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The algebras of Lewis's counterfactuals: axiomatizations and algebraizability

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Abstract

The logico-algebraic study of Lewis's hierarchy of variably strict conditional logics has been essentially unexplored, hindering our understanding of their mathematical foundations, and the connections with other logical systems. This work starts filling this gap by providing a logico-algebraic analysis of Lewis's logics. We begin by introducing novel finite axiomatizations for Lewis's logics on the syntactic side, distinguishing between global and local consequence relations on Lewisian sphere models on the semantical side, in parallel to the case of modal logic. As first main results, we prove the strong completeness of the calculi with respect to the corresponding semantical consequence on spheres, and a deduction theorem. We then demonstrate that the global calculi are strongly algebraizable in terms of a variety of Boolean algebras with a binary operator representing the counterfactual implication; in contrast, we show that the local ones are generally not algebraizable, although they can be characterized as the degree-preserving logic over the same algebraic models. This yields the strong completeness of all the logics with respect to the algebraic models.

1 Introduction

A counterfactual conditional (or simply a counterfactual) is a conditional statement of the form "If *antecedent* were the case, then *consequent* would be the case", where the antecedent is usually assumed to be false. Counterfactuals have been studied in different fields, such as linguistics, artificial intelligence, and philosophy. The logical analysis of counterfactuals is rooted in the work of Lewis [17, 16] and Stalnaker [29, 30] who have introduced what has become the standard semantics for counterfactual conditionals based on particular Kripke models (called *sphere models*) equipped with a similarity relation among the possible worlds. In Lewis's language, a counterfactual is formalized as a formula of the kind $\varphi \Box\!\!\rightarrow \psi$ which is intended to mean that if φ were the case, then ψ would be the case. Lewis develops a hierarchy of logics meant to deal with different kinds of conditionals that have had a notable impact, and are quite well-known and studied; surprisingly, the literature on these logics (quite vast: Lewis's book counts thousands of citations at the present date) essentially lacks a logico-algebraic treatment.

The reader needs not be reminded that the role of algebra has been pivotal in the formalization and understanding of correct reasoning; indeed, modern logic really flourishes with the rise of the formal methods of mathematical logic, which moves its first steps with George Boole's intuition of using the symbolic language of algebra as a mean to formalize how sentences connect together via logical connectives [7]. More recently, the advancements of the discipline of (abstract) algebraic logic have been one of the main drivers behind the surge of systems of *nonclassical logics* in the 20th century, in particular via the notion of *equivalent algebraic semantics* of a logic introduced by Blok and Pigozzi [6]. In this framework, the deductions of a logic are fully and faithfully interpreted by the semantical consequence of the related algebras, and powerful *bridge theorems* allow one to study properties of the logics in the corresponding algebraic framework and vice versa [10].

While a few works present a semantics in terms of algebraic structures for Lewis's conditional logics ([31, 23, 28, 26]), the results therein are either partial or fall outside the framework of the abstract algebraic analysis (however see [24] for a logico-algebraic work on a conditional logic outside of Lewis's framework). On a different note, the proof-theoretic perspective on Lewis's conditional logics is instead more developed, in particular it is carried out in a series of recent works [12, 11, 21]. However, although the research on Lewis's conditional logics has been and still is very prolific, a foundational work that carries Lewis's hierarchy within the realm of the well-developed discipline of (abstract) algebraic logic is notably missing in the literature; the present manuscript aims at filling this void.

To this end, we start by considering Lewis's logics as consequence relations, instead of just sets of theorems; this brings us to consider two different kinds of derivation, depending on whether the deductive rules are applied only to theorems (giving a relatively weaker calculus) or to all derivations (i.e., yielding a stronger calculus). We stress that this distinction, although relevant, is often blurred in the literature. As it is the case for modal logic (see [4, 32]), these two choices turn out to correspond to considering two different consequence relations on the intended sphere models: a *local* and a *global* one; the latter, to the best of our knowledge, has not been considered in the literature. The strong completeness of the local consequence with respect to the weaker calculi essentially follows by Lewis's work; we provide an analogous result for the global consequence relation with respect to the stronger calculi (Theorem 3.23). It is worth stressing that the axiomatizations we introduce here are novel, and simpler than the original ones in [17].

Inspired by some results connecting modal operators and Lewis's counterfactuals (see [17, 33]), our work unveils a deep relationship between Lewis's logics and modal logic, which will guide the groundwork of this investigation. Specifically, we demonstrate how several model-theoretic techniques commonly used in standard Kripke semantics for modal logic can be successfully applied to Lewis's sphere semantics, thanks to a modal operator $\Box$ that can be term-defined in the language. This allows us to prove, for example, a deduction theorem for the strong calculus (Theorem 3.25), whereas the weak calculus is known to have the classical deduction theorem [17].

One should however keep in mind that the binary operator $\Box\!\!\rightarrow$ does not straightforwardly inherit the plethora of results on modal operators; for the reader more expert on the algebraic perspective, the models are not Boolean algebras with an operator in the usual sense (see [14]), since $\Box\!\!\rightarrow$ is not additive on both arguments (more precisely, it only distributes over infima on the right) and it cannot be recovered from a unary modal operator.

The other main results show that the stronger calculi, associated to the global consequence relation, are strongly algebraizable in the sense of Blok-Pigozzi (Corollary 4.7), i.e. there is a class of algebras axiomatized by means of equations (a variety of Boolean algebras with an extra binary operator $\Box\!\!\rightarrow$) that serve as the equivalent algebraic semantics; the weaker calculi, associated to the local consequence relation, are shown to not be algebraizable in general (there is no class of algebras whose consequence relation "corresponds" to the deduction of the logic), but they

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are the logics *preserving the degrees of truth* of the same algebraic models (Corollaries 4.19 and 4.22). Therefore, the same class of algebras can be meaningfully used to study both versions of Lewis's logics; precisely, we have strong completeness of both calculi with respect to the algebraic models. We also initiate the study of the structure theory of the algebraic models; interestingly, we demonstrate that the congruences of the algebras, which are in one-one correspondence with the deductive filters inherited by the logics, can be characterized by means of the congruences of their modal reducts (Proposition 4.12 and Corollary 4.13).

2 Lewis's Logics: new axiomatizations

This section lays out the groundwork for a logico-algebraic study of the hierarchy of logics introduced by Lewis in his seminal book [17] to reason with counterfactuals conditionals, and their intended models, i.e., sphere models. All the logics in the hierarchy are axiomatic extensions of the system $\mathtt{V}$, which according to Lewis is the "weakest system that has any claim to be called the logic of conditionals" [16, p. 80]; therefore our investigation starts with $\mathtt{V}$, and all our results will carry through its axiomatic extensions[1]. In particular, we will give a new and simpler axiomatization of $\mathtt{V}$ with respect to the original ones ([17]); we will take the counterfactual connective as a primitive symbol in the language, and we will distinguish between two different consequence relations: a weak one, where the rules of the calculus only apply to theorems (which is the one usually considered in the literature), and a strong one, where the rules instead apply to all deductions. We will see in the next section that these two choices correspond to considering two different consequence relations over sphere models: a local and a global one, in analogy with the case of modal logic.

While often in the literature $\mathtt{V}$ is presented as a set of theorems, we are interested in studying logics as consequence relations; let us be more precise. Given an algebraic language $\mathcal{L}$, we denote the set of its formulas built from a denumerable set of variables by $Fm_{\mathcal{L}}$ and the corresponding *algebra of formulas* by $\mathbf{Fm}_{\mathcal{L}}$. A *consequence relation* on $\mathbf{Fm}_{\mathcal{L}}$ is a relation $\vdash \subseteq \mathscr{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ (and we write $\Sigma \vdash \gamma$ for $(\Sigma, \gamma) \in \vdash$) such that:

1. if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;
2. if $\Gamma \vdash \delta$ for all $\delta \in \Delta$ and $\Delta \vdash \beta$, then $\Gamma \vdash \beta$.

We call *substitution* any endomorphism of $\mathbf{Fm}_{\mathcal{L}}$ (i.e., any function on itself that respects all operations); $\vdash$ is *substitution invariant* (also called *structural*) if $\Gamma \vdash \alpha$ implies $\{\sigma(\gamma) : \gamma \in \Gamma\} \vdash \sigma(\alpha)$ for each substitution σ. Finally, $\vdash$ is *finitary* if $\Gamma \vdash \alpha$ implies that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \alpha$. A *logic* $\mathtt{L}$ is an ordered pair $\mathtt{L} = (\mathcal{L}, \vdash_{\mathtt{L}})$ given by a language $\mathcal{L}$ and a substitution-invariant consequence relation $\vdash_{\mathtt{L}}$ on $\mathbf{Fm}_{\mathcal{L}}$; in this work we will actually only be interested in *finitary* logics.

Let us now fix the language $\mathcal{L}_{\Box\!\!\rightarrow}$ to be the one obtained from a denumerable set of variables and expanding the language of propositional classical logic $\{\wedge, \vee, \to, 0, 1\}$ with a binary connective $\Box\!\!\rightarrow$, where $\varphi \Box\!\!\rightarrow \psi$ should be read as

"if it were the case that φ, then it would be the case that ψ".

As usual, one can define further classical connectives by: $\neg x := x \to 0$, $\; x \leftrightarrow y := (x \to y) \wedge (y \to x)$.

[1] The system $\mathtt{V}$ from [17] is equivalent to the system $\mathtt{C0}$ in [16], essentially differing in minor differences in the language.

The following derived connectives will be also considered:

$$x \diamond\!\!\rightarrow y := \neg(x \Box\!\!\rightarrow \neg y);\quad \Box x := \neg x \Box\!\!\rightarrow x;\quad \Diamond x := \neg\Box\neg x;$$
$$x \preccurlyeq y := ((x \vee y) \diamond\!\!\rightarrow (x \vee y)) \rightarrow ((x \vee y) \diamond\!\!\rightarrow x);$$
$$x \prec y := \neg(y \preccurlyeq x);\ x \approx y := (x \preccurlyeq y) \wedge (y \preccurlyeq x).$$

Notation 2.1. We denote with $\mathbf{Fm}_{\Box\!\rightarrow}$ the algebra of $\mathcal{L}_{\Box\!\rightarrow}$-formulas over a fixed denumerable set of variables Var.

We will hence distinguish two different logics, GV and LV, which arise depending on whether we apply the rules of Lewis calculus only to theorems (for the weaker logic LV) or to all derivations (for the stronger logic GV). We remark that this distinction, although significant, is often blurred in the literature. The two systems GV and LV share the same axioms, that is, given $\varphi, \psi, \gamma \in Fm_{\Box\!\rightarrow}$ we have:

(L0) the reader's favorite Hilbert-style axioms of classical logic [2];

(L1) $\vdash \varphi \Box\!\!\rightarrow \varphi$;

(L2) $\vdash ((\varphi \Box\!\!\rightarrow \psi) \wedge (\psi \Box\!\!\rightarrow \varphi)) \rightarrow ((\varphi \Box\!\!\rightarrow \gamma) \leftrightarrow (\psi \Box\!\!\rightarrow \gamma))$;

(L3) $\vdash ((\varphi \vee \psi) \Box\!\!\rightarrow \varphi) \vee ((\varphi \vee \psi) \Box\!\!\rightarrow \psi) \vee (((\varphi \vee \psi) \Box\!\!\rightarrow \gamma) \leftrightarrow ((\varphi \Box\!\!\rightarrow \gamma) \wedge (\psi \Box\!\!\rightarrow \gamma)))$;

(L4) $\vdash (\varphi \Box\!\!\rightarrow (\psi \wedge \gamma)) \leftrightarrow ((\varphi \Box\!\!\rightarrow \psi) \wedge (\varphi \Box\!\!\rightarrow \gamma))$;

Moreover, both GV and LV satisfy modus ponens:

(MP) $\varphi, \varphi \rightarrow \psi \vdash \psi$.

While GV satisfies the following rule involving the counterfactual implication:

(C) $\varphi \rightarrow \psi \vdash (\gamma \Box\!\!\rightarrow \varphi) \rightarrow (\gamma \Box\!\!\rightarrow \psi)$,

LV satisfies the following weaker version of the rule:

(wC) $\vdash \varphi \rightarrow \psi$ implies $\vdash (\gamma \Box\!\!\rightarrow \varphi) \rightarrow (\gamma \Box\!\!\rightarrow \psi)$.

Definition 2.2. The logic GV is given by the pair $(\mathcal{L}_{\Box\!\rightarrow}, \vdash_{\mathtt{GV}})$, where $\vdash_{\mathtt{GV}}$ is the smallest finitary consequence relation satisfying all axioms [(L1)]–[(L4)], (MP), and (C). The logic LV is given by $(\mathcal{L}_{\Box\!\rightarrow}, \vdash_{\mathtt{LV}})$, where $\vdash_{\mathtt{LV}}$ is the smallest finitary consequence relation satisfying all axioms [(L1)]–[(L4)], (MP), and (wC).

Lewis's conditional logics are the axiomatic extensions of the above systems with the axioms:

(W) $\vdash (\varphi \Box\!\!\rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$; (weak centering)

(C) $\vdash ((\varphi \Box\!\!\rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)) \wedge ((\varphi \wedge \psi) \rightarrow (\varphi \Box\!\!\rightarrow \psi))$ (centering)

(N) $\vdash \Box\varphi \rightarrow \Diamond\varphi$ (normality)

(T) $\vdash \Box\varphi \rightarrow \varphi$ (total reflexivity)

[2] The reader can find some in this work by Łukasiewicz [18].

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(S) $\vdash (\varphi \boxright \psi) \vee (\varphi \boxright \neg\psi)$ (Stalnakerian)

(U) $\vdash (\Diamond\varphi \to \Box\Diamond\varphi) \wedge (\Box\varphi \to \Box\Box\varphi)$ (uniformity)

(A) $\vdash ((\varphi \preccurlyeq \psi) \to \Box(\varphi \preccurlyeq \psi)) \wedge ((\varphi \prec \psi) \to \Box(\varphi \prec \psi))$ (absoluteness)

We indicate a certain system in the family of Lewis's conditional logics by just juxtaposing to GV or LV the corresponding letter for axioms. For instance, LVCA indicates the axiomatic extension of the logic LV with the axioms C and A. Among these axiomatic extensions, it is worth mentioning the system LVC which is considered by Lewis to be the "correct logic of counterfactual conditionals" [16], while LVCS essentially corresponds to Stalnaker's logic of conditionals [29, 30].

It is clear from the definition that GV is a stronger deductive system than LV, i.e.:

Lemma 2.3. *For any set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}_{\boxright}$-formulas, $\Gamma \vdash_{\mathtt{LV}} \varphi$ implies $\Gamma \vdash_{\mathtt{GV}} \varphi$.*

While GV is strictly stronger than LV, e.g. the latter does not satisfy (*C*), they do have the same theorems; the following proof is standard.

Theorem 2.4. GV *and* LV *have the same theorems.*

Proof. It follows from Lemma 2.3 that if a formula φ is a theorem of LV, it is also a theorem of GV; we show the converse. Let φ be a theorem of GV, we show by induction on the length of the proof that φ is a theorem of LV. The base case is for φ being an axiom, thus the thesis holds given that GV and LV share the same axioms. Assume now that φ is obtained by an application of a rule of GV, i.e., either by modus ponens or (C). But such rule is applied to theorems or axioms of GV, that by inductive hypothesis are theorems of LV; therefore, one can obtain the same conclusion by applying (MP) or (wC). ◻

We will now see that the axiomatization we have given is equivalent to the one given by Lewis in [16]; with respect to the latter, we have added axioms (L4) and removed the denumerable set of rules describing "deductions within conditionals". Let us present the latter in the two versions, the strong ones:

$$\psi \vdash \varphi \boxright \psi, \qquad (\mathrm{DWC}_0)$$

$$(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi \vdash ((\gamma \boxright \varphi_1) \wedge \cdots \wedge (\gamma \boxright \varphi_n)) \to (\gamma \boxright \psi) \qquad (\mathrm{DWC}_n)$$

for each $n \in \mathbb{N}, n \geq 1$, and the weaker versions:

$$\vdash \psi \quad \text{implies} \quad \vdash \varphi \boxright \psi, \qquad (\mathrm{wDWC}_0)$$

$$\vdash (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi \quad \text{implies} \quad \vdash ((\gamma \boxright \varphi_1) \wedge \cdots \wedge (\gamma \boxright \varphi_n)) \to (\gamma \boxright \psi) \qquad (\mathrm{wDWC}_n)$$

for each $n \in \mathbb{N}, n \geq 1$. We start by noting that the monotonicity of $\boxright$ on the consequent can be shown to be a consequence of the axioms.

Lemma 2.5. *The following holds for all $\mathcal{L}_{\boxright}$-formulas φ, ψ, γ:*

1. $\vdash_{\mathtt{GV}} (\varphi \boxright \psi) \to (\varphi \boxright (\psi \vee \gamma))$.

Proof. Observe first that by the axioms and rules of classical logic, it holds that $\vdash_{\mathtt{GV}} \psi \leftrightarrow (\psi \wedge (\psi \vee \gamma))$. Therefore, by using (C) and (L4) we get

$$\vdash_{\mathtt{GV}} (\varphi \boxright \psi) \leftrightarrow (\varphi \boxright (\psi \wedge (\psi \vee \gamma))) \quad \vdash_{\mathtt{GV}} \quad (\varphi \boxright \psi) \leftrightarrow ((\varphi \boxright \psi) \wedge (\varphi \boxright (\psi \vee \gamma)))$$

from which we can derive $\vdash_{\mathtt{GV}} (\varphi \boxright \psi) \to (\varphi \boxright (\psi \vee \gamma))$, which concludes the proof. ◻

Moreover:

Lemma 2.6. *Consider a logic* $(\mathcal{L}_{\Box\!\!\rightarrow}, \vdash')$ *satisfying the axioms of classical logic, (MP), and* (DWC_2)*. Then:*

1. $\varphi \to \psi \vdash' (\gamma \Box\!\!\rightarrow \varphi) \to (\gamma \Box\!\!\rightarrow \psi)$;
2. $\varphi \leftrightarrow \psi \vdash' (\gamma \Box\!\!\rightarrow \varphi) \leftrightarrow (\gamma \Box\!\!\rightarrow \psi)$;
3. $\vdash' (\varphi \Box\!\!\rightarrow \psi) \to (\varphi \Box\!\!\rightarrow (\psi \vee \gamma))$.

Proof. For (1), from (DWC_2) we get:

$$\varphi \to \psi \vdash' (\varphi \wedge \varphi) \to \psi \vdash' ((\gamma \Box\!\!\rightarrow \varphi) \wedge (\gamma \Box\!\!\rightarrow \varphi)) \to (\gamma \Box\!\!\rightarrow \psi) \vdash' (\gamma \Box\!\!\rightarrow \varphi) \to (\gamma \Box\!\!\rightarrow \psi).$$

(2) is a direct consequence of (1). For (3) we have the following:

$$\vdash' (\psi \wedge \psi) \to (\psi \vee \gamma) \vdash' [(\varphi \Box\!\!\rightarrow \psi) \wedge (\varphi \Box\!\!\rightarrow \psi)] \to (\varphi \Box\!\!\rightarrow (\psi \vee \gamma)),$$

which, given that $\vdash' \psi \leftrightarrow (\psi \wedge \psi)$, proves the claim. □

Proposition 2.7. *Consider a logic* $(\mathcal{L}_{\Box\!\!\rightarrow}, \vdash')$ *satisfying the axioms of classical logic and (MP). The following are equivalent.*

1. $\vdash'$ *satisfies* (L1)–(L4) *and* (C);
2. $\vdash'$ *satisfies* (L1)–(L3) *and the rule* (DWC_2).

The same holds replacing (C)*, and* (DWC_2) *with their weaker versions* (wC)*, and* (wDWC_2).

Proof. Let us first show that (1) implies (2), i.e., we derive the rule (DWC_2) using (L4) and (C). Using (C) we obtain that $(\varphi_1 \wedge \varphi_2) \to \psi \vdash' [\gamma \Box\!\!\rightarrow (\varphi_1 \wedge \varphi_2)] \to (\gamma \Box\!\!\rightarrow \psi)$; using now (L4) we get

$$(\varphi_1 \wedge \varphi_2) \to \psi \vdash' [(\gamma \Box\!\!\rightarrow \varphi_1) \wedge (\gamma \Box\!\!\rightarrow \varphi_2)] \to (\gamma \Box\!\!\rightarrow \psi)$$

which is exactly (DWC_2).

Conversely, let us prove that (DWC_2) implies (C) and (L4); (C) can be derived by (DWC_2) by setting $\varphi_1 = \varphi_2 := \varphi$; for (L4), by the axioms and rules of classical logic and (DWC_2) we get:

$$\vdash' (\psi \wedge \gamma) \to (\psi \wedge \gamma) \ \vdash' \ ((\varphi \Box\!\!\rightarrow \psi) \wedge (\varphi \Box\!\!\rightarrow \gamma)) \to (\varphi \Box\!\!\rightarrow (\psi \wedge \gamma));$$

moreover, one can use monotonicity of $\Box\!\!\rightarrow$ (Lemma 2.5) and the fact that classically equivalent formulas can be substituted in the consequent of $\Box\!\!\rightarrow$ (Lemma 2.6) and obtain

$$\vdash' (\varphi \Box\!\!\rightarrow (\psi \wedge \gamma)) \to (\varphi \Box\!\!\rightarrow \psi) \quad \text{and} \quad \vdash' (\varphi \Box\!\!\rightarrow (\psi \wedge \gamma)) \to (\varphi \Box\!\!\rightarrow \gamma),$$

and therefore $\vdash' (\varphi \Box\!\!\rightarrow (\psi \wedge \gamma)) \to ((\varphi \Box\!\!\rightarrow \psi) \wedge (\varphi \Box\!\!\rightarrow \gamma))$, which shows (L4).

The proof for the statement involving the weaker rules goes similarly, with the proviso that every derivation starts from theorems and axioms. □

The next theorem shows that both the alternative axiomatizations we have introduced are equivalent to the one presented by Lewis's in [16].

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Theorem 2.8. *Consider a logic* $(\mathcal{L}_{\Box\!\!\rightarrow}, \vdash')$ *satisfying the axioms of classical logic and (MP). The following are equivalent.*

1. $\vdash'$ *satisfies* (L1)–(L3) *and* (DWC$_n$) *for all* $n \in \mathbb{N}$;
2. $\vdash'$ *satisfies* (L1)–(L3) *and* (DWC$_2$);
3. $\vdash'$ *coincides with* $\vdash_{\mathtt{GV}}$.

The same holds replacing (DWC$_2$), (DWC$_n$), *and* $\vdash_{\mathtt{GV}}$ *with their weakened versions* (wDWC$_2$), (wDWC$_n$), *and* $\vdash_{\mathtt{LV}}$.

Proof. (2) and (3) are equivalent by Proposition 2.7; moreover, it is obvious that (1) implies (2); let us show the converse. (DWC$_1$) follows from (DWC$_2$) by setting $\varphi_1 = \varphi_2$. Now, consider $n \in \mathbb{N}, n \geq 2$, then with (DWC$_2$) we obtain immediately:

$$((\varphi_1 \wedge \cdots \wedge \varphi_{n-1}) \wedge \varphi_n) \to \psi \vdash [((\gamma \Box\!\!\rightarrow (\varphi_1 \wedge \ldots \wedge \varphi_{n-1})) \wedge (\gamma \Box\!\!\rightarrow \varphi_n)] \to (\gamma \Box\!\!\rightarrow \psi),$$

which using the fact that (L4) holds by Proposition 2.7, yields that (DWC$_n$) holds for all $n \geq 2$. In order to show that (DWC$_0$) holds, we first observe that $\varphi \Box\!\!\rightarrow 1$ is a theorem, indeed by (L4) we get

$$\vdash' (\varphi \Box\!\!\rightarrow (\varphi \wedge 1)) \leftrightarrow ((\varphi \Box\!\!\rightarrow \varphi) \wedge (\varphi \Box\!\!\rightarrow 1)) \vdash' (\varphi \Box\!\!\rightarrow \varphi) \leftrightarrow ((\varphi \Box\!\!\rightarrow \varphi) \wedge (\varphi \Box\!\!\rightarrow 1)) \vdash' 1 \leftrightarrow (\varphi \Box\!\!\rightarrow 1)$$

where in the derivations we used (L1) and substitution of classically equivalent formulas in the consequent of $\Box\!\!\rightarrow$ (Lemma 2.6). Finally, (DWC$_0$) is then a consequence of applying (DWC$_2$) with $\varphi_1 = \varphi_2 := 1, \psi := \psi, \gamma := \varphi$.

The proof for the weaker calculus is completely analogous. □

Lastly, let us observe that both GV and LV satisfy the substitution of logical equivalents, in the following sense.

Proposition 2.9. *The following hold for any* $\varphi, \psi, \gamma \in Fm_{\Box\!\!\rightarrow}$:

1. $(\varphi \leftrightarrow \psi) \vdash_{\mathtt{GV}} (\gamma \Box\!\!\rightarrow \varphi) \leftrightarrow (\gamma \Box\!\!\rightarrow \psi)$,
2. $(\varphi \leftrightarrow \psi) \vdash_{\mathtt{GV}} (\varphi \Box\!\!\rightarrow \gamma) \leftrightarrow (\psi \Box\!\!\rightarrow \gamma)$,
3. $\vdash_{\mathtt{LV}} (\varphi \leftrightarrow \psi)$ *implies* $\vdash_{\mathtt{LV}} (\gamma \Box\!\!\rightarrow \varphi) \leftrightarrow (\gamma \Box\!\!\rightarrow \psi)$ *and* $\vdash_{\mathtt{LV}} (\varphi \Box\!\!\rightarrow \gamma) \leftrightarrow (\psi \Box\!\!\rightarrow \gamma)$.

Proof. First, notice that (1) follows by (C). Moreover, since (DWC$_1$) holds by Theorem 2.8, one gets

$$\varphi \to \psi \vdash_{\mathtt{GV}} (\varphi \Box\!\!\rightarrow \varphi) \to (\varphi \Box\!\!\rightarrow \psi) \vdash_{\mathtt{GV}} \varphi \Box\!\!\rightarrow \psi$$

and its symmetric copy; thus it follows $\varphi \leftrightarrow \psi \vdash_{\mathtt{GV}} (\varphi \Box\!\!\rightarrow \psi) \wedge (\psi \Box\!\!\rightarrow \varphi)$, which via (L2) gives (2). Finally, (3) follows from the previous points, given that GV and LV have the same theorems (Theorem 2.4). □

3 Sphere models

Lewis bases his interpretation of the counterfactual connective $\Box\!\!\rightarrow$ on a neighbourhood-style semantics. The intuitive idea to evaluate the connective $\Box\!\!\rightarrow$ is that $\varphi \Box\!\!\rightarrow \psi$ is true at some world w if in the closest worlds to w in which φ is true, also ψ is true. This results in the definition of what Lewis calls a "variably strict conditional", where the word "variably" stresses the fact that to evaluate counterfactuals with different antecedents at some world w, one might need to evaluate the corresponding classical implication in different worlds; from another point of view, this also means that, in general, a counterfactual $\varphi \Box\!\!\rightarrow \psi$ does not arise as some $\Box(\varphi \to \psi)$, for some modality $\Box$. Lewis formalizes this intuition by assigning to each possible world w a nested set of *spheres*, which are subsets of possible worlds, meant to describe a similarity relationship with w; the smaller is the sphere to which a world w' belongs, the closer, and therefore more similar, it is to w.

In this section we will introduce two different consequence relations over sphere models: a local and a global one, in complete analogy with the case of modal logic. This parallel will continue and guide the investigation throughout the rest of this section. In particular, we will use the tool of generated submodels, borrowed from modal logic (see [4]), and apply it to sphere models to first characterize the global consequence relation via the local one, secondly to prove a deduction theorem, and finally to prove the strong completeness of the global consequence with respect to the strong version of the presented Hilbert-style calculus. Let us now be more precise.

Definition 3.1. A sphere model $\mathcal{M}$ is a tuple $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ where:

1. W is a set of possible worlds;
2. $\mathcal{S} : W \to \mathscr{P}(\mathscr{P}(W))$ is a function assigning to each $w \in W$ a non-empty[3] set $\mathcal{S}(w)$ of nested subsets of W, i.e., for all $X, Y \in \mathcal{S}(w)$, either $X \subseteq Y$ or $Y \subseteq X$.
3. $v : Var \to \mathscr{P}(W)$ is an assignment of the variables to subsets of W, extending to all $\mathcal{L}_{\Box\!\rightarrow}$-formulas as follows:

$$
\begin{array}{lcl}
v(0) & = & \emptyset \\
v(1) & = & W \\
v(\varphi \wedge \psi) & = & v(\varphi) \cap v(\psi) \\
v(\varphi \vee \psi) & = & v(\varphi) \cup v(\psi) \\
v(\varphi \to \psi) & = & (W \setminus v(\varphi)) \cup v(\psi) \\
v(\varphi \Box\!\!\rightarrow \psi) & = & \{w \in W : (\bigcup \mathcal{S}(w) \cap v(\varphi)) = \emptyset \text{ or } \exists S \in \mathcal{S}(w) \text{ such that } \emptyset \neq (S \cap v(\varphi)) \subseteq v(\psi)\}
\end{array}
$$

Given a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$, and a set of $\mathcal{L}_{\Box\!\rightarrow}$-formulas Γ, we set:

$$w \Vdash \Gamma \text{ iff } w \in \bigcap\{v(\gamma) : \gamma \in \Gamma\}; \tag{1}$$

$$\mathcal{M} \Vdash \Gamma \text{ iff for all } w \in W,\ w \Vdash \Gamma. \tag{2}$$

Notation 3.2. If $\Gamma = \{\gamma\}$, we drop the parentheses and write $w \Vdash \gamma$ (or $\mathcal{M} \Vdash \gamma$) instead of $w \Vdash \{\gamma\}$ (or $\mathcal{M} \Vdash \{\gamma\}$). Moreover, since in what follows we will be dealing with some *submodels*, it is sometimes convenient to stress to which universe a world belongs; given a sphere model $\mathcal{M} = (W, \mathcal{S}, v)$, $w \in W$, we then write

$$\mathcal{M}, w \Vdash \varphi \text{ iff } w \Vdash \varphi. \tag{3}$$

[3] Note that given $w \in W$, it can be that $S(w) = \{\emptyset\}$, since $\{\emptyset\} \subseteq \mathscr{P}(W)$.

The theorems of GV and LV (which are the same by Theorem 2.4) are exactly the set of formulas true in all sphere models, i.e. the set of $\mathcal{L}_{\Box\!\!\rightarrow}$-formulas φ such that $\mathcal{M} \Vdash \varphi$ for all sphere models $\mathcal{M}$ (see Theorem 3.8 below).

3.1 Local and global semantics

We now introduce two natural notions of semantical consequence, in close analogy with the local and global consequence relations of modal logic, and we will see by the end of this section that they are sound and (strongly) complete with respect to LV and GV respectively.

Definition 3.3. Let $\mathfrak{S}$ be a class of sphere models.

1. We define the *global* $\mathfrak{S}$-consequence relation on sphere models as: $\Gamma \vDash_{\mathfrak{S},g} \varphi$ if and only if for all sphere models $\mathcal{M} \in \mathfrak{S}$, $\mathcal{M} \Vdash \Gamma$ implies $\mathcal{M} \Vdash \varphi$.
2. We define the *local* $\mathfrak{S}$-consequence relation on sphere models as: $\Gamma \vDash_{\mathfrak{S},\ell} \varphi$ if and only if for all sphere models $\mathcal{M} = \langle W, \mathcal{S}, v\rangle \in \mathfrak{S}$ and all $w \in W$, $w \Vdash \Gamma$ implies $w \Vdash \varphi$.

Notation 3.4. When $\mathfrak{S}$ is the class of all sphere models we write $\vDash_g$ for $\vDash_{\mathfrak{S},g}$ and $\vDash_\ell$ for $\vDash_{\mathfrak{S},\ell}$.

The following is a direct consequence of the definitions.

Theorem 3.5. *Given any $\mathcal{L}_{\Box\!\!\rightarrow}$-formula φ, $\vDash_g \varphi$ if and only if $\vDash_l \varphi$.*

Lewis [17] considers the classes of sphere models corresponding to the main axiomatic extensions of the system V; those classes are listed in the following definition:

Definition 3.6. Let $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ be a sphere model.

1. $\mathcal{M}$ is *normal* if for each $w \in W$, $\bigcup \mathcal{S}(w) \neq \emptyset$.
2. $\mathcal{M}$ is *totally reflexive* if for each $w \in W$, $w \in \bigcup \mathcal{S}(w)$.
3. $\mathcal{M}$ is *weakly centered* if for each $w \in W$, $w \in S$ for each nonempty $S \in \mathcal{S}(w)$, and there is a nonempty $S \in \mathcal{S}(w)$.
4. $\mathcal{M}$ is *centered* if for each $w \in W$, $\{w\} \in \mathcal{S}(w)$.
5. $\mathcal{M}$ is *Stalnakerian* if for any $w \in W$, and any $\mathcal{L}_{\Box\!\!\rightarrow}$-formula φ such that $v(\varphi) \cap \bigcup S(w) \neq \emptyset$, there is some $S \in S(w)$ and $y \in W$ such that $v(\varphi) \cap S = \{y\}$.
6. $\mathcal{M}$ is *locally uniform* if for any $w \in W$ and $v \in \bigcup \mathcal{S}(w)$, $\bigcup \mathcal{S}(w) = \bigcup \mathcal{S}(v)$.
7. $\mathcal{M}$ is *locally absolute* if for any $w \in W$ and $v \in \bigcup \mathcal{S}(w)$, $\mathcal{S}(w) = \mathcal{S}(v)$.
8. $\mathcal{M}$ is *weakly trivial* if for each $w \in W$, W is the only non-empty member of $\mathcal{S}(w)$.
9. $\mathcal{M}$ is *trivial* if W is a singleton $\{w\}$, and $\mathcal{S}(w) = \{\emptyset, \{w\}\}$.

Notice that if $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ is centered, $\{w\}$ is the smallest (nonempty) sphere of $S(w)$ for all $w \in W$. Stalnakerian sphere models are complete with respect to Stalnaker's logic of conditionals [29, 17].

Notation 3.7. It will be convenient to set the following notation. Given an axiomatic extension of $\mathtt{LV}$ by Σ, where Σ is a subset of the axioms $\{\mathtt{W},\mathtt{C},\mathtt{N},\mathtt{T},\mathtt{S},\mathtt{U},\mathtt{A}\}$, we denote by $\mathfrak{S}_\Sigma$ the corresponding class of sphere models, defined by the corresponding conditions in Definition 3.6.

In particular then, we denote the class of spheres that are: normal by $\mathfrak{S}_{\mathtt{N}}$, totally reflexive by $\mathfrak{S}_{\mathtt{T}}$, weakly centered by $\mathfrak{S}_{\mathtt{W}}$, centered by $\mathfrak{S}_{\mathtt{C}}$, Stalnakerian by $\mathfrak{S}_{\mathtt{S}}$, locally uniform by $\mathfrak{S}_{\mathtt{U}}$, absolute by $\mathfrak{S}_{\mathtt{A}}$. Moreover, Lewis shows that weak triviality corresponds to the combination of axioms $\mathtt{WA}$ thus we denote the class by $\mathfrak{S}_{\mathtt{WA}}$, and triviality is analogously linked to $\mathtt{CA}$ so the corresponding class of models is $\mathfrak{S}_{\mathtt{CA}}$.

Lewis demonstrates in [17, §6.1] that the logic $\mathtt{LV}$ and its extensions by the axioms $\{\mathtt{W},\mathtt{C},\mathtt{N},\mathtt{T},\mathtt{S},\mathtt{U},\mathtt{A}\}$ are sound and complete with respect to the corresponding classes of sphere models with the local consequence relation. This means that the theorems in a logic precisely correspond to the validities over the associated class of sphere models. Actually, Lewis's proof can be straightforwardly extended to show *strong completeness*, i.e., to consider derivations instead of just theorems. Indeed, Lewis proves in particular the following fact. Given any logic $\mathtt{LV}\Sigma$, for $\Sigma \subseteq \{\mathtt{W},\mathtt{C},\mathtt{N},\mathtt{T},\mathtt{S},\mathtt{U},\mathtt{A}\}$, and any set of $\mathcal{L}_{\Box\!\!\rightarrow}$-formulas Γ that is $\mathtt{LV}\Sigma$-*consistent*, i.e., such that $\Gamma \nvdash_{\mathtt{LV}\Sigma} 0$, there is a sphere model $\mathcal{M}$ in $\mathfrak{S}_\Sigma$ such that $\mathcal{M} \Vdash \Gamma$.[4] This is enough to show not only completeness, but strong completeness with standard arguments (see e.g., [4, Proposition 4.12]).

Theorem 3.8 (Strong Completeness and Soundness of the Local Consequence [17])**.** *Consider any logic* $\mathtt{LV}\Sigma$ *for* $\Sigma \subseteq \{\mathtt{W},\mathtt{C},\mathtt{N},\mathtt{T},\mathtt{S},\mathtt{U},\mathtt{A}\}$*. Then for all* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_{\Box\!\!\rightarrow}}$*,*

$$\Gamma \vdash_{\mathtt{LV}\Sigma} \varphi \Leftrightarrow \Gamma \vDash_{\mathfrak{S}_\Sigma, l} \varphi.$$

We will show that the global consequence relation can be characterized by means of the local one. In order to do that, we introduce a useful tool. In close analogy to the case of Kripke models, we will see how to manipulate a sphere model in order to obtain a new model, preserving the satisfaction of formulas. Namely, we will prove some invariance results for Lewis's sphere semantics of counterfactuals.

Definition 3.9. Let $\Sigma = \langle W, \mathcal{S}, v\rangle$ and $\Sigma' = \langle W', \mathcal{S}', v'\rangle$ two sphere models. We say that Σ' is a *submodel* of Σ if and only if:

1. $W' \subseteq W$
2. $\mathcal{S}'$ is the restriction of $\mathcal{S}$ to W', i.e. for all $w \in W'$, $\mathcal{S}'(w) = \mathcal{S}(w) \cap \mathscr{P}(\mathscr{P}(W'))$.
3. v' is the restriction of v to W', i.e. for any $\mathcal{L}_{\Box\!\!\rightarrow}$-formula φ, $v'(\varphi) = v(\varphi) \cap W'$.

We now consider a special class of submodels, namely *generated submodels*.

Definition 3.10. Let $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ and $\mathcal{M}' = \langle W', \mathcal{S}', v'\rangle$ be two sphere models. We say that $\mathcal{M}'$ is a *generated submodel* of $\mathcal{M}$ if $\mathcal{M}'$ is a submodel of $\mathcal{M}$ such that for all $w' \in W'$, $\mathcal{S}'(w') = \mathcal{S}(w')$.

[4] More precisely, Lewis's proofs consider an alternative, but equivalent, presentation of his logics where a different connective, $\preceq$, is taken as primitive instead of $\Box\!\!\rightarrow$. Moreover, he uses a technique that is widely used to show completeness in modal logics, that is, he constructs the *canonical model* of each logic, showing that it belongs to the corresponding class of spheres. See [17, §6.1] for details, and [4] for more on this technique and its use to prove strong completeness in modal logics.

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In other words, in order to obtain a generated submodel of some sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ one needs to select a subset $W' \subseteq W$ in such a way that, for each $w' \in W'$, all the worlds belonging to the spheres of w' also belong to W'. This particular type of generated submodel will play a key role in our analysis. Let us show how one can effectively construct one such submodel.

Consider a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ and a subset $X \subseteq W$. Let us define a binary relation $R_{\mathcal{S}} \subseteq W \times W$ as follows: for all $w, u \in W$,

$$w\, R_{\mathcal{S}}\, u \Leftrightarrow u \in \bigcup \mathcal{S}(w) \tag{4}$$

Thus, $wR_{\mathcal{S}}u$ if and only if u appears in the system of spheres associated to w. Now, for all $n \in \mathbb{N}$, we inductively define a relation $R^n_{\mathcal{S}} \subseteq W \times W$ in the following way:

- $wR^0_{\mathcal{S}}u \Leftrightarrow w = u$
- $wR^{n+1}_{\mathcal{S}}u \Leftrightarrow$ there is $z \in W$ such that $wR^n_{\mathcal{S}}z$ and $zR_{\mathcal{S}}u$.

We refer to $R_{\mathcal{S}}$ as the *accessibility relation of* $\mathcal{M}$. Intuitively, the relation $wR^n_{\mathcal{S}}u$ indicates that there are n steps needed to reach the world u starting from w, where every steps is given by checking a set of spheres. Now, consider the subset $W_X \subseteq W$ defined as follows:

$$W_X = \{w \in W : uR^n_{\mathcal{S}}w \text{ for some } n \in \mathbb{N} \text{ and } u \in X\} \tag{5}$$

Namely, W_X is the set of all worlds in W that are reachable from a member of X by a finite number of steps via the accessibility relation $R_{\mathcal{S}}$. We shall now define the sphere model

$$\mathcal{M}_X = \langle W_X, \mathcal{S}_X, v_X\rangle \tag{6}$$

where:

- $\mathcal{S}_X$ is the restriction of $\mathcal{S}$ to W_X, i.e. for all $w \in W_X$, $\mathcal{S}_X(w) = \mathcal{S}(w) \cap \mathscr{P}(\mathscr{P}(W_X))$.
- v_X is the restriction of v to W_X, i.e. $v_X(\varphi) = v(\varphi) \cap W_X$ for all $\mathcal{L}_{\Box\!\!\rightarrow}$-formulas φ.

We say that a submodel $\mathcal{M}'$ of $\mathcal{M}$ is *smaller* than another submodel $\mathcal{M}^*$ if the domain of $\mathcal{M}'$ is contained in the domain of $\mathcal{M}^*$. Then:

Lemma 3.11. *Let $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ be a sphere model, $X \subseteq W$, and let $\mathcal{M}_X$ be defined as above. Then $\mathcal{M}_X$ is the smallest generated submodel of $\mathcal{M}$ whose domain contains X.*

Proof. First observe that $\mathcal{M}_X$ is a submodel of $\mathcal{M}$ by definition; it is also easily seen to be a generated submodel of $\mathcal{M}$. Indeed by the definition of W_X the following holds: if $w \in W_X$ and $wR_{\mathcal{S}}u$, then $u \in W_X$. Equivalently,

$$\text{if } w \in W_X \text{ and } u \in \bigcup \mathcal{S}(w), \text{ then } u \in W_X,$$

i.e. $\mathcal{S}_X(w) = \mathcal{S}(w) \cap \mathscr{P}(\mathscr{P}(W_X)) = \mathcal{S}_X(w)$ and then $\mathcal{M}_X$ is a generated submodel of $\mathcal{M}$ by definition. By the very same equivalence and the definition of W_X, it follows that if $\mathcal{M}^* = \langle W^*, \mathcal{S}^*, v^*\rangle$ is any other generated submodel of $\mathcal{M}$ such that $X \subseteq W^*$, necessarily W_X is contained in W^*. Therefore, $\mathcal{M}_X$ is the smallest submodel of $\mathcal{M}$ generated by X. □

Definition 3.12. Consider a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$, and let $X \subseteq W$; we call *submodel generated by X* the smallest submodel of $\mathcal{M}$ whose domain contains X. Moreover, we call *centered* or *point-generated* a submodel of $\mathcal{M}$ generated by a singleton.

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Notice that by Lemma 3.11 above, the submodel of $\mathcal{M}$ generated by X is exactly $\mathcal{M}_X$. Importantly, all generated submodels preserve the validity of formulas, as the following lemma shows.

Lemma 3.13. *Let $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ be a sphere model, and let $\mathcal{M}' = \langle W', \mathcal{S}', v'\rangle$ be a generated submodel of $\mathcal{M}$. The following holds for all $w \in W'$, and all $\mathcal{L}_{\Box\!\!\rightarrow}$-formulas φ:*

$$\mathcal{M}, w \Vdash \varphi \Leftrightarrow \mathcal{M}', w \Vdash \varphi$$

Proof. The statement can be easily proved by induction on the construction of the formula φ. In particular, the base case where φ is a variable and the inductive cases given by the classical connectives (i.e. $\varphi = \psi * \gamma$ for $* \in \{\wedge, \vee, \rightarrow\}$) directly follow from the fact that v' is the restriction of v to W'. The inductive case $\varphi = \psi \Box\!\!\rightarrow \gamma$ follows from the fact that v' is the restriction of v to W' and that $\mathcal{S}'(w) = \mathcal{S}(w)$ for all $w \in W'$. □

Moreover, the following is a direct consequence of the definitions.

Lemma 3.14. *All the classes of spheres in Definition 3.6 are closed under generated submodels.*

Before delving into the relationship between local and global consequences, we provide a first application of generated submodels. Lewis in [17] considers three additional classes of sphere models, that we have not included in Definition 3.6; let us consider them now.

Definition 3.15. Let $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ be a sphere model, then:

1. $\mathcal{M}$ is *uniform* if for any $w, v \in W$, $\bigcup \mathcal{S}(w) = \bigcup \mathcal{S}(v)$.
2. $\mathcal{M}$ is *absolute* if for any $w, v \in W$, $\mathcal{S}(w) = \mathcal{S}(v)$.
3. $\mathcal{M}$ is *universal* if for each $w \in W$, $\bigcup \mathcal{S}(w) = W$.

We denote by $\mathfrak{S}_{\mathtt{U}^+}$ the class of *uniform* models, by $\mathfrak{S}_{\mathtt{A}^+}$ the class of *absolute* models, and by $\mathfrak{S}_{\mathtt{UT}}$ the class of *universal* models.

As the reader can easily check, uniformity implies local uniformity, and absoluteness implies local absoluteness. Lewis observes in [17] that the validity of formulas does not change between the classes of, respectively, locally uniform and uniform models, and locally absolute and absolute ones. We demonstrate that such classes of models are indistinguishable also from the point of view of derivations.

Proposition 3.16. *The following hold for any $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_{\Box\!\!\rightarrow}}$:*

1. $\Gamma \vDash_{\mathfrak{S}_{\mathtt{U},l}} \varphi \Leftrightarrow \Gamma \vDash_{\mathfrak{S}_{\mathtt{U}^+,l}} \varphi$ *and* $\Gamma \vDash_{\mathfrak{S}_{\mathtt{U},g}} \varphi \Leftrightarrow \Gamma \vDash_{\mathfrak{S}_{\mathtt{U}^+,g}} \varphi$
2. $\Gamma \vDash_{\mathfrak{S}_{\mathtt{A},l}} \varphi \Leftrightarrow \Gamma \vDash_{\mathfrak{S}_{\mathtt{A}^+,l}} \varphi$ *and* $\Gamma \vDash_{\mathfrak{S}_{\mathtt{A},g}} \varphi \Leftrightarrow \Gamma \vDash_{\mathfrak{S}_{\mathtt{A}^+,g}} \varphi$

Proof. We prove (1), the proof of (2) being similar. Let us focus on the local consequence first. The ($\Rightarrow$) direction is straightforward since every *uniform* models is also *locally uniform*. For the ($\Leftarrow$) direction, we reason by contraposition and assume that $\Gamma \nvDash_{\mathfrak{S}_{\mathtt{U},l}} \varphi$, namely there is a locally uniform sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ and a $w \in W$ such that $w \Vdash \Gamma$ and $w \nVdash \varphi$. Now, consider the submodel generated by $\{w\}$, $\mathcal{M}_w = \langle W_w, \mathcal{S}_w, v_w\rangle$ and observe that all $x \in W_w$ are such that $wR^n_{\mathcal{S}}x$ for some $n \in \mathbb{N}$, by Definition 3.12. We prove by induction on n that for all $x \in W_w$ such that $wR^n_{\mathcal{S}}x$, $\bigcup \mathcal{S}(x) = \bigcup \mathcal{S}(w)$. For the base case, if $wR^0_{\mathcal{S}}x$ we have that $x = w$ by definition of $R^0_{\mathcal{S}}$, and

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then clearly $\bigcup \mathcal{S}(x) = \bigcup \mathcal{S}(w)$. For the inductive step, by inductive hypothesis we have that for all y such that $wR^n_{\mathcal{S}}y$, $\bigcup \mathcal{S}(w) = \bigcup \mathcal{S}(y)$. Assume that $wR^{n+1}_{\mathcal{S}}x$; then by definition of $R^{n+1}_{\mathcal{S}}$, we have that $wR^n_{\mathcal{S}}yR_{\mathcal{S}}x$ for some y, and so $x \in \bigcup \mathcal{S}(y)$. Since the original model $\mathcal{M}$ is locally uniform, we have that $\bigcup \mathcal{S}(y) = \bigcup \mathcal{S}(x)$, which yields that $\bigcup \mathcal{S}(w) = \bigcup \mathcal{S}(x)$. This proves that $\mathcal{M}_w$ is uniform. Moreover, by Lemma 3.13, we have that $\mathcal{M}_w, w \Vdash \Gamma$ and $\mathcal{M}_w, w \nVdash \varphi$, hence $\Gamma \nvDash_{\mathfrak{S}_{\mathrm{U}^+},l} \varphi$. The proof for the global consequence proceeds analogously. ◻

Additionally, as Lewis [17, p.120] himself noted, the class of universal sphere models corresponds to the class of models that are both uniform and totally reflexive. Therefore, for the purpose of the present work, it is sufficient to confine our attention to the class of models presented in Definition 3.6. Nonetheless, in light of the previous observations, is important to note that all of our results can be easily extended to include these other classes of models as well.

After this brief digression, we can continue towards the main focus of this section. Before presenting the characterization of the global consequence relation by means of the local consequence, we need another technical result. Observe that, given a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ and $w \in W$, $w \Vdash \neg\varphi \mathrel{\Box\!\!\rightarrow} \varphi$ if and only if $\bigcup \mathcal{S}(w) \cap v(\neg\varphi) = \emptyset$, or equivalently $\bigcup \mathcal{S}(w) \subseteq v(\varphi)$. Recall that $\Box\varphi := \neg\varphi \mathrel{\Box\!\!\rightarrow} \varphi$. It is then straightforward that, given a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$, $\Box$ can be characterized by means of the relation $R_{\mathcal{S}}$ defined in (4) above.

Lemma 3.17. *Let $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ be a sphere model; given any $w \in W$ and $\mathcal{L}_{\Box\rightarrow}$-formula φ, the following are equivalent:*

1. *$w \Vdash \Box\varphi$;*
2. *$\bigcup \mathcal{S}(w) \subseteq v(\varphi)$;*
3. *$wR_{\mathcal{S}}u$ implies $u \Vdash \varphi$.*

One can then easily show that $\Box$ is a modal operator, in the following sense.

Proposition 3.18. *The following hold for all $\varphi, \psi \in Fm_{\Box\rightarrow}$:*

1. *$\vDash_g \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;*
2. *$\varphi \vDash_g \Box\varphi$;*
3. *$\vDash_l \varphi$ implies $\vDash_l \Box\varphi$;*
4. *$\vDash_g \Box(\varphi \land \psi) \leftrightarrow (\Box\varphi \land \Box\psi)$.*

Proof. Let us start with (1); consider any sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$, and let $w \in W$. Suppose $w \Vdash \Box(\varphi \to \psi)$, i.e. by Lemma 3.17 $\bigcup \mathcal{S}(w) \subseteq v(\varphi \to \psi)$. Therefore, if $w \Vdash \Box\varphi$, or equivalently $\bigcup \mathcal{S}(w) \subseteq v(\varphi)$, it follows that $\bigcup \mathcal{S}(w) \subseteq v(\psi)$; applying Lemma 3.17 again, we get that $w \Vdash \Box\psi$, which proves the claim.

Let us now prove (2); one needs to show that for all sphere models $\mathcal{M} = (W, S, v)$, $\mathcal{M} \Vdash \varphi$ implies $\mathcal{M} \Vdash \Box\varphi$, which is an easy consequence of Lemma 3.17, since if $\mathcal{M} \Vdash \varphi$, then for every w, $w \in v(\varphi)$. (3) can be proved analogously, while (4) follows from the previous points. ◻

Let us define inductively an operator $\Box^n$, that iterates $\Box$, for $n \in \mathbb{N}$:

$$\Box^0\varphi := \varphi; \qquad \Box^{n+1}\varphi := \Box\Box^n\varphi. \tag{7}$$

We are now ready to characterize the connection between local and global consequence relations.

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Theorem 3.19. *Let $\mathfrak{S}$ be a class of sphere models closed under generated submodels. For all sets of $\mathcal{L}_{\Box\to}$-formulas Γ and $\mathcal{L}_{\Box\to}$-formula φ the following are equivalent:*

1. $\Gamma \vDash_{\mathfrak{S},g} \varphi$;
2. $\{\Box^n\gamma : n \in \mathbb{N}, \gamma \in \Gamma\} \vDash_{\mathfrak{S},l} \varphi$.

Proof. We verify (2) implies (1) by contraposition. Assume $\Gamma \nvDash_{\mathfrak{S},g} \varphi$; i.e., there is a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle \in \mathfrak{S}$ such that $w \Vdash \gamma$ for all $w \in W, \gamma \in \Gamma$, and for some $u \in W$, $u \nVdash \varphi$. By the definition of $\Box^n$ and Lemma 3.17, it follows that $w \Vdash \Box^n\gamma$ for all $\gamma \in \Gamma, n \in \mathbb{N}$ and $w \in W$. Thus in particular $u \Vdash \Box^n\gamma$ for all $\gamma \in \Gamma, n \in \mathbb{N}$, but $u \nVdash \varphi$. Therefore $\{\Box^n\gamma \mid n \in \mathbb{N} \text{ and } \gamma \in \Gamma\} \nvDash_{\mathfrak{S},l} \varphi$.

We now prove that (1) implies (2), again by contraposition; assume that $\{\Box^n\gamma \mid n \in \mathbb{N} \text{ and } \gamma \in \Gamma\} \nvDash_{\mathfrak{S},l} \varphi$. Thus, there is a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle \in \mathfrak{S}$ and $x \in W$ such that $x \Vdash \Box^n\gamma$ for all $n \in \mathbb{N}$ and $\gamma \in \Gamma$ but $x \nVdash \varphi$. Consider the submodel generated by $\{x\}$, $\mathcal{M}_x = \langle W_x, \mathcal{S}_x, v_x\rangle$, where

$$W_x = \{w \in W : xR^n_{\mathcal{S}}w \text{ for some } n \in \mathbb{N}\}.$$

By Lemma 3.11, $\mathcal{M}_x$ is a sphere model and it is in $\mathfrak{S}$ since $\mathfrak{S}$ is closed under generated submodels by assumption; moreover, by Lemma 3.13 we have that for all $w \in W_x$

$$\mathcal{M}, w \Vdash \varphi \text{ if and only if } \mathcal{M}_x, w \Vdash \varphi.$$

Hence, in particular, $\mathcal{M}_x, x \Vdash \Box^n\gamma$ for all $n \in \mathbb{N}, \gamma \in \Gamma$ but $\mathcal{M}_x, x \nVdash \varphi$. We now prove that for all $w \in W_x$, $w \Vdash \Gamma$, which will conclude the proof by showing that $\mathcal{M}_x \Vdash \Gamma$ but $\mathcal{M}_x \nVdash \varphi$ (since $\mathcal{M}_x, x \nVdash \varphi$). By definition, all elements $w \in W_x$ are such that $xR^m_{\mathcal{S}}w$ for some $m \in \mathbb{N}$; we show by induction on k that $xR^k_{\mathcal{S}}w$ implies $w \Vdash \Box^n\gamma$ for all $n \in \mathbb{N}, \gamma \in \Gamma$.

- If $k = 0$, we get $w = x$ and thus by assumption $x \Vdash \Box^n\gamma$ for all $\gamma \in \Gamma, n \in \mathbb{N}$.
- Assume that the inductive hypothesis holds for k, we show it for $k+1$. Suppose $xR^{k+1}_{\mathcal{S}}w$, i.e. by definition of $R^{k+1}_{\mathcal{S}}$, there is some $z \in W_x$ such that $x\, R^k_{\mathcal{S}}\, z\, R_{\mathcal{S}}\, w$. By inductive hypothesis $z \Vdash \Box^n\gamma$ for all $n \in \mathbb{N}, \gamma \in \Gamma$. Thus Lemma 3.17 implies that also $w \Vdash \Box^n\gamma$ for all $n \in \mathbb{N}, \gamma \in \Gamma$.

Therefore, we have shown that, in particular, all elements $w \in W_x$ are such that $w \Vdash \Box^0\gamma = \gamma$ for all $\gamma \in \Gamma$, which concludes the proof. ◻

We will now use the last result to prove a deduction theorem and a strong completeness result for the global consequence relation.

3.2 Completeness and deduction theorem

Lewis proves soundness and (strong) completeness with respect to sphere models of what we called the local consequence relation with respect to the logic $\vdash_{\mathsf{LV}}$ (Theorem 3.8); moreover he proves a deduction theorem for the local consequence with respect to the classical implication:

Theorem 3.20 (Deduction Theorem of the local consequence)**.** *For all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\Box\to}$ the following holds: $\Gamma, \psi \vDash_l \varphi$ if and only if $\Gamma \vDash_l \psi \to \varphi$.*

In this subsection we prove the analogous results for the stronger deductive systems and the corresponding global consequence relation; but first, some technical results.

Lemma 3.21. *Consider any $\mathcal{L}_{\Box\!\!\to}$-formula φ, then $\varphi \vdash_{\mathtt{GV}} \Box^n\varphi$ for all $n \in \mathbb{N}$.*

Proof. The claim is easily shown by induction on n; indeed the case $n = 0$ is obvious, and the inductive case is given by one application of (DWC_0): $\varphi \vdash_{\mathtt{GV}} \neg\varphi \Box\!\!\to \varphi$, which holds for GV by Proposition 2.7 and Theorem 2.8. □

Proposition 3.22. *Let* L *be any axiomatic extension of* GV. *For all $\Gamma \cup \{\varphi\} \subseteq Fm_{\Box\!\!\to}$ the following are equivalent:*

1. $\Gamma \vdash_{\mathtt{L}} \varphi$;
2. $\{\Box^n\gamma \mid \gamma \in \Gamma$ *and* $n \in \mathbb{N}\} \vdash_{\mathtt{L}} \varphi$.
3. *There exist a finite subset $\Gamma_0 \subseteq \Gamma$ and $n_0 \in \mathbb{N}$ such that $\{\Box^n\gamma \mid \gamma \in \Gamma_0$ and $n \leq n_0\} \vdash_{\mathtt{L}} \varphi$.*

Proof. The fact that (1) implies (2) is obvious, since $\Gamma \subseteq \{\Box^n\gamma \mid \gamma \in \Gamma$ and $n \in \mathbb{N}\}$. For the converse, let us assume that $\{\Box^n\gamma \mid \gamma \in \Gamma$ and $n \in \mathbb{N}\} \vdash_{\mathtt{L}} \varphi$; By Lemma 3.21, we have that $\Gamma \vdash_{\mathtt{L}} \Box^n\gamma$ for all $\gamma \in \Gamma$ and $n \in \mathbb{N}$, and thus also $\Gamma \vdash_{\mathtt{L}} \varphi$. Lastly, (2) and (3) are equivalent since $\vdash_{\mathtt{L}}$ is a finitary consequence relation. □

We now have all the ingredients to prove our completeness result.

Theorem 3.23 (Soundness and strong completeness of the global consequence)**.** *Let Σ be a subset of the axioms* {W, C, N, T, S, U, A}. *For all subsets $\Gamma \cup \{\varphi\} \subseteq Fm_{\Box\!\!\to}$,*

$$\Gamma \vdash_{\mathtt{GV}\Sigma} \varphi \Leftrightarrow \Gamma \vDash_{\mathfrak{S}_\Sigma, g} \varphi.$$

Proof. The soundness follows from the facts that: (MP) and (C) are easily seen to be sound with respect to sphere models, and the axioms of GVΣ are the same axioms of LVΣ, which is sound with respect to the same class of models for the local consequence relation (Theorem 3.8), and the latter has the same valid formulas as the global one (Theorem 3.5).

We prove completeness by contraposition; assume $\Gamma \nvdash_{\mathtt{GV}\Sigma} \varphi$. By Proposition 3.22, we have that $\{\Box^n\gamma \mid \gamma \in \Gamma$ and $n \in \mathbb{N}\} \nvdash_{\mathtt{GV}\Sigma} \varphi$. By the fact that all deductions of $\vdash_{\mathtt{LV}\Sigma}$ are deductions of $\vdash_{\mathtt{GV}\Sigma}$ (Lemma 2.3), it follows that $\{\Box^n\gamma \mid \gamma \in \Gamma$ and $n \in \mathbb{N}\} \nvdash_{\mathtt{LV}\Sigma} \varphi$. By the strong completeness of $\vdash_{\mathtt{LV}\Sigma}$ in Theorem 3.8, we get that $\{\Box^n\gamma \mid \gamma \in \Gamma$ and $n \in \mathbb{N}\} \nvDash_{\mathfrak{S}_\Sigma, l} \varphi$. Theorem 3.19 then yields that $\Gamma \nvDash_{\mathfrak{S}_\Sigma, g} \varphi$ and the proof is complete. □

We will now show another relevant fact, i.e., that the global consequence relation has a deduction theorem. However, it generally does not have the classical deduction theorem, as the following example show.

Example 3.24. By Lemma 3.21 and the strong completeness in Theorem 3.23, $\varphi \vDash_g \Box\varphi$ for any $\mathcal{L}_{\Box\!\!\to}$-formula φ. However, it is easily seen that in general $\nvDash_g \varphi \to \Box\varphi$. Consider the following sphere model $\mathcal{M} = (W, \mathcal{S}, v)$ such that

- $W = \{w_1, w_2\}$;
- $\mathcal{S}(w_1) = \{\{w_2\}\}$; $\mathcal{S}(w_2) = \{w_2, \{w_1, w_2\}\}$;
- v is such that it maps a propositional variable p to $v(p) = \{w_1\}$.

Note that then $v(\Box p) = \emptyset$, and therefore $w \Vdash p$ but $w \nVdash \Box p$; hence $\mathcal{M} \nVdash p \to \Box p$. One can easily adapt this same example for any class of sphere models that allow at least two different worlds in W (thus, except for the trivial class of models).

Nonetheless:

Theorem 3.25 (Deduction Theorem of the global consequence)**.** *For all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\Box\!\!\to}$, the following are equivalent:*

1. $\Gamma, \psi \vDash_g \varphi$,
2. *there is $n \in \mathbb{N}$ such that* $\Gamma \vDash_g (\bigwedge_{m \leq n} \Box^m \psi) \to \varphi$,

Proof. Let us start by proving that (1) implies (2); assume $\Gamma, \psi \vDash_{\mathfrak{S},g} \varphi$. By Theorem 3.19, this is equivalent to the fact that

$$\{\Box^n \gamma : \gamma \in \Gamma, n \in \mathbb{N}\} \cup \{\Box^n \psi : n \in \mathbb{N}\} \vDash_{\mathfrak{S},l} \varphi$$

By the deduction theorem for the local consequence, and the fact that this is finitary (given that it is strongly complete with respect to a finitary logic by Theorem 3.8), it follows that there is $n_0 \in \mathbb{N}$

$$\{\Box^n \gamma : \gamma \in \Gamma, n \in \mathbb{N}\} \vDash_{\mathfrak{S},l} (\bigwedge_{k \leq n_0} \Box^k \psi) \to \varphi.$$

Using Theorem 3.19 again, we have that $\Gamma \vDash_{\mathfrak{S},g} (\bigwedge_{k \leq n_0} \Box^k \psi) \to \varphi$.

We now show that (2) implies (1); assume that there is a $n \in \mathbb{N}$ such that $\Gamma \vDash_{\mathfrak{S},g} (\bigwedge_{m \leq n} \Box^m \psi) \to \varphi$. Equivalently, for all sphere models $\mathcal{M} = \langle W, \mathcal{S}, v\rangle \in \mathfrak{S}$, we have that if $\mathcal{M} \Vdash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \Vdash (\bigwedge_{m \leq n} \Box^m \psi) \to \varphi$. Consider then a sphere model $\mathcal{M} = (W, \mathcal{S}, v)$ such that $\mathcal{M} \Vdash \gamma$ for all $\gamma \in \Gamma$ and $\mathcal{M} \Vdash \psi$. By assumption, we then have that for any world $w \in W$, $w \Vdash \psi$ and $w \Vdash (\bigwedge_{m \leq n} \Box^m \psi) \to \varphi$. By Proposition 3.18, this implies that for all $w \in W$, $w \Vdash \bigwedge_{m \leq n} \Box^m \psi$. Therefore, by modus ponens, we have that for all $w \in W$, $w \Vdash \varphi$ as well; i.e. $\mathcal{M} \Vdash \varphi$. Hence $\Gamma, \psi \vDash_{\mathfrak{S},g} \varphi$. ◻

By the strong completeness in Theorem 3.23, the theorem above can also be read as follows:

Corollary 3.26. *For all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\Box\!\!\to}$, $\Gamma, \psi \vdash_{\mathsf{GV}} \varphi$ if and only if $\Gamma \vdash_{\mathsf{GV}} (\bigwedge_{m \leq n} \Box^m \psi) \to \varphi$ for some $n \in \mathbb{N}$.*

We are now ready to proceed our investigation towards an algebraic study of Lewis's hierarchy of logics for counterfactual conditionals.

4 Algebraic semantics

In this section we show that the stronger calculi are algebraizable in the sense of Blok-Pigozzi, and we study the equivalent algebraic semantics, given by varieties of Boolean algebras with an extra operator $\Box\!\!\to$. Moreover, we show that such algebras give a semantics for the weaker logics as well, in the sense that the latter are the logics *preserving the degrees of truth* of the algebras. For the sake of the reader, we first recall the basics of the Blok-Pigozzi machinery [6], that connects *algebraizable logics* with their *equivalent algebraic semantics*. For the omitted details we refer to [6, 10].

Let us set some notation; Roman bold letters will be used to represent algebras, while the corresponding Roman standard letters will denote their underlying domains. For instance, if $\mathbf{A}$ is an algebra, then the symbol A will refer to its domain. Given an algebraic language $\mathcal{L}$, recall that we write $\mathbf{Fm}_{\mathcal{L}}$ for its *algebra of formulas* written over a denumerable set of variables. An *equation* of the language $\mathcal{L}$ (or an $\mathcal{L}$-equation for short) is a pair (p, q) of $\mathcal{L}$-formulas (i.e. elements of $Fm_{\mathcal{L}}$) that we write suggestively as $p \approx q$. We write $Eq_{\mathcal{L}}$ for the set of all $\mathcal{L}$-equations. A *quasi-equation* of $\mathcal{L}$ is a *first-order* formula of the form $\&_{i=1}^{n} p_i \approx q_i \Rightarrow p \approx q$ where $\{p_i \approx q_i : i = 1, \ldots, n\} \cup \{p \approx q\} \subseteq Eq_{\mathcal{L}}$, and & and $\Rightarrow$ are, respectively, first order conjunction and implication. It is understood that this expression also covers the case of an empty antecedent, so that equations can be seen as particular cases of quasi-equations.

An *assignment* is a homomorphism (i.e., a function which commutes with all the operations) from the algebra of formulas $\mathbf{Fm}_{\mathcal{L}}$ to some $\mathcal{L}$-algebra $\mathbf{A}$. An $\mathcal{L}$-algebra $\mathbf{A}$ satisfies an $\mathcal{L}$-equation $p \approx q$ with an assignment h (and we write $\mathbf{A}, h \vDash p \approx q$) if $h(p) = h(q)$ in $\mathbf{A}$. An $\mathcal{L}$-equation $p \approx q$ is *valid* in $\mathbf{A}$ (and we write $\mathbf{A} \vDash p \approx q$) if for all assignments h to $\mathbf{A}$, $\mathbf{A}, h \vDash p \approx q$; if Σ is a set of $\mathcal{L}$-equations then $\mathbf{A} \vDash \Sigma$ if $\mathbf{A} \vDash \sigma$ for all $\sigma \in \Sigma$. An $\mathcal{L}$-equation $p \approx q$ is valid in a class of $\mathcal{L}$-algebras K, and we write $\mathsf{K} \vDash p \approx q$ or $\vDash_{\mathsf{K}} p \approx q$, if $\mathbf{A} \vDash p \approx q$ for all $\mathbf{A} \in \mathsf{K}$. With respect to quasi-equations, an $\mathcal{L}$-algebra $\mathbf{A}$ satisfies an $\mathcal{L}$-quasi-equation $\&_{i=1}^{n} p_i \approx q_i \Rightarrow p \approx q$ with an assignment h if $h(p_i) = h(q_i)$ for all $i = 1, \ldots, n$ implies $h(p) = h(q)$; the other notions of validity extend to quasi-equations in the obvious way.

Moreover, given any set of $\mathcal{L}$-equations $\Sigma \cup \{p \approx q\}$, and any class of $\mathcal{L}$-algebras K, we write

$$\Sigma \vDash_{\mathsf{K}} p \approx q$$

if for every algebra $\mathbf{A} \in \mathsf{K}$, and any assignment h to $\mathbf{A}$, if $h(p') = h(q')$ for all $p' \approx q' \in \Sigma$, then $h(p) = h(q)$. $\Sigma \vDash_{\mathsf{K}} \Delta$, for Σ, Δ sets of $\mathcal{L}$-equations, is interpreted as $\Sigma \vDash_{\mathsf{K}} \delta$ for all $\delta \in \Delta$. $\vDash_{\mathsf{K}}$ is called the *equational consequence relative to* K. We write $\Sigma \mathrel{=\!\!\mid\vDash}_{\mathsf{K}} \Delta$ as a shortening of $\Sigma \vDash_{\mathsf{K}} \Delta$ *and* $\Delta \vDash_{\mathsf{K}} \Sigma$.

Intuitively, in order to establish the algebraizability of a logic L with respect to a class of algebras K_{L} over the same language $\mathcal{L}$, one wants to be able to fully and faithfully interpret the consequence relation of L into the equational consequence relative to K_{L}.

Let us be precise. Fix a language $\mathcal{L}$; a *transformer from formulas to (sets of) equations* is any function τ mapping each $\mathcal{L}$-formula to a set of $\mathcal{L}$-equations, $\tau : Fm_{\mathcal{L}} \to \mathscr{P}(Eq_{\mathcal{L}})$. This extends to sets of formulas by setting, for any set of formulas $\Gamma \subseteq Fm_{\mathcal{L}}$, $\tau(\Gamma) = \bigcup_{\gamma \in \Gamma} \tau(\gamma)$. Similarly, a *transformer of equations into (sets of) formulas* is a function ρ mapping each $\mathcal{L}$-equation into a set of $\mathcal{L}$-formulas, $\rho : Eq_{\mathcal{L}} \to \mathscr{P}(Fm_{\mathcal{L}})$, that extends to sets of equations by taking unions. In particular, one wants to consider *structural* transformers: a transformer is structural when it commutes with substitutions.

A logic L is *algebraizable* when there is a class of algebras K and structural transformers τ, ρ (from formulas into equations and from equations into formulas, respectively) such that the following conditions are satisfied, for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and $\mathcal{L}$-equation $p \approx q$:

$$\Gamma \vdash_{\mathsf{L}} \varphi \quad \text{iff} \quad \tau(\Gamma) \vDash_{\mathsf{K}_{\mathsf{L}}} \tau(\varphi) \tag{8}$$

$$(p \approx q) \mathrel{=\!\!\mid\vDash}_{\mathsf{K}_{\mathsf{L}}} \tau(\rho(p \approx q)). \tag{9}$$

The transformers τ and ρ are said to *witness* the algebraizability of L with respect to the class K. The equations $E(x) := \tau(x)$ are called the *defining equations* and the formulas in $\Delta(x, y) := \rho(x \approx y)$ are called the *equivalence formulas*. Given L an algebraizable logic, its *equivalent algebraic semantics* is the largest class of algebras K such that L is algebraizable with respect to it. In particular, if L is algebraizable with respect to a *quasivariety* (i.e., a class of models of a set of quasi-equations) K, K is the equivalent algebraic semantics of L, and every finitary logic has a quasivariety as its equivalent algebraic semantics. When the equivalent algebraic semantics can be axiomatized by equations (i.e., it is a *variety*), we say that a finitary logic is *strongly algebraizable*.

Example 4.1. Classical logic is (strongly) algebraizable with respect to the variety of Boolean algebras, as testified by $\tau(x) = \{x \approx 1\}$ and $\rho(x \approx y) = \{x \leftrightarrow y\}$.

While the conditions (8) and (9) above are necessary and sufficient to show the algebraizability of a logic, in some cases there are easier ways to check whether a logic is algebraizable. In fact, many of the well-known algebraizable logics belong to the class of *implicative logics*, that is, they have a well-behaved binary connective $\to$ which allows to show that (8) and (9) hold.

Definition 4.2. An *implicative logic* is a logic L in a language $\mathcal{L}$ with a binary term $\to$ such that:

1. $\vdash x \to x$
2. $x \to y, y \to z \vdash x \to z$
3. $x_1 \to y_1, \dots, x_n \to y_n, y_1 \to x_1, \dots, y_n \to x_n \vdash \lambda(x_1, \dots, x_n) \to \lambda(y_1, \dots, y_n)$ for each term $\lambda \in \mathcal{L}$ of arity $n > 0$
4. $x, x \to y \vdash y$
5. $x \vdash y \to x$.

Classical logic is an example of an implicative logic. In an implicative logic that does not have a constant 1 that is a theorem, one can always define $1 := x \to x$ for a fixed variable x, and 1 is a theorem by the above definition.

Theorem 4.3 ([10]). *All implicative logics are algebraizable, with defining equation $\tau(x) := \{x \approx x \to x\}$ and equivalence formulas $\Delta(x, y) := \{x \to y, y \to x\}$. If there is a constant 1 that is a theorem, $\tau(x) := \{x \approx 1\}$. If the logic is finitary, the quasivariety that is its equivalent algebraic semantics can be presented by the equations and quasi-equations that result by applying the transformation τ to the axioms and rules of any Hilbert-style presentation of the logic.*

From the more strictly algebraic perspective, in a class of algebras that is the equivalent algebraic semantics of a logic L over a language $\mathcal{L}$, congruences are in one-one correspondence with the *deductive filters* induced by the logic. An L-deductive filter F of an algebra $\mathbf{A}$ is a subset of the domain of $\mathbf{A}$ that is closed under the deductions of the logic L; that is, for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, if $\Gamma \vdash_{\mathsf{L}} \varphi$, for every homomorphism f from $\mathbf{Fm}_{\mathcal{L}}$ to $\mathbf{A}$, if $f[\Gamma] \subseteq F$, then $f(\varphi) \in F$.

Theorem 4.4 ([10, Theorem 3.51]). *Let $\mathsf{L} = (\mathcal{L}, \vdash)$ be a finitary logic with equivalent algebraic semantics a quasivariety* K*, and let $\mathbf{A}$ be an $\mathcal{L}$-algebra. Then the $\mathcal{L}$- deductive filters of $\mathbf{A}$ are in bijection with the* K*-relative congruences of $\mathbf{A}$.*

We mention that the L-deductive filters of the algebras of formulas are the *theories* of the logic L. In particular, if L is an implicative logic and K_{L} its equivalent algebraic semantics, for every $\mathbf{A}$ in K_{L} the correspondence between congruences and deductive filters is given by the following maps:

$$\theta \mapsto F_\theta = \{a \in A : (a, a \to a) \in \theta\}, \qquad F \mapsto \theta_F = \{(a, b) : a \to b, b \to a \in F\}$$

where θ is any congruence of $\mathbf{A}$ and F is any deductive filter of $\mathbf{A}$. If there is a constant 1 in the language of L that is a theorem, as it is the case for classical logic, then congruences are totally determined by their 1-blocks, i.e. the first map above becomes:

$$\theta \mapsto F_\theta = \{a \in A : (a, 1) \in \theta\} = 1/\theta.$$

This means that $\mathsf{K}_\mathtt{L}$ is *ideal-determined* with respect to 1, and so in particular also *1-regular* ($1/\theta = 1/\gamma$ implies $\theta = \gamma$ for all congruences θ, γ of every algebra $\mathbf{A} \in \mathsf{K}_\mathtt{L}$), see [1, 13].

4.1 Global equivalent algebraic semantics

The algebraizability of GV follows from the fact that it is an implicative logic.

Theorem 4.5. *Let* L *be any axiomatic extension of* GV*. Then* L *is an implicative logic.*

Proof. We need to show that the conditions of Definition 4.2 hold; (3) follows from Proposition 2.9 and the others follow from the fact that $\to$ is a Boolean implication. □

Moreover, since 1 is a theorem of GV (see Theorem 4.3), we get the following.

Theorem 4.6. GV *is algebraizable with defining equation $\tau(x) = \{x \approx 1\}$ and equivalence formula $\Delta(x, y) = \{x \leftrightarrow y\}$.*

An important consequence of algebraizability is that axiomatic extensions of algebraizable logics are also algebraizable with the same equivalence formulas and defining equations. Moreover, the lattice of axiomatic extensions of the logic is dually isomorphic to the subvariety lattice of the equivalent algebraic semantics whenever the latter is a variety [10], as it is the case here. Therefore, we obtain the algebraizability of GV and all its axiomatic extensions with respect to the corresponding class of algebras.

Corollary 4.7 (Algebraizability)**.** *Let* L *be any axiomatic extension of* GV*, axiomatized relatively to* GV *by the set of axioms Φ. Then the equivalent algebraic semantic of* L*, $\mathsf{K}_\mathtt{L}$, is axiomatized relative to $\mathsf{K}_\mathtt{GV}$ by $\tau(\Phi) = \{\varphi \approx 1 : \varphi \in \Phi\}$. In particular given any set of formulas Γ and formulas ψ:*

$$\Gamma \vdash_\mathtt{L} \psi \quad \Leftrightarrow \quad \tau(\Gamma) \vDash_{\mathsf{K}_\mathtt{L}} \tau(\psi).$$

By direct application of the Blok-Pigozzi machinery, we get an axiomatization of $\mathsf{K}_\mathtt{GV}$ made of equations (coming from the axioms) and quasi-equations (coming from rules); we now show that actually equations suffice, in particular because the rules (MP) and (C) are translated to quasi-equations that already hold as a consequence of the other axioms. In other words, the equivalent algebraic semantics is a *variety* of algebras.

Definition 4.8. A *Lewis variably strict conditional algebra*, or V*-algebra* for short, is an algebra $\mathbf{C} = (C, \wedge, \vee, \to, \boxright, 0, 1)$ where $(C, \wedge, \vee, \to, 0, 1)$ is a Boolean algebra and $\boxright$ is a binary operation such that, for all $x, y, z \in C$:

1. $x \boxright x = 1$
2. $(x \boxright y) \wedge (y \boxright x) \leq (x \boxright z) \leftrightarrow (y \boxright z)$
3. $((x \vee y) \boxright x) \vee ((x \vee y) \boxright y) \vee (((x \vee y) \boxright z) \leftrightarrow ((x \boxright z) \wedge (y \boxright z))) = 1$

4. $x \boxright (y \wedge z) = (x \boxright y) \wedge (x \boxright z)$

We denote the variety of V-algebras with VA.

Theorem 4.9. VA *is the equivalent algebraic semantics of* GV.

Proof. Note that since algebras in VA have a Boolean algebra reduct, for $\mathbf{C} \in$ VA, and $x, y \in C$, $x \to y = 1$ iff $x \leq 1$; then by Corollary 4.7, the equivalent algebraic semantics of GV is axiomatized by the axioms of VA plus the quasi-equations:

$$(x \approx 1 \ \& \ x \to y \approx 1) \Rightarrow y \approx 1, \quad \text{and} \quad x \to y \approx 1 \Rightarrow (z \boxright x) \to (z \boxright y) \approx 1.$$

The result then follows from the fact that the latter are easily seen to hold given the other axioms; in particular, note that the fact that $\boxright$ is order-preserving on the right follows from the distributivity over the meet operation on the right: if $x \leq y$ then $x = x \wedge y$, and so $z \boxright x = z \boxright (x \wedge y) = (z \boxright x) \wedge (z \boxright y)$ and so $z \boxright x \leq z \boxright y$. ◻

Recall that $x \diamondright y := \neg(x \boxright \neg y)$. Let VCA be the subvariety of VA further satisfying:

$$x \wedge y \leq x \boxright y \leq x \to y \tag{10}$$

and VCSA the subvariety of VCA of *Lewis conditional algebras*, satisfying:

$$(x \boxright y) \vee (x \boxright \neg y) = 1. \tag{11}$$

Corollary 4.10. VCA *and* VCSA *are the equivalent algebraic semantics of, respectively,* GVC *and* GVCS.

Let us consider again the unary connective $\Box$ in the language as $\Box\varphi := \neg\varphi \boxright \varphi$, and its iteration $\Box^n\varphi$. We can show that, analogously to the case of modal algebras, the operator $\Box$ can be used to characterize congruence filters.

Definition 4.11. Let $\mathbf{A} \in$ VA; a nonempty lattice filter $F \subseteq A$ is said to be *open* if $x \in F$ implies $\Box x \in F$.

Proposition 4.12. *Let* $\mathbf{A} \in$ VA*; a nonempty lattice filter* $F \subseteq A$ *is a congruence filter if and only if it is open.*

Proof. The proof is based on the fact that, as a consequence of the fact that VA is the equivalent algebraic semantics of GV, congruence filters coincide with the deductive filters induced by the logic. In other words, for every rule $\Gamma \vdash \varphi$ and every homomorphism f from $\mathbf{Fm}_{\boxright}$ to $\mathbf{A}$, if $f[\Gamma] \subseteq F$, then $f(\varphi) \in F$. It is clear that every deductive filter is an open lattice filter. For the converse, consider an open lattice filter F; F respects the axioms because it is nonempty (i.e. it contains 1, where all the instances of the axioms are mapped as a direct consequence of the algebraizability result, Corollary 4.7), and it respects modus ponens since it is a lattice filter. We only need to prove that it respects the rule $\varphi \to \psi \vdash (\gamma \boxright \varphi) \to (\gamma \boxright \psi)$.

Suppose there is an assignment h to $\mathbf{A} \in$ VA such that $h(\varphi) = a, h(\psi) = b, h(\gamma) = c$, with $a, b, c \in A$, and assume that $a \to b \in F$. From the fact that $\varphi \to \psi \vdash (\gamma \boxright \varphi) \to (\gamma \boxright \psi)$ holds, by the deduction theorem of the global consequence (Theorem 3.25) and the strong completeness (Theorem 3.23) we obtain that there is some $n \in \mathbb{N}$ such that

$$\vdash_{\mathtt{GV}} (\bigwedge_{m \leq n} \Box^m(\varphi \to \psi)) \to ((\gamma \boxright \varphi) \to (\gamma \boxright \psi)).$$

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This implies that the element $(\bigwedge_{m\leq n}\Box^m(a\to b))\to((c \mathrel{\Box\!\!\to} a)\to(c \mathrel{\Box\!\!\to} b)) = 1 \in F$. Now, since $a\to b\in F$ and F is open, $\Box^k(a\to b)\in F$ for all $k\in\mathbb{N}$. Thus, since filters are closed under finitary meets, the element $(\bigwedge_{m\leq n}\Box^m(a\to b))\in F$. Since lattice filters respect modus ponens, also $((c \mathrel{\Box\!\!\to} a)\to(c \mathrel{\Box\!\!\to} b))\in F$, which shows that open lattice filters respect the rule (C).

We have then shown that open filters coincide with deductive filters, and therefore with congruence filters. □

We remind the reader that the proposition above describes the Gumm-Ursini ideals [13] of VA, which are also the GV-deductive filters of the algebras in VA. The following is an interesting observation:

Corollary 4.13. *Let* $\mathbf{A}\in\mathsf{VA}$*; then the congruence filters of* $\mathbf{A}$ *are exactly the congruence filters of its modal reduct* $(A,\wedge,\vee,\to,\Box,0,1)$.

The corollary above entails for instance that one can characterize subdirectly irreducible algebras in VA by the subdirect irreducibility of their modal reduct; we leave this purely algebraic investigation of V-algebras to future work.

4.2 Local algebraic semantics

In this section we focus on the weaker logic LV; in particular, we will show that the latter is not algebraizable, however it still can be studied by means of V-algebras. We will indeed see that $\vdash_{\mathtt{LV}}$ coincides with the logic *preserving degrees of truth* of VA. Let us be more precise.

We call an *ordered algebra* a pair $(\mathbf{A},\leq)$, where $\mathbf{A}$ is an algebra and $\leq$ is a partial order on its universe. Note that all algebras with a (semi)lattice reduct can be seen as ordered algebras, and thus in particular Boolean algebras and V-algebras are ordered algebras.

Definition 4.14. Let K be a class of ordered algebras over a language $\mathcal{L}$; the logic *preserving degrees of truth* of K, in symbols $\mathsf{L}^{\leq}_{\mathsf{K}} = (\mathcal{L},\vdash^{\leq}_{\mathsf{K}})$, is defined as follows: for every set $\Gamma\cup\{\varphi\}$ of formulas in the language of K, $\Gamma\vdash^{\leq}_{\mathsf{K}}\varphi$ if and only if for all $(\mathbf{A},\leq)\in\mathsf{K}$, and assignment $h:\mathbf{Fm}_{\mathcal{L}}\to\mathbf{A}$, $a\in A$,

$$a\leq h(\gamma) \text{ for every } \gamma\in\Gamma \quad\Rightarrow\quad a\leq h(\varphi).$$

Remark 4.15. If K is an elementary class of algebras (in particular, if K is a variety) with a lattice reduct, one can rephrase the above definition and say that

$$\Gamma\vdash^{\leq}_{\mathsf{K}}\varphi \quad\text{iff}\quad \mathsf{K}\vDash\gamma_1\wedge\ldots\wedge\gamma_n\leq\varphi$$

for some $\{\gamma_1,\ldots,\gamma_n\}\subseteq\Gamma$ and $n\in\mathbb{N}$ (see [20, Remark 2.4]).

Example 4.16. Intuitionistic logic is the logic preserving degrees of truth of Heyting algebras, and the local consequence of the modal logic K is the logic preserving degrees of truth of modal algebras.

Logics preserving the degrees of truth have been studied in generality in [22, 9], and in residuated structures in [8]. In order to prove the analogous result for LV, let us first state a useful lemma.

Lemma 4.17. *For all* $\mathcal{L}_{\Box\!\to}$*-formulas* φ,ψ,γ*, and* $n\in\mathbb{N}$*:*

1. $\vdash_{\mathtt{LV}} \Box^{n+1}(\varphi \to \psi) \to \Box^n((\gamma \boxright \varphi) \to (\gamma \boxright \psi))$;

2. $\vdash_{\mathtt{LV}} \Box^{n+1}(\varphi \leftrightarrow \psi) \to \Box^n((\varphi \boxright \gamma) \to (\psi \boxright \gamma))$.

Proof. We will show that the two statements hold in sphere models, which implies the claim by the completeness Theorem 3.8. We start with (1), and proceed by induction on n. Let $n = 0$, we want to prove that:

$$\vDash_l \Box(\varphi \to \psi) \to ((\gamma \boxright \varphi) \to (\gamma \boxright \psi));$$

consider a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$ and let $w \in W$. Suppose $w \Vdash \Box(\varphi \to \psi)$, or equivalently via Lemma 3.17 $\bigcup \mathcal{S}(w) \subseteq v(\varphi \to \psi)$. Now, if $w \Vdash \gamma \boxright \varphi$, it means that there is $S \in \mathcal{S}(w)$ such that $\emptyset \neq S \cap v(\gamma) \subseteq v(\varphi)$; but since $\bigcup \mathcal{S}(w) \subseteq v(\varphi \to \psi)$, $S \cap v(\varphi) \subseteq v(\psi)$, i.e. there is $S \in \mathcal{S}(w)$ such that $\emptyset \neq S \cap v(\gamma) \subseteq v(\psi)$, and thus $w \Vdash \gamma \boxright \psi$. The inductive step follows from the fact that $\Box$ is a modal operator, more precisely that theorems are closed under $\Box$ and $\Box$ distributes over the implication (Proposition 3.18). Let us now show (2), again by induction on n; for $n = 0$, we prove:

$$\vDash_l \Box(\varphi \leftrightarrow \psi) \to ((\varphi \boxright \gamma) \to (\psi \boxright \gamma)).$$

Consider then a sphere model $\mathcal{M} = \langle W, \mathcal{S}, v\rangle$, let $w \in W$, and suppose $w \Vdash \Box(\varphi \leftrightarrow \psi)$, or equivalently by Lemma 3.17, $\bigcup \mathcal{S}(w) \subseteq v(\varphi \leftrightarrow \psi)$. If $w \Vdash \varphi \boxright \gamma$, it means that there is $S \in \mathcal{S}(w)$ such that $\emptyset \neq S \cap v(\varphi) \subseteq v(\gamma)$; but since $\bigcup \mathcal{S}(w) \subseteq v(\varphi \leftrightarrow \psi)$, we get that $S \cap v(\varphi) = S \cap v(\psi)$, and therefore there is $S \in \mathcal{S}(w)$ such that $\emptyset \neq S \cap v(\psi) \subseteq v(\gamma)$, and thus $w \Vdash \psi \boxright \gamma$. The inductive step follows again from the fact that $\Box$ is a modal operator. ◻

We are now ready to prove that LV can also be studied by means of V-algebras, via the following (strong) completeness result.

Theorem 4.18. LV *is the logic preserving degrees of truth of* VA*; i.e., for all* $\Gamma \cup \{\varphi\} \subseteq Fm_{\boxright}$*:*

$$\Gamma \vdash_{\mathtt{LV}} \varphi \;\Leftrightarrow\; \Gamma \vdash^{\leq}_{\mathsf{VA}} \varphi.$$

Proof. The forward direction is a usual soundness proof; note that the axioms are mapped to 1 by the algebraizability result (Corollary 4.7), (MP) holds in the form $x \wedge (x \to y) \leq y$, and (wC) becomes $x \to y = 1$ implies $(z \boxright x) \leq (z \boxright y)$ which holds in VA.

For the converse, we reason by contraposition; assume $\Gamma \nvdash_{\mathtt{LV}} \varphi$. Then consider the relation θ defined as follows:

$$\theta := \{(\psi, \gamma) \in Fm_{\boxright} \times Fm_{\boxright} : \Gamma \vdash_{\mathtt{LV}} \Box^n(\psi \to \gamma) \text{ and } \Gamma \vdash_{\mathtt{LV}} \Box^n(\gamma \to \psi) \text{ for all } n \in \mathbb{N}\}.$$

We can show that θ is a congruence relation. In particular, while reflexivity and symmetry are trivial, transitivity follows the fact that $\Box$ distributes over $\to$ (Proposition 3.18); the Boolean operations are also easily shown to be respected: for $\wedge$, as a consequence of the fact that $\Box$ distributes over $\wedge$ (Proposition 3.18), and for $\neg$, by the observation that $\neg\Box^n x = \Box^n(\neg x)$. Let us then prove that θ preserves the binary operation $\boxright$. Assume $\psi\theta\gamma$, it suffices to show that:

$$(\delta \boxright \psi, \delta \boxright \gamma), (\psi \boxright \delta, \gamma \boxright \delta) \in \theta;$$

that is to say, we need prove that:

$$\text{(i)} \quad \Gamma \vdash_{\mathtt{LV}} \Box^n((\delta \boxright \psi) \to (\delta \boxright \gamma)); \qquad \text{(ii)} \quad \Gamma \vdash_{\mathtt{LV}} \Box^n((\delta \boxright \gamma) \to (\delta \boxright \psi));$$

$$\text{(iii)}\quad \Gamma \vdash_{\mathtt{LV}} \Box^n((\psi \,\Box\!\!\rightarrow \delta) \to (\gamma \,\Box\!\!\rightarrow \delta)); \quad \text{(iv)}\quad \Gamma \vdash_{\mathtt{LV}} \Box^n((\gamma \,\Box\!\!\rightarrow \delta) \to (\psi \,\Box\!\!\rightarrow \delta)).$$

Given the assumption that $\Gamma \vdash_{\mathtt{LV}} \Box^n(\psi \to \gamma)$ and $\Gamma \vdash_{\mathtt{LV}} \Box^n(\gamma \to \psi)$ for all $n \in \mathbb{N}$, and thus also $\Gamma \vdash_{\mathtt{LV}} \Box^n(\varphi \leftrightarrow \psi)$ (since $\Box$ distributes over $\wedge$ by Proposition 3.18), (i)–(iv) follow by Lemma 4.17. Therefore, θ is a congruence, and we can consider the quotient $\mathbf{Fm}_{\Box\!\rightarrow}/\theta$; let us verify that the latter is an algebra in VA. Consider the axiomatization of VA, we show that for every equation $\varepsilon \approx \delta$ appearing in it, $(\varepsilon, \delta) \in \theta$. By the algebraizability of VA, for each such $\varepsilon \approx \delta$ and the completeness result of Theorem 3.23, $\vdash_{\mathtt{GV}} \varepsilon \leftrightarrow \delta$; thus also $\vdash_{\mathtt{LV}} \varepsilon \leftrightarrow \delta$ (since they have the same theorems by Theorem 2.4), and then $\vdash_{\mathtt{LV}} \Box^n(\varepsilon \leftrightarrow \delta)$ for all $n \in \mathbb{N}$ by the fact that theorems are closed under $\Box$ (Proposition 3.18). Thus, each $(\varepsilon, \delta) \in \theta$, and $\mathbf{Fm}_{\Box\!\rightarrow}/\theta \in \mathsf{VA}$.

We now show that for all (nonempty) finite subsets $\Delta \subseteq \Gamma$, $\Delta \nvdash^{\leq}_{\mathsf{VA}} \varphi$. By Remark 4.15, this implies that $\Gamma \nvdash^{\leq}_{\mathsf{VA}} \varphi$ which would complete the proof. Given that $\Delta \vdash^{\leq}_{\mathsf{VA}} \varphi$ iff $\mathsf{VA} \vDash \bigwedge \Delta \leq \varphi$, it is enough to show that in particular $\mathbf{Fm}_{\Box\!\rightarrow}/\theta \nvDash \bigwedge \Delta \leq \varphi$. Consider $\Delta = \{\delta_1, \ldots, \delta_n\} \subseteq \Gamma$, and let π be the natural epimorphism $\pi : \mathbf{Fm}_{\Box\!\rightarrow} \to \mathbf{Fm}_{\Box\!\rightarrow}/\theta$; assume by way of contradiction that $\pi(\delta_1) \wedge \cdots \wedge \pi(\delta_n) \leq \pi(\varphi)$; thus $\pi((\delta_1 \wedge \cdots \wedge \delta_n) \to \varphi) = 1$. By the definition of θ, this implies that in particular $\Gamma \vdash_{\mathtt{LV}} (\delta_1 \wedge \cdots \wedge \delta_n) \to \varphi$. But since $\Delta \subseteq \Gamma$, it follows that $\Gamma \vdash_{\mathtt{LV}} \delta_1 \wedge \cdots \wedge \delta_n$; by modus ponens we would get $\Gamma \vdash_{\mathtt{LV}} \varphi$, a contradiction. This completes the proof. □

Actually, the same proof works for any axiomatic extension of LV.

Corollary 4.19. *Let Γ be a set of formulas;* LV $+ \Gamma$ *is the logic preserving degrees of truth of* $\mathsf{VA} + \tau(\Gamma)$.

As a corollary of the above theorem, we get that:

Corollary 4.20. *Let Γ be a set of formulas. For all $\varphi, \psi \in Fm_{\Box\!\rightarrow}$, $\vDash_{\mathsf{VA}+\tau(\Gamma)} \varphi \approx \psi$ if and only if $\varphi \vdash_{\mathtt{LV}+\Gamma} \psi$ and $\psi \vdash_{\mathtt{LV}+\Gamma} \varphi$.*

We are now going to conclude that the local calculus is not algebraizable. In fact, it follows from a general theory regarding (semi)lattice-based logics that have an algebraizable *assertional* companion [5]. We recall the relevant notions from [10, §7].

Given a class of algebras K that have a constant 1 in their language $\mathcal{L}$, we define the *1-assertional logic of* K as the logic $\mathtt{L}^1_{\mathsf{K}} = (\mathcal{L}, \vdash^1_{\mathsf{K}})$ such that for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$

$$\Gamma \vdash^1_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{K}} \varphi \approx 1.$$

Let now K be a variety of algebras with semilattice reducts, and such that the associated order in algebras of K has a maximum defined by some term 1. The two associated logics $\mathtt{L}^{\leq}_{\mathsf{K}}$ and $\mathtt{L}^1_{\mathsf{K}}$ are said to be *companions* of each other, i.e., $\mathtt{L}^1_{\mathsf{K}}$ is the *assertional companion* of $\mathtt{L}^{\leq}_{\mathsf{K}}$, and the latter in turn is the *semilattice-based companion* of $\mathtt{L}^1_{\mathsf{K}}$. Moreover, $\mathtt{L}^{\leq}_{\mathsf{K}}$ is algebraizable if and only if it coincides with $\mathtt{L}^1_{\mathsf{K}}$ ([10, Theorem 4.2]).

It is clear that $\mathsf{K} = \mathsf{VA}$ satisfies the above assumptions; moreover, it follows from Theorem 4.6 that GV is the (algebraizable) 1-assertional logic of VA, and by Theorem 4.18 that LV is its semilattice based companion. Since GV and LV do not coincide, we get at once:

Corollary 4.21. LV *is not algebraizable.*

Given Corollary 4.19, we can actually extend this result to all the axiomatic extensions of Lewis's logics where global and local consequence do not collapse.

[5]We thank an anonymous referee for bringing this general approach to our attention, which helped simplify the presentation of the following results.

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Corollary 4.22. *If* Γ *is a set of axioms such that* LV + Γ *does not coincide with* GV + Γ*, then* LV + Γ *is not algebraizable.*

One can actually show that all the local variably strict conditional logics considered by Lewis's in his hierarchy are not algebraizable, since the global and local calculi do not coincide, except for LVCA, which collapses to classical logic ([17]).

Corollary 4.23. *No axiomatic extension of* LV *by the axioms in* {W, C, N, T, S, U, A} *is algebraizable, except (the ones coinciding with)* LVCA*. The strong and weak calculi* GVCA *and* LVCA *coincide, and they are both algebraizable with respect to the subvariety of* VA *where the identity* $x \Box\!\!\rightarrow y \approx x \rightarrow y$ *holds.*

5 Conclusions

The main objective of this paper is to provide a logico-algebraic analysis of Lewis's variably strict conditional logics, a subject that has been notably lacking in the literature. Our efforts have clarified several ambiguities surrounding these logics, explicitly defined and refined their properties, brought to light the deep connection with the modal logic framework, and introduced a novel general algebraic framework for their technical analysis. By doing so, this work aims to foster a fruitful synergy between a classical theme in formal philosophy and the advancements in abstract algebraic logic.

To the best of our knowledge, the model-theoretic tools and the techniques proper of the abstract algebraic logic framework we have used were not employed before to analyze these logics; these powerful tools proved instrumental in establishing several logical results, e.g. a deduction theorem, the representation of one logical consequence in terms of the other, and the strong completeness with respect to the algebraic structures for both versions of the calculi. These results collectively offer a deeper and more comprehensive understanding of the properties of Lewis's logics and the features of their models.

Moreover, while filling a notable void in the literature, this work is just the start of a formal investigation of Lewis's logics. The logico-algebraic machinery indeed offers a variety of tools to study logical properties from the algebraic point of view, the so-called *bridge theorems*. These are the core results of the field of algebraic logic; they allow one to study *metalogical* properties algebraically, concerning in particular the entailment of the logic, answering questions about its expressivity, or formalizing notions that can be relevant in applications [10, 15, 25]. These allow the study of, e.g.,: the lattice of (axiomatic) extensions of the logics; *definability* properties, i.e. to what extent implicit properties of the logic can be made explicit [5] and interpolation properties [19]; *admissible* rules [27], those that added to the logic do not change the theorems, and whether a logic is *structurally complete* (i.e., whether its admissible rules are also derivable) like classical logic but unlike intuitionistic logic [3, 2].

This work also lays the foundation for a deeper conceptual understanding of Lewis's logics beyond the technical sphere. In a forthcoming work, we will focus on the fact that to properly consider infinite models one should not simply consider sets of worlds, but *topological spaces*; i.e., the subsets of the universe that are meant to represent the formulas (the clopen sets of the topology) play a special role. In technical terms, we will show that the variety of algebras introduced in this work enjoys a *categorical duality* with respect to topological spaces based on Lewis's sphere models. From a conceptual standpoint, this will provide a fresh perspective into the similarity relationship among worlds and spheres, and the nature of the so-called *limit assumption*, stating the existence of most similar antecedent worlds.

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