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The asymptotic behaviour near the boundary of periodic homogenization problems via two-scale convergence

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The usual asymptotic expansion for the solutions of an elliptic linear problem with oscillatory periodic coefficients is known to not be accurate near the boundary. In order to obtain a better approximation it is necessary to add to this expansion a boundary-layer term. This term has been obtained by other authors in the case of a plane boundary, such that its normal is proportional to some period. We consider the case where the normal is arbitrary.

1. Introduction

Let us deal with the asymptotic behaviour when $\varepsilon \to 0$ of the solutions of

$$\left.\begin{aligned} -\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon = f \quad &\text{in } \Omega,\\ u_\varepsilon = 0 \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geqslant 2$, A is a matrix-valued function, uniformly elliptic, bounded and periodic, of period the unit cube Y in $\mathbb{R}^N$ and f is a given function in Ω. This homogenization problem arises for example in the study of the electric behaviour of a periodic medium with small period. The study of the asymptotic behaviour of the solution of (1.1) has been considered by several authors (see, for example, [1–3, 9–15]). Usually, we search for an asymptotic expansion of u_ε of the form

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots, \tag{1.2}$$

where $u_1, u_2, \dots$ are periodic of period Y in the second variable. Substituting this expression for u_ε in (1.1), introducing $\Gamma^1, \dots, \Gamma^N$ by

$$\left.\begin{aligned} -\operatorname{div} A(\nabla\Gamma^i + e_i) = 0 \quad \text{in } \mathbb{R}^N,\\ \Gamma^i \text{ is periodic of period } Y \end{aligned}\right\} \tag{1.3}$$

(they are defined up to a constant) with $e_1, \dots, e_N$ the usual basis of $\mathbb{R}^N$, and defining $A_H \in \mathcal{M}_N$ (the homogenized matrix) by

$$A_H e_i = \int_Y A(\nabla\Gamma^i + e_i)\,\mathrm{d}y, \tag{1.4}$$

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we find that u_0 is the solution of

$$\left.\begin{aligned} -\operatorname{div} A_H\nabla u_0 = f \quad &\text{in } \Omega, \\ u_0 = 0 \quad &\text{on } \partial\Omega \end{aligned}\right\} \tag{1.5}$$

and (up to a function which depends only on x) u_1 is given by

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u_0}{\partial x_i}(x)\Gamma^i(y). \tag{1.6}$$

It is well known (see, for example, [1–3,9–15]) that the solution u_ε of (1.1) converges weakly to u_0 in $H_0^1(\Omega)$, and assuming that u_1 is sufficiently smooth, $u_\varepsilon - u_0 - \varepsilon u_1(x,x/\varepsilon)$ converges strongly to zero in $H^1(\Omega)$. This justifies (1.2), but we remark that in general ∇u_0, and thus $u_1(x,x/\varepsilon)$, does not vanish on $\partial\Omega$. Thus, since $u_\varepsilon = 0$ on $\partial\Omega$, we deduce that (1.2) cannot give a good description of the behaviour of u_ε near the boundary. Deriving (1.2), we formally obtain

$$\nabla u_\varepsilon(x) \sim \nabla_x u_0(x) + \nabla_y u_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon\left(\nabla_x u_1\left(x,\frac{x}{\varepsilon}\right) + \nabla_y u_2\left(x,\frac{x}{\varepsilon}\right)\right) + \cdots.$$

So, we hope to have the estimate

$$\left\| u_\varepsilon - u_0 - u_1\left(x,\frac{x}{\varepsilon}\right)\right\|_{H^1(\Omega)} \leqslant C\varepsilon. \tag{1.7}$$

Assuming sufficient smoothness, estimate (1.7) holds if ∇u_0 and then the u_1 vanish on $\partial\Omega$. When ∇u_0 does not vanish on the boundary, we still have [11]

$$\left\| u_\varepsilon - u_0 - u_1\left(x,\frac{x}{\varepsilon}\right)\right\|_{H^1(\Omega')} \leqslant C_{\Omega'}\varepsilon, \tag{1.8}$$

for every open set Ω' with $\bar{\Omega}' \subset \Omega$, where $C_{\Omega'}$ depends on Ω'. However, in the whole set Ω, we only have (see, for example, [3, 12])

$$\left\| u_\varepsilon - u_0 - u_1\left(x,\frac{x}{\varepsilon}\right)\right\|_{H^1(\Omega)} \leqslant C\sqrt{\varepsilon}. \tag{1.9}$$

In order to obtain a better approximation of u_ε such that $u_0 + \varepsilon u_1(x,x/\varepsilon)$, it is necessary to add a boundary-layer term. To the best of our knowledge, this has been carried out only for a plane boundary with unit normal ν proportional to an element of $\mathbb{Z}^N$ (see, for example, [3, 12, 13]). The goal of the present paper is to study the case where ν is arbitrary. Our results will be applied in [5] to study the case for a curved boundary. The organization of the paper and the main results contained in it, are as follows.

In § 2, we introduce some notation and we recall some results about almost-periodic functions which will be used in this paper. In § 3 we obtain some estimates related to the behaviour of the sequence $u_0 + \varepsilon u_1(x,x/\varepsilon)$. Then, we show that obtaining a boundary-layer term which describes the behaviour of u_ε near $\partial\Omega$ is equivalent to studying the asymptotic behaviour of the solution b_ε of

$$\left.\begin{aligned} -\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla b_\varepsilon = 0 \quad \text{in } \Omega,\\ b_\varepsilon = u_1\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.10}$$

In § 4, we study the asymptotic behaviour of the solutions of (1.10) near a plane boundary $\Lambda = \{a\nu + z : z \in \omega\} \subset \partial\Omega$, where $a \in \mathbb{R}$ and ν is the inner unit normal to Ω on Λ. For this purpose, we obtain the two-scale limit $\check{b}(z, s, \zeta)$ (see [1,6,8,15–18]) of the sequence $\check{b}_\varepsilon(z, s)$, obtained from b_ε by the change of variables

$$x = z + (a + \varepsilon s)\nu, \quad z \in \omega, \ s \in \left(0, \frac{\delta}{\varepsilon}\right).$$

This is given in theorem 4.5, assuming that $\Omega \in C^{1,1}$, $A \in L^\infty_\sharp(Y; \mathcal{M}_N)$ and $f \in L^{N+\tau}(\Omega)$, $\tau > 0$. When ν is proportional to an element of $\mathbb{Z}^N$, $\check{b}(z, s, \zeta)$ is periodic in $\zeta \in \{\nu\}^\perp$. Then, assuming $u_0 \in W^{2,\infty}(\Omega)$ and defining $\check{b}^\varepsilon$ (it works better than $\check{b}$) by (4.15), we can use the sequence

$$\check{b}^\varepsilon\left(Px, \frac{|x - Px - a\nu|}{\varepsilon}, \frac{Px}{\varepsilon}\right)$$

with P the orthogonal projection from $\mathbb{R}^N$ onto $\{\nu\}^\perp$ to obtain an approximation of order ε of u_ε in H^1 of a neighbourhood of Λ (see [3, 12, 13] for related results). When ζ is not proportional to an element of $\mathbb{Z}^N$, $\check{b}$ (and $\check{b}^\varepsilon$) is only almost periodic in ζ in the Besicovitch sense. Thus, it is not a true function but a class of functions, and so, the sequence

$$\check{b}^\varepsilon\left(Px, \frac{|x - Px - a\nu|}{\varepsilon}, \frac{Px}{\varepsilon}\right)$$

is meaningless. Thus, although for composite materials it is usual to deal with a discontinuous matrix-valued function A, the only assumption $A \in L^\infty_\sharp(Y; \mathcal{M}_N)$ does not appear to be sufficient to obtain a good approximation of u_ε when ν is not proportional to an element of $\mathbb{Z}^N$. We remark that $\check{b}$ is obtained rigorously and so this difficulty is inherent to the problem.

In § 5, assuming $\Omega \in C^{1,1}$ and $A \in L^\infty_\sharp(Y; \mathcal{M}_N)$ are uniformly elliptic and $\partial_{y_\nu} A \in L^\infty_\sharp(Y; \mathcal{M}_N)$, $u_0 \in W^{2;\infty}(\Omega)$, we study the decay to infinity and some smoothness properties for $\check{b}^\varepsilon$. As a consequence, we get an approximation of order $\varepsilon\sqrt{|\log u_\varepsilon|}$ of u_ε in H^1 of a neighbourhood of Λ.

2. General notation

We take $N \in \mathbb{N}$, $N \geqslant 2$. We define $\{e_1, \dots, e_N\}$ as the usual basis of $\mathbb{R}^N$. The space of matrices of dimension $N \times N$ is denoted by $\mathcal{M}_N$.

We denote by Y the unit cube in $\mathbb{R}^N$. We will use the index $\sharp$ to mean Y-periodic, for example, $L^p_\sharp(Y)$, with $1 \leqslant p \leqslant +\infty$, denotes the space of functions of $L^p_{\text{loc}}(\mathbb{R}^N)$, which are Y-periodic.

The orthogonal set in $\mathbb{R}^N$ of a set $S \subset \mathbb{R}^N$ is denoted by $S^\perp$. For a function $u = u(x)$, defined in an open set of $\mathbb{R}^N$ and $\nu \in \mathbb{R}^N$, we denote by $\partial_{x_\nu} u$ the derivative of u in the direction ν.

We will denote by C a generic positive constant which can change from one line to another and does not depend on the parameter ε which appears in (1.1).

Given a subspace V of $\mathbb{R}^N$ and an additive group $\mathcal{G} \subset V$, we define $\mathcal{G}^*$ by

$$\mathcal{G}^* = \{g = (j_1, \dots, j_N) \in \mathcal{G} \setminus \{0\} : j_{i_0} > 0 \text{ with } i_0 = \min\{i : j_i \neq 0\}\}.$$

We denote by $CAP(\mathcal{G})$ the space of almost-periodic functions in the Bohr sense relating to $\mathcal{G}$. It is defined as the space of uniformly continuous functions $u : V \to \mathbb{R}$ such that, for every $\varepsilon > 0$, there exist $g^1, \dots, g^n \in \mathcal{G}^*$, $\alpha^0, \alpha^1, \dots, \alpha^n, \beta^1, \dots, \beta^n \in \mathbb{R}$ with

$$\left\| u - \alpha^0 - \sum_{i=1}^{n} (\alpha^i \cos(g^i \zeta) + \beta^i \sin(g^i \zeta)) \right\|_{L^\infty(V)} < \varepsilon.$$

We denote by $CAP^\infty(\mathcal{G})$ the space of functions of $CAP(\mathcal{G})$ with partial derivatives of arbitrary order in $CAP(\mathcal{G})$.

For $u \in L^1_{\mathrm{loc}}(V)$, we define $M(u)$ by

$$M(u) = \limsup_{R\to\infty} \frac{1}{|\{|\zeta| < R\}|} \int_{\{|\zeta|<R\}} u \, \mathrm{d}x, \tag{2.1}$$

where the measure $|\{|\zeta| < R\}|$ and the integral are taken in the sense of V.

The space of the almost-periodic functions in the Besicovitch sense relating to $\mathcal{G}$, $\mathcal{B}(\mathcal{G})$, is defined as the space of functions $u : V \to \mathbb{R}$ such that, for every $\varepsilon > 0$, there exists an almost-periodic function v in the sense of Bohr which satisfies $M(|u - v|^2) < \varepsilon$. We recall that, for every u in $\mathcal{B}(\mathcal{G})$, the supremum limit in (2.1) is in fact a limit, and it is finite.

In the space $\mathcal{B}(\mathcal{G})$ we define

$$\|u\|^2_{\mathcal{B}(\mathcal{G})} = M(|u|^2) \quad \text{for all } u \in \mathcal{B}(\mathcal{G}).$$

However, $\|\cdot\|_{\mathcal{B}(\mathcal{G})}$ is not a norm but a semi-norm. Note, for example, that every $u \in L^2(V)$ satisfies $\|u\|_{\mathcal{B}(\mathcal{G})} = 0$. In order to have a Hilbert space it is necessary to consider a quotient space. Thus, the elements of $\mathcal{B}(\mathcal{G})$ are not functions in V but rather are a class of functions. Two functions in the same class can differ in every point of V.

The space $CAP(\mathcal{G})$ is dense in $\mathcal{B}(\mathcal{G})$. Moreover, as elements of $\mathcal{B}(\mathcal{G})$, the functions of $CAP(\mathcal{G})$ have a unique representative which is uniformly continuous in V.

The space $\mathcal{B}(\mathcal{G})$ can be identified with the space (which we also denote by $\mathcal{B}(\mathcal{G})$)

$$\left\{ \alpha^0 + \sum_{g\in\mathcal{G}^*} (\alpha^g \cos(g\zeta) + \beta^g \sin(g\zeta)) : |\alpha^0|^2 + \sum_{g\in\mathcal{G}^*} (|\alpha^g|^2 + |\beta^g|^2) < +\infty \right\}, \tag{2.2}$$

endowed with the norm

$$\left\| \alpha^0 + \sum_{g\in\mathcal{G}^*} (\alpha^g \cos(g\zeta) + \beta^g \sin(g\zeta)) \right\|^2_{\mathcal{B}(\mathcal{G})} = |\alpha^0|^2 + \frac{1}{2} \sum_{g\in\mathcal{G}^*\setminus\{0\}} (|\alpha^g|^2 + |\beta^g|^2).$$

Using this representation of the elements of $\mathcal{B}(\mathcal{G})$, we can also define the gradient of a function of $\mathcal{B}(\mathcal{G})$ by

$$\nabla\biggl(\alpha^0 + \sum_{g\in\mathcal{G}^*}(\alpha^g\cos(g\zeta) + \beta^g\sin(g\zeta))\biggr) = \sum_{g\in\mathcal{G}^*} g(-\alpha^g\sin(g\zeta) + \beta^g\cos(g\zeta)) \quad (2.3)$$

(see also [6, 7, 16, 17]).

3. Global error estimates

As stated in § 1, we will study the asymptotic behaviour near the boundary of the solutions of (1.1), where Ω is a bounded open set of $\mathbb{R}^N$, $A \in L^\infty_\sharp(Y, \mathcal{M}_N)$ is uniformly elliptic, and f is an element of $H^{-1}(\Omega)$ (stronger smoothness properties for Ω, A and f will be needed later). We start this section by recalling some well-known results relating to the homogenization of (1.1).

For $1 \leqslant i \leqslant N$, we define Γ^i as the solutions of (1.3) or, more rigourously, as the solutions of the variational problem (the mean value of Γ^i is taken as zero in order to have uniqueness)

$$\left.\begin{aligned} \int_Y \Gamma^i\,\mathrm{d}y = 0, \quad \Gamma^i \in H^1_\sharp(Y),\\ \int_Y A(y)(\nabla\Gamma^i + e_i)\nabla v\,\mathrm{d}y = 0, \quad \forall v \in H^1_\sharp(Y).\end{aligned}\right\} \quad (3.1)$$

The homogenized matrix $A_H \in \mathcal{M}_N$ of A is defined by (1.4). We have the following theorem (see [1–3, 9, 10, 12–15]).

Theorem 3.1. *For $A \in L^\infty_\sharp(Y, \mathcal{M}_N)$ uniformly elliptic, and $f \in H^{-1}(\Omega)$, the solution u_ε of (1.1) converges weakly in $H^1_0(\Omega)$ to the unique solution u_0 of*

$$\left.\begin{aligned} -\operatorname{div} A_H\nabla u_0 = f \quad \text{in } \Omega,\\ u_0 = 0 \quad \text{on } \partial\Omega.\end{aligned}\right\} \quad (3.2)$$

Moreover, if $u_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}$ given by

$$u_1(x, y) = \sum_{i=1}^N \frac{\partial u_0}{\partial x_i}(x)\Gamma^i(y) \quad \text{for a.e. } (x, y) \in \Omega \times Y, \quad (3.3)$$

is sufficiently smooth, we have

$$u_\varepsilon - u_0 - \varepsilon u_1\biggl(x, \frac{x}{\varepsilon}\biggr) \to 0 \quad \text{in } H^1(\Omega). \quad (3.4)$$

Let us now improve assertion (3.4) by obtaining some estimates for the difference $u_\varepsilon - u_0 - \varepsilon u_1(x, x/\varepsilon)$. This will be a consequence of the following theorem, which we prove later, showing that $u_0 + \varepsilon u_1(x, x/\varepsilon)$ satisfies an equation similar to that satisfied by u_ε.

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Theorem 3.2. *Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, and $\Omega \in C^{1,1}$, $f \in L^N(\Omega)$. Then, for u_1 defined by (3.3), the application $x \in \Omega \to u_1(x, x/\varepsilon)$ belongs to $H^1(\Omega)$, and there exists $C > 0$, independent of ε, such that*

$$\left\| u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leqslant C, \qquad \left\| u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{H^1(\Omega)} \leqslant \frac{C}{\varepsilon}. \tag{3.5}$$

Moreover, the function $u_0 + \varepsilon u_1(x, x/\varepsilon)$ satisfies

$$-\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla\left(u_0 + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)\right) = f + \varepsilon g_\varepsilon \quad \text{in } \Omega, \tag{3.6}$$

in the sense of the distributions, where g_ε is bounded in $H^{-1}(\Omega)$.

Remark 3.3. If, in addition to the hypotheses of lemma 3.2, we assume that ∇u_0 vanishes on the boundary of Ω, then, from (3.3), we obtain that $u_1(x, x/\varepsilon)$ is in $H^1_0(\Omega)$, and so, taking $u_\varepsilon - u_0 - \varepsilon u_1(x, x/\varepsilon)$ as a test function in the difference of (1.1) and (3.6), we deduce the existence of $C > 0$ such that

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)\right\|_{H^1_0(\Omega)} \leqslant C\varepsilon. \tag{3.7}$$

When ∇u_0 does not vanish on $\partial\Omega$, all we can prove is estimate (3.8), below. This estimate was obtained in [3] (see also [12]). It can also be deduced from (3.6), taking into account the fact that if $u_0 \in W^{1,\infty}(\Omega)$ (for it is sufficient to assume $f \in L^{N+\tau}(\Omega)$, $\tau > 0$), then, for $\phi \in C^\infty([0, +\infty))$ with $\phi(0) = 0$, $\phi(s) = 1$ for $s > 1$, we have

$$\left\| \varepsilon\phi\left(\frac{\mathrm{d}(x, \partial\Omega)}{\varepsilon}\right)u_1\left(x, \frac{x}{\varepsilon}\right) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)\right\|_{H^1(\Omega)} \leqslant C\sqrt{\varepsilon}$$

and

$$u_0(x) + \varepsilon\phi\left(\frac{\mathrm{d}(x, \partial\Omega)}{\varepsilon}\right)u_1\left(x, \frac{x}{\varepsilon}\right) = 0 \quad \text{on } \partial\Omega.$$

Theorem 3.4. *Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic and that $\Omega \in C^{1,1}$, $f \in L^{N+\tau}(\Omega)$, $\tau > 0$. Then, taking u_ε as the solution of (1.1) and defining u_0, u_1 by (3.2) and (3.3), we get*

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)\right\|_{H^1(\Omega)} \leqslant C\sqrt{\varepsilon}. \tag{3.8}$$

In order to obtain an approximation of u_ε better than $u_0 + \varepsilon u_1(x, x/\varepsilon)$ we need to search for a boundary-layer term. In this sense, we have the following result.

Corollary 3.5. *Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, $\Omega \in C^{1,1}$, $f \in L^{N+\tau}(\Omega)$, $\tau > 0$, and define u_ε, u_0 and u_1 by (1.1), (3.2) and (3.3), respectively. We introduce b_ε as the solution of*

$$\left.\begin{aligned} -\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla b_\varepsilon = 0 \quad \text{in } \Omega,\\ b_\varepsilon = u_1\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.9}$$

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Then, we have

$$\left\| u_\varepsilon - \left(u_0 + \varepsilon \left(u_1\left(x, \frac{x}{\varepsilon}\right) - b_\varepsilon \right) \right) \right\|_{H_0^1(\Omega)} \leqslant C\varepsilon \tag{3.10}$$

and

$$\|b_\varepsilon\|_{H^1(\Omega)} \leqslant \frac{C}{\sqrt{\varepsilon}}. \tag{3.11}$$

Proof. From (3.6) and (3.9), $u_0 + \varepsilon(u_1(x, x/\varepsilon) - b_\varepsilon)$ satisfies

$$\left.\begin{aligned} -\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla\left(u_0 + \varepsilon\left(u_1\left(x, \frac{x}{\varepsilon}\right) - b_\varepsilon\right)\right) &= f + \varepsilon g_\varepsilon \quad \text{in } \Omega, \\ u_0 + \varepsilon\left(u_1\left(x, \frac{x}{\varepsilon}\right) - b_\varepsilon\right) &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.12}$$

with g_ε bounded in $H^{-1}(\Omega)$. Taking $u_\varepsilon - (u_0 + \varepsilon(u_1(x, x/\varepsilon) - b_\varepsilon))$ as a test function in the difference of (1.1) and (3.12), we deduce (3.10). From (3.8), we then obtain (3.11). □

Remark 3.6. In the hypotheses of corollary 3.5, b_ε is bounded in $L^\infty(\Omega)$ by the maximum principle. Then, taking $b_\varepsilon\varphi^2$, with $\varphi \in C_c^\infty(\Omega)$, as a test function in (3.9), we may deduce that $b_\varepsilon\varphi$ is bounded in $H^1(\Omega)$, i.e. b_ε is bounded in $H^1_{\mathrm{loc}}(\Omega)$. From (3.10), we then get

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) \right\|_{H^1(\Omega')} \leqslant C_{\Omega'}\varepsilon,$$

for every open set Ω' with $\bar{\Omega}' \subset \Omega$, where $C_{\Omega'}$ depends on Ω'. A better result has been obtained in [11], where it is assumed only that $f \in L^2(\Omega')$. In this case, it is necessary to replace u_1 by a regularization of it.

Proof of theorem 3.2. Since $\Omega \in C^{1,1}$ and $f \in L^N(\Omega)$, the solution u_0 of (3.2) is in $W^{2,N}(\Omega)$. Moreover, since $-\operatorname{div}_y A(y)e_i$, $1 \leqslant i \leqslant N$, belongs to $W^{-1,\infty}(\mathbb{R}^N)$, we deduce (see lemma A.1) that the solutions Γ^i of (3.1) are in $C_\sharp^{0,\alpha}(Y)$, for every $\alpha \in [0, 1)$. In particular, they are in $L_\sharp^\infty(Y)$. By Meyer's regularity theorem, we also know that there exists $p > 2$ such that Γ^i belongs to $W_\sharp^{1,p}(Y)$. From these smoothness properties of u_0 and Γ^i, and using (3.3), we deduce (3.5).

To prove (3.6), we consider $v \in H_0^1(\Omega)$, which, by taking its extension by zero, we assume to be defined in the whole space $\mathbb{R}^N$. Since Ω is $C^{1,1}$ and u_0 is in $W^{2,N}(\Omega)$, we also know that there exists an extension of u_0, still denoted by u_0, in $W^{2,N}(\mathbb{R}^N)$. Using the definitions (1.4) and (3.3) of A_H and u_1, respectively, we have

$$\begin{aligned} &\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\left(u_0 + \varepsilon\nabla\left[u_1\left(x, \frac{x}{\varepsilon}\right)\right]\right)\nabla v \,\mathrm{d}x \\ &\quad = \int_\Omega A_H \nabla u_0 \nabla v \,\mathrm{d}x + \sum_{i=1}^N \int_\Omega h_i\left(\frac{x}{\varepsilon}\right)\nabla v \frac{\partial u_0}{\partial x_i} \,\mathrm{d}x \\ &\quad\quad + \varepsilon \sum_{i=1}^N \int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\left(\frac{\partial u_0}{\partial x_i}\right)\nabla v \Gamma^i\left(\frac{x}{\varepsilon}\right) \mathrm{d}x, \end{aligned} \tag{3.13}$$

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with

$$h_i(y) = A(y)(\nabla\Gamma^i(y) + e_i) - \int_Y A(\rho)(\nabla\Gamma^i(\rho) + e_i)\,\mathrm{d}\rho$$
$$= A(y)(\nabla\Gamma^i(y) + e_i) - \int_Y A(y+\rho)(\nabla\Gamma^i(y+\rho) + e_i)\,\mathrm{d}\rho \quad \text{for a.e. } y \in \mathbb{R}^N.$$

Since u_0 is in $W^{2,N}(\Omega)$ (it would be sufficient for u_0 to be in $H^2(\Omega)$) and Γ^i in $L^\infty_\sharp(Y)$, $1 \leqslant i \leqslant N$, the third term on the right-hand side of (3.13) satisfies

$$\left|\varepsilon \sum_{i=1}^N \int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\left(\frac{\partial u_0}{\partial x_i}\right)\nabla v \Gamma^i\left(\frac{x}{\varepsilon}\right)\mathrm{d}x\right| \leqslant C\varepsilon\|v\|_{H^1_0(\Omega)}, \tag{3.14}$$

By (3.2), the first term on the right-hand side of (3.13) satisfies

$$\int_\Omega A_H \nabla u_0 \nabla v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x. \tag{3.15}$$

In order to estimate the second term on the right-hand side of (3.13), we use the fact that h_i has mean value zero in Y. Thus, for $1 \leqslant i \leqslant N$, we have

$$\begin{aligned}
&\int_\Omega h_i\left(\frac{x}{\varepsilon}\right)\nabla v \frac{\partial u_0}{\partial x_i}\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\left(\nabla v \frac{\partial u_0}{\partial x_i} - \frac{1}{\varepsilon^N}\int_{\varepsilon Y} \nabla v(x+\rho)\frac{\partial u_0}{\partial x_i}(x+\rho)\,\mathrm{d}\rho\right)\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\nabla v\left(\frac{\partial u_0}{\partial x_i} - \frac{1}{\varepsilon^N}\int_{\varepsilon Y} \frac{\partial u_0}{\partial x_i}(x+\rho)\,\mathrm{d}\rho\right)\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\frac{1}{\varepsilon^N}\int_{\varepsilon Y}\frac{\partial u_0}{\partial x_i}(x+\rho)(\nabla v(x) - \nabla v(x+\rho))\,\mathrm{d}\rho\,\mathrm{d}x.
\end{aligned} \tag{3.16}$$

For the first term on the right-hand side of this equality we use

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\nabla v\left(\frac{\partial u_0}{\partial x_i} - \frac{1}{\varepsilon^N}\int_{\varepsilon Y}\frac{\partial u_0}{\partial x_i}(x+\rho)\,\mathrm{d}\rho\right)\mathrm{d}x\right| \\
&\qquad \leqslant \sum_{k\in\mathbb{Z}^N}\int_{\varepsilon k+\varepsilon Y}\left|h_i\left(\frac{x}{\varepsilon}\right)\right||\nabla v|\left|\frac{\partial u_0}{\partial x_i} - \frac{1}{\varepsilon^N}\int_{\varepsilon Y}\frac{\partial u_0}{\partial x_i}(x+\rho)\,\mathrm{d}\rho\right|\mathrm{d}x.
\end{aligned}$$

Now, using the fact that h_i is in $L^p_\sharp(Y)$ for some $p > 2$, the periodicity of h_i, and the inequality (which can easily be proved by using a translation and a dilatation that transforms $\varepsilon k + \varepsilon Y$ in Y)

$$\begin{aligned}
&\int_{\varepsilon k+\varepsilon Y}\int_{\varepsilon Y}|w(x) - w(x+\rho)|^q\,\mathrm{d}\rho\,\mathrm{d}x \\
&\qquad \leqslant C_{q,r}\varepsilon^{2N-((N-r)q)/r}\left(\int_{\varepsilon k+2\varepsilon Y}|\nabla w|^r\,\mathrm{d}x\right)^{q/r}, \quad \forall k \in \mathbb{N}, \\
&\qquad\qquad \forall r \in [1,+\infty), \quad \forall q \in \left[1, \frac{Nr}{N-r}\right), \quad \forall w \in W^{1,r}(\varepsilon k + 2\varepsilon Y),
\end{aligned} \tag{3.17}$$

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applied to $q = 2p/(p-2)$, $r = N$, $w = \partial u_0/\partial x_i$, we get

$$\begin{aligned}\int_{\varepsilon k+\varepsilon Y} \left|h_i\left(\frac{x}{\varepsilon}\right)\right| |\nabla v| \left|\frac{\partial u_0}{\partial x_i} - \frac{1}{\varepsilon^N}\int_{\varepsilon Y} \frac{\partial u_0}{\partial x_i}(x+\rho)\,\mathrm{d}\rho\right| \mathrm{d}x \\ \leqslant \left(\int_{\varepsilon k+\varepsilon Y} \left|h_i\left(\frac{x}{\varepsilon}\right)\right|^p \mathrm{d}x\right)^{1/p} \left(\int_{\varepsilon k+\varepsilon Y} |\nabla v|^2\,\mathrm{d}x\right)^{1/2} \\ \times \left(\frac{1}{\varepsilon^N}\int_{\varepsilon k+\varepsilon Y}\int_{\varepsilon Y} \left|\frac{\partial u_0}{\partial x_i}(x) - \frac{\partial u_0}{\partial x_i}(x+\rho)\right|^{2p/(p-2)} \mathrm{d}\rho\,\mathrm{d}x\right)^{(p-2)/2p} \\ \leqslant C\varepsilon^{N/2}\left(\int_Y |h_i|^p\,\mathrm{d}y\right)^{1/p}\left(\int_{\varepsilon k+\varepsilon Y} |\nabla v|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\varepsilon k+2\varepsilon Y} |D^2 u_0|^N\,\mathrm{d}x\right)^{1/N}.\end{aligned}$$

So, from Hölder's inequality and $\nabla v = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^N}\int_{\varepsilon k+\varepsilon Y} \left|h_i\left(\frac{x}{\varepsilon}\right)\right| |\nabla v| \left|\frac{\partial u_0}{\partial x_i} - \frac{1}{\varepsilon^N}\int_{\varepsilon Y} \frac{\partial u_0}{\partial x_i}(x+\rho)\,\mathrm{d}\rho\right| \mathrm{d}x \\ \leqslant C\varepsilon^{N/2}\left(\sum_{k\in\mathbb{Z}^N}\int_{\varepsilon k+\varepsilon Y} |\nabla v|^2\,\mathrm{d}x\right)^{1/2} \\ \times \left(\sum_{k\in\mathbb{Z}^N}\int_{\varepsilon k+2\varepsilon Y} |D^2 u_0|^N\,\mathrm{d}x\right)^{1/N}\left(\sum_{(\varepsilon k+\varepsilon Y)\cap\Omega\neq\emptyset} 1\right)^{(N-2)/2N} \\ \leqslant C\varepsilon\|v\|_{H_0^1(\Omega)}.\end{aligned}$$

Thus, we have

$$\left|\int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\nabla v\left(\frac{\partial u_0}{\partial x_i} - \frac{1}{\varepsilon^N}\int_{\varepsilon Y}\frac{\partial u_0}{\partial x_i}(x+\rho)\,\mathrm{d}\rho\right)\mathrm{d}x\right| \leqslant C\varepsilon\|v\|_{H_0^1(\Omega)}. \tag{3.18}$$

To estimate the last term of (3.16) we use the fact that (3.1) and lemma A.1 imply that the divergence of h_i is zero in $\mathbb{R}^N$. So,

$$\begin{aligned}\int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\frac{1}{\varepsilon^N}\int_{\varepsilon Y}\frac{\partial u_0}{\partial x_i}(x+\rho)(\nabla v(x) - \nabla v(x+\rho))\,\mathrm{d}\rho\,\mathrm{d}x \\ = \int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\nabla\left(\frac{1}{\varepsilon^N}\int_{\varepsilon Y}\frac{\partial u_0}{\partial x_i}(x+\rho)(v(x) - v(x+\rho))\,\mathrm{d}\rho\right)\mathrm{d}x \\ - \int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\left(\frac{1}{\varepsilon^N}\int_{\varepsilon Y}\nabla\left(\frac{\partial u_0}{\partial x_i}\right)(x+\rho)(v(x) - v(x+\rho))\,\mathrm{d}\rho\right)\mathrm{d}x \\ = -\int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\left(\frac{1}{\varepsilon^N}\int_{\varepsilon Y}\nabla\left(\frac{\partial u_0}{\partial x_i}\right)(x+\rho)(v(x) - v(x+\rho))\,\mathrm{d}\rho\right)\mathrm{d}x.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} h_i\left(\frac{x}{\varepsilon}\right)\left(\frac{1}{\varepsilon^N}\int_{\varepsilon Y}\nabla\left(\frac{\partial u_0}{\partial x_i}\right)(x+\rho)(v(x) - v(x+\rho))\,\mathrm{d}\rho\right)\mathrm{d}x\right| \\ \leqslant \sum_{k\in\mathbb{Z}^N}\int_{\varepsilon k+\varepsilon Y}\left|h_i\left(\frac{x}{\varepsilon}\right)\right|\frac{1}{\varepsilon^N}\int_{\varepsilon Y}|D^2 u_0(x+\rho)||v(x) - v(x+\rho)|\,\mathrm{d}\rho\,\mathrm{d}x.\end{aligned}$$

Now, by (3.17), with $w = v$, $r = 2$, $q = Np/(N(p-1)-p)$, for every $k \in \mathbb{Z}^N$, we have

$$\begin{aligned}
&\int_{\varepsilon k+\varepsilon Y}\left|h_i\left(\frac{x}{\varepsilon}\right)\right|\frac{1}{\varepsilon^N}\int_{\varepsilon Y}|D^2u_0(x+\rho)||v(x)-v(x+\rho)|\,\mathrm{d}\rho\,\mathrm{d}x\\
&\quad\leqslant\left(\varepsilon^N\int_Y|h_i|^p\,\mathrm{d}y\right)^{1/p}\\
&\qquad\times\left(\int_{\varepsilon k+\varepsilon Y}\frac{1}{\varepsilon^N}\int_{\varepsilon Y}|D^2u_0(x+\rho)|^{p/(p-1)}|v(x)-v(x+\rho)|^{p/(p-1)}\,\mathrm{d}\rho\,\mathrm{d}x\right)^{(p-1)/p}\\
&\quad\leqslant\varepsilon^{-N(p-2)/p}\left(\int_Y|h_i|^p\,\mathrm{d}y\right)^{1/p}\left(\int_{\varepsilon k+\varepsilon Y}\int_{\varepsilon Y}|D^2u_0(x+\rho)|^N\,\mathrm{d}\rho\,\mathrm{d}x\right)^{1/N}\\
&\qquad\times\left(\int_{\varepsilon k+\varepsilon Y}\int_{\varepsilon Y}|v(x)-v(x+\rho)|^{Np/(N(p-1)-p)}\,\mathrm{d}\rho\,\mathrm{d}x\right)^{(N(p-1)-p)/Np}\\
&\quad\leqslant C\varepsilon^{N/2}\left(\int_Y|h_i|^p\,\mathrm{d}y\right)^{1/p}\left(\int_{\varepsilon k+2\varepsilon Y}|D^2u_0|^N\,\mathrm{d}x\right)^{1/N}\left(\int_{\varepsilon k+2\varepsilon Y}|\nabla v|^2\,\mathrm{d}x\right)^{1/2}.
\end{aligned}$$

So, using Hölder's inequality and $\nabla v = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}h_i\left(\frac{x}{\varepsilon}\right)\left(\frac{1}{\varepsilon^N}\int_{\varepsilon Y}\nabla\left(\frac{\partial u_0}{\partial x_i}\right)(x+\rho)(v(x)-v(x+\rho))\,\mathrm{d}\rho\right)\mathrm{d}x\right|\\
&\quad\leqslant C\varepsilon^{N/2}\left(\int_\Omega|D^2u_0|^N\,\mathrm{d}x\right)^{1/N}\left(\int_\Omega|\nabla v|^2\,\mathrm{d}x\right)^{1/2}\left(\sum_{(\varepsilon k+\varepsilon Y)\cap\Omega\neq\emptyset}1\right)^{(N-2)/2N}\\
&\quad\leqslant C\varepsilon\|v\|_{H_0^1(\Omega)}.
\end{aligned}\tag{3.19}$$

From (3.13)–(3.19), we then deduce that there exists $C > 0$, which does not depend on ε, such that, for every $v \in H_0^1(\Omega)$, we have

$$\left|\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\left(u_0+\varepsilon\nabla\left[u_1\left(x,\frac{x}{\varepsilon}\right)\right]\right)\nabla v\,\mathrm{d}x-\int_\Omega fv\,\mathrm{d}x\right|\leqslant C\varepsilon\|v\|_{H_0^1(\Omega)}.$$

This proves (3.6). □

4. Two-scale convergence for the expansion near a plane boundary

In the present section, let us study the asymptotic behaviour of the sequence b_ε defined by (3.9) in the case of a plane boundary.

In the rest of the paper, we consider an unit vector ν of $\mathbb{R}^N$, and we denote by P the orthogonal projection of $\mathbb{R}^N$ onto $\{\nu\}^\perp$. We assume there exist an open set ω of $\{\nu\}^\perp$, $a \in \mathbb{R}$, and $\delta > 0$ such that

$$\Lambda = \omega + a\nu \subset \partial\Omega, \tag{4.1}$$

$$\Lambda_\delta = \{z + (a+t)\nu : z \in \omega,\ 0 < t < \delta\} \subset \Omega. \tag{4.2}$$

Thus, $\Lambda \subset \Omega$ is an open subset of a hyperplane and ν is the inner unit normal to Ω in Λ. To study the asymptotic behaviour of the solution b_ε of (3.9) near Λ, let us first realize a dilatation of Λ_δ in the direction of ν. For this purpose, we introduce the change of variables

$$z+(a+t)\nu \in \Lambda_\delta \to (z,s)=\left(z,\frac{t}{\varepsilon}\right) \in \omega \times \left(0,\frac{\delta}{\varepsilon}\right). \tag{4.3}$$

We introduce the following definitions.

DEFINITION 4.1. Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, and that $\Omega \in C^{1,1}$, $f \in L^{N+\tau}(\Omega)$, $\tau>0$.

We define $\check{A}_\varepsilon : \mathbb{R}^+ \times \{\nu\}^\perp \to \mathcal{M}_N$ by

$$\check{A}_\varepsilon(s,\zeta)=A\left(\zeta+\left(\frac{a}{\varepsilon}+s\right)\nu\right). \tag{4.4}$$

For the solution b_ε of (3.9), we define $\check{b}_\varepsilon : \omega \times (0,\delta/\varepsilon) \to \mathbb{R}$ by

$$\check{b}_\varepsilon(z,s)=b_\varepsilon(z+(a+\varepsilon s)\nu), \quad \text{a.e. } (z,s) \in \omega \times \left(0,\frac{\delta}{\varepsilon}\right). \tag{4.5}$$

Also, we introduce the differential operator D_ε as

$$D_\varepsilon v = \varepsilon \nabla_z v + \frac{\partial v}{\partial s}\nu \quad \text{for all } v \in H^1\left(\omega \times \left(0,\frac{\delta}{\varepsilon}\right)\right).$$

Here $\nabla_z v$ is the unique vector in the direction of $\{\nu\}^\perp$, which is such that, for every $\mu \in \{\nu\}^\perp$, the derivative of v in the direction $(\mu,0) \in \{\nu\}^\perp \times \mathbb{R}$ coincides with $\nabla_z v \mu$.

From (3.9) and (3.11), we easily deduce that $\check{b}_\varepsilon$ satisfies the following proposition.

PROPOSITION 4.2. *Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, and $\Omega \in C^{1,1}$, $f \in L^{N+\tau}(\Omega)$, $\tau>0$. Then, defining $\check{A}_\varepsilon$ by (4.4) and $\check{b}_\varepsilon$ by (4.5), we have*

$$\|D_\varepsilon \check{b}_\varepsilon\|_{L^2(\omega\times(0,\delta/\varepsilon))^N} \leqslant C, \tag{4.6}$$

$$\left.\begin{aligned} \check{b}_\varepsilon(z,0)=u_1\left(z+a\nu,\frac{z+a\nu}{\varepsilon}\right), \quad \check{b}_\varepsilon \in H^1\left(\omega\times\left(0,\frac{\delta}{\varepsilon}\right)\right),\\ \int_{\omega\times(0,\delta/\varepsilon)} \check{A}_\varepsilon\left(s,\frac{z}{\varepsilon}\right) D_\varepsilon \check{b}_\varepsilon D_\varepsilon v \,\mathrm{d}z\,\mathrm{d}t = 0,\\ \textit{for all } v \in H^1_0\left(\omega\times\left(0,\frac{\delta}{\varepsilon}\right)\right). \end{aligned}\right\} \tag{4.7}$$

REMARK 4.3. Using the fact that A is periodic, we find that it can be obtained as a limit in $L^2(\Omega,\mathcal{M}_N)$ of trigonometrical matrices of the form

$$R^0_n + \sum_{k\in\mathbb{Z}^N,|k|\leqslant n} [R^k_n \cos(2k\pi x) + S^k_n \sin(2k\pi x)].$$

We then find that $\check{A}_\varepsilon(s,\zeta)$ can be obtained as the limit of the matrices

$$\begin{aligned} R_n^0 &+ \sum_{k\in\mathbb{Z}^N,|k|\leqslant n}\biggl[R_n^k\cos\biggl(2k\pi\zeta+\biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr)+S_n^k\sin\biggl(2k\pi\zeta+\biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr)\biggr] \\ &= R_n^0+\sum_{k\in\mathbb{Z}^N,|k|\leqslant n}\biggl[\biggl(R_n^k\cos\biggl(\biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr)+S_n^k\sin\biggl(\biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr)\biggr)\cos(2\pi Pk\zeta) \\ &\qquad + \biggl(-R_n^k\sin\biggl(\biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr)+S_n^k\cos\biggl(\biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr)\biggr)\sin(2\pi Pk\zeta)\biggr]\end{aligned}$$

and hence, as a function of ζ, it is in $\mathcal{B}(G)$ with

$$\mathcal{G}=\{2\pi Pk : k\in\mathbb{Z}^N\}\subset\{\nu\}^\perp. \tag{4.8}$$

Moreover, we have

$$\check{A}_\varepsilon(s,\zeta)=A\biggl(\zeta+\biggl[\frac{a\nu}{\varepsilon}\biggr]+s\nu\biggr),\quad \biggl[\frac{a\nu}{\varepsilon}\biggr]\in[0,1)^N,\quad \frac{a\nu}{\varepsilon}-\biggl[\frac{a\nu}{\varepsilon}\biggr]\in\mathbb{Z}^N.$$

In particular, $[a\nu/\varepsilon]$ is bounded and then, extracting a subsequence if necessary, we can assume that

$$\exists\varrho=\lim_{\varepsilon\to 0}\biggl[\frac{a\nu}{\varepsilon}\biggr]. \tag{4.9}$$

Since $\check{A}_\varepsilon(s,\zeta)$ is almost periodic in ζ, let us use the two-scale convergence theory (see [1, 6, 8, 15–17]) to study the asymptotic behaviour of the solutions of (4.7). A definition of two-scale convergence which is useful for our purpose is as follows.

Definition 4.4. We say that a sequence $\check{v}_\varepsilon\in L^2(\omega\times(a,b))$ two-scale converges to a function $\check{v}\in L^2(\omega\times(a,b);\mathcal{B}(\mathcal{G}))$, and we write

$$\check{v}_\varepsilon \overset{2e}{\rightharpoonup} \check{v}\quad \text{in } \omega\times(a,b),$$

if and only if we have

$$\left.\begin{aligned}\lim_{\varepsilon\to0}\int_{\omega\times(a,b)}\check{v}_\varepsilon(z,s)\psi\biggl(z,s,\frac{z}{\varepsilon}\biggr)\,\mathrm{d}z\,\mathrm{d}s=\int_{\omega\times(a,b)}M_\zeta(\check{v}(z,s,\zeta)\psi(z,s,\zeta))\,\mathrm{d}z\,\mathrm{d}s \\ \forall\psi\in C_{\mathrm{c}}^\infty(\omega\times(a,b);CAP^\infty(\mathcal{G})).\end{aligned}\right\} \tag{4.10}$$

With this definition the main result of the present section is the following theorem, which we will prove later.

Theorem 4.5. *Assume that $A\in L_\sharp^\infty(Y;\mathcal{M}_N)$ is uniformly elliptic, and $\Omega\in C^{1,1}$, $f\in L^{N+\tau}(\Omega)$, $\tau>0$. Consider a subsequence of ε, such that there exists the limit ϱ which appears in (4.9), and define $\check{b}$ as the unique solution of the variational problem:*

$$\left.\begin{aligned}\check{b}(z,0,\zeta)=u_1(z+a\nu,\zeta+\varrho),\ D_0\check{b}\in L^2(\omega\times\mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N \\ \int_{\mathbb{R}^+}M_\zeta(A(\zeta+\varrho+s\nu)D_0\check{b}D_0\check{v})\,\mathrm{d}s=0 \\ \forall\check{v}\ \text{with}\ \check{v}(0,\zeta)=0,\quad D_0\check{v}\in L^2(\mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N,\ \text{a.e.}\ z\in\omega.\end{aligned}\right\} \tag{4.11}$$

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Then, the sequence $\check{b}_\varepsilon$ *defined by (4.5) satisfies*

$$\check{b}_\varepsilon \overset{2\text{e}}{\rightharpoonup} \check{b} \qquad \text{in } \omega \times (0, M) \quad \text{for all } M > 0, \tag{4.12}$$

$$D_\varepsilon \check{b}_\varepsilon \chi_{\omega\times\mathbb{R}^+} \overset{2\text{e}}{\rightharpoonup} D_0 \check{b} \quad \text{in } \omega \times \mathbb{R}^+. \tag{4.13}$$

Remark 4.6. If we assume that ν belongs to $\mathbb{R}\cdot\mathbb{Z}^N$, then the elements of $\mathcal{B}(\mathcal{G})$ are periodic in $\{\nu\}^\perp$. So, using the results which appear in [4] (related results have been obtained by L. Tartar (personal communication) and G. Weiske [19]), we find that the solution $\check{b}$ of (4.11) satisfies that $D_0\check{b}$ decreases exponentially as $s \to \infty$, in the sense that there exists $\lambda > 0$ with $\mathrm{e}^{\lambda s} M_\zeta(D_0\check{b}) \in L^2(\omega \times \mathbb{R}^+)^N$. So, in this case $\check{b}$ coincides with the boundary-value term which was used in [3, 12, 13] to study the asymptotic behaviour of the solution u_ε of (1.1) near a boundary plane with unit normal proportional to an element of $\mathbb{Z}^N$.

Remark 4.7. From theorem 4.5, $D_\varepsilon\check{b}_\varepsilon$ two-scale converges to $D_0\check{b}$. Thus, assuming sufficient smoothness, we expect (see, for example, [1, 8, 15, 16]) the approximation $D_\varepsilon\check{b}_\varepsilon(z, s) \sim D_0\check{b}(z, s, z/\varepsilon)$. From the definition (4.4) of $\check{b}_\varepsilon$ and (3.10), this implies that

$$\nabla u_\varepsilon \sim \nabla u_0 + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) - D_0\check{b}\left(Px, \frac{|x - Px - a\nu|}{\varepsilon}, \frac{Px}{\varepsilon}\right) \tag{4.14}$$

near Λ. But this only holds for a subsequence such that (4.9) holds. To avoid the extraction of this subsequence we can replace $\check{b}$ by the solution $\check{b}^\varepsilon$ of (cf. the definition (4.11) of $\check{b}$)

$$\left.\begin{aligned} &\check{b}^\varepsilon(z, 0, \zeta) = u_1\left(z + a\nu, \zeta + \frac{a}{\varepsilon}\nu\right), \quad D_0\check{b}^\varepsilon \in L^2(\omega\times\mathbb{R}^+; \mathcal{B}(\mathcal{G}))^N \\ &\int_{\mathbb{R}^+} M_\zeta\left(A\left(\zeta + \left(\frac{a}{\varepsilon} + s\right)\nu\right) D_0\check{b}^\varepsilon D_0\check{v}\right) \mathrm{d}z\, \mathrm{d}s = 0 \\ &\text{for all } \check{v} \text{ with } \check{v}(0, \zeta) = 0, D_0\check{v} \in L^2(\mathbb{R}^+; \mathcal{B}(\mathcal{G}))^N, \text{ a.e. } z \in \omega. \end{aligned}\right\} \tag{4.15}$$

In the case where ν is proportional to an element of $\mathbb{Z}^N$, we know that $\check{b}^\varepsilon$ is periodic in ζ and decreases exponentially to infinity. In this case, assuming $u_0 \in W^{2,\infty}(\Omega)$ (take, for example, $f \in C^{0,\alpha}(\bar{\Omega})$, $\alpha > 0$) we can prove that, for every compact set $K \subset \Lambda$, every $\psi \in C^1(\bar{\Omega})$, with $\operatorname{supp}(\psi) \subset \bar{\Lambda}_\delta$, $\psi = 1$ on K, and every $\phi \in C^\infty(\bar{\Omega})$, with $\phi = 0$ on $\partial\Omega \setminus K$, we have

$$\left\|\left(u_\varepsilon - u_0 - \varepsilon\left(u_1\left(x, \frac{x}{\varepsilon}\right) - \psi\check{b}^\varepsilon\left(Px, \frac{|x - Px - a\nu|}{\varepsilon}, \frac{Px}{\varepsilon}\right)\right)\right)\phi\right\|_{H^1(\Omega)} \leqslant C\varepsilon \tag{4.16}$$

(ϕ can be taken to equal 1 if $\nabla u_0 = 0$ outside K). We do not give the proof of this result because it is very similar to that of theorem 5.4, below. Moreover, we are more interested in the case where ν is not proportional to an element of $\mathbb{Z}^N$, which is the main novelty of the present paper. As we stated in § 1, to obtain a similar result when ν is not proportional to an element of $\mathbb{Z}^N$ (then the functions $\check{b}^\varepsilon$ are only almost periodic in ζ, and so they are not true functions) we will need to consider stronger smoothness assumptions on A.

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The sequence $\check{b}^\varepsilon$ depends on ε, but we will see in § 5 how it can be obtained from a fixed function.

Remark 4.8. Defining $\check{\gamma}_\varepsilon^i$, $1 \leqslant i \leqslant N$, by

$$\left.\begin{aligned}&\check{\gamma}_\varepsilon^i(0,\zeta) = \Gamma^i\biggl(\zeta + \frac{a}{\varepsilon}\nu\biggr), \quad D_0\check{\gamma}_\varepsilon^i \in L^2(\mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N\\&\int_0^{+\infty} M_\zeta\biggl(A\biggl(\zeta + \biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr)D_0\check{\gamma}_\varepsilon^i D_0\check{v}\biggr)\,\mathrm{d}s = 0\\&\text{for all } \check{v} \text{ with } \check{v}(0,\zeta)=0, D_0\check{v} \in L^2(\mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N,\end{aligned}\right\} \tag{4.17}$$

with Γ^i given by (3.1), we find from (3.3) that the solution $\check{b}^\varepsilon$ of (4.15) satisfies

$$\check{b}^\varepsilon(z,s,\zeta) = \sum_{i=1}^N \frac{\partial u_0}{\partial x_i}(z+a\nu)\check{\gamma}_\varepsilon^i(s,\zeta). \tag{4.18}$$

In order to prove theorem 4.5 let us first obtain some compactness results for the two-scale convergence defined in definition 4.4. The following theorem follows from [6] by taking $\mathbb{R} \otimes CAP(\mathcal{G})$ as an algebra (see also [16, 17]). Indeed, since $\mathcal{G}$ is countable, the result can also easily be proved by using the arguments of the classical periodic two-scale convergence compactness theorem (see [1, 15]).

Theorem 4.9. *Let $\check{v}_\varepsilon$ be a bounded sequence in $L^2(\omega \times (a,b))$. Then there exist $\check{v} \in L^2(\omega \times (a,b);\mathcal{B}(\mathcal{G}))$, and a subsequence of ε, still denoted by ε, such that (for this subsequence) $\check{v}_\varepsilon$ two-scale converges to $\check{v}$.*

Let us apply the above result to the sequence $\check{b}_\varepsilon$ defined by (4.5). Since this sequence satisfies (4.6), proposition 4.11, below, is more interesting than the previous result.

Definition 4.10. For $\check{v} \in L^2(\omega \times (0,M);\mathcal{B}(\mathcal{G}))$ for every $M > 0$ ($\check{v} = \check{v}(z,s,\zeta)$) with $\partial\check{v}/\partial s \in L^2(\omega \times \mathbb{R}^+;\mathcal{B}(\mathcal{G}))$, we denote

$$D_0\check{v} = \nabla_\zeta \check{v} + \frac{\partial \check{v}}{\partial s}\nu, \tag{4.19}$$

where the gradient with respect to ζ is given in the sense of (2.3).

Proposition 4.11. *Let $\check{v}_\varepsilon \in H^1(\omega \times (0,1/\varepsilon))$ such that the application $z \in \omega \to \check{v}_\varepsilon(z,0)$ is bounded in $L^2(\omega)$ and $D_\varepsilon\check{v}_\varepsilon\chi_{\omega\times(0,1/\varepsilon)}$ is bounded in $L^2(\omega\times\mathbb{R}^+)^N$. Then there exist $\check{v} \in L^2(\omega \times (0,M);\mathcal{B}(\mathcal{G}))$ for every $M > 0$, with $D_0\check{v} \in L^2(\omega \times \mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N$, and a subsequence of ε, still denoted by ε, such that (for this subsequence) we have*

$$\check{v}_\varepsilon \stackrel{2\mathrm{e}}{\rightharpoonup} \check{v} \qquad \text{in } \omega \times (0,M) \quad \text{for all } M > 0, \tag{4.20}$$

$$D_\varepsilon\check{v}_\varepsilon\chi_{\omega\times\mathbb{R}^+} \stackrel{2\mathrm{e}}{\rightharpoonup} D_0\check{v} \quad \text{in } \omega\times\mathbb{R}^+. \tag{4.21}$$

Proof. Since $\check{v}_\varepsilon(z,0)$ and $\partial\check{v}_\varepsilon/\partial s\chi_{\omega\times(0,1/\varepsilon)}$ are bounded in $L^2(\omega)$ and $L^2(\omega\times\mathbb{R}^+)$, respectively, we have that $\check{v}_\varepsilon$ is bounded in $L^2(\omega\times(0,M))$, for every $M > 0$.

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Thus, applying theorem 4.9 to $\check{v}_\varepsilon$ in $\omega \times (0,n)$, for every $n \in \mathbb{N}$, and using a diagonal procedure, we deduce the existence of $\check{v} \in L^2(\omega \times (0,M); \mathcal{B}(\mathcal{G}))$, for every $M > 0$, such that (4.20) holds. On the other hand, since $D_\varepsilon \check{v}_\varepsilon \chi_{\omega\times(0,1/\varepsilon)}$ is bounded in $L^2(\omega \times \mathbb{R}^+)^N$, theorem 4.9 also gives the existence of $\check{W} \in L^2(\omega \times \mathbb{R}^+; \mathcal{B}(\mathcal{G}))^N$ such that

$$D_\varepsilon \check{v}_\varepsilon \chi_{\omega\times\mathbb{R}^+} \overset{2\mathrm{e}}{\rightharpoonup} \check{W} \quad \text{in } \omega \times \mathbb{R}^+.$$

It remains to prove that $\check{W}$ satisfies $\check{W} = D_0 \check{v}$. Taking

$$\varPsi \in C_\mathrm{c}^\infty(\omega \times \mathbb{R}^+; CAP^\infty(\mathcal{G}))^N,$$

we have

$$\begin{aligned}
\int_{\omega\times\mathbb{R}^+} & M_\zeta(\check{W}\varPsi)\,\mathrm{d}z\,\mathrm{d}s \\
&= \lim_{\varepsilon\to 0} \int_{\omega\times(0,1/\varepsilon)} D_\varepsilon \check{v}_\varepsilon \varPsi\Big(z, s, \frac{z}{\varepsilon}\Big)\,\mathrm{d}z\,\mathrm{d}s \\
&= \lim_{\varepsilon\to 0} \int_{\omega\times(0,1/\varepsilon)} \Big(\varepsilon\nabla_z \check{v}_\varepsilon P\varPsi\Big(z, s, \frac{z}{\varepsilon}\Big) + \frac{\partial \check{v}_\varepsilon}{\partial s}\nu\varPsi\Big(z, s, \frac{z}{\varepsilon}\Big)\Big)\,\mathrm{d}z\,\mathrm{d}s \\
&= -\lim_{\varepsilon\to 0} \int_{\omega\times(0,1/\varepsilon)} \check{v}_\varepsilon\Big(\varepsilon\,\mathrm{div}_z(P\varPsi)\Big(z, s, \frac{z}{\varepsilon}\Big) \\
&\qquad\qquad + \mathrm{div}_\zeta(P\varPsi)\Big(z, s, \frac{z}{\varepsilon}\Big) + \frac{\partial(\varPsi\nu)}{\partial s}\Big(z, s, \frac{z}{\varepsilon}\Big)\Big)\,\mathrm{d}z\,\mathrm{d}s \\
&= -\int_{\omega\times\mathbb{R}^+} M_\zeta\Big(\check{v}\Big(\mathrm{div}_\zeta(P\varPsi) + \frac{\partial(\varPsi\nu)}{\partial s}\Big)\Big)\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_{\omega\times\mathbb{R}^+} M_\zeta(D_0\check{v}\varPsi)\,\mathrm{d}z\,\mathrm{d}s,
\end{aligned}$$

for every $\varPsi \in C_0^\infty(\omega \times \mathbb{R}^+; CAP^\infty(\mathcal{G}))^N$. This proves that $\check{W} = D_0\check{v}$. □

In order to apply proposition 4.11 to study the asymptotic behaviour of $\check{b}_\varepsilon$, we also need the following density result.

Lemma 4.12. *For every $\check{v}$ such that $\check{v}(z,0,\zeta) = 0$, $D_0\check{v} \in L^2(\omega \times \mathbb{R}^+; \mathcal{B}(\mathcal{G}))^N$, there exists a sequence $\check{v}_n \in C_\mathrm{c}^\infty(\omega \times \mathbb{R}^+; CAP^\infty(\mathcal{G}))$ such that $D_0\check{v}_n$ converges strongly to $D_0\check{v}$ in $L^2(\omega \times \mathbb{R}^+; \mathcal{B}(\mathcal{G}))^N$.*

Proof. It is clear that the result holds true if $\check{v}$ is such that $\check{v}(z,0,\zeta) = 0$, $\check{v} \in L^2(\omega \times \mathbb{R}^+ : CAP^\infty(\mathcal{G}))^N$, $D_0\check{v} \in L^2(\omega \times \mathbb{R}^+ : CAP^\infty(\mathcal{G}))^N$, and there exists $M > 0$, with $\check{v}(z,s,\zeta) = 0$ for $s > M$. So, in order to prove lemma 4.12, it is sufficient to prove that, for every $\check{v}$ with $\check{v}(z,0,\zeta) = 0$, $D_0\check{v} \in L^2(\omega \times \mathbb{R}^+ : \mathcal{B}(\mathcal{G}))^N$, there exists $\check{v}_n$ with $\check{v}_n(z,0,\zeta) = 0$, $\check{v}_n \in L^2(\omega \times \mathbb{R}^+; CAP^\infty(\mathcal{G}))^N$, $D_0\check{v}_n \in L^2(\omega \times \mathbb{R}^+ : CAP^\infty(\mathcal{G}))^N$, $\check{v}_n(z,s,\zeta) = 0$ for $s > 2n$, such that $D_0\check{v}_n$ converges to $D_0\check{v}$ in $L^2(\omega \times \mathbb{R}^+; \mathcal{B}(\mathcal{G}))$. For such $\check{v}$, using Fourier's representation, we have

$$\check{v}(z,s,\zeta) = \alpha^0(z,s) + \sum_{g\in\mathcal{G}^*} (\alpha^g(z,s)\cos(g\zeta) + \beta^g(z,s)\sin(g\zeta))$$

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with $\alpha^0(z,0)=0$, $\alpha^g(z,0)=\beta^g(z,0)=0$ for every $g\in\mathcal{G}^*$ and

$$\int_{\omega\times\mathbb{R}^+}\biggl(\biggl|\frac{\partial\alpha^0}{\partial s}\biggr|^2+\sum_{g\in\mathcal{G}^*}\biggl(|g|^2(|\alpha^g|^2+|\beta^g|^2)+\biggl|\frac{\partial\alpha^g}{\partial s}\biggr|^2+\biggl|\frac{\partial\beta^g}{\partial s}\biggr|^2\biggr)\biggr)\,\mathrm{d}z\,\mathrm{d}s<+\infty.$$

Using the fact that $\mathcal{G}^*$ is countable, $\mathcal{G}^*=\{g_i\}_{i\geqslant1}$, we define $\check{v}_n(z,s,\zeta)$ as

$$\alpha^0(z,s)+\sum_{i=1}^{n}(\alpha^{g_i}(z,s)\cos(g_i\zeta)+\beta^{g_i}(z,s)\sin(g_i\zeta))$$

if $0\leqslant s\leqslant n$,

$$\biggl(\alpha^0(z,s)+\sum_{i=1}^{n}(\alpha^{g_i}(z,s)\cos(g_i\zeta)+\beta^{g_i}(z,s)\sin(g_i\zeta))\biggr)\biggl(2-\frac{s}{n}\biggr)$$

if $n\leqslant s\leqslant 2n$ and 0 if $2n\leqslant s$.

Let us prove that $D_0\check{v}_n$ converges to $D_0\check{v}$ in $L^2(\omega\times\mathbb{R}^+;\mathcal{B}(\mathcal{G}))$. We have

$$\begin{aligned}
&\|D_0\check{v}-D_0\check{v}_n\|^2_{L^2(\omega\times\mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N}\\
&\quad\leqslant C\int_{\omega\times\mathbb{R}^+}\biggl(\sum_{i=n+1}^{\infty}\biggl(|g_i|^2(|\alpha^{g_i}|^2+|\beta^{g_i}|^2)+\biggl|\frac{\partial\alpha^{g_i}}{\partial s}\biggr|^2+\biggl|\frac{\partial\beta^{g_i}}{\partial s}\biggr|^2\biggr)\biggr)\,\mathrm{d}z\,\mathrm{d}s\\
&\qquad+C\int_{\omega\times(n,\infty)}\biggl(\biggl|\frac{\partial\alpha^0}{\partial s}\biggr|^2+\sum_{i=1}^{n}\biggl(|g_i|^2(|\alpha^{g_i}|^2+|\beta^{g_i}|^2)\\
&\qquad\qquad+\biggl|\frac{\partial\alpha^{g_i}}{\partial s}\biggr|^2+\biggl|\frac{\partial\beta^{g_i}}{\partial s}\biggr|^2\biggr)\biggr)\,\mathrm{d}z\,\mathrm{d}s\\
&\qquad+\frac{C}{n^2}\int_{\omega\times(n,2n)}\biggl(|\alpha^0|^2+\sum_{i=1}^{n}(|\alpha^{g_i}|^2+|\beta^{g_i}|^2)\biggr)\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}$$

In the right-hand side of this inequality, the first and second terms clearly tend to zero when $n\to\infty$. To estimate the third term, we use the fact that, for every $t<n$ and $s\in(n,2n)$, we have

$$|\alpha^{g_i}(z,s)|^2\leqslant 2|\alpha^{g_i}(z,t)|^2+2(2n-t)\int_t^{+\infty}\biggl|\frac{\partial\alpha^{g_i}}{\partial r}(z,r)\biggr|^2\,\mathrm{d}r,$$

for a.e. $z\in\omega$, and similarly for β^{g_i}. Then, we have

$$\begin{aligned}
&\frac{1}{n^2}\int_{\omega\times(n,2n)}\biggl(|\alpha^0|^2+\sum_{i=1}^{n}(|\alpha^{g_i}|^2+|\beta^{g_i}|^2)\biggr)\,\mathrm{d}z\,\mathrm{d}s\\
&\quad\leqslant\frac{2}{n}\int_\omega\biggl(|\alpha^0(z,t)|^2+\sum_{i=1}^{n}(|\alpha^{g_i}(z,t)|^2+|\beta^{g_i}(z,t)|^2)\biggr)\,\mathrm{d}z\\
&\qquad+2\frac{2n-t}{n}\int_{\omega\times(t,\infty)}\biggl(\biggl|\frac{\partial\alpha^0}{\partial s}\biggr|^2+\sum_{i=1}^{n}\biggl(\biggl|\frac{\partial\alpha^{g_i}}{\partial s}\biggr|^2+\biggl|\frac{\partial\beta^{g_i}}{\partial s}\biggr|^2\biggr)\biggr)\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}$$

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So

$$\limsup_{n\to\infty}\frac{1}{n^2}\int_{\omega\times(n,2n)}\biggl(|\alpha^0|^2+\sum_{i=1}^{n}(|\alpha^{g_i}|^2+|\beta^{g_i}|^2)\biggr)\,\mathrm{d}z\,\mathrm{d}s \leqslant 4\int_{\omega\times(t,\infty)}\biggl(\biggl|\frac{\partial\alpha^0}{\partial s}\biggr|^2+\sum_{g\in\mathcal{G}^*}\biggl(\biggl|\frac{\partial\alpha^g}{\partial s}\biggr|^2+\biggl|\frac{\partial\beta^g}{\partial s}\biggr|^2\biggr)\biggr)\,\mathrm{d}z\,\mathrm{d}s,$$

for every $t>0$. This proves

$$\lim_{n\to\infty}\frac{1}{n^2}\int_{\omega\times(n,2n)}\biggl(|\alpha^{g_0}|^2+\sum_{i=1}^{n}(|\alpha^{g_i}|^2+|\beta^{g_i}|^2)\biggr)\,\mathrm{d}z\,\mathrm{d}s=0.$$

□

Proof of theorem 4.5. Since $\check{b}_\varepsilon$ satisfies (4.6), we can apply proposition 4.11 in order to deduce the existence of $\check{b}\in L^2(\omega\times(0,M);\mathcal{B}(\mathcal{G}))$, for every $M>0$, with $D_0\check{b}\in L^2(\omega\times\mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N$ such that (4.12) and (4.13) hold. Once we have proved that $\check{b}$ is the solution of (4.11), we may deduce by uniqueness that it is not necessary to extract any subsequence.

For $\delta>0$, we consider $A^\delta\in C^0_\sharp(\bar{Y};\mathcal{M}_N)$ such that

$$\|A-A^\delta\|_{L^2_\sharp(Y;\mathcal{M}_N)}<\delta. \tag{4.22}$$

For $\check{v}\in C^\infty_{\mathrm{c}}(\omega\times\mathbb{R}^+;CAP^\infty(\mathcal{G}))$, we take $v_\varepsilon(z,s)=\check{v}(z,s,z/\varepsilon)$ as a test function in (4.7). Adding and subtracting

$$\check{A}^\delta_\varepsilon\biggl(s,\frac{z}{\varepsilon}\biggr)=A^\delta\biggl(\frac{z}{\varepsilon}+\biggl(\frac{a}{\varepsilon}+s\biggr)\nu\biggr),$$

we get

$$\int_{\omega\times\mathbb{R}^+}\biggl(\check{A}_\varepsilon\biggl(s,\frac{z}{\varepsilon}\biggr)-\check{A}^\delta_\varepsilon\biggl(s,\frac{z}{\varepsilon}\biggr)\biggr)D_\varepsilon\check{b}_\varepsilon D_\varepsilon v_\varepsilon\,\mathrm{d}z\,\mathrm{d}s + \int_{\omega\times\mathbb{R}^+}\check{A}^\delta_\varepsilon\biggl(s,\frac{z}{\varepsilon}\biggr)D_\varepsilon\check{b}_\varepsilon D_\varepsilon v_\varepsilon\,\mathrm{d}z\,\mathrm{d}s=0. \tag{4.23}$$

Since $\check{A}^\delta_\varepsilon$ converges uniformly to $\check{A}^\delta$ defined by

$$\check{A}^\delta(s,\zeta)=A^\delta(\zeta+\varrho+s\nu)\quad\text{for all }\zeta\in\{\nu\}^\perp\text{ and all }s\geqslant 0,$$

and

$$D_\varepsilon v_\varepsilon-D_0\check{v}\biggl(z,s,\frac{z}{\varepsilon}\biggr)=\varepsilon\nabla_z\check{v}\biggl(z,s,\frac{z}{\varepsilon}\biggr)$$

converges strongly to zero in $L^2(\omega\times\mathbb{R}^+)^N$, we deduce

$$\lim_{\varepsilon\to 0}\int_{\omega\times\mathbb{R}^+}\biggl(\check{A}^\delta_\varepsilon\biggl(s,\frac{z}{\varepsilon}\biggr)D_\varepsilon\check{b}_\varepsilon D_\varepsilon v_\varepsilon-\check{A}^\delta\biggl(s,\frac{z}{\varepsilon}\biggr)D_\varepsilon\check{b}_\varepsilon D_0\check{v}\biggl(z,s,\frac{z}{\varepsilon}\biggr)\biggr)\,\mathrm{d}z\,\mathrm{d}s=0. \tag{4.24}$$

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But, by the definition of two-scale convergence, we have

$$\lim_{\varepsilon\to 0}\int_{\omega\times\mathbb{R}^+}\check{A}^\delta\Big(s,\frac{z}{\varepsilon}\Big)D_\varepsilon\check{b}_\varepsilon D_0\check{v}\Big(z,s,\frac{z}{\varepsilon}\Big)\,\mathrm{d}z\,\mathrm{d}s=\int_{\omega\times\mathbb{R}^+}M_\zeta(\check{A}^\delta(s,\zeta)D_0\check{b}D_0\check{v})\,\mathrm{d}z\,\mathrm{d}s. \tag{4.25}$$

In order to estimate the first term in (4.23), we use (4.6), $D_\varepsilon v_\varepsilon$ bounded in $L^\infty(\omega\times\mathbb{R}^+)$ and the existence of $S>0$ such that $\check{v}(z,s,\zeta)$ is zero for $s>S$. Thus, we deduce that there exists $C>0$ (which does not depend on ε and δ) such that

$$\begin{aligned}\bigg|\int_{\omega\times\mathbb{R}^+}\Big(\check{A}_\varepsilon\Big(s,\frac{z}{\varepsilon}\Big)-\check{A}^\delta_\varepsilon\Big(s,\frac{z}{\varepsilon}\Big)\Big)D_\varepsilon\check{b}_\varepsilon D_\varepsilon v_\varepsilon\,\mathrm{d}z\,\mathrm{d}s\bigg|&\\ \leqslant C\Big\|\check{A}_\varepsilon\Big(s,\frac{z}{\varepsilon}\Big)-\check{A}^\delta_\varepsilon\Big(s,\frac{z}{\varepsilon}\Big)\Big\|_{L^2(\omega\times(0,S);\mathcal{M}_N)}.&\end{aligned}\tag{4.26}$$

On the other hand, using the change of variables

$$y=\frac{z}{\varepsilon}+\Big(\frac{a}{\varepsilon}+s\Big)\nu,$$

we get

$$\Big\|\check{A}_\varepsilon\Big(s,\frac{z}{\varepsilon}\Big)-\check{A}^\delta_\varepsilon\Big(s,\frac{z}{\varepsilon}\Big)\Big\|^2_{L^2(\omega\times(0,S);\mathcal{M}_N)}=\varepsilon^{N-1}\int_{(a\nu+\omega)/\varepsilon+(0,S)\nu}|A-A^\delta|^2\,\mathrm{d}y\leqslant C\delta,$$

for every $\varepsilon>0$. So, taking the limit in (4.23) first in ε and then in δ by (4.24)–(4.26), we deduce

$$\int_{\omega\times\mathbb{R}^+}M_\zeta(\check{A}(\zeta+\varrho+s\nu)D_0\check{b}D_0\check{v})\,\mathrm{d}z\,\mathrm{d}s=0\quad\text{for all }\check{v}\in C^\infty_{\mathrm{c}}(\omega\times\mathbb{R}^+;CAP^\infty(\mathcal{G})).$$

By lemma 4.12, this equality holds in fact for every $\check{v}$ such that $\check{v}(z,0,s)=0$ and $D_0\check{v}$ belongs to $L^2(\omega\times\mathbb{R}^+;\mathcal{B}(\mathcal{G}))^N$. In order to show that $\check{b}$ is the solution of (4.11), it remains only to prove $\check{b}(z,0,\zeta)=u_1(z+a\nu,\zeta+\varrho)$. For this purpose we use the fact that

$$\check{b}_\varepsilon(z,0)=u_1\Big(z+a\nu,\frac{z+a\nu}{\varepsilon}\Big).$$

Thus, for every $\rho>0$ and every $\psi\in C^\infty_{\mathrm{c}}(\omega;CAP^\infty(\mathcal{G}))$ we have

$$\begin{aligned}\int_\omega\Big(\frac{1}{\rho}\int_0^\rho\check{b}_\varepsilon(z,s)\,\mathrm{d}s-u_1\Big(z+a\nu,\frac{z+a\nu}{\varepsilon}\Big)\Big)\psi\Big(z,\frac{z}{\varepsilon}\Big)\,\mathrm{d}z&\\ \leqslant C\rho\Big\|\frac{\partial\check{b}_\varepsilon}{\partial s}\Big\|_{L^2(\omega\times(0,\rho))}\|\psi\|_{C^0(\bar{\omega};CAP(\mathcal{G}))}.&\end{aligned}$$

Using (4.12), the fact that $u_1(z+a\nu,(z+a\nu)/\varepsilon)$ two-scale converges to $u_1(z+a\nu,\zeta+\varrho)$ (see, for example, [1, 8, 15, 16]), and (4.6), we easily deduce that

$$\int_\omega M_\zeta((\check{b}(z,0,\zeta)-u_1(z+a\nu,\zeta+\varrho))\psi(z,\zeta))\,\mathrm{d}z=0\quad\text{for all }\psi\in C^\infty_{\mathrm{c}}(\omega;CAP^\infty(\mathcal{G})),$$

which proves $\check{b}(z,0,\zeta)=u_1(z+a\nu,\zeta+\varrho)$. □

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5. Expansion near a plane boundary with irrational normal

In this section, we obtain an estimate of the left-hand side of (4.16) for an irrational normal. For this purpose we will use the following result, showing that the functions $\check{\gamma}_\varepsilon^i$ of (4.17), which are almost periodic in the variable ζ, can be obtained from a periodic problem.

Proposition 5.1. *Assume $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$, uniformly elliptic. For $1 \leqslant i \leqslant N$, we define $\Upsilon^i : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ by*

$$\left.\begin{array}{c} \Upsilon^i \text{ periodic in } y, \quad \Upsilon^i(0,y) = \Gamma^i(y), \quad P\nabla_y \Upsilon^i + \dfrac{\partial \Upsilon^i}{\partial s}\nu \in L^2(\mathbb{R}^+; L^2_\sharp(Y))^N, \\ \displaystyle\int_{Y\times\mathbb{R}^+} A(y+s\nu)\left(P\nabla_y\Upsilon^i + \frac{\partial \Upsilon^i}{\partial s}\nu\right)\left(P\nabla_y\varphi + \frac{\partial\varphi}{\partial s}\nu\right)\mathrm{d}s\,\mathrm{d}y = 0 \\ \text{for all } \varphi \text{ periodic in } y,\ P\nabla_y\varphi + \dfrac{\partial\varphi}{\partial s}\nu \in L^2(\mathbb{R}^+; L^2_\sharp(Y))^N,\ \varphi(0,y) = 0. \end{array}\right\} \tag{5.1}$$

We suppose that (at least) one of the following assumptions holds:

(i) $\nu \notin \mathbb{R}\cdot\mathbb{Z}^N$;

(ii) $A \in C^0_\sharp(Y;\mathcal{M}_N)$, $P\nabla_y\Upsilon^j + \nu(\partial\Upsilon^j/\partial s) \in L^2(\mathbb{R}^+; C^0_\sharp(Y)^N)$.

Then, for every $\varepsilon > 0$ and every $i \in \{1,\dots,N\}$, the solution $\check{\gamma}_\varepsilon^i$ of (4.17) satisfies

$$\check{\gamma}_\varepsilon^i(s,\zeta) = \Upsilon^i\left(s, \frac{a}{\varepsilon}\nu + \zeta\right). \tag{5.2}$$

Better than Υ^i, we will use the following functions $T^i = T^i(r,\mu)$ which follows from Υ^i by using the change of variables

$$y = P\mu + r\nu, \quad s = \mu\nu \quad \Longleftrightarrow \quad \mu = Py + s\nu, \quad r = y\nu. \tag{5.3}$$

Definition 5.2. Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, that $\partial_{y_\nu}A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ and define Υ^i by (5.1). Denoting by H the half-space of $\mathbb{R}^N$,

$$H = \{\mu \in \mathbb{R}^N : \mu\nu > 0\},$$

we define $T^i : \mathbb{R} \times H \to \mathbb{R}$ by

$$T^i(r,\mu) = \Upsilon^i(\mu\nu, P\mu + r\nu) \quad \text{a.e. in } \mathbb{R}\times H. \tag{5.4}$$

Remark 5.3. From (5.2), we have $\check{\gamma}_\varepsilon^i(s,\zeta) = T^i(a/\varepsilon, \zeta + s\nu)$.

The main result of this section is the following theorem.

Theorem 5.4. *Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, that $\partial_{y_\nu}A \in L^\infty_\sharp(Y;\mathcal{M}_N)$, u_0 is a solution of (3.2) in $W^{2,\infty}(\Omega)$, $\psi \in C^\infty(\bar{\Omega})$ with $\psi = 1$ on a compact subset K of ω, $\operatorname{supp}(\psi) \subset \bar{\Lambda}_\delta$, and define*

$$\tilde{u}_\varepsilon(x) = u_0(x) + \varepsilon\sum_{i=1}^N\left(\frac{\partial u_0}{\partial x_i}(x)\Gamma^i\left(\frac{x}{\varepsilon}\right) - \frac{\partial u_0}{\partial x_i}(a\nu + Px)T^i\left(\frac{a}{\varepsilon}, \frac{x - a\nu}{\varepsilon}\right)\psi(x)\right). \tag{5.5}$$

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Then there exists $C > 0$ such that if u_ε is the solution of (1.1), we have

$$\|(u_\varepsilon - \tilde{u}_\varepsilon)\phi\|_{H_0^1(\Omega)} \leqslant C\|\phi\|_{W^{1,\infty}(\Omega)}\varepsilon\sqrt{|\log\varepsilon|} \quad \textit{for all } \varepsilon > 0, \tag{5.6}$$

for every $\phi \in C^\infty(\Omega)$, with $\phi\nabla u_0 = 0$ on $\partial\Omega \setminus K$.

In order to prove proposition 5.1, we start by observing that, by the definition of $CAP(\mathcal{G})$, with $\mathcal{G}$ given by (4.8), and Fejer's theorem, the restriction to the hyperplane $c\nu + \{\nu\}^\perp$, $c \in \mathbb{R}$, of a function of $C^0_\sharp(\bar{Y})$ is in $CAP(\mathcal{G})$. Related to this result, we also have the following proposition.

Proposition 5.5. *We assume that ν does not belong to $\mathbb{R}\cdot\mathbb{Z}^N$, and we define $\mathcal{G}$ by (4.8). Then, for every $c \in \mathbb{R}$, the application $J : L^2_\sharp(Y) \to \mathcal{B}(\mathcal{G})$ defined by*

$$J(u) = u_{|c\nu+\{\nu\}^\perp}$$

is an isometric isomorphism.

Proof. Let us first show that P satisfies the property

$$Pk_1 = Pk_2 \iff k_1 = k_2 \quad \text{for all } k_1, k_2 \in \mathbb{Z}^N, \tag{5.7}$$

for it is sufficient to use

$$Pk_1 = Pk_2 \iff P(k_1 - k_2) = 0 \iff \exists\rho \in \mathbb{R} \text{ such that } k_1 - k_2 = \rho\nu.$$

However, since $\nu \notin \mathbb{R}\cdot\mathbb{Z}^N$, this is equivalent to $k_1 = k_2$.

Using Fourier's formulation, J is defined by (for $y \in c\nu + \{\nu\}^\perp$, we use the decomposition $y = c\nu + \zeta$ with $\zeta \in \{\nu\}^\perp$)

$$\begin{aligned} J\bigg(\alpha^0 + \sum_{k\in(\mathbb{Z}^N)^*}(\alpha^k\cos(2\pi ky) + \beta^k\sin(2\pi ky))\bigg) &= \alpha^0 + \sum_{k\in(\mathbb{Z}^N)^*}(\alpha^k\cos(2\pi(ck\nu + Pk\zeta)) + \beta^k\sin(2\pi(ck\nu + Pk\zeta))) \\ &= \alpha^0 + \sum_{k\in(\mathbb{Z}^N)^*}(\alpha^k\cos(2\pi ck\nu) + \beta^k\sin(2\pi ck\nu))\cos(2\pi Pk\zeta) \\ &\quad + \sum_{k\in(\mathbb{Z}^N)^*}(-\alpha^k\sin(2\pi ck\nu) + \beta^k\cos(2\pi ck\nu))\sin(2\pi Pk\zeta). \end{aligned}$$

Thus, from (5.7), we easily obtain that J is bijective and

$$\begin{aligned} \bigg\|J\bigg(\alpha^0 + \sum_{k\in(\mathbb{Z}^N)^*}(\alpha^k\cos(2\pi ky) + \beta^k\sin(2\pi ky))\bigg)\bigg\|^2_{\mathcal{B}(\mathcal{G})} &= |\alpha^0|^2 + \frac{1}{2}\sum_{k\in(\mathbb{Z}^N)^*}(|\alpha^k|^2 + |\beta^k|^2) \\ &= \bigg\|\alpha^0 + \sum_{k\in(\mathbb{Z}^N)^*}(\alpha^k\cos(2\pi ky) + \beta^k\sin(2\pi ky))\bigg\|^2_{L^2_\sharp(Y)}, \end{aligned}$$

which shows that the linear application J is an isomorphism isometric. □

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REMARK 5.6. Proposition 5.5 can be surprising because it gives a meaning to the restriction of $u \in L^2_\sharp(Y)$ to a hyperplane such that its unit normal is not proportional to an element of $\mathbb{Z}^N$. However, $J(u)$ is only in $\mathcal{B}(\mathcal{G})$ and thus it is not a function but a class of functions.

Proof of proposition 5.1. This is immediate, using the fact that the restriction to the hyperplane $c\nu + \{\nu\}^\perp$ of a function in $C_\sharp(\bar{Y})$ is in $CAP(\mathcal{G})$ and proposition 5.5. □

Taking the derivative of problem (5.1) with respect to y in the direction of ν, which we denote by ∂_{y_ν} we easily have the following proposition.

PROPOSITION 5.7. *Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, and that $\partial_{y_\nu} A \in L^\infty_\sharp(Y;\mathcal{M}_N)$. Then, for every $i \in \{1,\dots,N\}$, the derivative (in the sense of the distributions) $\partial_{y_\nu}\Upsilon^i$ of the solution Υ^i of (5.1) is given by*

$$\left.\begin{aligned} &\partial_{y_\nu}\Upsilon^i \text{ is periodic in } y, \quad \partial_{y_\nu}\Upsilon^i(0,y) = \partial_{y_\nu}\Gamma^i(y),\\ &P\nabla_y(\partial_{y_\nu}\Upsilon^i) + \frac{\partial}{\partial s}(\partial_{y_\nu}\Upsilon^i)\nu \in L^2(\mathbb{R}^+; L^2_\sharp(Y))^N,\\ &\int_{\mathbb{R}^+\times Y}\biggl(A(y+s\nu)\Bigl(P\nabla_y(\partial_{y_\nu}\Upsilon^i) + \frac{\partial}{\partial s}(\partial_{y_\nu}\Upsilon^i)\nu\Bigr)\\ &\qquad + \partial_{y_\nu}A(y+s\nu)\Bigl(P\nabla_y\Upsilon^i + \frac{\partial\Upsilon^i}{\partial s}\nu\Bigr)\biggr)\Bigl(P\nabla_y\varphi + \frac{\partial\varphi}{\partial s}\nu\Bigr)\,\mathrm{d}s\,\mathrm{d}y = 0\\ &\text{for all } \varphi \text{ periodic in } y,\ P\nabla_y\varphi + \frac{\partial\varphi}{\partial s}\nu \in L^2(\mathbb{R}^+; L^2_\sharp(Y))^N,\ \varphi(0,y) = 0. \end{aligned}\right\} \tag{5.8}$$

Reasoning as in the proof of lemma A.1, we deduce from (5.1) and (5.7) that Υ^i and $\partial_{y_\nu}\Upsilon^i$ satisfy the following proposition.

PROPOSITION 5.8. *Assume that $A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, that $\partial_{y_\nu} A \in L^\infty_\sharp(Y;\mathcal{M}_N)$ and define Υ^i by (5.1). Then, Υ^i and $\partial_{y_\nu}\Upsilon^i$ respectively satisfy*

$$\left.\begin{aligned} &\int_{\mathbb{R}^+\times\mathbb{R}^N} A(y+s\nu)\Bigl(P\nabla_y\Upsilon^i + \frac{\partial\Upsilon^i}{\partial s}\nu\Bigr)\Bigl(P\nabla_y\varphi + \frac{\partial\varphi}{\partial s}\nu\Bigr)\,\mathrm{d}y\,\mathrm{d}s = 0\\ &\qquad \text{for all } \varphi \text{ with } P\nabla_y\varphi + \frac{\partial\varphi}{\partial s}\nu \in L^2(\mathbb{R}^+\times\mathbb{R}^N)^N,\\ &\varphi(0,y) = 0 \text{ on } \mathbb{R}^N, \quad \exists R>0 \text{ with } \varphi(s,y) = 0 \text{ a.e. in } |y|>R,\ s>0 \end{aligned}\right\} \tag{5.9}$$

and

$$\left.\begin{aligned} &\int_{\mathbb{R}^+\times\mathbb{R}^N} A(y+s\nu)\Bigl(P\nabla_y(\partial_{y_\nu}\Upsilon^i) + \frac{\partial}{\partial s}(\partial_{y_\nu}\Upsilon^i)\nu\\ &\qquad + \partial_{y_\nu}A(y+s\nu)\Bigl(P\nabla_y\Upsilon^i + \frac{\partial\Upsilon^i}{\partial s}\nu\Bigr)\Bigr)\Bigl(P\nabla_y\varphi + \frac{\partial\varphi}{\partial s}\nu\Bigr)\,\mathrm{d}y\,\mathrm{d}s = 0\\ &\qquad \text{for all } \varphi \text{ with } P\nabla_y\varphi + \frac{\partial\varphi}{\partial s}\nu \in L^2(\mathbb{R}^+\times\mathbb{R}^N)^N,\\ &\varphi(0,y) = 0 \text{ on } \mathbb{R}^N, \quad \exists R>0 \text{ with } \varphi(s,y) = 0 \text{ a.e. in } |y|>R, s>0. \end{aligned}\right\} \tag{5.10}$$

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REMARK 5.9. Proposition 5.8 shows in particular that Υ^i and $\partial_{y_\nu}\Upsilon^i$ respectively satisfy the following partial differential equations in the sense of the distributions:

$$-\frac{\partial}{\partial s}\bigg(A(y+s\nu)\bigg(P\nabla_y\Upsilon^i+\frac{\partial\Upsilon^i}{\partial s}\nu\bigg)\nu\bigg) \\ -\operatorname{div}_y\bigg(PA(y+s\nu)\bigg(P\nabla_y\Upsilon^i+\frac{\partial\Upsilon^i}{\partial s}\nu\bigg)\bigg)=0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^N, \tag{5.11}$$

and

$$-\frac{\partial}{\partial s}\bigg(A(y+s\nu)\bigg(P\nabla_y(\partial_{y_\nu}\Upsilon^i)+\frac{\partial}{\partial s}(\partial_{y_\nu}\Upsilon^i)\nu\bigg)\nu\bigg) \\ -\operatorname{div}_y\bigg(PA(y+s\nu)\bigg(P\nabla_y(\partial_{y_\nu}\Upsilon^i)+\frac{\partial}{\partial s}(\partial_{y_\nu}\Upsilon^i)\nu\bigg)\bigg) \\ =\frac{\partial}{\partial s}\bigg(\partial_{y_\nu}A(y+s\nu)\bigg(P\nabla_y\Upsilon^i+\frac{\partial\Upsilon^i}{\partial s}\nu\bigg)\nu\bigg) \\ +\operatorname{div}_y\bigg(P\partial_{y_\nu}A(y+s\nu)\bigg(P\nabla_y\Upsilon^i+\frac{\partial\Upsilon^i}{\partial s}\nu\bigg)\bigg) \quad \text{in } \mathbb{R}^+\times\mathbb{R}^N. \tag{5.12}$$

Using these properties for Υ^i, we can now show that the T^i given by (5.4) satisfies the following proposition.

PROPOSITION 5.10. *Assume that $A\in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, and that $\partial_{y_\nu}A\in L^\infty_\sharp(Y;\mathcal{M}_N)$. Then, for every $i\in\{1,\dots,N\}$ and every $r\in\mathbb{R}$, the function $\mu\mapsto T^i(r,\mu)$ is in $H^1(B(0,R)\cap H)$, for every $R>0$, and there exists $C>0$ such that, for every $r\in\mathbb{R}$, we have*

$$T^i(r,\mu)=\Gamma^i(\mu+r\nu) \quad \textit{for } \mu\in\{\nu\}^\perp, \tag{5.13}$$

$$\|T^i(r,\mu)\|_{L^\infty(H)}\leqslant C, \tag{5.14}$$

$$\int_{\{\mu\in H:|P\mu-\zeta|<1\}}|\nabla_\mu T^i(r,\mu)|^2\,\mathrm{d}\mu\leqslant C \quad \textit{for all } \zeta\in\nu^\perp, \tag{5.15}$$

$$-\operatorname{div}A(\mu+r\nu)\nabla_\mu T^i(r,\mu)=0 \quad \textit{in } H. \tag{5.16}$$

Proof. We consider $i\in\{1,\dots,N\}$. Statement (5.13) is immediate from $\Upsilon^i(y,0)=\Gamma^i(y)$ for a.e. $y\in\mathbb{R}^N$ and the definition, (5.4), of T^i. Since Γ^i is in $L^\infty_\sharp(Y)$, taking $(\Upsilon^i-\|\Gamma^i\|_{L^\infty_\sharp(Y)})^+$, $(\Upsilon^i+\|\Gamma^i\|_{L^\infty_\sharp(Y)})^-$ as a test functions in (5.1), we deduce that Υ^i is in $L^\infty(\mathbb{R}^+\times\mathbb{R}^N)$. From (5.1) and (5.7) we also have Υ^i in $H^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^N)$. Thus, from (5.4) we get (5.14). Using the fact that

$$P\nabla_y\Upsilon^i+\frac{\partial\Upsilon^i}{\partial s}\nu \quad\text{and}\quad P\nabla_y(\partial_{y_\nu}\Upsilon^i)+\frac{\partial}{\partial y_s}(\partial_{y_\nu}\Upsilon^i)\nu$$

belong to $L^2(\mathbb{R}^+;L^2_\sharp(Y))^N$, definition 5.4 of T^i and the inequality

$$|v(r_0)|^2\leqslant 2\int_{r_0-(1/2)}^{r_0+(1/2)}|v(r)|^2\,\mathrm{d}r \\ +2\int_{r_0-(1/2)}^{r_0+(1/2)}\left|\frac{\mathrm{d}v}{\mathrm{d}r}(r)\right|^2\mathrm{d}r \quad \text{for all } v\in H^1(r_0-\tfrac{1}{2},r_0+\tfrac{1}{2}),$$

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we deduce that for every $r_0 \in \mathbb{R}$, and every $\zeta_0 \in \{\nu\}^{\perp}$, we have

$$
\begin{aligned}
&\int_{\{\mu\in H:|P\mu-\zeta_0|<1\}} |\nabla_\mu T^i(r_0,\mu)|^2 \,\mathrm{d}\mu \\
&\quad = \int_{\{\mu\in H:|P\mu-\zeta_0|<1\}} \left| P\nabla_y \Upsilon^i(\mu\nu, P\mu + r_0\nu) + \frac{\partial \Upsilon^i}{\partial s}(\mu\nu, P\mu + r_0\nu)\nu \right|^2 \mathrm{d}\mu \\
&\quad = \int_0^{+\infty} \int_{\{\zeta\in\{\nu\}^\perp:|\zeta-\zeta_0|<1\}} \left| P\nabla_y \Upsilon^i(s, \zeta + r_0\nu) + \frac{\partial \Upsilon^i}{\partial s}(s, \zeta + r_0\nu)\nu \right|^2 \mathrm{d}\zeta\,\mathrm{d}s \\
&\quad \leqslant 2\int_0^{+\infty} \int_{r_0-(1/2)}^{r_0+(1/2)} \int_{\{\zeta\in\{\nu\}^\perp:|\zeta-\zeta_0|<1\}} \bigg| P\nabla_y \Upsilon^i(s, \zeta + r\nu) \\
&\qquad\qquad + \frac{\partial \Upsilon^i}{\partial s}(s, \zeta + r\nu)\nu \bigg|^2 \mathrm{d}\zeta\,\mathrm{d}r\,\mathrm{d}s \\
&\qquad + 2\int_0^{+\infty} \int_{r_0-(1/2)}^{r_0+(1/2)} \int_{\{\zeta\in\{\nu\}^\perp:|\zeta-\zeta_0|<1\}} \bigg| P\nabla_y (\partial_{y_\nu}\Upsilon^i)(s, \zeta + r\nu) \\
&\qquad\qquad + \frac{\partial}{\partial s}(\partial_{y_\nu}\Upsilon^i)(s, \zeta + r\nu)\nu \bigg|^2 \mathrm{d}\zeta\,\mathrm{d}r\,\mathrm{d}s \\
&\quad \leqslant 2\int_0^{+\infty} \int_Y \left(\left| P\nabla_y \Upsilon^i + \frac{\partial \Upsilon^i}{\partial s}\nu \right|^2 + \left| P\nabla_y (\partial_{y_\nu}\Upsilon^i) + \frac{\partial}{\partial s}(\partial_{y_\nu}\Upsilon^i)\nu \right|^2 \right) \mathrm{d}y\,\mathrm{d}s \\
&\quad \leqslant C.
\end{aligned}
$$

This proves (5.15).

In order to prove (5.16), we consider $v \in C_c^\infty(H)$. Then, for $\psi \in C_c^\infty(\mathbb{R})$, we define $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi(s,y) = v(Py + s\nu)\psi(y\nu), \quad \text{a.e. } (y,s) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Taking φ as a test function in (5.9) and using the change of variables (5.3), we get

$$\int_{-\infty}^{+\infty} \int_H A(\mu + r\nu)\nabla T^i \nabla v \,\mathrm{d}\mu \psi(r)\,\mathrm{d}r = 0. \tag{5.17}$$

Moreover, since $P\nabla_y(\partial_{y_\nu}\Upsilon^i) + (\partial/\partial s)(\partial_{y_\nu}\Upsilon^i)\nu$ belongs to $L^2(\mathbb{R}^+; L^2_\sharp(Y))^N$, we deduce that the application

$$r \mapsto \int_H A(\mu + r\nu)\nabla_\mu T^i \nabla v \,\mathrm{d}\mu$$

is continuous. Thus, since (5.17) holds for every $\psi \in C_c^\infty(\mathbb{R})$, we deduce (5.16). □

Let us now estimate, for $r > 0$, the decay to zero of $\nabla_\mu T^i(r,\mu)$, when

$$\operatorname{dist}(\mu, \{\nu\}^\perp) \to \infty.$$

This will follow from the following result.

LEMMA 5.11. *There exists $C_N > 0$, which depends only on N, such that, for every $w \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{N-1}) \cap H^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^{N-1})$, and every $B \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{N-1}; \mathcal{M}_N)$*

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such that

$$\exists M>0 \quad \textit{with} \int_{\mathbb{R}^+\times Q} |\nabla w|^2\,\mathrm{d}x \leqslant M \textit{ for all } Q\subset\mathbb{R}^{N-1},\ \textit{cube of side } 1, \tag{5.18}$$

$$\exists\alpha,\beta>0 \quad \textit{with } \alpha|\xi|^2\leqslant B\xi\xi,\quad |B\xi|\leqslant\beta|\xi| \textit{ for all } \xi\in\mathbb{R}^N,\ \textit{a.e. in } \mathbb{R}^+\times\mathbb{R}^{N-1}, \tag{5.19}$$

$$-\operatorname{div} B\nabla w = 0 \textit{ in } \mathbb{R}^+\times\mathbb{R}^{N-1}, \tag{5.20}$$

we have

$$\int_S^{2S}\int_{\{|x'|<R\}} |\nabla w|^2\,\mathrm{d}x'\,\mathrm{d}x_1 \leqslant C_N\biggl(\frac{M}{1+S}+\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^{N-1}\times\mathbb{R}^+)}\biggr)\frac{R^{N-1}}{1+S} \tag{5.21}$$

and

$$\int_0^R\int_{\{|x'|<R\}} |\nabla w|^2(1+x_1)\,\mathrm{d}x'\,\mathrm{d}x_1 \leqslant C_N\biggl(M+\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^{N-1}\times\mathbb{R}^+)}\log R\biggr)R^{N-1}, \tag{5.22}$$

for every $R>2$ and every $S\in[0,R]$.

Proof. Throughout the proof we will denote by C_N a generic positive constant which depends only on N and which can change from one line to another. The points $x\in\mathbb{R}^+\times\mathbb{R}^{N-1}$ will be decomposed as $x=(x_1,x')$, with $x_1\in\mathbb{R}^+$, $x'\in\mathbb{R}^{N-1}$.

For $R>2$, we define $\eta:\mathbb{R}^+\times\mathbb{R}^{N-1}\to\mathbb{R}$ by

$$\eta(x)=\begin{cases}\dfrac{1}{(x_1+1)^{N-1}} & \text{if } |x'|<R,\\[2ex] \dfrac{\mathrm{e}^{-(|x'|-R)/(x_1+1)}}{(x_1+1)^{N-1}} & \text{if } |x'|>R.\end{cases}$$

Let us first estimate

$$\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}} |\nabla w|^2\eta\,\mathrm{d}x,$$

for which we consider a cut-off function $\zeta\in C^\infty([0,+\infty))$ such that $\zeta(0)=0$, $\zeta=1$ in $(1,+\infty)$, $0\leqslant\zeta\leqslant1$ in $(0,1)$, $|\mathrm{d}\zeta/\mathrm{d}s|\leqslant2$. Also, for $T>0$, we take a cut-off function $\varphi_T\in C^\infty(\mathbb{R}^N)$ such that $\varphi_T=1$ in $(-T,T)^N$, $\varphi_T=0$ in $\mathbb{R}^N\setminus(-2T,2T)^N$, $0\leqslant\varphi_T\leqslant1$, $|\nabla\varphi_T|\leqslant2/T$, in $\mathbb{R}^N$. Taking $\zeta(x_1)w\eta\varphi_T^2$ as the test function in (5.20), we obtain

$$\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}} B\nabla w\nabla w\zeta(x_1)\eta\varphi_T^2\,\mathrm{d}x+\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}} B\nabla we_1\frac{\mathrm{d}\zeta}{\mathrm{d}x_1}(x_1)w\eta\varphi_T^2\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}} B\nabla w\nabla\eta\zeta(x_1)w\varphi_T^2\,\mathrm{d}x+2\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}} B\nabla w\nabla\varphi_T\zeta(x_1)w\eta\varphi_T\,\mathrm{d}x=0.$$

Using (5.19), $w \in L^\infty(\mathbb{R}^{N-1} \times \mathbb{R}^+)$ and the properties of φ_T and ζ, we deduce

$$\begin{aligned}\alpha \int_{\mathbb{R}^+ \times \mathbb{R}^{N-1}} |\nabla w|^2 \zeta(x_1) \eta \varphi_T^2 \, dx \\ &\leqslant 2\beta \|w\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^{N-1})} \int_0^1 \int_{\mathbb{R}^{N-1}} |\nabla w| \eta \, dx' \, dx_1 \\ &\quad + \beta \|w\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^{N-1})} \int_{(0,2T)\times(-2T,2T)^{N-1}} |\nabla w||\nabla \eta| \zeta(x_1) \varphi_T^2 \, dx \\ &\quad + \frac{4\beta}{T} \|w\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^{N-1})} \int_{(0,2T)\times(-2T,2T)^{N-1}} |\nabla w| \zeta(x_1) \eta \varphi_T \, dx,\end{aligned}$$

which, by Young's inequality and $\zeta, \varphi_T \leqslant 1$, gives

$$\begin{aligned}\frac{\alpha}{2} \int_{\mathbb{R}^+ \times \mathbb{R}^{N-1}} |\nabla w|^2 \zeta(x_1) \eta \varphi_T^2 \, dx \\ \leqslant 2\beta \|w\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^{N-1})} \int_0^1 \int_{\mathbb{R}^{N-1}} |\nabla w| \eta \, dx' \, dx_1 \\ + \frac{\beta^2}{\alpha} \|w\|^2_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^{N-1})} \biggl(\int_{\mathbb{R}^+ \times \mathbb{R}^{N-1}} \frac{|\nabla \eta|^2}{\eta} \, dx + \frac{16}{T^2} \int_{(0,2T)\times(-2T,2T)^{N-1}} \eta \, dx \biggr).\end{aligned} \tag{5.23}$$

Let us estimate the first term on the right-hand side of (5.23). Denoting $Y' = (-\frac{1}{2}, \frac{1}{2})^{N-1}$, $|k|_\infty = \max_{1 \leqslant i \leqslant N-1} |k_i|$, for every $k = (k_1, \dots, k_{N-1}) \in \mathbb{Z}^{N-1}$ and taking into account (5.18) and the definition of η, we have

$$\begin{aligned}&\int_0^1 \int_{\mathbb{R}^{N-1}} |\nabla w| \eta \, dx' \, dx_1 \\ &= \sum_{j=0}^{+\infty} \sum_{k \in \mathbb{Z}^{N-1}, |k|_\infty = j} \int_0^1 \int_{k+Y'} |\nabla w| \eta \, dx' \, dx_1 \\ &\leqslant \sum_{k \in \mathbb{Z}^{N-1}, |k|_\infty \leqslant [R]+1} \int_0^1 \int_{k+Y'} |\nabla w| \, dx' \, dx_1 \\ &\quad + \sum_{j=[R]+2}^{+\infty} \sum_{k \in \mathbb{Z}^{N-1}, |k|_\infty = j} \int_0^1 \int_{k+Y'} |\nabla w| \exp\biggl\{ \frac{-\sqrt{N-1}(j+\frac{1}{2}) + R}{2} \biggr\} \, dx' \, dx_1 \\ &\leqslant \biggl((2[R]+3)^{N-1} + \sum_{j=[R]+2}^{+\infty} ((2j+1)^{N-1} - (2j-1)^{N-1}) \\ &\qquad \times \exp\biggl\{ \frac{-\sqrt{N-1}(j+\frac{1}{2}) + R}{2} \biggr\} \biggr) \sqrt{M} \\ &\leqslant C_N (R+1)^{N-1} \sqrt{M}.\end{aligned} \tag{5.24}$$

Additionally, using

$$\int_{\mathbb{R}^+ \times \mathbb{R}^{N-1}} \frac{|\nabla \eta|^2}{\eta} \, dx \leqslant C_N (R+1)^{N-2}$$

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and

$$\lim_{T\to\infty}\frac{1}{T^2}\int_{(0,2T)\times(-2T,2T)^{N-1}}\eta\,\mathrm{d}x=0,$$

we deduce from (5.23) that

$$\begin{aligned}\int_{(1,+\infty)\times\mathbb{R}^{N-1}}|\nabla w|^2\eta\,\mathrm{d}x &\leqslant \lim_{T\to\infty}\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}}|\nabla w|^2\zeta(x_1)\eta\varphi_T^2\,\mathrm{d}x\\ &\leqslant C_N\frac{\beta}{\alpha}\|w\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\sqrt{M}(R+1)^{N-1}\\ &\quad + C_N\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}(R+1)^{N-2}.\end{aligned}$$

Then, since

$$\int_{(0,1)\times\mathbb{R}^{N-1}}|\nabla w|^2\eta\,\mathrm{d}x\leqslant C_N(R+1)^{N-1}M,$$

which can be proved similarly to (5.24), we conclude

$$\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}}|\nabla w|^2\eta\,\mathrm{d}x\leqslant C_N(R+1)^{N-1}\bigg(M+\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\bigg).\tag{5.25}$$

In order to show (5.21), we consider $\varphi\in C_{\mathrm{c}}^\infty(\mathbb{R}^+)$ and $\phi_n\in C_{\mathrm{c}}^\infty(\mathbb{R}^{N-1})$, which converges to 1 everywhere and satisfies $\|\phi_n\|_{L^\infty(\mathbb{R}^{N-1})}\leqslant 1$, $\|\nabla\phi_n\|_{L^\infty(\mathbb{R}^{N-1})^{N-1}}$ converging to zero. Taking $w\eta\phi_n(x')\varphi(x_1)$ as a test function in (5.20), we get (we define $\nabla_{x'}\phi_n=(\nabla\phi_n,0)$)

$$\begin{aligned}&\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}}B\nabla w\nabla(w\eta)\phi_n(x')\varphi(x_1)\,\mathrm{d}x\\ &\qquad+\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}}B\nabla w\nabla_{x'}\phi_n(x')w\eta\varphi(x_1)\,\mathrm{d}x\\ &\qquad\qquad+\int_{\mathbb{R}^+\times\mathbb{R}^{N-1}}B\nabla we_1\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(x_1)w\eta\phi_n(x')\,\mathrm{d}x=0.\end{aligned}\tag{5.26}$$

Using the inequality

$$\begin{aligned}&\bigg|\int_{\mathbb{R}^{N-1}\times\mathbb{R}^+}B\nabla w\nabla_{x'}\phi_n(x')w\eta\varphi(x_1)\,\mathrm{d}x\bigg|\\ &\quad\leqslant\beta\|w\nabla\phi_n\|_{L^\infty(\mathbb{R}^{N-1})^{N-1}}\bigg(\int_{\mathbb{R}^{N-1}\times\mathbb{R}^+}|\nabla w|^2\eta\,\mathrm{d}x\bigg)^{1/2}\bigg(\int_{\mathbb{R}^{N-1}\times\mathbb{R}^+}\eta|\varphi(x_1)|^2\,\mathrm{d}x\bigg)^{1/2}\end{aligned}\tag{5.27}$$

in the second term of (5.26), we deduce that this term tends to zero when $n\to\infty$. To pass to the limit in the other two terms of (5.26), we use the fact that

$$B\nabla w\nabla(w\eta)\varphi(x_1)\quad\text{and}\quad B\nabla we_1\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(x_1)w\eta$$

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are in $L^1(\mathbb{R}^+ \times \mathbb{R}^{N-1})$ and the Lebesgue dominated convergence theorem. We get

$$\int_0^{+\infty}\int_{\mathbb{R}^{N-1}} B\nabla w\nabla(w\eta)\,\mathrm{d}x'\varphi(x_1)\,\mathrm{d}x_1 + \int_0^{+\infty}\int_{\mathbb{R}^{N-1}} B\nabla we_1 w\eta\,\mathrm{d}x'\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(x_1)\,\mathrm{d}x_1 = 0,$$

for every $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^+)$, which, by definition of the weak derivative, implies

$$\frac{\mathrm{d}}{\mathrm{d}S}\biggl(\int_{\{x_1=S\}} B\nabla we_1 w\eta\,\mathrm{d}x'\biggr) = \int_{\{x_1=S\}} B\nabla w\nabla(w\eta)\,\mathrm{d}x' \quad \text{in } \mathbb{R}^+,$$

in the sense of distributions. Since on the other hand, we know that

$$\frac{\mathrm{d}}{\mathrm{d}S}\biggl(\int_S^{+\infty}\int_{\mathbb{R}^{N-1}} B\nabla w\nabla(w\eta)\,\mathrm{d}x'\,\mathrm{d}x_1\biggr) = -\int_{\{x_1=S\}} B\nabla w\nabla(w\eta)\,\mathrm{d}x',$$

we deduce that there exists a constant c such that

$$\int_S^{+\infty}\int_{\mathbb{R}^{N-1}} B\nabla w\nabla(w\eta)\,\mathrm{d}x'\,\mathrm{d}x_1 = -\int_{\{x_1=S\}} B\nabla we_1 w\eta\,\mathrm{d}x' + c, \quad \text{a.e. } S \in \mathbb{R}^+.$$

Integrating the above equality in $(T, T+1)$, with $T > 0$, we get

$$\int_T^{T+1}\int_S^{+\infty}\int_{\mathbb{R}^{N-1}} B\nabla w\nabla(w\eta)\,\mathrm{d}x'\,\mathrm{d}x_1\,\mathrm{d}S = -\int_T^{T+1}\int_{\mathbb{R}^{N-1}} B\nabla we_1 w\eta\,\mathrm{d}x'\,\mathrm{d}x_1 + c. \tag{5.28}$$

Since $B\nabla w\nabla(w\eta)$ belongs to $L^1(\mathbb{R}^+ \times \mathbb{R}^{N-1})$, the first term on the left-hand side of this equality tends to zero when $T \to \infty$. For the second term, we use

$$\biggl|\int_T^{T+1}\int_{\mathbb{R}^{N-1}} B\nabla we_1 w\eta\,\mathrm{d}x'\,\mathrm{d}x_1\biggr| \leqslant \beta\|w\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\biggl(\int_T^{T+1}\int_{\mathbb{R}^{N-1}} |\nabla w|^2\eta\,\mathrm{d}x'\,\mathrm{d}x_1\biggr)^{1/2} \times \biggl(\int_T^{T+1}\int_{\mathbb{R}^{N-1}} \eta\,\mathrm{d}x'\,\mathrm{d}x_1\biggr)^{1/2},$$

which tends to zero. Thus, from (5.28), we get $c = 0$, and so we have

$$\int_S^{+\infty}\int_{\mathbb{R}^{N-1}} B\nabla w\nabla(w\eta)\,\mathrm{d}x'\,\mathrm{d}x_1 = -\int_{\{x_1=S\}} B\nabla we_1 w\eta\,\mathrm{d}x', \quad \text{a.e. } S > 0.$$

Using the inequality

$$|\nabla\eta| \leqslant \frac{\eta}{x_1+1}\biggl(N + \frac{|x'|-R}{x_1+1}\chi_{\{|x'|>R\}}\biggr), \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R}^{N-1}$$

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here, we get

$$\int_S^{+\infty}\int_{\mathbb{R}^{N-1}} B\nabla w\nabla w\eta\,\mathrm{d}x' \leqslant \int_{\{x_1=S\}} |B||\nabla w||w|\eta\,\mathrm{d}x' + \int_S^{+\infty}\int_{\mathbb{R}^{N-1}} |B||\nabla w||w| \times \frac{\eta}{x_1+1}\left(N + \frac{|x'|-R}{x_1+1}\chi_{\{|x'|>R\}}\right)\mathrm{d}x'\,\mathrm{d}x_1.$$

So, from (5.19), w in $L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})$ and Young's inequality, we obtain

$$\begin{aligned}\frac{\alpha}{2}\int_S^{+\infty}&\int_{\mathbb{R}^{N-1}} |\nabla w|^2\eta\,\mathrm{d}x'\,\mathrm{d}x_1\\ &\leqslant \beta\|w\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\left(\int_{\{x_1=S\}}|\nabla w|^2\eta\,\mathrm{d}x'\right)^{1/2}\left(\int_{\{x_1=S\}}\eta\,\mathrm{d}x'\right)^{1/2}\\ &\quad+\frac{\beta^2}{\alpha}\|w\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\\ &\quad\times\int_S^{+\infty}\int_{\mathbb{R}^{N-1}}\frac{\eta}{(x_1+1)^2}\left(N^2+\frac{(|x'|-R)^2}{(x_1+1)^2}\chi_{\{|x'|>R\}}\right)\mathrm{d}x'\,\mathrm{d}x_1.\end{aligned}$$

Estimating the integrals which do not depend on w, and assuming $S < R$, we deduce the existence of $C_N > 0$ such that

$$\begin{aligned}\frac{\alpha}{2}\int_S^{+\infty}\int_{\mathbb{R}^{N-1}}|\nabla w|^2\eta\,\mathrm{d}x'\,\mathrm{d}x_1 &\leqslant C_N\frac{\beta^2}{\alpha}\|w\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\frac{R^{N-1}}{(S+1)^N}\\ &+\beta C_N\|w\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\frac{R^{(N-1)/2}}{(S+1)^{(N-1)/2}}\left(\int_{\{x_1=S\}}|\nabla w|^2\eta\,\mathrm{d}x'\right)^{1/2},\quad \text{a.e. } S>0.\end{aligned}\tag{5.29}$$

Now, we define

$$\Psi(S) = \int_S^{+\infty}\int_{\mathbb{R}^{N-1}}|\nabla w|^2\eta\,\mathrm{d}x'\,\mathrm{d}x_1\quad\text{for all } S>0.$$

Then, applying Young's inequality in (5.29) we deduce that

$$\Psi(S)\leqslant C_N\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^{N-1}\times\mathbb{R}^+)}\frac{R^{N-1}}{(S+1)^N}-\frac{S+1}{N+1}\Psi'(S),\ \text{a.e. } S>0,$$

which gives

$$(N+1)(S+1)^N\Psi(S)+(S+1)^{N+1}\Psi'(S)\leqslant C_N\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}R^{N-1},$$

for a.e. $S\in(0,R)$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}S}((S+1)^{N+1}\Psi)\leqslant C_N\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}R^{N-1},\quad\text{a.e. } S>0.$$

Integrating this inequality in $(0,S)$, we get

$$\Psi(S)\leqslant\frac{\Psi(0)}{(S+1)^{N+1}}+C_N\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}^+\times\mathbb{R}^{N-1})}\frac{R^{N-1}S}{(S+1)^{N+1}}\quad\text{for all } S\in[0,R].$$

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So, from (5.25) and the inequality

$$\frac{1}{(S+1)^{N-1}}\int_S^{2S}\int_{\{|x'|<R\}}|\nabla w|^2\,\mathrm{d}x'\,\mathrm{d}x_1\leqslant \Psi(S),$$

we conclude (5.21).

To prove (5.22), we take $n\in\mathbb{N}$ such that $2^{n-1}\leqslant R\leqslant 2^n$. From (5.21) we get

$$\begin{aligned}\int_0^R\int_{\{|x'|<R\}}&|\nabla w|^2(1+x_1)\,\mathrm{d}x'\,\mathrm{d}x_1\\ &\leqslant\sum_{j=1}^{n}(1+2^j)\int_{2^{j-1}}^{2^j}\int_{\{|x'|<R\}}|\nabla w|^2\,\mathrm{d}x'\,\mathrm{d}x_1\\ &\leqslant C_N\biggl(M\sum_{j=1}^{n}\frac{1+2^j}{(1+2^{j-1})^2}+\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}\times\mathbb{R}^{N-1})}\sum_{j=1}^{n}\frac{1+2^j}{1+2^{j-1}}\biggr)R^{N-1}\\ &\leqslant C_N\biggl(M+\frac{\beta^2}{\alpha^2}\|w\|^2_{L^\infty(\mathbb{R}\times\mathbb{R}^{N-1})}\log R\biggr)R^{N-1}.\end{aligned}$$

This proves (5.22). □

Using a rotation which transforms H in $\mathbb{R}^+\times\mathbb{R}^{N-1}$, and taking into account proposition 5.10, we can apply lemma 5.11 to T^i. This gives the following corollary.

Corollary 5.12. *Assume that $A\in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, and $\partial_{y_\nu}A\in L^\infty_\sharp(Y;\mathcal{M}_N)$. Then there exists a constant $C>0$ such that, for every $R>2$, every $r\in\mathbb{R}$, and every $i\in\{1,\dots,N\}$, we have*

$$\int_0^R\int_{B(0,R)\cap\{\nu\}^\perp}|\nabla_\mu T^i(r,\rho+s\nu)|^2(1+s)\,\mathrm{d}\rho\,\mathrm{d}s\leqslant CR^{N-1}\log R. \tag{5.30}$$

Remark 5.13. We do not know whether (5.30) is optimal. Indeed, if we consider an algebra generated by a sequence $s^i\in\mathbb{R}^{N-1}$ such that $|s^i|$ strictly decreases to zero, then, taking $\phi^0(x')=\sum\alpha^i\sin(s^ix')$, with $\sum_{i=0}^{+\infty}|s_i||\alpha^i|^2<+\infty$, the solution of the problem (it has a similar structure to problem (4.11) and so it is closely related to the problem defining Υ^i and then T^i)

$$\phi(0,x')=\phi^0(x'),\quad \nabla u\in L^2(\mathbb{R}^+,\mathcal{B}(\mathcal{S}))^N,$$
$$\int_{\mathbb{R}^+}M_{x'}(\nabla\phi\nabla v)\,\mathrm{d}x_1=0\quad\text{for all } v \text{ with } v(0,x_2)=0,\quad \nabla v\in L^2(\mathbb{R}^+,\mathcal{B}(\mathcal{S}))^N,$$

is given by

$$\phi(x)=\sum_{i=0}^{+\infty}\alpha_i\mathrm{e}^{-|s^i|x_1}\sin(s^ix'),$$

and so it satisfies the condition that

$$\int_{\mathbb{R}^+}M_{x'}(|\nabla\phi|^2)x_1\,\mathrm{d}x_1<+\infty,$$

while, by lemma 5.11, we just expect (this estimate can be proved for the solution of problem (4.11)) that

$$\int_0^R M_{x'}(|\nabla\phi|^2)x_1\,\mathrm{d}x_1 \leqslant C(1+\log R) \quad \text{for all } R>1.$$

We also observe in this example that, for every $\gamma>1$, we can choose φ^0 such that

$$\int_0^{+\infty} M_{x'}(|\nabla\phi|^2)x_1^\gamma\,\mathrm{d}x_1 = +\infty.$$

This means in particular that we cannot expect an exponential decay for T^i.

From corollary 5.12 we have the following result.

LEMMA 5.14. *Assume that $A\in L^\infty_\sharp(Y;\mathcal{M}_N)$ is uniformly elliptic, that $\partial_{y_\nu}A\in L^\infty_\sharp(Y;\mathcal{M}_N)$, and consider $\varphi\in W^{1,\infty}(\Omega)$ such that* $\operatorname{supp}(\varphi)\subset\bar{\Lambda}_\delta$*, with Λ_δ defined by (4.2). Then, for every $i\in\{1,\dots,N\}$, we have*

$$-\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla\left(\varepsilon\varphi T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\right) = h^i_\varepsilon \quad \text{in } \Omega, \tag{5.31}$$

where h^i_ε is such that there exists $C>0$ (which depends on φ) with

$$\|h^i_\varepsilon\|_{H^{-1}(\Omega)} \leqslant C\varepsilon\sqrt{|\log\varepsilon|}. \tag{5.32}$$

Proof. For $v\in H^1_0(\Omega)$, we have

$$\begin{aligned}
\int_\Omega & A\left(\frac{x}{\varepsilon}\right)\nabla\left(\varepsilon\varphi T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\right)\nabla v\,\mathrm{d}x \\
&= \int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\nabla(\varphi v)\,\mathrm{d}x \\
&\quad - \int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\nabla\varphi v\,\mathrm{d}x \\
&\quad + \varepsilon\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\varphi\nabla v T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\mathrm{d}x.
\end{aligned} \tag{5.33}$$

The first term on the right-hand side of (5.33) can be estimated easily by using (5.16) and the change of variables $x=a\nu+\varepsilon\mu$. We have

$$\begin{aligned}
\int_\Omega & A\left(\frac{x}{\varepsilon}\right)\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\nabla(\varphi v)\,\mathrm{d}x \\
&= \varepsilon^{N-1}\int_H A\left(\frac{a\nu}{\varepsilon}+\mu\right)\nabla_\mu T^i\left(\frac{a}{\varepsilon},\mu\right)\nabla_\mu\left(\varphi\left(\frac{a\nu}{\varepsilon}+\mu\right)v\left(\frac{a\nu}{\varepsilon}+\mu\right)\right)\mathrm{d}x = 0.
\end{aligned} \tag{5.34}$$

To estimate the second term on the right-hand side of (5.33) we use $\varphi=0$ outside Λ_δ and $v=0$ on Λ. Thus, the change of variables $x=z+(a+t)\nu$, with $z\in\omega$,

$0 < t < \delta$, shows that

$$
\begin{aligned}
&\left|\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\nabla\varphi v\,\mathrm{d}x\right| \\
&\quad\leqslant C\int_{\Lambda_\delta}\left|\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\right||v|\,\mathrm{d}x \\
&\quad\leqslant C\int_0^\delta\left(\int_\omega\left|\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{z+t\nu}{\varepsilon}\right)\right|^2\mathrm{d}z\right)^{1/2}\left(\int_\omega|v(z+(a+t)\nu)|^2\,\mathrm{d}z\right)^{1/2}\mathrm{d}t \\
&\quad= C\int_0^\delta\left(\int_\omega\left|\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{z+t\nu}{\varepsilon}\right)\right|^2\mathrm{d}z\right)^{1/2} \\
&\qquad\quad\times\left(\int_\omega\left|\int_0^t\partial_{x_\nu}v(z+(a+\tau)\nu)\,\mathrm{d}\tau\right|^2\mathrm{d}z\right)^{1/2}\mathrm{d}t \\
&\quad\leqslant C\int_0^\delta\left(\int_\omega\left|\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{z+t\nu}{\varepsilon}\right)\right|^2\mathrm{d}z\right)^{1/2}\left(t\int_\omega\int_0^\delta|\nabla v(z+(a+\tau)\nu)|^2\,\mathrm{d}\tau\,\mathrm{d}z\right)^{1/2}\mathrm{d}t \\
&\quad\leqslant C\left(\int_0^\delta\int_\omega\left|\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{z+t\nu}{\varepsilon}\right)\right|^2 t\,\mathrm{d}z\,\mathrm{d}t\right)^{1/2}\|v\|_{H_0^1(\Omega)} \\
&\quad= C\left(\varepsilon^{N+1}\int_0^{\delta/\varepsilon}\int_{B(0,R/\varepsilon)\cap\{\nu\}^\perp}\left|\nabla_\mu T^i\left(\frac{a}{\varepsilon},\rho+s\nu\right)\right|^2 s\,\mathrm{d}\rho\,\mathrm{d}s\right)^{1/2}\|v\|_{H_0^1(\Omega)},
\end{aligned}
$$

with $R > \delta$ and such that $\omega \subset B(0,R)\cap\{\nu\}^\perp$. So, by (5.30), we get

$$
\left|\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla_\mu T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\nabla\varphi v\,\mathrm{d}x\right| \leqslant C\varepsilon\sqrt{|\log\varepsilon|}\|v\|_{H_0^1(\Omega)}. \tag{5.35}
$$

For the third term on the right-hand side of (5.33) we merely use the fact that T^i is bounded in $L^\infty(\Omega)$, which easily gives

$$
\varepsilon\left|\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\varphi\nabla v T^i\left(\frac{a}{\varepsilon},\frac{x-a\nu}{\varepsilon}\right)\mathrm{d}x\right| \leqslant C\varepsilon\|v\|_{H_0^1(\Omega)}. \tag{5.36}
$$

From (5.33)–(5.36), we deduce (5.31), where h_ε^i satisfies (5.32). □

Proof of theorem 5.4. From (3.3), (3.6) and lemma 5.14 applied to

$$
\varphi = \frac{\partial u_0}{\partial x_i}(a\nu + Px)\psi, \quad i \in \{1,\dots,N\},
$$

we see that there exists $r_\varepsilon \in H^{-1}(\Omega)$, with $\|r_\varepsilon\|_{H^{-1}(\Omega)} \leqslant C\varepsilon\sqrt{|\log\varepsilon|}$, such that $\tilde{u}_\varepsilon$ satisfies

$$
-\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla\tilde{u}_\varepsilon = f + r_\varepsilon \quad \text{on } \Omega. \tag{5.37}
$$

Now, analogously to the solution b_ε of (3.9), we define $\tilde{b}_\varepsilon$ by

$$
\left.\begin{aligned}
-\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla\tilde{b}_\varepsilon &= 0 \quad \text{in } \Omega,\\
\tilde{b}_\varepsilon &= \tilde{u}_\varepsilon \quad \text{on } \partial\Omega.
\end{aligned}\right\} \tag{5.38}
$$

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From the maximum principle, we know that $\|\tilde{b}_\varepsilon\|_{L^\infty(\Omega)} \leqslant \|\tilde{u}_\varepsilon\|_{L^\infty(\partial\Omega)}$. Thus, since u_0 vanishes on $\partial\Omega$, we deduce that there exists $C > 0$ such that

$$\|\tilde{b}_\varepsilon\|_{L^\infty(\Omega)} \leqslant C\varepsilon. \tag{5.39}$$

From (5.37) and (5.38), the sequence $\tilde{u}_\varepsilon - \tilde{b}_\varepsilon$ satisfies

$$\left.\begin{aligned} -\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla(\tilde{u}_\varepsilon - \tilde{b}_\varepsilon) &= f + r_\varepsilon \quad \text{in } \Omega, \\ \tilde{u}_\varepsilon - \tilde{b}_\varepsilon &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.40}$$

Taking $u_\varepsilon - \tilde{u}_\varepsilon + \tilde{b}_\varepsilon$ as a test function in the difference of (1.1) and (5.37) we deduce

$$\|u_\varepsilon - \tilde{u}_\varepsilon + \tilde{b}_\varepsilon\|_{H^1(\Omega)} \leqslant C\varepsilon\sqrt{|\log\varepsilon|},$$

and thus, for every $\phi \in C^\infty(\Omega)$, with $\phi = 0$ on $\partial\Omega \setminus K$, we have

$$\|(u_\varepsilon - \tilde{u}_\varepsilon + \tilde{b}_\varepsilon)\phi\|_{H^1(\Omega)} \leqslant C\varepsilon\sqrt{|\log\varepsilon|}.$$

Inequality (5.6) will be then proved if we show that

$$\|\tilde{b}_\varepsilon\phi\|_{H^1(\Omega)} \leqslant C\|\varphi\|_{W^{1,\infty}(\Omega)}\varepsilon\sqrt{|\log\varepsilon|}. \tag{5.41}$$

For this purpose, we use

$$\|\tilde{b}_\varepsilon\phi\|_{H^1(\Omega)} \leqslant C(\|\tilde{b}_\varepsilon\phi\|_{L^2(\Omega)} + \|\tilde{b}_\varepsilon\nabla\phi\|_{L^2(\Omega)^N} + \|\nabla\tilde{b}_\varepsilon\phi\|_{L^2(\Omega)}), \tag{5.42}$$

where, by (5.39), the first and second terms on the right-hand side satisfy

$$\|\tilde{b}_\varepsilon\phi\|_{L^2(\Omega)} + \|\tilde{b}_\varepsilon\nabla\phi\|_{L^2(\Omega)^N} \leqslant C\|\varphi\|_{W^{1,\infty}(\Omega)}\varepsilon. \tag{5.43}$$

To estimate the last term in (5.42), we take $\tilde{b}_\varepsilon\phi^2 \in H_0^1(\Omega)$ as a test function in (5.38). This gives

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\tilde{b}_\varepsilon\nabla\tilde{b}_\varepsilon\phi^2\,\mathrm{d}x + 2\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla\tilde{b}_\varepsilon\nabla\phi\tilde{b}_\varepsilon\phi\,\mathrm{d}x = 0.$$

Owing to A being bounded and uniformly elliptic, and the Cauchy–Schwarz inequality and (5.39), we have $\|\nabla\tilde{b}_\varepsilon\phi\|_{L^2(\Omega)^N} \leqslant C\|\varphi\|_{W^{1,\infty}(\Omega)}\varepsilon$, which on combination with (5.42) and (5.43) shows that $\|\tilde{b}_\varepsilon\phi\|_{H^1(\Omega)} \leqslant C\|\varphi\|_{W^{1,\infty}(\Omega)}\varepsilon$, and thus we get (5.41). □

Remark 5.15. If we assume $\nabla u_0 = 0$ outside of K in theorem 5.4, then we can take $\phi = 1$.

Remark 5.16. By (3.2) and $\Omega \in C^{1,1}$, assuming $f \in C^{0,\alpha}(\bar{\Omega})$, $\alpha \in (0, 1]$, we then have that $u_0 \in C^{2,\alpha}(\bar{\omega})$ and thus is also in $W^{2,\infty}(\Omega)$.

Appendix A.

The following result permits us to show that the solution of a standard variational problem for an elliptic equation with periodic conditions (see, for example, equation (3.1)) is in fact a solution in the sense of the distributions. We refer the reader to [9] for related results.

LEMMA A.1. *Assume $G \in L^1_\sharp(Y)^N$ such that*

$$\int_Y G\nabla v \,\mathrm{d}y = 0 \quad \textit{for all } v \in C^\infty_\sharp(Y). \tag{A 1}$$

Then, G satisfies the equation

$$-\operatorname{div} G = 0 \quad \textit{in } \mathbb{R}^N, \tag{A 2}$$

in the sense of the distributions.

Proof. We have to prove

$$\int_{\mathbb{R}^N} G\nabla\varphi \,\mathrm{d}y = 0 \quad \text{for all } \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N). \tag{A 3}$$

The periodicity of G gives

$$\begin{aligned}\int_{\mathbb{R}^N} G\nabla\varphi \,\mathrm{d}y &= \sum_{k\in\mathbb{Z}^N}\int_{k+Y} G(y)\nabla\varphi(y)\,\mathrm{d}y = \sum_{k\in\mathbb{Z}^N}\int_Y G(y+k)\nabla\varphi(y+k)\,\mathrm{d}y \\ &= \int_Y G(y)\nabla\biggl(\sum_{k\in\mathbb{Z}^N}\varphi(y+k)\biggr)\mathrm{d}y \quad \text{for all } \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N).\end{aligned}$$

However, $\sum_{k\in\mathbb{Z}^N}\varphi(y+k)$ belongs to $C^\infty_\sharp(Y)$, for every $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$. Thus, by (A 1) we get (A 3). □

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References

1 G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Analysis* **23** (1992), 1482–1518.
2 T. Arbogast, J. Douglas and U. Hornung. Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Analysis* **21** (1990), 823–836.
3 A. Bensoussan, J. L. Lions and G. Papanicolau. *Asymptotic analysis for periodic structures*. Studies in Mathematics and Its Applications, vol. 5 (Amsterdam: North-Holland, 1978).
4 J. Casado-Díaz. Exponential decay for the solutions of nonlinear elliptic problems posed in unbounded cylinders. *J. Math. Analysis Applic.* **328** (2007), 151–169.
5 J. Casado-Díaz. The asymptotic behavior of oscillatory periodic problems near a curved boundary. (In preparation.)
6 J. Casado-Díaz and I. Gayte. A general compactness result and its application to the two-scale convergence of almost periodic functions. *C. R. Acad. Sci. Paris Sér. I* **323** (1996), 329–334.
7 J. Casado-Díaz and I. Gayte. A derivation theory for generalized Besicovitch spaces and its application for partial differential equations. *Proc. R. Soc. Edinb.* A **132** (2002), 283–315.
8 J. Casado-Díaz and I. Gayte. The two-scale convergence method applied to generalized Besicovitch spaces. *Proc. R. Soc. Lond.* A **458** (2002), 2925–2946.
9 D. Cioranescu and P. Donato. *An introduction to homogenization*. Oxford Lecture Series in Mathematics and Its Applications, vol. 7 (Oxford University Press, 1999).
10 C. Conca and M. Vanninathan. Homogenization of periodic structures via Bloch decomposition. *SIAM J. Appl. Math.* **57** (1997), 1639–1659.

11 G. Griso. Interior error estimate for periodic homogenization. *C. R. Acad. Sci. Paris Sér. I* **340** (2005), 251–254.

12 V. V. Jikov, S. M. Kozlov and O. A. Oleinik. *Homogenization of differential operators and integral functionals* (Springer, 1994).

13 J. L. Lions. *Some methods in the mathematical analysis of systems and their control* (Beijing: Science Press, 1981).

14 F. Murat and L. Tartar. *H*-convergence. In *Topics in the mathematical modelling of composite materials* (ed. A. Cherkaev and R. Kohn), Progress in Nonlinear Differential Equations and Their Applications, pp. 21–43 (Basel: Birkhäuser, 1997).

15 G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Analysis* **20** (1989), 608–623.

16 G. Nguetseng. Homogenization structures and applications. I. *Z. Analysis Anwend.* **22** (2003), 73–107.

17 G. Nguetseng. Homogenization structures and applications. II. *Z. Analysis Anwend.* **23** (2004), 483–508.

18 G. Nguetseng and H. Nnang. Homogenization of nonlinear monotone operators beyond the periodic setting. *Electron. J. Diff. Eqns* **36** (2003), 1–24.

19 G. Weiske. On some problems related to linear elasticity, optimal design and homogenisation. PhD thesis, Carnegie Mellon University, Pittsburgh, PA (1997).

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