilon > 0$ such that

$$\begin{aligned}
&\bigg\| (I+\omega_1^2)^{r/2} \bigg(I + \sum_{i=2}^n z_i^2\bigg)^{s/2} \mathcal{F}_1 f \bigg\|_{C^{0,\epsilon}(\mathbb{R}^n)} \\
&\leq C_\epsilon \bigg\| \bigg(I - \frac{\partial^2}{\partial \omega_1^2} - \sum_{i=2}^n \frac{\partial^2}{\partial z_i^2}\bigg)^{(n/2+\epsilon)/2} (I+\omega_1^2)^{r/2} \bigg(I + \sum_{i=2}^n z_i^2\bigg)^{s/2} \mathcal{F}_1 f \bigg\|_{L^2(\mathbb{R}^n)} \\
&\leq C_\epsilon \bigg\| \bigg(I + z_1^2 - \sum_{i=2}^n \frac{\partial^2}{\partial z_i^2}\bigg)^{(n/2+\epsilon)/2} \bigg(I - \frac{\partial^2}{\partial z_1^2}\bigg)^{r/2} \bigg(I + \sum_{i=2}^n z_i^2\bigg)^{s/2} f \bigg\|_{L^2(\mathbb{R}^n)} \\
&\leq C_\epsilon |||f|||_{r+s+n/2+\epsilon}.
\end{aligned}$$

The estimate of (27) follows as above. □

2.3.2. *Invariant operators and cohomological equations.* For any $m \in \mathbb{Z}$, let $\pi_{m,\boldsymbol{\tau}}$ be the formal operator

$$\pi_{m,\boldsymbol{\tau}} f(z) := \mathcal{F}_1 f\bigg(\frac{m}{\tau}, z_2, \ldots, z_n\bigg). \tag{29}$$

Now we record a decay estimate of $|\pi_{m,\boldsymbol{\tau}}(f)|_s$ with respect to m, which will be used later in the splitting result, Theorem 1.3. Recall Definition 2.4 for the meaning of $|f|_s$.

COROLLARY 2.8. *For any* $\epsilon > 0$*, there is a constant* $C_\epsilon > 0$ *such that for any* $s, r \geq 0$ *and for any* $m \in \mathbb{Z}$*, the operator* $\pi_{m,\boldsymbol{\tau}}$ *satisfies the following estimate. For any*

$f \in W^{r+s+3n/2-1+\epsilon}(\mathbb{R}^n)$, we have

$$|\pi_{m,\boldsymbol{\tau}}(f)|_s \leq C_\epsilon \left(1 + \left|\frac{m}{\tau}\right|\right)^{-r} |||f|||_{r+s+3n/2-1+\epsilon}.$$

Proof. First let $f \in \mathscr{S}(\mathbb{R}^n)$. Because $\mathcal{F}_1$ commutes with $(I - \sum_{i=2}^n (\partial^2/\partial z_i^2) + z_i^2)^{s/2}$, for any $m \in \mathbb{Z}$, we have

$$\begin{aligned} |\pi_{m,\boldsymbol{\tau}}(f)|_s &= \left\| \left(I - \sum_{i=2}^n \frac{\partial^2}{\partial z_i^2} + z_i^2\right)^{s/2} \mathcal{F}_1 f_{\mathbf{z}_3}\left(\frac{m}{\tau}, z_2\right) \right\|_{L^2(\mathbb{R}^{n-1})} \\ &= \left\| \mathcal{F}_1\left(\left(I - \sum_{i=2}^n \frac{\partial^2}{\partial z_i^2} + z_i^2\right)^{s/2} f_{\mathbf{z}_3}\right)\left(\frac{m}{\tau}, z_2\right) \right\|_{L^2(\mathbb{R}^{n-1})}. \end{aligned} \tag{30}$$

Then (28) gives

$$\begin{aligned} &\left| \mathcal{F}_1\left(\left(I - \sum_{i=2}^n \frac{\partial^2}{\partial z_i^2} + z_i^2\right)^{s/2} f_{\mathbf{z}_3}\right)\left(\frac{m}{\tau}, z_2\right) \right| \leq \frac{C_\epsilon}{(1 + (m/\tau)^2)^{r/2}\big(1 + \sum_{i=2}^n z_i^2\big)^{(n-1+\epsilon)/2}} \\ &\quad \times \left|\left|\left| \left(I - \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2} + z_i^2\right)^{s/2} f_{\mathbf{z}_3} \right|\right|\right|_{r+3n/2-1+2\epsilon} \\ &\quad \leq \frac{C_\epsilon}{(1 + (m/\tau)^2)^{r/2}\big(1 + \sum_{i=2}^n z_i^2\big)^{(n-1+\epsilon)/2}} |||f|||_{s+r+3n/2-1+2\epsilon}, \end{aligned}$$

so

$$(30) \leq C_\epsilon \left(1 + \left|\frac{m}{\tau}\right|\right)^{-r} |||f|||_{r+s+3n/2-1+2\epsilon}. \qquad \square$$

The next lemma shows that for any $m \in \mathbb{Z}$, $\pi_{m,\boldsymbol{\tau}}$ are invariant operators for $\tilde{\mu}(\exp(Y_{\boldsymbol{\tau}}))$ on sufficiently regular functions.

LEMMA 2.9. *For any $m \in \mathbb{Z}$ and for any $\epsilon > 0$,*

$$\pi_{m,\boldsymbol{\tau}} L_{\boldsymbol{\tau}} = 0$$

holds on $W^{n/2+\epsilon}(\mathbb{R}^n)$.

Proof. Lemma 2.7 shows that for any $m \in \mathbb{Z}$ and any $f \in W^{n/2+\epsilon}(\mathbb{R}^n)$, $\pi_{m,\boldsymbol{\tau}} f$ is continuous on $\mathbb{R}^{n-1}$. Moreover,

$$\begin{aligned} \pi_{m,\boldsymbol{\tau}} \tilde{\mu}(\exp(Y_{\boldsymbol{\tau}})) f(z) &= \pi_{m,\boldsymbol{\tau}} f(z - (\tau, 0, \ldots, 0)) \\ &= \pi_{m,\boldsymbol{\tau}} f(z). \end{aligned} \qquad \square$$

For any $s > 2$, define

$$\mathrm{Ann}_{\boldsymbol{\tau}} := \{f \in \mathscr{S}(\mathbb{R}^n) : \pi_{m,\boldsymbol{\tau}}(f) \equiv 0 \text{ for all } m \in \mathbb{Z}\}.$$

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PROPOSITION 2.10. *For any* $f \in \mathrm{Ann}_{\boldsymbol{\tau}}$, *the cohomological equation*

$$L_{\boldsymbol{\tau}} P = f \tag{31}$$

has a unique solution P *in* $L^2(\mathbb{R}^n)$ *and, moreover, for any* $\epsilon > 0$, *there is a constant* $C_\epsilon > 0$ *such that for any* $s \geq 0$,

$$\|P\|_s \leq \frac{C_\epsilon}{\tau} \|f\|_{s+3n/2+1+\epsilon}.$$

Proof. Let $s \geq 0$, $f \in \mathscr{S}(\mathbb{R}^n)$ and $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ be the linear map

$$Tf(z) = \sum_{m=1}^{\infty} f(z_1 + m\tau, z_2, \ldots, z_n), \tag{32}$$

which converges absolutely and uniformly on compact sets.

By (24), the cohomological equation (31) is

$$P(z - (\tau, 0, \ldots, 0)) - P(z) = f(z).$$

Clearly, there is at most one $L^2(\mathbb{R}^n)$ solution P to the above equation. Because Tf is a solution,

$$L_{\boldsymbol{\tau}} Tf = f$$

on $\mathbb{R}^n$.

Because $f \in \mathrm{Ann}_{\boldsymbol{\tau}}$, the Poisson summation formula gives that for any $z \in \mathbb{R}^n$,

$$\sum_{m \in \mathbb{Z}} f(z_1 + m\tau, z_2, \ldots, z_n) = 0.$$

By combining the above equality with (32), we get that

$$Tf(z) = \sum_{m=0}^{\infty} f(z_1 - m\tau, z_2, \ldots, z_n), \tag{33}$$

which is again convergent.

Now we estimate the homogeneous norm $|||Tf|||_s$. By (25) and formula (32), we get that for all $z \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R}^{n-1}$,

$$\left| \left(I + \sum_{i=1}^{n} z_i^2 - \frac{\partial}{\partial z_i^2} \right)^{s/2} Tf(z) \right| \tag{34}$$

$$\begin{aligned} &\leq \sum_{m=0}^{\infty} \left| \left(I + \sum_{i=1}^{n} z_i^2 - \frac{\partial}{\partial z_i^2} \right)^{s/2} f(z_1 + m\tau, z_2, \ldots, z_n) \right| \\ &\leq \sum_{m=0}^{\infty} \left| \left(I - \frac{\partial^2}{\partial (z_1 + m\tau)^2} + (z_1 + m\tau)^2 + \sum_{i=2}^{n} z_i^2 - \frac{\partial^2}{\partial z_i^2} \right)^{s/2} \right. \\ &\qquad \left. \times f(z_1 + m\tau, z_2, \ldots, z_n) \right|. \end{aligned} \tag{35}$$

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Let $r = n + 1 + 2\epsilon$, so equation (25) gives

$$(35) \leq C_\epsilon \sum_{m=0}^{\infty} \left(1 + (z_1 + m\tau)^2 + \sum_{i=2}^{n} z_i^2\right)^{-r/2} \left\|\left(I + \sum_{i=1}^{n} z_i^2 - \frac{\partial^2}{\partial z_i^2}\right)^{s/2} f\right\|_{r+n/2+\epsilon}$$
$$= C_\epsilon \sum_{m=0}^{\infty} \left(1 + (z_1 + m\tau)^2 + \sum_{i=2}^{n} z_i^2\right)^{-(n+1+2\epsilon)/2} |||f|||_{s+3n/2+1+3\epsilon}$$
$$\leq \frac{C_\epsilon}{\tau} \left(1 + \sum_{i=1}^{n} z_i^2\right)^{-(n/2+\epsilon)} |||f|||_{s+3n/2+1+3\epsilon}.$$

Using (33), we get by a completely analogous argument that for all $z \in \mathbb{R}^- \times \mathbb{R}^{n-1}$,

$$\left|\left(I + \sum_{i=1}^{n} z_i^2 - \frac{\partial}{\partial z_i^2}\right)^{s/2} Tf(z)\right| \leq \frac{C_\epsilon}{\tau} \left(1 + \sum_{i=1}^{n} z_i^2\right)^{-(n+\epsilon)/2} |||f|||_{s+3n/2+1+3\epsilon}.$$

It follows that

$$|||Tf|||_s \leq \frac{C_\epsilon}{\tau} |||f|||_{s+3n/2+1+3\epsilon}.$$

Because $s \geq 0$ was arbitrary, we have shown that the above estimate in terms of the norm that is homogeneous in h holds for any $s \geq 0$. Then we apply Lemma 2.3 to get the estimate with respect to the Laplacian. So,

$$\|Tf\|_s \leq \frac{C_\epsilon}{\tau} \|f\|_{s+3n/2+1+3\epsilon}.$$

The lemma is now proved with $P = Tf$. □

Now we find a solution with Sobolev estimates to the equation $L_\eta P = f$. For any $m \in \mathbb{Z}$, define $\pi_{m,\eta}$ to be the formal operator

$$\pi_{m,\eta} f(z_1, z_3, \ldots, z_n) := f\left(z_1, \frac{2\pi m}{\nu_2}, z_3, \ldots, z_n\right).$$

We get as in Corollary 2.8 that for any $s \geq 0$ and $\epsilon > 0$, $\pi_{m,\eta} : W^{s+3n/2-1+\epsilon}(\mathbb{R}^n) \to W^s(\mathbb{R}^{n-1})$ and, by Lemma 2.7 that for any $f \in W^{n/2+\epsilon}(\mathbb{R}^n)$, $\pi_{m,\eta} f$ is continuous on $\mathbb{R}^{n-1}$.

As in Lemma 2.9, it can be immediately verified that for any $m \in \mathbb{Z}$, $\pi_{m,\eta}$ is invariant for the operator $\tilde{\mu}(\exp(Y_\eta))$. Define

$$\mathrm{Ann}_\eta := \{f \in \mathscr{S}(\mathbb{R}^n) : \pi_{m,\eta}(f) \equiv 0 \text{ for all } m \in \mathbb{Z}\}.$$

We have a corresponding estimate for the cohomological equation $L_\eta P = f$.

COROLLARY 2.11. *For any $f \in \mathrm{Ann}_\eta$, the equation*

$$L_\eta P = f$$

has a unique solution P in $L^2(\mathbb{R}^n)$ and, moreover, for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that for any $s \geq 0$,

$$\|P\|_s \leq \frac{C_\epsilon}{\nu_2}\|f\|_{s+3n/2+1+\epsilon}.$$

Proof. Writing (24) as a Fourier transform, we get

$$\mathcal{F}_2 L_{\boldsymbol{\eta}} f_{\mathbf{z}_3}(z_1, \omega_2) = \mathcal{F}_2 f_{\mathbf{z}_3}\bigg(z_1, \omega_2 - \frac{\nu_2}{2\pi}\bigg) - \mathcal{F}_2 f_{\mathbf{z}_3}(z_1, \omega_2). \tag{36}$$

Then, setting $\tau = (\nu_2/2\pi)$, the corollary follows in the same way as Proposition 2.10, using the decay estimate (26) in place of (25). □

Next, we prove Theorem 1.2 for Schrödinger representations.

THEOREM 2.12. *For any $f, g \in \mathscr{S}(\mathbb{R}^n)$ that satisfy $L_{\boldsymbol{\tau}} g = L_{\boldsymbol{\eta}} f$, there is a solution $P \in \mathscr{S}(\mathbb{R}^n)$ such that*

$$L_{\boldsymbol{\tau}} P = f \quad \text{and} \quad L_{\boldsymbol{\eta}} P = g.$$

Moreover, for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that for any $s \geq 0$,

$$\|P\|_s \leq \frac{C_\epsilon}{\tau}\|f\|_{s+3n/2+1+\epsilon}.$$

Proof. Let $m \in \mathbb{Z}$. Because $\pi_{m,\boldsymbol{\tau}}$ is invariant for $\tilde{\mu}(\exp(Y_{\boldsymbol{\tau}}))$, we have that

$$0 \equiv \pi_{m,\boldsymbol{\tau}} L_{\boldsymbol{\tau}} g = \pi_{m,\boldsymbol{\tau}} L_{\boldsymbol{\eta}} f.$$

From the formulas for $\pi_{m,\boldsymbol{\tau}}$ and $L_{\boldsymbol{\eta}}$, see (29) and (24), respectively, we get

$$[L_{\boldsymbol{\eta}}, \pi_{m,\boldsymbol{\tau}}] = 0.$$

Moreover, for any $(z_2, \ldots, z_n) \in \mathbb{R}^{n-1}$,

$$0 = L_{\boldsymbol{\eta}} \pi_{m,\boldsymbol{\tau}} f(z_2, \ldots, z_n) = [\exp(i\nu_2 z_2) - 1]\mathcal{F}_1 f\bigg(\frac{m}{\tau}, z_2, \ldots, z_n\bigg).$$

So, we get that off a countable set of $z_2 \in \mathbb{R}$,

$$\mathcal{F}_1 f\bigg(\frac{m}{\tau}, z_2, \ldots, z_n\bigg) = 0.$$

Lemma 2.7 shows that $\mathcal{F}_1 f$ is continuous, which implies that $\pi_{m,\boldsymbol{\tau}} f \equiv 0$. Because $m \in \mathbb{Z}$ was arbitrary, we conclude that $f \in \mathrm{Ann}_{\boldsymbol{\tau}}$.

Proposition 2.10 now implies that there is a unique function P in $L^2(\mathbb{R}^n)$ that is a solution to

$$L_{\boldsymbol{\tau}} P = f$$

and, for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$\|P\|_s \leq \frac{C_\epsilon}{\tau}\|f\|_{s+3n/2+1+\epsilon}.$$

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Finally, because $[Y_{\boldsymbol{\eta}}, Y_{\boldsymbol{\tau}}] = 0$, we use $L_{\boldsymbol{\tau}} g = L_{\boldsymbol{\eta}} f$ and get

$$L_{\boldsymbol{\tau}} g = L_{\boldsymbol{\eta}} L_{\boldsymbol{\tau}} P = L_{\boldsymbol{\tau}} L_{\boldsymbol{\eta}} P.$$

So, $L_{\boldsymbol{\tau}}(g - L_{\boldsymbol{\eta}} P) = 0$ and, because $g - L_{\boldsymbol{\eta}} P \in L^2(\mathbb{R}^n)$, it follows that

$$g = L_{\boldsymbol{\eta}} P$$

in $L^2(\mathbb{R}^n)$. □

Now we will prove Theorem 1.3 in the case of Schrödinger representations. Recall from Lemma 2.6 that $\nu_2 \neq 0$.

THEOREM 2.13. *For any $f, g, \phi \in \mathscr{S}(\mathbb{R}^n)$ that satisfy $L_{\boldsymbol{\eta}} f - L_{\boldsymbol{\tau}} g = \phi$, there exists a non-constant function $P \in \mathscr{S}(\mathbb{R}^n)$ such that the following holds. For any $s \geq 0$ and for any $\epsilon > 0$, there is a constant $C_{s,\epsilon} > 0$ such that*

$$\begin{aligned}
\|g - L_{\boldsymbol{\eta}} P\|_s &\leq C_{s,\epsilon}(\tau^{-1} + \tau^{\epsilon})\|\phi\|_{s+3n+1+2\epsilon}, \\
\|f - L_{\boldsymbol{\tau}} P\|_s &\leq \frac{C_{s,\epsilon}}{\nu_2}(1 + \tau^{1+\epsilon})\|\phi\|_{s+3n+1+2\epsilon}, \\
\|P\|_s &\leq C_{s,\epsilon}(\tau^{-1} + \tau^{\epsilon})(\|f\|_{s+3n+1+\epsilon} + \|g\|_{s+3n+1+2\epsilon}).
\end{aligned}$$

Proof. Notice that if $f = g = 0$, then $\phi = 0$ and the above statement holds trivially. Without loss of generality, we assume that $f \neq 0$.

Let $\psi \in \mathscr{S}(\mathbb{R})$, whose Fourier transform is compactly supported on $[-(1/2\tau), (1/2\tau)]$ and satisfies $\mathcal{F}\psi(0) = 1$. For each $m \in \mathbb{Z}$, define the functional $\Pi_{m,\boldsymbol{\tau}}$ on $L^2(\mathbb{R}^{n-1})$ by

$$\Pi_{m,\boldsymbol{\tau}} F(z_2, \ldots, z_n) = e^{2\pi i z_1 m/\tau} \psi(z_1) F(z_2, \ldots, z_n). \tag{37}$$

Below, $\Pi_{m,\boldsymbol{\tau}}$ will be a component of an operator that maps sufficiently regular functions to coboundaries. See (39) and Lemma 2.16 for details. For now, we have the following estimate.

LEMMA 2.14. *For any $s \in 2\mathbb{N}$, for any $m \in \mathbb{Z}$ and for any $F \in W^s(\mathbb{R}^{n-1})$, there is a constant $C_s > 0$ such that*

$$|||\Pi_{m,\boldsymbol{\tau}} F|||_s \leq C_s \left\| \left(I + z_1^2 - \frac{\partial^2}{\partial z_1^2} \right)^{s/2} \psi \right\|_{L^\infty(\mathbb{R})} \sum_{k=0}^{s/2} \left(1 + \left| \frac{m}{\tau} \right| \right)^{2k} |F|_{s-2k}. \quad \square$$

Proof. Because ψ is supported on $[-1/2, 1/2]$, we have

$$|||\Pi_{m,\tau} F|||_s = \left\| \left(\left(z_1^2 - \frac{\partial^2}{\partial z_1^2} \right) + \left(I + \sum_{i=2}^{n} z_i^2 - \frac{\partial^2}{\partial z_i^2} \right) \right)^{s/2} (e^{-2\pi i z_1 m/\tau} \psi F) \right\|_{L^2(\mathbb{R}^n)}. \tag{38}$$

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Then, because $z_1^2 - (\partial^2/\partial z_1^2)$ and $(I + \sum_{i=2}^n z_i^2 - (\partial^2/\partial z_i^2))$ commute, the triangle inequality gives

$$(38) \leq C_s \sum_{k=0}^{s/2} \left\| \left(z_1^2 - \frac{\partial^2}{\partial z_1^2}\right)^k \left(I + \sum_{i=2}^n z_i^2 - \frac{\partial^2}{\partial z_i^2}\right)^{s/2-k} (e^{-2\pi i z_1 m/\tau} \psi F) \right\|_{L^2(\mathbb{R}^n)}$$
$$\leq C_s \sum_{k=0}^{s/2} \left\| \left(z_1^2 - \frac{\partial^2}{\partial z_1^2}\right)^k e^{-2\pi i z_1 m/\tau} \psi \right\|_{L^\infty(\mathbb{R})} \left\| \left(I + \sum_{i=2}^n z_i^2 - \frac{\partial^2}{\partial z_i^2}\right)^{s/2-k} F \right\|_{L^2(\mathbb{R}^{n-1})}.$$

□

Note that $\Pi_{m,\boldsymbol{\tau}}$ depends on ψ. Then we formally define the operator R_ψ on $L^2(\mathbb{R}^n)$ by

$$R_\psi := I - \sum_{m\in\mathbb{Z}} \Pi_{m,\boldsymbol{\tau}} \pi_{m,\boldsymbol{\tau}}. \tag{39}$$

Over the next two lemmas, we describe properties of R_ψ.

LEMMA 2.15. *For any $s \geq 0$ and for any $\epsilon > 0$, there is a constant $C_{s,\epsilon} > 0$ such that for any non-zero $f \in W^{s+n/2+\epsilon}(\mathbb{R}^n)$, we can choose ψ such that $R_\psi f \neq 0$ and*

$$\|R_\psi f\|_s \leq C_{s,\epsilon}(1 + \tau^{s+1+\epsilon})\|f\|_{s+3n/2+\epsilon}.$$

Proof. We first claim that we can choose ψ such that $R_\psi f \neq 0$ and, for some universal constant $C_s^{(0)} > 0$,

$$\left\| \left(I + z_1^2 - \frac{\partial^2}{\partial z_1^2}\right)^{s/2} \psi \right\|_{L^\infty(\mathbb{R})} \leq C_s^{(0)}(1 + \tau^s).$$

Fix ψ. So, for some $C_s^{(0)} > 0$, the above estimate holds. If $R_\psi f \neq 0$, then the claim is holds, so suppose that $R_\psi f = 0$. Hence,

$$f(z) = \psi(z_1) \sum_{m\in\mathbb{Z}} \exp(2\pi i z_1 m/\tau) \mathcal{F}_1 f\left(\frac{m}{\tau}, z_2, \ldots, z_n\right).$$

So, we can perturb ψ to a smooth function compactly supported in $[-(1/2\tau), (1/2\tau)]$ satisfying $\mathcal{F}\psi_0(0) = 1$, $\psi_0 \neq \psi$ and $R_\psi f \neq 0$, where also

$$\left\| \left(I + z_1^2 - \frac{\partial^2}{\partial z_1^2}\right)^{s/2} \psi_0 \right\|_{L^\infty(\mathbb{R})} \leq (C_s^{(0)} + 1)(1 + \tau^s). \tag{40}$$

This proves the claim.

Now we say that $s \in \mathbb{N}$ is even and let $R_\psi f \neq 0$, where ψ satisfies (40). By the triangle inequality and Lemma 2.14, we get a constant $C_s^{(1)} > C_s^{(0)} + 1$ such that

$$|||R_\psi f|||_s \leq |||f|||_s + \sum_{m\in\mathbb{Z}} |||\Pi_{m,\tau}\pi_{m,\tau} f|||_s$$
$$\leq |||f|||_s + C_s^{(1)}(1 + \tau^s) \sum_{m\in\mathbb{Z}} \sum_{k=0}^{s/2} \left(1 + \left|\frac{m}{\tau}\right|\right)^{2k} |\pi_{m,\tau} f|_{s-2k}. \tag{41}$$

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By Corollary 2.8, there is a constant $C_\epsilon > 0$ such that for any $m \in \mathbb{Z}$,

$$\bigg(1+\bigg|\frac{m}{\tau}\bigg|\bigg)^{2k}|\pi_{m,\tau} f|_{s-2k} \le C_\epsilon\bigg(1+\bigg|\frac{m}{\tau}\bigg|\bigg)^{2k}\bigg(1+\bigg|\frac{m}{\tau}\bigg|\bigg)^{-(2k+1+\epsilon)}|||f|||_{s+3n/2+\epsilon}$$
$$\le C_\epsilon(1+\tau)^{s+1+\epsilon}(1+|m|)^{-(1+\epsilon)}|||f|||_{s+3n/2+\epsilon}.$$

Hence, there is a constant $C_{s,\epsilon} > 0$ such that

$$(41) \le |||f|||_s + C_{s,\epsilon}(1+\tau)^{s+1+\epsilon}\sum_{m\in\mathbb{Z}}(1+m^2)^{-(1+\epsilon)/2}|||f|||_{s+3n/2+\epsilon}$$
$$\le C_{s,\epsilon}(1+\tau)^{s+1+\epsilon}|||f|||_{s+3n/2+\epsilon}.$$

By interpolation, the above estimate holds for all $s \ge 0$. Because R_ψ is a linear operator and the estimate holds for all s, Lemma 2.3 gives the estimate for $s \ge 0$. □

Next we show that the operator R_ψ is a projection into $\mathrm{Ann}_{\boldsymbol{\tau}}$ and it commutes with $L_{\boldsymbol{\eta}}$.

LEMMA 2.16. *Let ψ be as in the previous lemma. Then*

$$R_\psi : \mathscr{S}(\mathbb{R}^n) \to \mathrm{Ann}_{\boldsymbol{\tau}}$$

and

$$R_\psi L_{\boldsymbol{\tau}} = L_{\boldsymbol{\tau}}, \quad [R_\psi, L_{\boldsymbol{\eta}}] = 0$$

on $\mathscr{S}(\mathbb{R}^n)$.

Proof. Let $f \in \mathscr{S}(\mathbb{R}^n)$. By the previous lemma, $R_\psi f \in \mathscr{S}(\mathbb{R}^n)$, so we need to show that $R_\psi f$ is in the kernel of every $\pi_{m,\boldsymbol{\tau}}$. Using the property that $\mathcal{F}\psi$ is supported on the interval $[-(1/2\tau), (1/2\tau)]$ and $\mathcal{F}\psi(0) = 1$, we get that for any $m \in \mathbb{Z}$,

$$\begin{aligned}\pi_{m,\tau}R_\psi f &= \pi_{m,\tau}f - \sum_{k\in\mathbb{Z}}\pi_{m,\tau}\Pi_{k,\tau}\pi_{k,\tau}f\\ &= \pi_{m,\tau}f - \pi_{m,\tau}\Pi_{m,\tau}\pi_{m,\tau}f\\ &= \pi_{m,\tau}f - \pi_{m,\tau}f\\ &= 0.\end{aligned}\tag{42}$$

This implies that $R_\psi : \mathscr{S}(\mathbb{R}^n) \to \mathrm{Ann}_{\boldsymbol{\tau}}$.

By Lemma 2.9, for any $m \in \mathbb{Z}$, $\pi_{m,\boldsymbol{\tau}}L_{\boldsymbol{\tau}} = 0$. We have

$$\begin{aligned}R_\psi L_{\boldsymbol{\tau}} &= \bigg(I - \sum_{m\in\mathbb{Z}}\Pi_{m,\tau}\pi_{m,\tau}\bigg)L_{\boldsymbol{\tau}}\\ &= L_{\boldsymbol{\tau}}.\end{aligned}$$

Finally, we prove that $[L_{\boldsymbol{\eta}}, R_\psi] = 0$. We have

$$\begin{aligned}\Pi_{m,\boldsymbol{\tau}}\pi_{m,\boldsymbol{\tau}}L_{\boldsymbol{\eta}}f(z) &= \exp(2\pi i z_1 m/\tau)\ \psi(z_1)[\pi_{m,\boldsymbol{\tau}}L_{\boldsymbol{\eta}}f](z_2,\ldots,z_n)\\ &= \exp(2\pi i z_1 m/\tau)\ \psi(z_1)\mathcal{F}_1[L_{\boldsymbol{\eta}}f]\bigg(\frac{m}{\tau}, z_2,\ldots,z_n\bigg)\\ &= (\exp(i\nu_2 z_2)-1)\exp(2\pi i z_1 m/\tau)\ \psi(z_1)\mathcal{F}_1 f\bigg(\frac{m}{\tau}, z_2,\ldots,z_n\bigg)\end{aligned}$$

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$$= L_{\eta} \exp(2\pi i z_1 m/\tau)\ \psi(z_1)\mathcal{F}_1 f\left(\frac{m}{\tau}, z_2, \ldots, z_n\right)$$
$$= L_{\eta}\Pi_{m,\tau}\mathcal{F}_1 f\left(\frac{m}{\tau}, z_2, \ldots, z_n\right)$$
$$= L_{\eta}\Pi_{m,\tau}\pi_{m,\tau} f.$$

This proves that $[L_{\eta}, R_{\psi}] = 0$ and finishes the proof of the lemma. □

Now we prove Theorem 2.13. Let

$$\phi = L_{\eta} f - L_{\tau} g$$

be as in the theorem and recall from the beginning of its proof that we take $f \neq 0$. By Lemmas 2.15 and 2.16, we can choose ψ such that there is a non-constant function P that is a solution to $R_{\psi} f = L_{\tau} P$ and, for a fixed constant $C_s^{(1)} > 0$,

$$\left\| \left(I - \frac{\partial^2}{\partial z_1^2}\right)^{s/2} \psi \right\|_{L^{\infty}(\mathbb{R})} \leq C_s^{(1)}.$$

In particular, Lemma 2.16 implies that

$$\begin{aligned} R_{\psi}\phi &= R_{\psi} L_{\eta} f - R_{\psi} L_{\tau} g \\ &= L_{\eta} R_{\psi} f - L_{\tau} g \\ &= L_{\eta} L_{\tau} P - L_{\tau} g \\ &= L_{\tau}(L_{\eta} P - g). \end{aligned} \tag{43}$$

Then, by Proposition 2.10 and by Lemmas 2.15 and 2.5, we get that for any $s \geq 0$ and for any $\epsilon > 0$, there is a constant $C_{s,\epsilon} > 0$ such that

$$\|L_{\eta} P - g\|_s = \|L_{\eta} P - g\|_s \tag{44}$$

$$\begin{aligned} &\leq \frac{C_{s,\epsilon}}{\tau} \|R_{\psi}\phi\|_{s+3n/2+1+\epsilon} \\ &\leq C_{s,\epsilon}(\tau^{-1} + \tau^{s+\epsilon})\|\phi\|_{s+3n+1+2\epsilon}. \end{aligned} \tag{45}$$

To estimate $\|L_{\tau} P - f\|_s$, because $R_{\psi} f = L_{\tau} P$ and by Lemma 2.5,

$$\begin{aligned} \|L_{\tau} P - f\|_s &= \|L_{\tau} P - f\|_s \\ &= \|L_{\tau} P - R_{\psi} f + R_{\psi} f - f\|_s \\ &= \|(R_{\psi} - I) f\|_s. \end{aligned} \tag{46}$$

Notice that by Lemma 2.16,

$$(R_{\psi} - I) L_{\tau} g = 0.$$

Then, using $L_{\eta} f - L_{\tau} g = \phi$, we get

$$\begin{aligned} (R_{\psi} - I) L_{\eta} f &= (R_{\psi} - I)\phi + (R_{\psi} - I) L_{\tau} g \\ &= (R_{\psi} - I)\phi. \end{aligned} \tag{47}$$

Then, by Lemma 2.16 again, we get

$$L_{\eta}(R_{\psi} - I)f = (R_{\psi} - I)\phi.$$

We conclude by Corollary 2.11 and Lemmas 2.15 and 2.5 that

$$\begin{aligned}(46) &\leq \frac{C_{s,\epsilon}}{\nu_2}\|(R_{\psi} - I)\phi\|_{s+3n/2+1+\epsilon} \\ &\leq \frac{C_{s,\epsilon}}{\nu_2}(1 + \tau^{s+1+\epsilon})\|\phi\|_{s+3n+1+2\epsilon}. \end{aligned} \tag{48}$$

Finally, because $L_{\tau} P = R_{\psi} f$, Proposition 2.10 and Lemma 2.15 give

$$\begin{aligned}\|P\|_s &\leq \frac{C_{s,\epsilon}}{\tau}\|R_{\psi} f\|_{s+3n/2+1+\epsilon} \\ &\leq C_{s,\epsilon}(\tau^{-1} + \tau^{s+\epsilon})\|f\|_{s+3n+1+2\epsilon}.\end{aligned}$$

At the start of the proof of Theorem 2.13, we assumed that $f \neq 0$. If we instead choose $g \neq 0$, then by first applying the Fourier transform, the same argument proves the above estimates in terms of $\|\phi\|_{s+3n+1+\epsilon}$ and

$$\|P\|_s \leq C_{s,\epsilon}(\tau^{-1} + \tau^{s+\epsilon})\|g\|_{s+3n+1+2\epsilon}.$$

This completes the proof of Theorem 2.13. □

Proof of Theorem 1.2. The regular representation of H on $L^2(M)$ decomposes as

$$L^2(M) = \mathbb{C}\langle 1\rangle \oplus \bigoplus_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} \mathcal{P}_{\mathbf{m}} \oplus \bigoplus_{h\in\mathbb{Z}} \mathcal{P}_h,$$

where each $\mathcal{P}_{\mathbf{m}}$ is an abelian representation of $\mathbb{R}^{2n}$ equivalent to a character given by (11), and each $\mathcal{P}_h$ is equivalent to a countable collection of Schrödinger representations μ_h of H on $L^2(\mathbb{R}^n)$ given by (13). The subspace of zero-average functions in $L^2(M)$ is denoted $L^2_0(M)$, which therefore decomposes as

$$\begin{aligned}L^2_0(M) &= \bigoplus_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} \mathcal{P}_{\mathbf{m}} \oplus \bigoplus_{h\in\mathbb{Z}} \mathcal{P}_h \\ &= L^2_0(\mathbb{T}^{2n}) \oplus \bigoplus_{h\in\mathbb{Z}} \mathcal{P}_h.\end{aligned}$$

As indicated in §2.1, vector fields in $\mathfrak{h}$ split into the unitary components in the above Hilbert space. Then the decomposition of the Sobolev space of s-differentiable, zero-average functions is

$$W^s_0(M) = W^s_0(\mathbb{T}^{2n}) \oplus \bigoplus_{h\in\mathbb{Z}} W^s(\mathcal{P}_h), \tag{49}$$

where $W^s_0(\mathbb{T}^{2n})$ and $W^s(\mathcal{P}_h)$ are s-order Sobolev spaces on the torus $\mathbb{T}^{2n}$ and of the representation $\mathcal{P}_h$, respectively.

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Now, in Theorem 1.2, we are given zero-average functions $f, g \in C^\infty(M)$ that satisfy $L_{\boldsymbol{\tau}} g = L_{\boldsymbol{\eta}} f$ and we aim to find a solution $P \in C^\infty(M)$ such that

$$L_{\boldsymbol{\tau}} P = f, \quad L_{\boldsymbol{\eta}} P = g.$$

Write

$$f = f_t \oplus \bigoplus_{h\in\mathbb{Z}} f_h, \quad g = g_t \oplus \bigoplus_{h\in\mathbb{Z}} g_h,$$

where $f_t, g_t \in W_0^\infty(\mathbb{T}^{2n})$ and, for each $h \in \mathbb{Z}$, $f_h, g_h \in \mathcal{P}_h$. Proposition 2.1 and Theorem 2.12 give smooth solutions $P_t \in W_0^\infty(\mathbb{T}^{2n})$ and $\{P_h\}_{h\in\mathbb{Z}} \subset W^\infty(\mathcal{P}_h)$ satisfying the estimate of Theorem 1.2 in the finite- and infinite-dimensional representations, respectively. Define $P \in C^\infty(M)$ by

$$P := P_t \oplus \bigoplus_{h\in\mathbb{Z}} P_h.$$

So, there exists a constant $C_{s,\epsilon} := C_{s,\epsilon,\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ such that

$$\begin{aligned} \|P\|^2_{W^s(M)} &= \|P_t\|_s^2 + \sum_{h\in\mathbb{Z}} \|P_h\|_s^2 \\ &\le C_{s,\epsilon}(\|f_t\|_{s+2\gamma} + \|g_t\|_{s+2\gamma})^2 + C_{s,\epsilon} \sum_{h\in\mathbb{Z}} \|f_h\|^2_{s+3n/2+1+\epsilon} \\ &\le C_{s,\epsilon}(\|f\|_{s+\max\{2\gamma,\, 3n/2+1+\epsilon\}} + \|g\|_{s+2\gamma})^2. \end{aligned} \tag{50}$$

□

Proof of Theorem 1.3. This follows from Proposition 2.2 and Theorem 2.13 as in the proof of Theorem 1.2. □

3. *Proof of Theorem 1.7*

We fix now $Y_{\boldsymbol{\tau}}$ and $Y_{\boldsymbol{\eta}}$ with $\boldsymbol{\tau} \cdot \boldsymbol{\eta} = 0$ and $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ Diophantine, as in the main setting. We denote by ρ the $\mathbb{Z}^2$ action generated by $Y_{\boldsymbol{\tau}}$ and $Y_{\boldsymbol{\eta}}$ as described in (7) in §1.1.

In this section we prove Theorem 1.7. We will apply here a similar method which was applied in [7]. The method consists in taking successive iterations and adjustment of the parameter λ at each step. The procedure is outlined in a general theorem which was proved in [2]. There, a set of conditions in cohomology is given, which imply transversal local rigidity of a finite-dimensional family of Lie group actions. This general theorem was then used in [3] to obtain transversal local rigidity of certain $\mathbb{R}^2$ actions on two-step nilmanifolds. Even though we have a similar situation here, we cannot unfortunately use the general theorem from [2] because that theorem is for *Lie group* actions, and here we have a *discrete group* action. This is the only difference though; the method of successive iterations is completely parallel to that used in the above-mentioned papers.

We write the proof of Theorem 1.7 here in the case where the manifold is the five-dimensional Heisenberg nilmanifold, that is, in the case $n = 2$. This is the lowest dimensional case in which our result holds. We chose to present the proof for concrete n for the benefit of the reader because computations are more clear and notation is simpler. Otherwise, the proof is clearly completely parallel for any $n \ge 2$. We stress the points in computation of cohomology where dimension matters, and how it affects the computation.

We will first compute in §3.1 the cohomology with coefficients in constant vector fields (that is, in the Lie algebra $\mathfrak{h}$) for the action ρ. Then we describe in §3.2 the finite-dimensional family ρ^λ of algebraic actions to which this action belongs, where $\rho^0 = \rho$. This family is completely determined by the cocycles (with values in $\mathfrak{h}$) over ρ. Then we move on to analyze the conjugacy operator and the commutator operator in §§3.3 and 3.4 and their linearized operators. The linearizations of these two operators are corresponding to the first and the second coboundary operators for the cohomology over ρ with coefficients in smooth vector fields $\mathrm{Vect}^\infty M$. Using the results from the previous part of the paper (specifically Theorem 1.3), we show in §3.5 that this cohomology sequence splits and that the first cohomology with coefficients in $\mathrm{Vect}^\infty M$ is the same as the cohomology with coefficients in $\mathfrak{h}$. This allows us to prove Theorem 1.7 by showing convergence of successive iterations in §3.6.

For a vector field $H \in \mathrm{Vect}^\infty M$, we denote by H_c its component in the center direction and by H_T the remainder, that is, the component of H in the off-center directions. We denote by $\mathrm{Ave}(H)$ the constant vector field (that is, an element in $\mathfrak{h}$) which is obtained by taking the average of H with respect to the Haar measure.

For two vector fields $F, G \in \mathrm{Vect}^\infty M$, we use the notation $\|F, G\|_r := \max\{\|F\|_r, \|G\|_r\}$, where $\|\cdot\|_r$ denotes the C^r norm.

3.1. *Constant cohomology for the discrete-time action.* We have

$$De^{Y_\tau}(X_1) = X_1, \quad De^{Y_\tau}(X_2) = X_2, \quad De^{Y_\tau}(Z) = Z.$$

Furthermore,

$$\begin{aligned} e^{Y_\tau} e^{t\Lambda_1} &= (e^{Y_\tau} e^{t\Lambda_1} e^{-Y_\tau}) e^{Y_\tau} \\ &= \exp(e^{\mathrm{ad}_{Y_\tau}} t\Lambda_1) e^{Y_\tau} = \exp(t\Lambda_1 + t[Y_{\boldsymbol{\tau}}, \Lambda_1]) e^{Y_\tau} \\ &= \exp(t(\Lambda_1 + \tau_1 Z)) e^{Y_\tau}. \end{aligned}$$

Therefore, for a constant vector field

$$H = h_1 X_1 + h_2 X_2 + h_3 \Lambda_1 + h_4 \Lambda_2 + h_5 Z,$$

where h_i are constants, we have

$$De^{Y_\tau}(H) = h_1 X_1 + h_2 X_2 + h_3 \Lambda_1 + h_4 \Lambda_2 + (h_5 - h_3 \tau_1 - h_4 \tau_2) Z.$$

Another way to write this is

$$De^{Y_\tau}(H) = H + [Y_{\boldsymbol{\tau}}, H].$$

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A similar computation can be done for De^{Y_η}. In the matrix form, we have

$$De^{Y_\tau} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \tau_1 & \tau_2 & 1 \end{pmatrix}, \quad De^{Y_\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\eta_1 & -\eta_2 & 0 & 0 & 1 \end{pmatrix}.$$

A pair of constant vector fields $(F, G) \in \mathfrak{h} \times \mathfrak{h}$ is a cocycle over the $\mathbb{Z}^2$ action generated by $Y_{\boldsymbol{\tau}}$ and $Y_{\boldsymbol{\eta}}$ if

$$(De^{Y_\tau} - \mathrm{Id})G = (De^{Y_\eta} - \mathrm{Id})F,$$

which implies that

$$[Y_{\boldsymbol{\tau}}, G] = [Y_{\boldsymbol{\eta}}, F]. \tag{51}$$

Because H is two-step nilpotent, this condition is only on the off-center coordinates of F and G. More precisely, if $F = f_1X_1 + f_2X_2 + f_3\Lambda_1 + f_4\Lambda_2 + f_5Z$ and $G = g_1X_1 + g_2X_2 + g_3\Lambda_1 + g_4\Lambda_2 + g_5Z$, then (51) implies that g_i and f_i for $i = 1, 2, 3, 4$ satisfy the relation

$$\tau_1 g_3 + \tau_2 g_4 + \eta_1 f_1 + \eta_2 f_2 = 0. \tag{52}$$

Since the constants f_5 and g_5 are arbitrary, the space of constant cocycles has dimension nine.

A pair of constant vector fields (F, G) is a coboundary if

$$(De^{Y_\tau} - \mathrm{Id})H = F, \quad (De^{Y_\eta} - \mathrm{Id})H = G,$$

that is, if

$$[Y_{\boldsymbol{\tau}}, H] = F, \quad [Y_{\boldsymbol{\eta}}, H] = G. \tag{53}$$

This implies that off-center coordinates of both F and G must be zero, and the center ones must satisfy certain relations. More precisely, if $H = h_1X_1 + h_2X_2 + h_3\Lambda_1 + h_4\Lambda_2 + h_5Z$, the equations (53) imply that

$$f_5 = \tau_1 h_3 + \tau_2 h_4, \quad g_5 = -\eta_1 h_1 - \eta_2 h_2 \tag{54}$$

and these equations always have solutions for coefficients of H.

This implies that the first cohomology is seven dimensional, and each cohomology class is represented by cocycles (F, G) of the following form: $F = f_1X_1 + f_2X_2 + f_3\Lambda_1 + f_4\Lambda_2$ and $G = g_1X_1 + g_2X_2 + g_3\Lambda_1 + g_4\Lambda_2$, where the coefficients g_i and f_i for $i = 1, 2, 3, 4$ satisfy the relation (52).

It is clear from the above computation that the dimension of the constant cohomology over the action generated by $Y_{\boldsymbol{\tau}}$ and $Y_{\boldsymbol{\eta}}$ in the case where the manifold is a $(2n + 1)$-dimensional Heisenberg nilmanifold is parallel to what we wrote above in the case $n = 2$ and that the resulting cohomology has dimension $4n - 1$.

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3.2. *The finite-dimensional family of $\mathbb{Z}^2$ algebraic actions.* For easier notation, in the rest of the paper we let $Y_1 = Y_{\boldsymbol{\tau}}$ and $Y_2 = Y_{\boldsymbol{\eta}}$. In what follows we will use the fact that in $\mathfrak{h}$ we have $\exp(X + Y + \frac{1}{2}[X, Y]) = \exp(X)\exp(Y)$. In the remainder of the paper the brackets $[\cdot, \cdot]$ denote the bracket in the Lie algebra $\mathfrak{h}$, so each bracket which appears as a result has a vector field in the Z-direction only.

We define now a nine-dimensional family of $\mathbb{Z}^2$ actions ρ^λ on $M = \Gamma \setminus H$ generated by the following maps:

$$y_i^\lambda(x) = x \cdot \exp(Y_i + F_i^\lambda) \tag{55}$$

for $i = 1, 2$, subject to the commutativity relation $y_1^\lambda \circ y_2^\lambda = y_2^\lambda \circ y_1^\lambda$, where $F_i^\lambda \in \mathfrak{h}$. In particular, at the parameter λ equal to 0, $F_i^0 = 0$ and $y_i^0 = y_i$, where $y_1 = \exp Y_1$ and $y_2 = \exp Y_2$ are generators of our original action ρ.

The following lemma is a simple computation.

LEMMA 3.1. *The two maps y_1^λ and y_2^λ commute if and only if one of the following equivalent conditions hold.*

(1) $[Y_1 + F_1^\lambda, Y_2 + F_2^\lambda] = 0$.

(2) $[Y_1, F_2^\lambda] - [Y_2, F_1^\lambda] + [F_1^\lambda, F_2^\lambda] = 0$.

If $F_i^\lambda = f_i^1 X_1 + f_i^2 X_2 + f_i^3 \Lambda_1 + f_i^4 \Lambda_2 + f_i^5 Z$, the coefficients $f_i^k \in \mathbb{R}$ for $i = 1, 2$ and $k = 1, 2, 3, 4, 5$, commutativity implies that the coefficients are subject to the relation

$$\tau_1 f_2^3 + \tau_2 f_2^4 + \eta_1 f_1^1 + \eta_2 f_1^2 = f_2^1 f_1^3 + f_2^2 f_1^4 - f_2^3 f_1^1 - f_2^4 f_1^2. \tag{56}$$

Therefore, the parameter λ is understood here as a vector in the nine-dimensional space:

$$\{f_i^k \in \mathbb{R}, i = 1, 2,\ k = 1, 2, 3, 4, 5,\ \text{subject to relation (56)}\}.$$

Within this nine-dimensional family of actions we impose identifications via conjugacies obtained by constant vector fields. More precisely, if

$$y_i^\lambda(x) = x \cdot \exp(Y_i + F_i^\lambda)$$

and, for some $H \in \mathfrak{h}$ and $h(x) := x \cdot \exp H$, we have

$$h \circ y_i^\lambda(x) = y_i \circ h,$$

then

$$x \cdot \exp(Y_i + F_i^\lambda) \exp H = x \cdot \exp H \exp Y_i.$$

This implies that

$$Y_i + F_i^\lambda + H + \tfrac{1}{2}[Y_i + F_i^\lambda, H] = Y_i + H + \tfrac{1}{2}[H, Y_i].$$

Thus,

$$F_i^\lambda + \tfrac{1}{2}[F_i^\lambda, H] + [Y_i, H] = 0.$$

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This implies that in the off-center direction the components of F_i^λ are trivial and, for the center direction, we have

$$(F_i^\lambda)_c = -[Y_i, H].$$

In particular, this defines a coordinate change H which produces conjugate algebraic actions in the family, and each conjugacy class is two dimensional, determined only by the values $(F_i^\lambda)_c$ $(i = 1, 2)$.

So, the nine-dimensional family of algebraic actions modulo the algebraic conjugacy classes gives a seven-dimensional family of non-conjugate algebraic actions. This is the family ρ^λ in Theorem 1.7.

In the case where the manifold is a $(2n+1)$-dimensional Heisenberg nilmanifold, the dimension of the family of non-conjugate algebraic actions in Theorem 1.7 is $4n-1$ and the family is described as in (55).

3.3. *The commutator operator.* Now we analyze the commutator operator for non-algebraic perturbations of translations which generate ρ. Recall that ρ is the $\mathbb{Z}^2$ action generated by the translation maps y_i, $i = 1, 2$, where $y_i(x) = x \cdot \exp(Y_i)$ and Y_i are the two commuting elements in $\mathfrak{H}$.

LEMMA 3.2. *Let $F, G \in \mathrm{Vect}^\infty M$ be two sufficiently small vector fields so that the maps $f(x) = x \cdot \exp(Y_1 + F(x))$ and $g(x) = x \cdot \exp(Y_2 + G(x))$ are in $\mathrm{Diff}^\infty M$. If f and g commute, then the vector fields F and G satisfy the following nonlinear equation:*

$$F \circ y_2 - F + \tfrac{1}{2}[Y_2, F \circ y_2 + F] - (G \circ y_1 - G + \tfrac{1}{2}[Y_1, G \circ y_1 + G]) + E(F, G) = 0, \tag{57}$$

where

$$\begin{aligned} E(F, G) = {} & (F \circ g - F \circ y_2) - (G \circ f - G \circ y_1) \\ & + \tfrac{1}{2}[Y_2, F \circ g - F \circ y_2] - \tfrac{1}{2}[Y_1, G \circ f - G \circ y_1] \\ & + \tfrac{1}{2}[G, F \circ g] - \tfrac{1}{2}[F, G \circ f]. \end{aligned} \tag{58}$$

Proof. The commutation $f \circ g = g \circ f$ implies that

$$x \cdot \exp(Y_2 + G) \exp(Y_1 + F \circ g) = x \cdot \exp(Y_1 + F) \exp(Y_2 + G \circ f).$$

Hence,

$$\begin{aligned} & x \cdot \exp(Y_1 + Y_2 + G + F \circ g + \tfrac{1}{2}[Y_2, F \circ g] - \tfrac{1}{2}[Y_1, G] + \tfrac{1}{2}[G, F \circ g]) \\ & \quad = x \cdot \exp(Y_1 + Y_2 + F + G \circ f + \tfrac{1}{2}[Y_1, G \circ f] - \tfrac{1}{2}[Y_2, F] + \tfrac{1}{2}[F, G \circ f]). \end{aligned}$$

The above implies the nonlinear equation directly due to the following very simple fact: $x \cdot \exp(Y + F) = x \cdot \exp(Y + G)$ with $Y \in \mathfrak{h}$ and $F, G \in \mathrm{Vect}^\infty M$ if and only if $F = G$. □

The following immediate consequence of the lemma above will be used later.

COROLLARY 3.3. *In the setting of Lemma 3.2,* $|\mathrm{Ave}[Y_2, F] - \mathrm{Ave}[Y_1, G]| \leq C\|F\|_1\|G\|_1$.

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3.4. *The conjugation operator.* Here we analyze the conjugation operator for conjugacies close to the identity, we derive the linear part of the conjugacy operator and estimate the error.

LEMMA 3.4. *Let $f(x) = x \cdot \exp(Y + F(x))$ be a diffeomorphism of M, where $Y \in \mathfrak{h}$ and $F \in \mathrm{Vect}^\infty M$ is a smooth vector field. Let $h \in \mathrm{Diff}^\infty M$ be a diffeomorphism close to the identity given by a small vector field $H \in \mathrm{Vect}^\infty M$ via $h(x) = x \cdot \exp H(x)$. Then $g := h^{-1} \circ f \circ h$ is a diffeomorphism close to f given by $G \in \mathrm{Vect}^\infty M$ via $g(x) = x \cdot \exp(Y + G(x))$ and*

$$G = H - H\circ g + \tfrac{1}{2}[H + H\circ g, Y] + F\circ h + [H, F\circ h] + \tfrac{1}{2}[H, H\circ g] - \tfrac{1}{2}[F\circ h, H\circ g].$$

Proof. We have

$$\begin{aligned} h^{-1}\circ f\circ h(x) &= x\cdot \exp H(x) \exp(Y + F\circ h(x)) \exp(-H\circ h^{-1}\circ f\circ h(x)) \\ &= x\cdot \exp(H + Y + F\circ h + \tfrac{1}{2}[H, Y] + [H, F\circ h])(x) \exp(-H\circ g)(x) \\ &= x\cdot \exp(H + Y - H\circ g + F\circ h + \tfrac{1}{2}[H, Y] + [H, F\circ h] \\ &\quad - \tfrac{1}{2}[H, -H\circ g] - \tfrac{1}{2}[Y, H\circ g] - \tfrac{1}{2}[F\circ h, H\circ g])(x). \end{aligned}$$

This implies the equality claimed for G. □

3.5. *Linearizations of the conjugacy and the commutator operators: first and second coboundary operators on vector fields, splitting.* The linear part of the nonlinear equation (57) defines the *second coboundary operator* on vector fields over the action ρ generated by y_1 and y_2.

Definition 3.5. Let $\mathbf{d}_2 : \mathrm{Vect}^\infty M \times \mathrm{Vect}^\infty M \to \mathrm{Vect}^\infty M$ be the linear operator defined by

$$\mathbf{d}_2(F, G) = F\circ y_2 - F + \tfrac{1}{2}[Y_2, F\circ y_2 + F] - (G\circ y_1 - G + \tfrac{1}{2}[Y_1, G\circ y_1 + G]).$$

We say that a pair of smooth vector fields (F, G) generates a cocycle over the action ρ if $(F, G) \in \mathrm{Ker}\, \mathbf{d}_2$.

The first coboundary operator on vector fields over the action ρ is given by the following definition.

Definition 3.6. Let $H \in \mathrm{Vect}^\infty M$. Then we define $\mathbf{d}_1 : \mathrm{Vect}^\infty M \to \mathrm{Vect}^\infty M \times \mathrm{Vect}^\infty M$ by

$$\mathbf{d}_1(H) = (H\circ y_1 - H + \tfrac{1}{2}[Y_1, H\circ y_1 + H], H\circ y_2 - H + \tfrac{1}{2}[Y_2, H\circ y_2 + H]).$$

It is an easy exercise to check that $\mathrm{Im}\, \mathbf{d}_1 \subset \mathrm{Ker}\, \mathbf{d}_2$. The first cohomology over ρ with coefficients in vector fields is the quotient space $H^1_\rho(\mathrm{Vect}^\infty M) := \mathrm{Ker}\, \mathbf{d}_2/\mathrm{Im}\, \mathbf{d}_1$. Notice that for constant vector fields $H \in \mathfrak{h}$ cocycles and coboundaries defined here coincide with those defined in §3.1. The subsequent proposition has as a corollary that for our fixed action ρ the cohomology $H^1_\rho(\mathrm{Vect}^\infty M)$ is the same as the cohomology $H^1_\rho(\mathfrak{h})$ with coefficients in the constant vector fields $\mathfrak{h}$ which was computed in §3.1.

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PROPOSITION 3.7. *If the two vector fields* $F, G \in \mathrm{Vect}^\infty M$ *satisfy* $\mathbf{d}_2(F, G) = \Phi$ *and* $(\mathrm{Ave}\, F, \mathrm{Ave}\, G)$ *is in the trivial cohomology class in* $H^1_\rho(\mathfrak{n})$*, then there exist* $H, \tilde{F}, \tilde{G} \in \mathrm{Vect}^\infty M$ *such that* $(F, G) = \mathbf{d}_1 H + (\tilde{F}, \tilde{G})$ *and the following estimates hold:*

$$\begin{aligned} \|\tilde{F}\|_s &\le C_s \|\Phi\|_{s+\sigma},\\ \|\tilde{G}\|_s &\le C_s \|\Phi\|_{s+\sigma},\\ \|H\|_s &\le C_{s,r,S,T,\Gamma} \|F\|_{s+\sigma}. \end{aligned} \tag{59}$$

Proof. Recall that we disintegrate an arbitrary vector field $H \in \mathrm{Vect}^\infty M$ into $H = H_c + H_T$, where H_c is the component of H in the direction of Z and H_T is the component of H in all the directions other than Z. So, one can view H_T as $H_T = \sum h^i_t X_i + \bar{h}^i_t \Lambda_i$, where $h^i_t, \bar{h}^i_t$ are smooth functions.

The equation $\mathbf{d}_2(F_T, G_T) = \Phi$ (since $\mathbf{d}_2$ is a linear operator) splits then in the off-center directions into finitely many functional equations each of which has a form

$$f \circ y_2 - f - (g \circ y_1 - g) = \phi.$$

Since by assumption $(\mathrm{Ave}\, F, \mathrm{Ave}\, G)$ is assumed to be in the trivial constant cohomology class, it implies in particular that all the off-center components are 0 (see §3.1).

Now we may apply Theorem 1.3, which for each of these finitely many equations gives as an output smooth functions $\tilde{f}, \tilde{g}, h$ such that

$$f = h \circ y_1 - f + \tilde{f}, \quad g = g \circ y_2 - g + \tilde{g}$$

such that the corresponding estimates hold. Putting these coordinate functions all together gives functions $\tilde{F}_T, \tilde{G}_T, H_t$ such that $(\mathbf{d}_2(\tilde{F}_T, \tilde{G}_T))_T = \Phi_T$, $(F_T, G_T) = (\mathbf{d}_1 H_T)_T + (\tilde{F}_T, \tilde{G}_T)$ and $\tilde{F}_T, \tilde{G}_T, H_T$ satisfy the estimates (59).

Now let F_c, G_c be the components of F and G in the center direction. Then, because the Z components within the brackets do not contribute, we have

$$\begin{aligned} \mathbf{d}_2(F_c, G_c) = F_c \circ y_2 - F_c + \tfrac{1}{2}[Y_2, F_T \circ y_2 + F_T]&\\ - \big(G_c \circ y_1 - G_c + \tfrac{1}{2}[Y_1, G_T \circ y_1 + G_T]\big) &= \Phi_c. \end{aligned}$$

Since we already have $(F_T, G_T) = (\mathbf{d}_1 H_T)_T + (\tilde{F}_T, \tilde{G}_T)$, we can substitute this in the above expression to obtain

$$\begin{aligned} &\big(F_c - \tfrac{1}{2}[Y_1, H_T \circ y_1 + H_T]\big) \circ y_2 - \big(F_c - \tfrac{1}{2}[Y_1, H_T \circ y_1 + H_T]\big)\\ &\quad - \Big(\big(G_c - \tfrac{1}{2}[Y_2, H_T \circ y_2 + H_T]\big) \circ y_1 - \big(G_c - \tfrac{1}{2}[Y_2, H_T \circ y_2 + H_T]\big)\Big) = \Phi'_c, \end{aligned} \tag{60}$$

where

$$\Phi'_c = \Phi_c - \tfrac{1}{2}[Y_2, \tilde{F}_T \circ y_2 + \tilde{F}_T] + \tfrac{1}{2}[Y_1, \tilde{G}_T \circ y_1 + \tilde{G}_T].$$

Clearly, since for any s we have $\|\tilde{F}_T\|_s, \|\tilde{G}_T\|_s \le \|\tilde{\Phi}_T\|_{s+\sigma}$, it follows that $\|\Phi'_c\|_s \le \|\tilde{\Phi}\|_{s+\sigma}$, where σ is re-defined to be $\sigma + 1$.

The vector field H_T is determined only up to a constant vector field, so we may choose H_T so that $\mathrm{Ave}\, F_c = \mathrm{Ave}[Y_1, H_T]$. This forces $F_c - \frac{1}{2}[Y_1, H_T \circ y_1 + H_T]$ to have the

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average 0. Moreover, because of the assumption that (Ave F, Ave G) is in the trivial cohomology class, we also have that Ave F_c = Ave G_c.

So, the equation (60) is again the same type of equation as in Theorem 1.3. By applying the theorem, we get $\tilde{F}_c, \tilde{G}_c, H_c$ such that

$$F_c - \tfrac{1}{2}[Y_1, H_T \circ y_1 + H_T] = H_c \circ y_1 - H_c + \tilde{F}_c,$$
$$G_c - \tfrac{1}{2}[Y_2, H_T \circ y_2 + H_T] = H_c \circ y_2 - H_c + \tilde{G}_c.$$

This clearly implies that

$$(F_c, G_c) = (\mathbf{d}_1 H)_c + (\tilde{F}_c, \tilde{G}_c).$$

Putting the c- and T-components together gives the solution. Estimates (59) are direct consequences of coordinate-wise estimates which are obtained already in Theorem 1.3. □

3.6. *Set-up of the perturbative problem and the iterative scheme.* We will frequently refer here to [7], so we recommend that the reader has that paper at hand.

We consider here a family of perturbations $\tilde{\rho}^\lambda$ of ρ^λ, which are generated by commuting maps $\tilde{y}_1^\lambda$ and $\tilde{y}_2^\lambda$, where, for $i = 1, 2$,

$$\tilde{y}_i^\lambda(x) = x \cdot \exp(Y_i + \tilde{F}_i^\lambda). \tag{61}$$

Here $\tilde{F}_i^\lambda$ are small vector fields such that $\tilde{y}_1^\lambda$ and $\tilde{y}_2^\lambda$ commute.

Now let h be a diffeomorphism of the manifold, close to the identity, defined via the smooth vector field H as follows:

$$h(x) = \Gamma\theta \cdot \exp H(x).$$

The iterative step consists of the following: given the perturbation $\tilde{\rho}^\lambda$ of ρ^λ, define a new perturbation $\bar{\rho}^\lambda$ which is a conjugation of $\tilde{\rho}^\lambda$ via h, so $\bar{\rho}^\lambda$ is generated by two diffeomorphisms $\bar{y}_i^\lambda$, $i \in \{1, 2\}$, defined by

$$\bar{y}_i^\lambda = h^{-1} \circ \tilde{y}_i^\lambda \circ h.$$

In each iterative step this is done for the λ parameter in some ball, and it is shown that in that ball there is a parameter for which the new family of perturbations is much (quadratically) closer to ρ for parameters in some smaller ball. The next proposition shows that that this process is controlled in the sequence of C^r norms.

We will need to control derivatives of each perturbed family $\tilde{\rho}^\lambda$ in the direction of the parameter λ as well, so we will use the following norms for a family of vector fields $\tilde{F}_i^\lambda$: $\|\tilde{F}_i^\lambda\|_{0,k}$ stands for the supremum of the C^k norms of $\tilde{F}_i^\lambda$ in the λ variable. $\|\tilde{F}_i^\lambda\|_{r,k}$ is the same only taken over all the derivatives of $\tilde{F}_i^\lambda$ in the manifold direction. As before, we reserve the notation $\|\tilde{F}_i^\lambda\|_r$ for the usual C^r norm on M of the vector field $\tilde{F}_i^\lambda \in \mathrm{Vect}^\infty M$ for a fixed parameter λ.

The following is an immediate corollary of the classical implicit function theorem and we will use it for the maps which compute averages of vector fields for actions in the perturbed family.

LEMMA 3.8. *There exists an open ball $\mathcal{O} = \mathcal{O}(\mathrm{Id}, R)$ in $C^1(\mathbb{R}^d, \mathbb{R}^d)$, there exist a neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^d$ and a C^1 map $\Psi : \mathcal{O} \to \mathcal{U}$ such that for every $G \in \mathcal{O}$, $G(\Psi(G)) = 0$.*

Now we state the main iterative step proposition where we show that one can obtain indeed estimates which are needed for the convergence of the process to a smooth conjugation map.

PROPOSITION 3.9. *There exist constants $\bar{C}$ and r_0 such that the following holds.*

Given the family $\tilde{\rho}_n^\lambda$ of perturbations of ρ^0 generated by $\tilde{y}_{i,n}^\lambda$ ($i = 1, 2$), assume that for all λ in a ball B centered at 0, for $r \in \mathbb{N}$ and $t > 0$:

(1) $\|\tilde{F}_{i,n}^\lambda\|_0 \leq \varepsilon_n < 1$;

(2) *the map* $\lambda \mapsto \tilde{F}_{i,n}^\lambda$ *is* C^2 *in* λ, $\|\tilde{F}_{i,n}^\lambda\|_{0,1} \leq \varepsilon_n$ *and* $\|\tilde{F}_{i,n}^\lambda\|_{0,2} \leq K_n$;

(3) *the map* $\Phi^n : \lambda \mapsto \mathrm{Ave}(\tilde{F}_{i,n}^\lambda)$ *is in* $\mathcal{O}$ *and has a zero at* λ_n;

(4) $\|\tilde{F}_{i,n}^\lambda\|_{r_0+r} \leq \delta_{r,n}$;

(5) $t^{r_0} \varepsilon_n^{1-(1/(r+r_0))} \delta_r^{(1/(r+r_0))} < \bar{C}$.

There exists a $H_n \in \mathrm{Vect}^\infty M$ such that h_n defined by $h(x) = x \cdot \exp H_n(x)$ is such that the newly formed family of perturbations $\tilde{\rho}_{n+1}^\lambda$ of ρ, generated by $\tilde{y}_{i,n+1}^\lambda = h^{-1} \circ \tilde{y}_{i,n}^\lambda \circ h$, with $\tilde{y}_{i,n+1}^\lambda(x) = x \exp(Y_i + \tilde{F}_{i,n+1}^\lambda(x))$, satisfies the following:

(a) $\|H_n\|_r \leq C_r t^{2r_0} \|\tilde{F}_i^{\lambda_n}\|_r$;

(b) $\|\tilde{F}_{i,n+1}^\lambda\|_0 \leq K_n \|\lambda - \lambda_n\| + \mathrm{Err}(t, r)$, *where*

$$\mathrm{Err}_{n+1}(t, r) := C\varepsilon_n^2 + C\delta_{r,n}^{(r_0+1)/(r_0+r)} \varepsilon_n^{2-((r_0+1)/(r_0+r))} + C_r t^{-r} \delta_{r,n} + C_{r_0} t^{2r_0} \varepsilon_n^{2-(1/(r_0+r))} \delta_{r,n}^{1/(r_0+r)} + C_{r_0} t^{2r_0} \varepsilon_n^{3-(1/(r_0+r))} \delta_{r,n}^{2/(r_0+r)};$$

(c) $\|\tilde{F}_{i,n+1}^\lambda\|_{r_0+r} \leq C_r t^{2r_0} \delta_{r,n} := \delta_{r,n+1}$;

(d) *the map* $\Phi^{n+1} := \lambda \mapsto \mathrm{Ave}(\tilde{F}_{i,n+1}^\lambda)$ *satisfies*

$$\|\Phi^{n+1} - \Phi^n\|_{(0)} \leq \mathrm{Err}_{n+1}(t, r),$$
$$\|\Phi^{n+1} - \Phi^n\|_{(1)} \leq K_n t^{r_0} \varepsilon_n + \mathrm{Err}_{n+1}(t, r).$$

If Φ^{n+1} is in $\mathcal{O}$, then it has a zero at $\lambda_{n+1} \in B$ which satisfies

$$\|\lambda_{n+1} - \lambda_n\| \leq C\mathrm{Err}_{n+1}(t, r) + CK_n(K_n t^{r_0} \varepsilon_n + \mathrm{Err}_{n+1}(t, r))^2;$$

(e) $\tilde{F}_{i,n+1}^\lambda$ *is* C^2 *in* λ *and*

$$\|\tilde{F}_{i,n+1}^\lambda\|_{0,2} \leq (1 + Ct^{r_0} \varepsilon_n^{1-(1/(r+r_0))} \delta_{r,n}^{1/(r+r_0)}) K_n =: K_{n+1}(t, r).$$

Proof. As was mentioned in [7, Remark 6.3], the proof of the iterative step is universal given tame splitting for vector fields (Proposition 3.7). We repeat the main points here for the sake of completeness with fewer details than in the proof of the corresponding proposition in [7, Proposition 6.2].

In this proof, as is customary whenever there is a loss of regularity for solutions of linearized equations, we will use the smoothing operators. For the construction of smoothing operators on $C^\infty(M)$, see the following in [10]: Example 1.1.2(2), Definition 1.3.2, Theorem 1.3.6 and Corollary 1.4.2. There exists a collection of smoothing operators $S_t : C^\infty(M) \to C^\infty(M)$, $t > 0$, such that the following holds:

$$\|S_t F\|_{s+s'} \le C_{s,s'} t^{s'} \|F\|_s,$$
$$\|(I - S_t)F\|_{s-s'} \le C_{s,s'} t^{-s'} \|F\|_s. \tag{62}$$

Smoothing operators on $C^\infty(M)$ clearly induce smoothing operators on $\mathrm{Vect}^\infty M$ via smoothing operators applied to coordinate maps.

It is easy to see that averages of F with respect to the Haar measure on M, in various directions in the tangent space, do not affect the properties of smoothing operators listed above, so without loss of generality we may assume that S_t are such that averages of $S_t F$ are the same as those of F.

Given $\tilde{F}^\lambda_{i,n}$, we first apply the smoothing operators to it and write $\tilde{F}^\lambda_{i,n} = S_t \tilde{F}^\lambda_{i,n} + (I - S_t)\tilde{F}^\lambda_{i,n}$. Now $\mathrm{Ave}(\tilde{F}^\lambda_{i,n}) = \mathrm{Ave}(S_t \tilde{F}^\lambda_{i,n})$. From the commutativity of $\tilde{F}^\lambda_{i,n}$ for $i = 1$ and $i = 2$ (see Corollary 3.3), it follows that $|\mathrm{Ave}[Y_2, \tilde{F}_{1,n}] - \mathrm{Ave}[Y_1, \tilde{F}_{2,n}]| \le C\|\tilde{F}_{1,n}\|_1 \|\tilde{F}_{2,n}\|_1 \le C\varepsilon_n^2$ and clearly the same holds after application of the corresponding smoothing operators. Now we can apply Proposition 3.7 to $S_t \tilde{F}^\lambda_{i,n} - \mathrm{Ave}(\tilde{F}^\lambda_{i,n})_T$, $i = 1, 2$ (recall that $\mathrm{Ave}(\tilde{F}^\lambda_{i,n})_T$ are averages in the off-center directions). Proposition 3.7 gives existence of H_n such that

$$\|(S_t \tilde{F}_{1,n} - \mathrm{Ave}(\tilde{F}_{1,n})_T, S_t \tilde{F}_{2,n} - \mathrm{Ave}(\tilde{F}_{2,n})_T) - \delta_1 H_n\|_r \le C\|\Phi\|_{r+\sigma},$$

where (see (58))

$$\Phi := E(\tilde{F}_{1,n}, \tilde{F}_{2,n}).$$

From the expression for E in (58), we have the following estimate for Φ:

$$\|\Phi\|_r \le C\|\tilde{F}^\lambda_{i,n}\|_r \|\tilde{F}^\lambda_{i,n}\|_{r+1},$$

where we use short notation $\|\tilde{F}^\lambda_{i,n}\|_r$ for the maximum of the norms for $i = 1$ and $i = 2$. Also, from Proposition 3.7, we have

$$\|H\|_r \le C\|S_t \tilde{F}^\lambda_{i,n} - \mathrm{Ave}(\tilde{F}_{1,n})_T\|_{r+\sigma} \le Ct^\sigma \|\tilde{F}^\lambda_{i,n}\|_r.$$

From Lemma 3.4, it follows that if we define h_n by $h_n(x) = x \cdot \exp H_n(x)$, and $\tilde{y}^\lambda_{i,n+1} = h^{-1} \circ \tilde{y}^\lambda_{i,n} \circ h$, with $\tilde{y}^\lambda_{i,n+1}(x) = x \cdot \exp(Y_i + \tilde{F}^\lambda_{i,n+1}(x))$, then $\tilde{F}^\lambda_{i,n+1}$ satisfy the following, after applying the interpolation estimates and the smoothing estimates and assumptions (2) and (3) (compare to [7, (6.7)]):

$$\begin{aligned}\|\tilde{F}^\lambda_{i,n+1}\|_0 \le{}& K_n\|\lambda - \lambda_n\| + C\varepsilon_n^2 + C\delta_{r,n}^{(r_0+1)/(r_0+r)} \varepsilon_n^{2-((r_0+1)/(r_0+r))} \\ &+ C_r t^{-r}\delta_{r,n} + C_{r_0} t^{2r_0} \varepsilon_n^{2-(1/(r_0+r))} \delta_{r,n}^{1/(r_0+r)} + C_{r_0} t^{2r_0} \varepsilon_n^{3-(1/(r_0+r))} \delta_{r,n}^{2/(r_0+r)}.\end{aligned} \tag{63}$$

For the C^{r_0+r} norm of the new error $\tilde{F}^\lambda_{i,n+1}$, as usual in this type of proofs, we only need a 'linear' bound with respect to the corresponding norm of the old error. This follows easily from the conjugacy relation and we obtain for any $s \geq 0$:

$$\|\tilde{F}^{\lambda}_{i,n+1}\|_s \leq C_s t^{2r_0} \|\tilde{F}^{\lambda}_{i,n}\|_s,$$

which, as in [7], implies that

$$\|\tilde{F}^{\lambda}_{i,n+1}\|_s \leq C_s t^{2r_0} \delta_{r,n}.$$

The remaining two statements (e) and (d) follow exactly in the same way as in the proof of [7, Proposition 6.2]. □

Given Proposition 3.9 (compare to [7, Proposition 6.2]), we can now apply the convergence of the successive iterative scheme proved in [7, §7]. Consequently, we obtain the following theorem, which is a more precise statement of our main transversal local rigidity result in Theorem 1.7.

THEOREM 3.10. *There exist $l > 0$, $\epsilon > 0$, $R > 0$ such that if a family $\tilde{\rho}^{\lambda}$ of perturbations of ρ generated by $\tilde{y}_i^{\lambda}$ is ϵ close to ρ in the C^l norm for parameters λ in an R-ball around 0, and in the C^1 norm in the parameter λ direction, then there exists a small parameter $\bar{\lambda}$ such that the action $\tilde{\rho}^{\bar{\lambda}}$ is conjugate to ρ via h, that is, for $i = 1, 2$, we have*

$$h \circ y_i = \tilde{y}_i^{\bar{\lambda}} \circ h,$$

where h is a smooth diffeomorphism order of ϵ close to the identity in the C^1 norm.

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A. *Appendix. Proof of Propositions 2.1 and 2.2*

The classical Diophantine condition (5) stated in §1.1 is clearly equivalent to the following condition: there are constants $c := c_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ and $\gamma := \gamma_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ such that for any $\mathbf{m} \in \mathbb{Z}^{2n}$ and $p \in \mathbb{Z}$, we have

$$\begin{aligned} |\boldsymbol{\tau} \cdot \mathbf{m} - p| &> c|\mathbf{m} \cdot \mathbf{m}|^{-\gamma} \quad \text{if } \mathbf{m}_1 \neq 0, \\ |\boldsymbol{\eta} \cdot \mathbf{m} - p| &> c|\mathbf{m} \cdot \mathbf{m}|^{-\gamma} \quad \text{if } \mathbf{m}_2 \neq 0. \end{aligned} \tag{A.1}$$

We will use the above version of the Diophantine condition to prove the splitting results for finite-dimensional representations in this section. The same splitting results were needed and used in three other works so far: [4, 13, 19] and they follow closely Moser's splitting construction on the circle in [12]. Our presentation here is somewhat different in that it follows a general splitting construction which applies to abelian actions where cohomological equations in irreducible representations have a finite-dimensional space of obstructions (as in [8], for example).

For any $\mathbf{m} \in \mathbb{Z}^{2n}$ and for any $\boldsymbol{\kappa} \in \{\boldsymbol{\tau}, \boldsymbol{\eta}\}$, define the constant $\zeta(\mathbf{m}, \boldsymbol{\kappa})$ by

$$\zeta(\mathbf{m}, \boldsymbol{\kappa}) := \exp(2\pi i(\mathbf{m} \cdot \boldsymbol{\kappa})) - 1.$$

The next lemma describes the operator $L_{\boldsymbol{\kappa}}$ on smooth functions in $L^2(\mathbb{T}^{2n})$. Its proof is straightforward and follows from the Diophantine condition (A.1).

LEMMA A.1. *Let* $h = \sum_{\mathbf{m}\in\mathbb{Z}^{2n}} h_{\mathbf{m}} \exp(2\pi i\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})) \in W_0^\infty(\mathbb{T}^{2n})$ *be a smooth, zero-average function with coefficients* $(h_{\mathbf{m}})$. *Then, for* $\boldsymbol{\kappa} \in \{\boldsymbol{\tau}, \boldsymbol{\eta}\}$,

$$L_{\boldsymbol{\kappa}} h(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} h_{\mathbf{m}}\zeta(\mathbf{m}, \boldsymbol{\kappa}) \exp(2\pi i\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})).$$

Moreover, there is a constant $C_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ *such that for any* $\mathbf{m} \in \mathbb{Z}^{2n} \setminus \{0\}$,

$$|\zeta(\mathbf{m}, \boldsymbol{\tau})|^{-1} \le C_{\boldsymbol{\tau},\boldsymbol{\eta}}|\mathbf{m} \cdot \mathbf{m}|^\gamma \quad \text{if } \mathbf{m}_1 \neq 0,$$
$$|\zeta(\mathbf{m}, \boldsymbol{\eta})|^{-1} \le C_{\boldsymbol{\tau},\boldsymbol{\eta}}|\mathbf{m} \cdot \mathbf{m}|^\gamma \quad \text{otherwise},$$

where, for $j = 1, 2$, $\mathbf{m}_j$ *is defined in (3) and* γ *is the exponent in the Diophantine condition for* $\boldsymbol{\tau}, \boldsymbol{\eta}$*; see (A.1).*

We now prove the estimates from Theorems 1.2 and 1.3 in the context of finite-dimensional representations.

Proof of Proposition 2.1. Because $f, g \in W^\infty(\mathbb{T}^{2n})$ are zero-average functions, there are coefficients $(g_{\mathbf{m}}), (f_{\mathbf{m}}) \in \ell^2(\mathbb{Z}^{2n})$, with $f_{\mathbf{0}} = g_{\mathbf{0}} = 0$, such that

$$g = \sum_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} g_{\mathbf{m}} \exp(2\pi i\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})),$$
$$f = \sum_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} f_{\mathbf{m}} \exp(2\pi i\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})). \tag{A.2}$$

By Lemma A.1, we get

$$L_{\boldsymbol{\tau}} g(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} g_{\mathbf{m}}\zeta(\mathbf{m}, \boldsymbol{\tau}) \exp(2\pi i\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})),$$
$$L_{\boldsymbol{\eta}} f(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} f_{\mathbf{m}}\zeta(\mathbf{m}, \boldsymbol{\eta}) \exp(2\pi i\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})).$$

Then, because $(\exp(2\pi i\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})))_{\mathbf{m}\in\mathbb{Z}^{2n}}$ is an orthogonal basis for $L^2(\mathbb{T}^{2n})$, $L_{\boldsymbol{\tau}} g = L_{\boldsymbol{\eta}} f$ implies that for any $\mathbf{m} \in \mathbb{Z}^{2n} \setminus \{0\}$,

$$g_{\mathbf{m}}\zeta(\mathbf{m}, \boldsymbol{\tau}) = f_{\mathbf{m}}\zeta(\mathbf{m}, \boldsymbol{\eta}).$$

From the definition of ζ and the Diophantine property for $\boldsymbol{\tau}$ and $\boldsymbol{\eta}$ (see (A.1)), we get that for any $\mathbf{m} \in \mathbb{Z}^{2n} \setminus \{0\}$,

$$\begin{cases} \zeta(\mathbf{m}, \boldsymbol{\tau}) \neq 0 & \text{if } \mathbf{m}_1 \neq 0, \\ \zeta(\mathbf{m}, \boldsymbol{\tau}) = 0 & \text{if } \mathbf{m}_1 = 0, \\ \zeta(\mathbf{m}, \boldsymbol{\eta}) \neq 0 & \text{if } \mathbf{m}_1 = 0. \end{cases} \tag{A.3}$$

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Hence,

$$g_{\mathbf{m}} = f_{\mathbf{m}} \frac{\zeta(\mathbf{m}, \boldsymbol{\eta})}{\zeta(\mathbf{m}, \boldsymbol{\tau})} \quad \text{if } \mathbf{m}_1 \neq 0, \qquad f_{\mathbf{n}} = 0, \quad \text{otherwise.} \tag{A.4}$$

Now define the sequence $(P_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^{2n}}$ by $P_{\mathbf{0}} = 0$ and, for any non-zero $\mathbf{m}$, set

$$P_{\mathbf{m}} := \begin{cases} \dfrac{f_{\mathbf{m}}}{\zeta(\mathbf{m}, \boldsymbol{\tau})} & \text{if } \mathbf{m}_1 \neq 0, \\ \dfrac{g_{\mathbf{n}}}{\zeta(\mathbf{m}, \boldsymbol{\eta})} & \text{otherwise.} \end{cases} \tag{A.5}$$

Let

$$P := \sum_{\mathbf{m}\in\mathbb{Z}^{2n}} P_{\mathbf{m}} \exp(2\pi i \mathbf{n} \cdot (\mathbf{x}, \boldsymbol{\xi})).$$

Then a calculation formally gives $L_{\boldsymbol{\tau}} P = f$, $L_{\boldsymbol{\eta}} P = g$, where the first equation follows from the second equalities in (A.3) and (A.4), and the second equation follows from the first equality in (A.4) and equation (A.5).

Now we estimate the Sobolev norm of P. Recall from (12) that for any $f \in W^\infty(\mathbb{T}^{2n})$ and for any $s \in \mathbb{N}$,

$$\|f\|_s = \|(1 + 4\pi^2(\mathbf{m} \cdot \mathbf{m}))^{s/2} f\|_{\ell^2(\mathbb{Z}^{2n})} < \infty.$$

Set $s \in \mathbb{N}$. By Lemma A.1 and formula (A.5), for any $\mathbf{m} \in \mathbb{Z}^{2n} \setminus \{0\}$ such that $\mathbf{m}_1 \neq 0$, we have

$$|P_{\mathbf{m}}| = \frac{|f_{\mathbf{m}}|}{|\zeta(\mathbf{m}, \boldsymbol{\tau})|} \le C_{\boldsymbol{\tau},\boldsymbol{\eta}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^\gamma |f_{\mathbf{m}}|.$$

On the other hand, when $\mathbf{m}_1 = 0$, we have

$$|P_{\mathbf{m}}| = \frac{|g_{\mathbf{m}}|}{|\zeta(\mathbf{m}, \boldsymbol{\eta})|} \le C_{\boldsymbol{\tau},\boldsymbol{\eta}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^\gamma |g_{\mathbf{m}}|.$$

Then, for any $s \in \mathbb{N}$, there is a constant $C_{\boldsymbol{\tau},\boldsymbol{\eta}} > 0$ such that

$$\begin{aligned} \|P\|_s^2 &= \sum_{\substack{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\} \\ \mathbf{m}_1 \neq 0}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^s |P_{\mathbf{m}}|^2 + \sum_{\substack{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\} \\ \mathbf{m}_1 = 0}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^s |P_{\mathbf{m}}|^2 \\ &\le C_{\boldsymbol{\tau},\boldsymbol{\eta}} \sum_{\mathbf{m}\in\mathbb{Z}^{2n}\setminus\{0\}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^{s+2\gamma} (|f_{\mathbf{m}}|^2 + |g_{\mathbf{m}}|^2). \end{aligned} \tag{A.6}$$

By interpolation, the above estimate holds for any $s \ge 0$. Hence, for any $s \ge 0$,

$$(A.6) = C_{\boldsymbol{\tau},\boldsymbol{\eta}} (\|f\|_{s+2\gamma}^2 + \|g\|_{s+2\gamma}^2) \le C_{\boldsymbol{\tau},\boldsymbol{\eta}} (\|f\|_{s+2\gamma} + \|g\|_{s+2\gamma})^2.$$

We conclude that

$$\|P\|_s \le C_{\boldsymbol{\tau},\boldsymbol{\eta}} (\|f\|_{s+2\gamma} + \|g\|_{s+2\gamma}). \qquad \square$$

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Proof of Proposition 2.2. Let $s \in \mathbb{N}$. Let f, g be given by (A.2) and write ϕ as

$$\phi = \sum_{\mathbf{m} \in \mathbb{Z}^{2n}} \phi_{\mathbf{m}} \exp(2\pi i \mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})),$$

where $(\phi_{\mathbf{m}}) \in \ell^2(\mathbb{Z}^{2n})$. Because $\phi = L_\eta f - L_{\boldsymbol{\tau}} g$, we get

$$\phi_{\mathbf{0}} = 0.$$

By assumption, f and g also have zero average, so

$$f_{\mathbf{0}} = g_{\mathbf{0}} = 0.$$

Define P by the sequence $(P_{\mathbf{m}})_{\mathbf{m} \in \mathbb{Z}}$ given in (A.5), where $P_{\mathbf{0}} = 0$.

Let R be the orthogonal projection in $L^2(\mathbb{T}^{2n})$ onto the space generated by

$$\bigcup_{\substack{\mathbf{m} \in \mathbb{Z}^{2n} \\ \mathbf{m}_1 \neq 0}} \{\exp(2\pi i \mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi}))\}.$$

That is, for any $h = \sum_{\mathbf{m} \in \mathbb{Z}^{2n}} h_{\mathbf{m}} \exp(2\pi i (\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})))$ in $L^2(\mathbb{T}^{2n})$,

$$Rh(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\substack{\mathbf{m} \in \mathbb{Z}^{2n} \\ \mathbf{m}_1 \neq 0}} h_{\mathbf{m}} \exp(2\pi i (\mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi}))). \tag{A.7}$$

A direct calculation gives the next lemma. □

LEMMA A.2. *The following equalities hold on $L^2(\mathbb{T}^{2n})$:*

$$RL_\eta = L_\eta R, \quad RL_{\boldsymbol{\tau}} = L_{\boldsymbol{\tau}}.$$

Now let P be defined by (A.5). Then

$$L_{\boldsymbol{\tau}} P = \sum_{\substack{\mathbf{m} \in \mathbb{Z}^{2n} \\ \mathbf{m}_1 \neq 0}} f_{\mathbf{m}} \exp(2\pi i \mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})) = Rf.$$

By the above equality and Lemma A.2, we get as in (43): $R\phi = L_{\boldsymbol{\tau}}(L_\eta P - g)$. From (A.7), it follows that for any $\mathbf{m} \in \mathbb{Z}^{2n}$ such that $\mathbf{m}_1 = 0$, $(R\phi)_{\mathbf{m}} = 0$. Moreover, for any $h \in L^2(\mathbb{T}^{2n})$, we get from the definition of $L_{\boldsymbol{\tau}}$ that for such $\mathbf{m}$,

$$(L_{\boldsymbol{\tau}} h)_{\mathbf{m}} = 0.$$

This means that

$$\sum_{\substack{\mathbf{m} \in \mathbb{Z}^{2n} \\ \mathbf{m}_1 \neq 0}} (R\phi)_{\mathbf{m}} \exp(2\pi i \mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})) = R\phi(\mathbf{x}, \boldsymbol{\xi}) = L_{\boldsymbol{\tau}}(L_\eta P - g)(\mathbf{x}, \boldsymbol{\xi})$$

$$= \sum_{\substack{\mathbf{m} \in \mathbb{Z}^{2n} \\ \mathbf{m}_1 \neq 0}} (L_\eta P - g)_{\mathbf{m}} \zeta(\mathbf{m}, \boldsymbol{\tau}) \exp(2\pi i \mathbf{m} \cdot (\mathbf{x}, \boldsymbol{\xi})).$$

By orthogonality, it follows that for all $\mathbf{m} \in \mathbb{Z}^{2n} \setminus \{0\}$ with $\mathbf{m}_1 \neq 0$,

$$(R\phi)_{\mathbf{m}} = (L_\eta P - g)_{\mathbf{m}} \zeta(\mathbf{m}, \boldsymbol{\tau}). \tag{A.8}$$

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Note that the definition of $P_{\mathbf{m}}$ gives

$$(L_{\boldsymbol{\eta}} P - g)_{\mathbf{m}} = P_{\mathbf{m}} \zeta(\mathbf{m}, \boldsymbol{\eta}) - g_{\mathbf{m}} = 0.$$

So, by the above equality, formula (A.8) and Lemma A.1, we get that for any $\mathbf{m} \in \mathbb{Z}^{2n}$,

$$|(L_{\boldsymbol{\eta}} P - g)_{\mathbf{m}}| = \frac{|(R\phi)_{\mathbf{m}}|}{|\zeta(\mathbf{m}, \boldsymbol{\tau})|} \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^{\gamma} |(R\phi)_{\mathbf{m}}|.$$

Hence,

$$\begin{aligned} \|L_{\boldsymbol{\eta}} P - g\|_s^2 &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}^{2n} \\ \mathbf{m}_1 \neq 0}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^s |(L_{\boldsymbol{\eta}} P - g)_{\mathbf{m}}|^2 \\ &\leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^{2n} \\ \mathbf{m}_1 \neq 0}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^{s+2\gamma} |(R\phi)_{\mathbf{m}}|^2 \\ &= C_{\boldsymbol{\tau},\boldsymbol{\eta}} \|R\phi\|_{s+2\gamma}^2 \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} \|\phi\|_{s+2\gamma}^2. \end{aligned}$$

Next, as in (46), we get

$$\|L_{\boldsymbol{\tau}} P - f\|_s = \|(R - I) f\|_s.$$

By Lemma A.2, it follows as in (47) that

$$L_{\boldsymbol{\eta}}(R - I) f = (R - I) L_{\boldsymbol{\eta}} f = (R - I)\phi.$$

Next, a calculation proves that for any $\mathbf{m} \in \mathbb{Z}^{2n} \setminus \{0\}$ such that $\mathbf{m}_1 = 0$,

$$|((R - I) f)_{\mathbf{m}}| \leq \frac{|((R - I)\phi)_{\mathbf{m}}|}{|\zeta(\mathbf{m}, \boldsymbol{\eta})|} \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} (1 + 4\pi^2 \mathbf{m} \cdot \mathbf{m})^{\gamma} |((R - I)\phi)_{\mathbf{m}}|.$$

Then, using Lemma A.1, we conclude that

$$\|(R - I) f\|_s \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} \|(R - I)\phi\|_{s+2\gamma} \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} \|\phi\|_{s+2\gamma}.$$

The third inequality in Proposition 2.2 holds because P is the same function from Proposition 2.1, which gives

$$\|P\|_s \leq C_{\boldsymbol{\tau},\boldsymbol{\eta}} (\|f\|_{s+2\gamma} + \|g\|_{s+2\gamma}).$$

Now, if P is non-constant, then we are done. So, suppose that P is constant and therefore zero. Notice that by the above estimate, $\phi = 0$ implies that $f = 0$, which contradicts the assumption that $f \neq 0$. So, we conclude that there is some $\mathbf{m}_0 \in \mathbb{Z}^{2n}$ such that

$$\phi_{\mathbf{m}_0} \neq 0.$$

Then define

$$\widetilde{P}(\mathbf{x}, \boldsymbol{\xi}) := \phi_{\mathbf{m}_0} \exp(2\pi i \mathbf{m}_0 \cdot (\mathbf{x}, \boldsymbol{\xi})).$$

By the orthogonal decomposition of ϕ, we have $\|\widetilde{P}\|_s \leq \|\phi\|_s$. So, the above estimates of $\|L_{\boldsymbol{\eta}} P - g\|_s$ and $\|L_{\boldsymbol{\tau}} P - f\|_s$ imply that

$$\begin{aligned}\|L_{\boldsymbol{\eta}} \widetilde{P} - g\|_s &= \|(L_{\boldsymbol{\eta}} P - g) + L_{\boldsymbol{\eta}} \widetilde{P}\|_s \\ &\leq \|L_{\boldsymbol{\eta}} P - g\|_s + \|L_{\boldsymbol{\eta}} \widetilde{P}\|_s \\ &\leq (C_{\boldsymbol{\tau},\boldsymbol{\eta}} + 1)\|\phi\|_{s+2\gamma}\end{aligned}$$

and, analogously, $\|L_{\boldsymbol{\tau}} \widetilde{P} - f\|_s \leq (C_{\boldsymbol{\tau},\boldsymbol{\eta}} + 1)\|\phi\|_{s+2\gamma}$.

This concludes the proof of Proposition 2.2. □

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