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# Codimension two spacelike submanifolds of the Lorentz-Minkowski spacetime into the light cone

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Following an original idea of Palmas, Palomo and Romero, recently developed in [**12**], we study codimension two spacelike submanifolds contained in the light cone of the Lorentz-Minkowski spacetime through an approach which allows us to compute their extrinsic and intrinsic geometries in terms of a single function $u$. As the first application of our approach, we classify the totally umbilical ones. For codimension two compact spacelike submanifolds into the light cone, we show that they are conformally diffeomorphic to the round sphere and that they are given by an explicit embedding written in terms of $u$. In the last part of the paper, we consider the case where the submanifold is (marginally, weakly) trapped. In particular, we derive some non-existence results for weakly trapped submanifolds into the light cone.

*Keywords:* spacelike submanifolds; Lorentz-Minkowski spacetime; light cone; mean curvature; trapped submanifolds

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## 1. Introduction

In this paper, we consider the study of the geometry of codimension two spacelike $n$-submanifolds immersed into the future component of the light cone of the $(n+2)$-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$, denoted here by $\Lambda^+$. At this respect, recall from [**3**] (see also corollary 7.6 in [**6**]) that an $n(\geqslant 3)$-dimensional Riemannian manifold is (locally) conformally flat if and only if it can be locally isometrically immersed in the light cone of $\mathbb{L}^{n+2}$. This fact is an important motivation for the study of spacelike hypersurfaces in the light cone of $\mathbb{L}^{n+2}$, that is, codimension two spacelike $n$-submanifolds of $\mathbb{L}^{n+2}$ into the light cone.

Following an original idea of Palmas, Palomo and Romero recently developed in [**12**] (see also [**13**] for the previous case of 2-dimensional spacelike surfaces into the light cone of $\mathbb{L}^4$), we know that if $\Sigma$ is such a submanifold there always exists a globally and naturally defined future-pointing normal null frame on $\Sigma$. As observed in [**12**], this allows us to compute the extrinsic and intrinsic geometry of the submanifold in terms of one single positive function defined on $\Sigma$ and denoted here by $u$. This is done in § 3 where following the approach in [**12**], we establish the basic equations for spacelike submanifolds into the light cone and compute their second fundamental form in terms of $u$ and its Hessian (propositions 3.1 and 3.2).

As the first application of our approach, § 4 is devoted to the study of totally umbilical submanifolds contained in the light cone. In particular, in theorem 4.2 we classify codimension two totally umbilical spacelike submanifolds in $\Lambda^+$. See [**8**] for a systematic study of the umbilicity and semi-umbilicity of spacelike submanifolds of the Lorentz-Minkowski spacetime. In § 5, we give a compactness criterion for complete submanifolds in terms of the growth of the positive function $u$, and we obtain that every codimension two compact spacelike submanifold contained in $\Lambda^+$ is conformally diffeomorphic to the round sphere (proposition 5.2). Even more, we prove that every codimension two compact spacelike submanifold into $\Lambda^+$ is given by an explicit embedding which can be written in terms of the single function $u$ (corollary 5.4).

In the last part of the paper, we are interested in the case where the submanifold $\Sigma$ is trapped. The original formulation of trapped surfaces was given by Penrose in terms of the signs or the vanishing of the null expansions. The notion of trapped surfaces was introduced early in General Relativity to study spacetime singularities and blackholes. We refer the reader to § 7 in [**5**] and references therein for a description of some of the recent mathematical developments in this field. In terms of its mean curvature vector field $\mathbf{H}$, a codimension two spacelike submanifold $\Sigma$ immersed into $\mathbb{L}^{n+2}$ is

(a) trapped if and only if either $\mathbf{H}$ is timelike and future-pointing (future trapped) or $\mathbf{H}$ is timelike and past-pointing (past trapped).

(b) marginally trapped if and only if either $\mathbf{H}$ is null and future-pointing (future marginally trapped) or $\mathbf{H}$ is null and past-pointing (past marginally trapped).

(c) weakly trapped if and only if either $\mathbf{H}$ is causal and future-pointing (future weakly trapped) or $\mathbf{H}$ is causal and past-pointing (past weakly trapped).

In particular, when $\Sigma$ is immersed into the future component of the light cone of $\mathbb{L}^{n+2}$ we obtain that (see corollary 6.2)

(a) $\Sigma$ is (necessarily past) trapped if and only if $2u\Delta u - n(1 + \|\nabla u\|^2) > 0$ on $\Sigma$.

(b) $\Sigma$ is (necessarily past) marginally trapped if and only if $2u\Delta u - n(1 + \|\nabla u\|^2) = 0$ on $\Sigma$.

(c) $\Sigma$ is (necessarily past) weakly trapped if and only if $2u\Delta u - n(1 + \|\nabla u\|^2) \geqslant 0$ on $\Sigma$.

It is already known [**1**, remark 4.2] that there exists no compact weakly trapped submanifolds in $\mathbb{L}^{n+2}$. In particular, there is no codimension two compact weakly trapped submanifold into the light cone of $\mathbb{L}^{n+2}$. In theorem 7.2, and as an application of the weak maximum principle, we extend this non-existence result to the more general case of stochastically complete submanifolds, under the assumption that the function $u$ is bounded from above. Related to this, in theorem 7.4 we also prove that there exists no codimension two complete weakly trapped submanifold $\Sigma$ into the light cone of $\mathbb{L}^{n+2}$ for which the function $u$ satisfies $u \in L^q(\Sigma)$, for any $q > 0$.

Finally, in § 8 we consider the case of submanifolds contained into a null hyperplane $\mathcal{L}$ of $\mathbb{L}^{n+2}$. In this case, there is no restriction on the sign of the function $u$, and we deduce that every codimension two spacelike submanifold $\Sigma$ contained into $\mathcal{L}$ is always marginally trapped except at points where $\Delta u = 0$ (if any), where $\Sigma$ is minimal (proposition 8.4). Moreover, we observe also that every codimension two complete spacelike submanifold contained into $\mathcal{L}$ is isometric to the flat Euclidean space $(\mathbb{R}^n, \langle,\rangle_{\mathbb{R}^n})$, and it is given by an explicit embedding which can be written in terms of the single function $u$. In particular, it is always embedded and marginally trapped, except at points where $\Delta u = 0$ (if any), where it is minimal (corollary 8.7). As an application in corollary 8.8 we characterize those having parallel mean curvature vector field.

## 2. Preliminaries

Let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz-Minkowski spacetime, that is, $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle,\rangle = -(\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + \ldots + (\mathrm{d}x_{n+2})^2$$

where $(x_1, x_2, \ldots, x_{n+2})$ are the canonical coordinates of $\mathbb{R}^{n+2}$. A smooth immersion $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ of an $n$-dimensional connected manifold $\Sigma$ is said to be spacelike if $\psi$ induces a Riemannian metric on $\Sigma$, which as usual is also denoted by $\langle,\rangle$. In other words, $\Sigma$ is a codimension two spacelike submanifold in the Lorentz-Minkowski spacetime. We will consider on $\mathbb{L}^{n+2}$ the time-orientation induced by the unit timelike vector

$$\mathbf{e}_1 = (1, 0, \ldots, 0).$$

In this paper, we are interested in the case where the submanifold $\Sigma$ is contained in the light cone of $\mathbb{L}^{n+2}$. In this case, the position vector field determines a globally defined null vector field on $\Sigma$ which is normal to the submanifold (and tangent to the light cone) and future-pointing. When this happens, it is not difficult to build a globally defined future-pointing normal frame on $\Sigma$. To see it, let $\Sigma$ be a codimension two spacelike submanifold in $\mathbb{L}^{n+2}$ (non necessarily contained in $\Lambda^+$) which admits a globally defined future-pointing null vector field $\xi \in \mathfrak{X}^\perp(\Sigma)$, so that $\langle \xi, \mathbf{e}_1 \rangle < 0$. Let $\mathbf{e}_1^\perp$ denote the normal component of $\mathbf{e}_1$ along the submanifold, that is, for every $p \in \Sigma$, we have the following orthogonal decomposition

$$\mathbf{e}_1 = \mathbf{e}_1^\top(p) + \mathbf{e}_1^\perp(p), \tag{2.1}$$

where $\mathbf{e}_1^\top \in \mathfrak{X}(\Sigma)$ and $\mathbf{e}_1^\perp \in \mathfrak{X}^\perp(\Sigma)$. In particular,

$$\langle \mathbf{e}_1^\perp, \mathbf{e}_1^\perp \rangle = -1 - \|\mathbf{e}_1^\top\|^2 \leqslant -1 < 0$$

and the vector field $\nu$ given by

$$\nu = \frac{\mathbf{e}_1^{\perp}}{\sqrt{1+\|\mathbf{e}_1^{\top}\|^2}} \tag{2.2}$$

determines, along the submanifold, a globally defined unit timelike vector which is normal to $\Sigma$ and future-pointing. In particular, we have $\langle \xi, \nu\rangle < 0$, and the vector field

$$\eta = -\frac{1}{2\langle \xi, \nu\rangle^2}\xi - \frac{1}{\langle \xi, \nu\rangle}\nu \tag{2.3}$$

provides us another globally defined normal null vector field along the submanifold which is future-pointing and it satisfies $\langle \xi, \eta\rangle = -1$.

Summing up, if $\Sigma$ is a codimension two spacelike submanifold which admits a globally defined future-pointing null vector field $\xi$, there always exists a globally defined future-pointing normal null frame $\{\xi, \eta\}$ and a globally defined future-pointing normal unit timelike vector field $\nu$ on $\Sigma$ satisfying

$$\langle \xi, \xi\rangle = \langle \eta, \eta\rangle = 0, \quad \langle \xi, \eta\rangle = -1, \quad \langle \xi, \nu\rangle < 0, \quad \langle \eta, \nu\rangle < 0.$$

At this respect, and as observed in [**12**, remark 4.2], it is worth pointing out that every codimension two spacelike submanifold in the light cone of $\mathbb{L}^{n+2}$ is orientable and, as a consequence, admits a globally defined normal null frame.

Let us denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connections of $\mathbb{L}^{n+2}$ and $\Sigma$, respectively, and let $\nabla^{\perp}$ denote the normal connection of $\Sigma$ in $\mathbb{L}^{n+2}$. Let

$$\mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^{\perp}(\Sigma)$$

stand for the second fundamental form of the submanifold. Observe that in this paper we will follow for II the usual convention in Relativity (and opposite to the one usually taken in Differential Geometry). With this convention, for instance, the mean curvature vector field of a round sphere in Euclidean space points outwards. Then, the Gauss and Weingarten formulas of the immersion $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - \mathrm{II}(X, Y), \tag{2.4}$$

for every $X, Y \in \mathfrak{X}(\Sigma)$, and

$$\overline{\nabla}_X \zeta = A_\zeta X + \nabla^{\perp}_X \zeta, \tag{2.5}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$ and normal vector field $\zeta \in \mathfrak{X}^{\perp}(\Sigma)$. Moreover, for every normal vector $\zeta \in \mathfrak{X}^{\perp}(\Sigma)$, $A_\zeta$ denotes the Weingarten endomorphism (or shape operator) associated to $\zeta$, that is, the symmetric operator $A_\zeta : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ given by

$$\langle A_\zeta X, Y\rangle = \langle \mathrm{II}(X, Y), \zeta\rangle$$

for every $X, Y \in \mathfrak{X}(\Sigma)$ and every $\zeta \in \mathfrak{X}^{\perp}(\Sigma)$.

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In terms of the globally defined normal null frame $\{\xi, \eta\}$, the second fundamental form is written as

$$\mathrm{II}(X,Y) = -\langle A_\eta X, Y\rangle \xi - \langle A_\xi X, Y\rangle \eta \tag{2.6}$$

for every $X, Y \in \mathfrak{X}(\Sigma)$. The mean curvature vector field of the submanifold is defined by

$$\mathbf{H} = \frac{1}{n}\mathrm{tr}(\mathrm{II}) \in \mathfrak{X}^\perp(\Sigma)$$

and, in terms of $\{\xi, \eta\}$, it is written as

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta, \tag{2.7}$$

where

$$\theta_\xi = \frac{1}{n}\,\mathrm{tr}(A_\xi) \quad \text{and} \quad \theta_\eta = \frac{1}{n}\,\mathrm{tr}(A_\eta)$$

define the null mean curvatures (or null expansion scalars) of $\Sigma$. In particular,

$$\langle \mathbf{H}, \mathbf{H}\rangle = -2\theta_\xi\theta_\eta. \tag{2.8}$$

Since $A_\zeta X = (\overline{\nabla}_X \zeta)^\top$ for every normal vector field $\zeta \in \mathfrak{X}^\perp(\Sigma)$, it follows that

$$\theta_\xi = \frac{1}{n}\mathrm{div}_\Sigma \xi \quad \text{and} \quad \theta_\eta = \frac{1}{n}\mathrm{div}_\Sigma \eta.$$

That means that, physically, $\theta_\xi$ (resp., $\theta_\eta$) measures the divergence of the light rays emanating from $\Sigma$ in the direction of $\xi$ (resp., $\eta$).

On the other hand, it follows from the Gauss equation of the submanifold that the curvature tensor $R$ of $\Sigma$ is given by

$$R(X,Y)Z = A_{\mathrm{II}(X,Z)}Y - A_{\mathrm{II}(Y,Z)}X$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where in our convention

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

In particular, the Ricci and the scalar curvature of $\Sigma$ are given by

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \mathrm{tr}(A_{\mathrm{II}(X,Y)}) - \sum_{i=1}^n \langle \mathrm{II}(X,E_i), \mathrm{II}(Y,E_i)\rangle \\ &= n\langle \mathbf{H}, \mathrm{II}(X,Y)\rangle + \langle (A_\xi \circ A_\eta + A_\eta \circ A_\xi)X, Y\rangle \end{aligned} \tag{2.9}$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on $\Sigma$, and

$$\mathrm{Scal} = \mathrm{tr}(\mathrm{Ric}) = n^2\langle \mathbf{H}, \mathbf{H}\rangle + 2\mathrm{tr}(A_\xi \circ A_\eta). \tag{2.10}$$

The two other fundamental equations of the submanifold $\Sigma$ are the Codazzi and the Ricci equations. In our case, the Codazzi equation of $\Sigma$ is given by

$$(\nabla_X \mathrm{II})(Y,Z) = (\nabla_Y \mathrm{II})(X,Z) \tag{2.11}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where as usual

$$(\nabla_X \mathrm{II})(Y, Z) = \nabla^\perp_X (\mathrm{II}(Y, Z)) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z).$$

Observe that, for every normal field $\zeta \in \mathfrak{X}^\perp(\Sigma)$, it holds

$$\langle (\nabla_X A_\zeta) Y, Z \rangle = \langle \nabla \mathrm{II}(Y, Z, X), \zeta \rangle + \langle \mathrm{II}(Y, Z)), \nabla^\perp_X \zeta \rangle,$$

where

$$(\nabla_X A_\zeta)\, Y = \nabla_X (A_\zeta Y) - A_\zeta (\nabla_X Y).$$

Using this into (2.11), Codazzi equation can be written equivalently as

$$(\nabla_X A_\zeta) Y = (\nabla_Y A_\zeta) X + A_{\nabla^\perp_X \zeta} Y - A_{\nabla^\perp_Y \zeta} X \tag{2.12}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$ and normal vector field $\zeta \in \mathfrak{X}^\perp(\Sigma)$. Finally, in our case the Ricci equation of $\Sigma$ is given by

$$\langle R^\perp(X, Y)\zeta_1, \zeta_2 \rangle = -\langle [A_{\zeta_1}, A_{\zeta_2}] X, Y \rangle, \tag{2.13}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$ and normal vector fields $\zeta_1, \zeta_2 \in \mathfrak{X}^\perp(\Sigma)$, where $R^\perp$ denotes the normal curvature,

$$R^\perp(X, Y)\zeta = \nabla^\perp_{[X,Y]} \zeta - [\nabla^\perp_X, \nabla^\perp_Y]\zeta,$$

and $[A_{\zeta_1}, A_{\zeta_2}] = A_{\zeta_1} \circ A_{\zeta_2} - A_{\zeta_2} \circ A_{\zeta_1}$.

## 3. Basic equations for submanifolds into the light cone

The light cone in $\mathbb{L}^{n+2}$ is the subset

$$\Lambda = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x \neq \mathbf{0}\},$$

and it corresponds to the subset of all points of de Lorentz-Minkowski spacetime which can be reached from $\mathbf{0} \in \mathbb{L}^{n+2}$ through a null (or lightlike) geodesic starting at $\mathbf{0} \in \mathbb{L}^{n+2}$. The future component of $\Lambda$ is

$$\Lambda^+ = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x_1 > 0\}.$$

Let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ a codimension two spacelike submanifold and assume that $\psi(\Sigma)$ is contained in the future connected component of the light cone, that is,

$$\psi(\Sigma) \subset \Lambda^+ = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x_1 > 0\}.$$

In other words,

$$\langle \psi, \psi \rangle = 0 \quad \text{and} \quad \langle \psi, \mathbf{e}_1 \rangle < 0.$$

In this case $\xi = \psi$ is a null vector field normal to the submanifold and future-pointing; it can therefore be chosen as the first vector field of our globally defined future-pointing normal null frame. Consider the positive function $u : \Sigma \to (0, +\infty)$ given by $u = -\langle \psi, \mathbf{e}_1 \rangle > 0$. It follows that

$$\nabla u = -\mathbf{e}_1^{\top},$$

where we are denoting

$$\mathbf{e}_1 = \mathbf{e}_1^{\top} + \mathbf{e}_1^{\perp}$$

as in (2.1). Thus, we get the expression

$$\mathbf{e}_1 = \mathbf{e}_1^{\perp} - \nabla u. \tag{3.1}$$

From this we have

$$\nu = \frac{1}{\sqrt{1 + \|\nabla u\|^2}} (\mathbf{e}_1 + \nabla u)$$

and

$$\langle \xi, \nu \rangle = -\frac{u}{\sqrt{1 + \|\nabla u\|^2}}.$$

Therefore, from (2.3) we get

$$\eta = -\frac{1 + \|\nabla u\|^2}{2u^2} \psi + \frac{1}{u} (\mathbf{e}_1 + \nabla u) = -\frac{1 + \|\nabla u\|^2}{2u^2} \xi + \frac{1}{u} \mathbf{e}_1^{\perp}. \tag{3.2}$$

Thus we have the following basic result, which is nothing but Lemma 4.1 in [**12**] written in our notation (observe that in our convention $\xi$ and $\eta$ are both future-pointing with $\langle \xi, \eta \rangle = -1$).

PROPOSITION 3.1. *Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the future component of the light cone $\Lambda^+$ of the Lorentz-Minkowski spacetime. Then,*

$$\xi = \psi \qquad \text{and} \qquad \eta = -\frac{1 + \|\nabla u\|^2}{2u^2} \xi + \frac{1}{u} \mathbf{e}_1^{\perp}$$

*are two globally normal null vector fields along the submanifold which are future-pointing and satisfy $\langle \xi, \eta \rangle = -1$.*

Next, as done in [**12**], we can compute the associated Weingarten endomorphisms, obtaining proposition 4.3 in [**12**].

PROPOSITION 3.2. *Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the future component of the light cone $\Lambda^+$ of the*

*Lorentz-Minkowski spacetime. Then,*

$$A_\xi = I \qquad \text{and} \qquad A_\eta = -\frac{1+\|\nabla u\|^2}{2u^2}I + \frac{1}{u}\nabla^2 u, \tag{3.3}$$

*where* $\nabla^2 : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ *stands for the Hessian operator* $\nabla^2 u(X) = \nabla_X \nabla u$. *In particular,*

$$\theta_\xi = \frac{1}{n}\operatorname{tr}(A_\xi) = 1 \quad \text{and} \quad \theta_\eta = \frac{1}{n}\operatorname{tr}(A_\eta) = \frac{2u\Delta u - n(1+\|\nabla u\|^2)}{2nu^2} \tag{3.4}$$

*where* $\Delta u = \operatorname{tr}(\nabla^2 u)$ *is the Laplacian of* $u$.

*Proof.* Taking into account Weingarten formula (2.5) we get

$$\overline{\nabla}_X \xi = X = A_\xi X + \nabla^\perp_X \xi$$

for every $X \in \mathfrak{X}(\Sigma)$, so that $A_\xi = I$ and $\nabla^\perp_X \xi = 0$. To obtain the expression of $A_\eta$ we observe from (3.2) that

$$A_\eta = -\frac{1+\|\nabla u\|^2}{2u^2}A_\xi + \frac{1}{u}A_{\mathbf{e}_1^\perp}.$$

Using (3.1) for every $X \in \mathfrak{X}(\Sigma)$,

$$0 = \overline{\nabla}_X \mathbf{e}_1 = \overline{\nabla}_X \mathbf{e}_1^\perp - \overline{\nabla}_X \nabla u. \tag{3.5}$$

On the other hand, we compute

$$\overline{\nabla}_X \mathbf{e}_1^\perp = A_{\mathbf{e}_1^\perp} X + \nabla^\perp_X \mathbf{e}_1^\perp$$

and

$$\overline{\nabla}_X \nabla u = \nabla_X \nabla u - \mathrm{II}(\nabla u, X),$$

where we have used the Gauss and Weingarten formulas. Inserting this into (3.5) we obtain

$$0 = A_{\mathbf{e}_1^\perp} X + \nabla^\perp_X \mathbf{e}_1^\perp - \nabla_X \nabla u + \mathrm{II}(\nabla u, X)$$

and from here we get

$$A_{\mathbf{e}_1^\perp} X = \nabla_X \nabla u. \tag{3.6}$$

Thus, we have

$$A_\eta = -\frac{1+\|\nabla u\|^2}{2u^2}I + \frac{1}{u}\nabla^2 u$$

as we wanted to prove. Finally, tracing the expressions for $A_\xi$ and $A_\eta$ we obtain (3.4). □

Remark 3.3. Observe that since $\langle \eta, \eta \rangle = 0$ we have $\langle \nabla^\perp_X \eta, \eta \rangle = 0$ and from $\langle \xi, \eta \rangle = -1$ we also infer $\langle \nabla^\perp_X \eta, \xi \rangle = 0$. Therefore $\nabla^\perp_X \eta = 0$. Since we already know that $\nabla^\perp_X \xi = 0$, the global null frame $\{\xi, \eta\}$ is parallel in the normal bundle and, in particular, the normal connection is flat. This was already observed in remark 3.3 (b) of [**13**] when $n = 2$, and in remark 4.2 (b) of [**12**] for general dimension $n$.

On the other hand, using proposition 3.2 and formulas (2.9) and (2.10), we easily see that the Ricci and scalar curvatures of $\Sigma$ are given by

$$\mathrm{Ric}(X,Y) = (n-1)\langle \mathbf{H}, \mathbf{H}\rangle\langle X,Y\rangle + \frac{(n-2)}{nu}(\Delta u\langle X,Y\rangle - n\mathrm{Hess}\, u(X,Y)), \quad (3.7)$$

and

$$\mathrm{Scal} = n(n-1)\langle \mathbf{H}, \mathbf{H}\rangle. \quad (3.8)$$

See also corollary 4.5 in [**12**]. Here, and in what follows, by Hess in (3.7) we are denoting the symmetric (0,2) tensor on $\Sigma$ which is metrically equivalent to the Hessian operator $\nabla^2$, that is,

$$\mathrm{Hess}\, u(X,Y) = \langle \nabla^2 u(X), Y\rangle = \langle \nabla_X \nabla u, Y\rangle$$

for every $X, Y \in \mathfrak{X}(\Sigma)$. Finally, since $\nabla^\perp \xi = \nabla^\perp \eta = 0$ and $A_\xi = I$, Codazzi equation (2.12) reduces to

$$(\nabla_X A_\eta)Y = (\nabla_Y A_\eta)X \quad (3.9)$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, and Ricci equations hold trivially since $R^\perp \equiv 0$ and $[A_\xi, A_\eta] = 0$.

## 4. Totally umbilical submanifolds into the light cone

In this section, and as the first main application of our approach, we derive a classification of the codimension two totally umbilical spacelike submanifolds contained in the light cone $\Lambda^+$ of $\mathbb{L}^{n+2}$. Recall that an $n$-dimensional submanifold $\Sigma$ is said to be totally umbilical if it is umbilical with respect to all possible normal directions $\zeta \in \mathfrak{X}^\perp(\Sigma)$. That is, for every $\zeta \in \mathfrak{X}^\perp(\Sigma)$ there exists a smooth function $\lambda_\zeta \in \mathcal{C}^\infty(\Sigma)$ such that

$$A_\zeta = \lambda_\zeta I$$

where $A_\zeta$ is the Weingarten endomorphism associated with $\zeta$.

Let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the future component of the light cone $\Lambda^+$ of $\mathbb{L}^{n+2}$ and consider $\{\xi, \eta\}$ the globally defined future-pointing normal null frame given in proposition 3.1. It follows from proposition 3.2 that $A_\xi = I$, so that $\Sigma$ is totally umbilical if and only if it is umbilical with respect to the normal direction $\eta$. Below we describe the following example of totally umbilical submanifolds into $\Lambda^+$.

EXAMPLE 4.1. Let $\mathbf{a} \in \mathbb{L}^{n+2}$ such that $\mathbf{a} \neq 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = c$ with $c \in \{-1, 0, 1\}$. We define

$$\Sigma(\mathbf{a}, \tau) = \{x \in \Lambda^+ : \langle x, \mathbf{a} \rangle = \tau\} \quad \text{for a certain } \tau \in \mathbb{R}, \tau > 0. \tag{4.1}$$

If we consider

$$\begin{aligned} F_{\mathbf{a}} : \mathbb{L}^{n+2} &\to \mathbb{R}^2 \\ x &\mapsto (\langle x, x \rangle, \langle x, \mathbf{a} \rangle) \end{aligned}$$

we can see $\Sigma(\mathbf{a}, \tau)$ as $\Sigma(\mathbf{a}, \tau) = F_{\mathbf{a}}^{-1}(0, \tau)$. An easy computation gives

$$d(F_{\mathbf{a}})_x(\mathbf{v}) = (2\langle x, \mathbf{v} \rangle, \langle \mathbf{a}, \mathbf{v} \rangle)$$

for every $x \in \mathbb{L}^{n+2}$ and for every $\mathbf{v} \in T_x\mathbb{L}^{n+2} = \mathbb{L}^{n+2}$. It follows from here that $d(F_{\mathbf{a}})_x$ is onto if and only if $x$ and $\mathbf{a}$ are linearly independent. In particular, $(0, \tau)$ is a regular value of $F_{\mathbf{a}}$ if and only if $\Sigma(\mathbf{a}, \tau) \neq \emptyset$ and $p$ and $\mathbf{a}$ are linearly independent for every $p \in \Sigma(\mathbf{a}, \tau)$. A detailed analysis of this condition shows that this is true for every $\tau > 0$ and $c \in \{-1, 0, 1\}$. In all those cases, $\Sigma(\mathbf{a}, \tau)$ is a codimension two spacelike submanifold contained in $\Lambda^+$ having $T_p^\perp \Sigma(\mathbf{a}, \tau) = \operatorname{span}\{p, \mathbf{a}\}$ for every $p \in \Sigma(\mathbf{a}, \tau)$.

Let $\Sigma = \Sigma(\mathbf{a}, \tau) \subset \Lambda^+$. Then, defining

$$\xi(p) = p \quad \text{and} \quad \eta(p) = \frac{c}{2\tau^2} p - \frac{1}{\tau}\mathbf{a}$$

we obtain a normal null frame $\{\xi, \eta\}$ such that $\langle \xi, \eta \rangle = -1$. Observe here that, for every $X \in \mathfrak{X}(\Sigma)$

$$\overline{\nabla}_X \xi = X,$$

which implies

$$A_\xi = I.$$

On the other hand, $\overline{\nabla}\mathbf{a} = 0$, so

$$\overline{\nabla}_X \eta = \frac{c}{2\tau^2} X$$

for every $X \in \mathfrak{X}(\Sigma)$, and

$$A_\eta = \frac{c}{2\tau^2} I.$$

As a consequence $\Sigma = \Sigma(\mathbf{a}, \tau)$ is a totally umbilical submanifold of $\mathbb{L}^{n+2}$ which is contained in the light cone $\Lambda^+$. We also see that, if $\mathbf{a}$ is null, then $\eta$ is a totally geodesic normal direction.

It is not difficult to see that if $c = 0$ then $\Sigma(\mathbf{a}, \tau)$ is isometric to the flat Euclidean space $\mathbb{R}^n$. On the other hand, when $c = -1$, $\Sigma(\mathbf{a}, \tau)$ is isometric to the Euclidean sphere $\mathbb{S}^n(\tau)$ with constant sectional curvature $1/\tau^2$. Finally, when $c = 1$, $\Sigma(\mathbf{a}, \tau)$ is isometric to the hyperbolic space $\mathbb{H}^n(-\tau)$ with constant sectional curvature $-1/\tau^2$. In particular, the only compact case occurs when $c = -1$.

Our next result characterizes $\Sigma(\mathbf{a},\tau)$ in the previous example as the only codimension two totally umbilical spacelike submanifolds contained in $\Lambda^+$. We refer also the reader to [**11**] for another approach to the characterization of such totally umbilical submanifolds.

THEOREM 4.2. *Let* $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ *be a codimension two totally umbilical spacelike submanifold contained in* $\Lambda^+$*. Then there exists* $\mathbf{a} \in \mathbb{L}^{n+2}$*,* $\mathbf{a} \neq 0$ *and* $\langle \mathbf{a},\mathbf{a}\rangle = c \in \{-1,0,1\}$*, and there exists* $\tau \in \mathbb{R}$*,* $\tau > 0$*, such that*

$$\psi(\Sigma) \subset \Sigma(\mathbf{a},\tau).$$

COROLLARY 4.3. *The only complete codimension two totally umbilical spacelike submanifolds contained in* $\Lambda^+$ *are the submanifolds*

$$\Sigma(\mathbf{a},\tau) = \{x \in \Lambda^+ : \langle x,\mathbf{a}\rangle = \tau\}$$

*with* $\mathbf{a} \in \mathbb{L}^{n+2}$*,* $\mathbf{a} \neq 0$ *and* $\langle \mathbf{a},\mathbf{a}\rangle = c \in \{-1,0,1\}$*, and* $\tau \in \mathbb{R}$*,* $\tau > 0$*. In particular, the only compact ones are the submanifolds* $\Sigma(\mathbf{a},\tau)$ *with* $\langle \mathbf{a},\mathbf{a}\rangle = -1$*.*

*Proof.* Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two totally umbilical spacelike submanifold contained in $\Lambda^+$ and consider $\{\xi,\eta\}$ the global normal null frame along the submanifold given by proposition 3.1. We know that $\Sigma$ is totally umbilical if and only if $\eta$ is an umbilical direction, that is, if and only if

$$A_\eta = \lambda I$$

for some function $\lambda$. A standard computation using Codazzi equation (3.9) implies that $\lambda$ is constant. Define $Q = -\eta + \lambda\xi$. Hence

$$\overline{\nabla}_X Q = -\overline{\nabla}_X \eta + \lambda \overline{\nabla}_X \xi = -\lambda X + \lambda X = 0 \tag{4.2}$$

for every $X \in \mathfrak{X}(\Sigma)$, which implies that $Q \in \mathbb{L}^{n+2}$ is a constant vector, $Q \neq 0$.

If $\lambda \neq 0$, let $\tau = 1/\sqrt{2|\lambda|} > 0$ and $\mathbf{a} = \tau Q$. Therefore, $\langle \mathbf{a},\mathbf{a}\rangle = c = \pm 1$ and $\langle \psi,\mathbf{a}\rangle = \tau > 0$, which means that $\psi(\Sigma) \subset \Sigma(\mathbf{a},\tau)$. If $\lambda = 0$, let $\mathbf{a} = Q$; hence $\langle \mathbf{a},\mathbf{a}\rangle = c = 0$ and $\langle \psi,\mathbf{a}\rangle = \tau = 1$, so that $\psi(\Sigma) \subset \Sigma(\mathbf{a},\tau)$. This finishes the proof. □

## 5. Compactness of submanifolds into the light cone

Let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the future component of the light cone $\Lambda^+$ of the Lorentz-Minkowski spacetime,

$$\psi(\Sigma) \subset \Lambda^+ = \{x \in \mathbb{L}^{n+2} : \langle x,x\rangle = 0, x_1 > 0\}.$$

As in the previous section, let us denote by $u$ the positive function on $\Sigma$ given by $u = -\langle \psi,\mathbf{e}_1\rangle = \psi_1 > 0$. As observed in proposition 5.1 of [**12**], if $\Sigma$ is compact then it is a topological $n$-sphere. In this section, we go further by showing that $\Sigma$ is in fact conformally diffeomorphic and given a compactness criteria under an appropriate bound on the growth of the function $u$.

First of all, we state the following technical lemma which is essentially lemma 5.2 in [**4**].

Lemma 5.1. *Let $g$ be a complete metric on a Riemannian manifold $\Sigma$ and let $r$ denote the Riemannian distance function from a fixed origin $o \in \Sigma$. If a function $w$ satisfies*

$$w^{2/(n-2)}(p) \geqslant \frac{C}{r(p)\log(r(p))}, \quad r(p) \gg 1, \tag{5.1}$$

*$C$ a positive constant, then the conformal metric $\tilde{g} = w^{4/(n-2)}g$ is also complete.*

In fact, lemma 5.2 in [4] is stated under the stronger hypothesis

$$w^{2/(n-2)}(p) \geqslant \frac{C}{r(p)}, \quad r(p) \gg 1,$$

but a detailed reading of the proof shows that the result holds true under the weaker hypothesis (5.1). The proof is the same as in [4], just replacing (5.5) in [4] by

$$L(\gamma;\alpha,\beta) \geqslant C_1[\log(\log(r(\gamma(\beta)))) - \log(\log(r(\gamma(\alpha))))]. \tag{5.2}$$

For the sake of completeness and for the reader's convenience, we include below a detailed proof.

*Proof.* Let $A = \{p \in \Sigma : r(p) < 1\}$ and suppose $\gamma : [0,b) \to \Sigma$ is a geodesic for $\tilde{g}$ with $\gamma(0) = o$ and which is not extendible to $b$. Since $(\Sigma, g)$ is complete, $\gamma$ cannot remain in any compact subset of $\Sigma$. In particular, with respect to $A$, there are two possibilities:

(i) $\gamma$ leaves $A$ in a finite time and does not return, or

(ii) $\gamma$ returns to $A$ infinitely many times.

In both cases, we are interested in the length of $\gamma$ outside of $A$, so suppose we have $0 < \alpha < \beta < b$ with $\gamma_{|_{[\alpha,\beta]}} \notin A$.

Consider the partition $\alpha < t_1 < t_2 < \ldots < t_N < \beta$ such that there exists a geodesic ball $B(\gamma(t_j), R_j)$ with centre $\gamma(t_j)$ and radius $R_j$ for which

$$\gamma(t_{j+1}) \in B(\gamma(t_j), R_j),$$

and that in $B(\gamma(t_j), R_j)$ there is a chart and coordinates in which we may write the metric $g$ as

$$g = (dr_j)^2 + r_j^2 g_\theta \tag{5.3}$$

where $r_j$ being the geodesic distance from $\gamma(t_j)$.

For $t_{j-1} < t < t_{j+1}$ and $h > 0$ small enough we have

$$r(\gamma(t+h)) \leqslant r(\gamma(t)) + r_j(\gamma(t+h)) - r_j(\gamma(t)),$$

which implies

$$\frac{\mathrm{d}r(\gamma(t))}{\mathrm{d}t} \leqslant \frac{\mathrm{d}r_j(\gamma(t))}{\mathrm{d}t}.$$

The length in $\tilde{g}$ of $\{\gamma(t) : t_j < t < t_{j+1}\}$ is given by

$$\begin{aligned} L_{t_j}^{t_{j+1}}(\gamma) &= \int_{t_j}^{t_{j+1}} (\tilde{g}(\gamma'(t), \gamma'(t))^{1/2}\mathrm{d}t \\ &= \int_{t_j}^{t_{j+1}} w^{\frac{2}{n-2}}(\gamma(t))(g(\gamma'(t), \gamma'(t))^{1/2}\mathrm{d}t. \end{aligned}$$

Using (5.1) and (5.3) we obtain

$$\begin{aligned} L_{t_j}^{t_{j+1}}(\gamma) &\geqslant \int_{t_j}^{t_{j+1}} \frac{C_1}{r(\gamma(t))\log(r(\gamma(t)))} \frac{\mathrm{d}r_j(\gamma(t))}{\mathrm{d}t} \\ &= C_1[\log(\log(r(\gamma(t_{j+1})))) - \log(\log(r(\gamma(t))))], \end{aligned}$$

where $C_1$ is a constant. Thus,

$$L_\alpha^\beta(\gamma) \geqslant C_1[\log(\log(r(\gamma(\beta)))) - \log(\log(r(\gamma(\alpha))))]. \tag{5.4}$$

Since $\gamma$ cannot remain in any compact set we can find a sequence $\{b_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} b_j = b \quad \text{and} \quad \lim_{j\to+\infty} r(\gamma(b_j)) = +\infty. \tag{5.5}$$

Now suppose that $\gamma$ satisfies $i$), that is, for some $a \in (0, b)$ we have $\gamma(t) \notin A$ for $a < t < b$. Then, by (5.4) we have

$$\lim_{j\to+\infty} L_a^{b_j}(\gamma) = +\infty$$

so that $\gamma$ has infinite length in $\tilde{g}$.

On the other hand, if $\gamma$ satisfies $ii$), then it returns to $A$ infinitely many times. We can now let $\{b_j\}_{j\in\mathbb{N}}$ satisfying (5.5) and $\{a_j\}_{j\in\mathbb{N}}$ such that

$$b_{j-1} < a_j < b_j, \quad \gamma(a_j) \in \partial A \quad \text{and} \quad \gamma(t) \notin A \quad \text{for } a_j < t < b_j.$$

By (5.4) we have

$$\begin{aligned} L_{a_j}^{b_j}(\gamma) &\geqslant C_1[\log(\log(r(\gamma(b_j)))) - \log(\log(r(\gamma(a_j))))] \\ &\geqslant C_1[\log(\log(r(\gamma(b_j)))) - \log(\log(d_0))], \end{aligned}$$

where $d_0 = \max\{r(p) : p \in \partial A\}$. Summing over $j$ we find that $\gamma$ has infinite length in $\tilde{g}$. Thus, $\tilde{g}$ is complete. □

Now we are ready to state the following result.

Proposition 5.2. *Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold contained in $\Lambda^+$. Assume that $\Sigma$ is complete and that the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ satisfies*

$$u(p) \leqslant C\, r(p) \log(r(p)), \quad r(p) \gg 1 \tag{5.6}$$

*where $C$ is a positive constant and $r$ denotes the Riemannian distance function from a fixed origin $o \in \Sigma$. Then $\Sigma$ is compact and conformally diffeomorphic to the sphere $\mathbb{S}^n$. In particular, this holds if $\sup_\Sigma u < +\infty$ and, more generally, if $\limsup_{r\to+\infty} u/(r \log(r)) < +\infty$.*

The upper bound (5.6) on the growth of $u$ is sharp as shown by the existence of complete and non-compact examples with $u(p) = r^2(p)$ in example 6.4.

*Proof.* Observe that, for every $p \in \Sigma$, $\psi(p) = (u(p), \psi_2(p), \dots, \psi_{n+2}(p))$, where

$$\sum_{i=2}^{n+2} \psi_i^2(p) = u^2(p) > 0. \tag{5.7}$$

Define the function $\Psi : \Sigma^n \to \mathbb{S}^n$ by

$$\Psi(p) = \frac{1}{u(p)} (\psi_2(p), \dots, \psi_{n+2}(p)).$$

For every $p \in \Sigma$ and $\mathbf{v} \in T_p\Sigma$ we have

$$\mathrm{d}\Psi_p(\mathbf{v}) = -\frac{\mathbf{v}(u)}{u^2(p)} (\psi_2(p), \dots, \psi_{n+2}(p)) + \frac{1}{u(p)} (\mathbf{v}(\psi_2), \dots, \mathbf{v}(\psi_{n+2})).$$

Denote by $\langle , \rangle_0$ the standard metric of the round sphere $\mathbb{S}^n$. Therefore, for every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we have

$$\begin{aligned}
\langle \mathrm{d}\Psi_p(\mathbf{v}), \mathrm{d}\Psi_p(\mathbf{w}) \rangle_0 =& \frac{\mathbf{v}(u)\mathbf{w}(u)}{u^4(p)} \sum_{i=2}^{n+2} \psi_i^2(p) + \frac{1}{u^2(p)} \sum_{i=2}^{n+2} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) \\
& - \frac{\mathbf{v}(u)}{2u^3(p)} \mathbf{w} \left( \sum_{i=2}^{n+2} \psi_i^2 \right) - \frac{\mathbf{w}(u)}{2u^3(p)} \mathbf{v} \left( \sum_{i=2}^{n+2} \psi_i^2 \right) \\
=& \frac{1}{u^2(p)} \left( -\mathbf{v}(u)\mathbf{w}(u) + \sum_{i=2}^{n+2} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) \right) \\
=& \frac{1}{u^2(p)} \langle \mathrm{d}\psi_p(\mathbf{v}), d\psi_p(\mathbf{w}) \rangle = \frac{1}{u^2(p)} \langle \mathbf{v}, \mathbf{w} \rangle.
\end{aligned}$$

In other words,

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2} \langle , \rangle, \tag{5.8}$$

where we recall that by $\langle , \rangle$ we denote the Riemannian metric on $\Sigma$ induced by the immersion $\psi$.

From (5.8) it follows that $\Psi$ is a local diffeomorphism. Assume now that $\Sigma$ is complete (that is, $\langle,\rangle$ is a complete Riemannian metric on $\Sigma$) and $u$ satisfies (5.6). Therefore, by lemma 5.1 applied to the function $w = u^{-(n-2)/2}$, we know that the conformal metric

$$\widetilde{\langle,\rangle} = \frac{1}{u^2}\langle,\rangle$$

is also complete on $\Sigma$. Then, equation (5.8) means that the map

$$\Psi : (\Sigma^n, \widetilde{\langle,\rangle}) \rightarrow (\mathbb{S}^n, \langle,\rangle_0)$$

is a local isometry between complete Riemannian manifolds. Now we recall that every isometry between complete (connected) Riemannian manifolds is a covering map (see, for instance, [7]). Hence, $\Psi$ is a covering map, but $\mathbb{S}^n$ being simply connected this means that $\Psi$ is, in fact, a global diffeomorphism between $\Sigma$ and $\mathbb{S}^n$. □

EXAMPLE 5.3. In this example, we observe that for each positive smooth function $f : \mathbb{S}^n \rightarrow (0, +\infty)$ we can construct an embedding $\psi_f : \mathbb{S}^n \rightarrow \Lambda^+ \subset \mathbb{L}^{n+2}$ by setting

$$\psi_f(p) = (f(p), f(p)p).$$

Clearly, for every $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^n$ we have

$$\mathrm{d}(\psi_f)_p(\mathbf{v}) = (\mathbf{v}(f), \mathbf{v}(f)p + f(p)\mathbf{v})$$

and

$$\langle \mathrm{d}(\psi_f)_p(\mathbf{v}), \mathrm{d}(\psi_f)_p(\mathbf{w})\rangle = f^2(p)\langle \mathbf{v}, \mathbf{w}\rangle_0.$$

That is

$$\psi_f^*(\langle,\rangle) = f^2\langle,\rangle_0, \tag{5.9}$$

what means that $\psi_f$ determines a spacelike immersion of $\mathbb{S}^n$ into $\Lambda^+$ whose induced metric is conformal to the standard metric of the round sphere.

In this case, $u = f$ and, from equation (3.3) in proposition 3.2, we can explicitly write the second fundamental form of $\psi_f$ in terms of the function $f$ and the gradient and the Hessian of $f$ with respect to the round metric $\langle,\rangle_0$. To see it first observe that, obviously, $A_\xi = I$ and $\theta_\xi = 1$. To compute $A_\eta$, let us denote by $\|\cdot\|_0^2$, $\nabla^0$ and $\mathrm{Hess}_0$ the norm, the gradient and the Hessian operator (as a symmetric (0,2) tensor) on $\mathbb{S}^n$ with respect to the standard metric $\langle,\rangle_0$. Then from (5.9) one has

$$\|\cdot\|^2 = f^2\|\cdot\|_0^2, \tag{5.10}$$

$$\nabla f = \frac{1}{f^2}\nabla^0 f, \tag{5.11}$$

$$\mathrm{Hess}\, f = \mathrm{Hess}_0 f - \frac{2}{f}\mathrm{d}f \otimes \mathrm{d}f + \frac{1}{f}\|\nabla^0 f\|_0^2\langle,\rangle_0. \tag{5.12}$$

Therefore by (5.10) and (5.11) we have

$$\|\nabla f\|^2 = \frac{1}{f^2}\|\nabla^0 f\|_0^2 \tag{5.13}$$

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and

$$\frac{1+\|\nabla f\|^2}{2f^2} = \frac{f^2+\|\nabla^0 f\|_0^2}{2f^4} \tag{5.14}$$

On the other hand, by (5.12) we also have, for every tangent vector fields $X, Y \in \mathfrak{X}(\mathbb{S}^n)$,

$$\begin{aligned}\mathrm{Hess}\,(X,Y) =& f^2\langle \nabla_X \nabla f, Y\rangle_0 \\ =& \langle \nabla^0_X \nabla^0 f, Y\rangle_0 - \frac{2}{f}\langle X, \nabla^0 f\rangle_0 \langle Y, \nabla^0 f\rangle_0 + \frac{1}{f}\|\nabla^0 f\|_0^2 \langle X, Y\rangle_0,\end{aligned}$$

which gives

$$\nabla_X \nabla f = \frac{1}{f^2}\nabla^0_X \nabla^0 f - \frac{2}{f^3}\langle X, \nabla^0 f\rangle_0 \nabla^0 f + \frac{1}{f^3}\|\nabla^0 f\|_0^2 X \tag{5.15}$$

for every tangent vector field $X \in \mathfrak{X}(\mathbb{S}^n)$. Therefore, using (5.14) and (5.15) in (3.3) we conclude after some computations that

$$A_\eta(X) = \frac{1}{f^3}\nabla^0_X \nabla^0 f - \frac{2}{f^4}\langle X, \nabla^0 f\rangle_0 \nabla^0 f + \frac{\|\nabla^0 f\|_0^2 - f^2}{2f^4} X \tag{5.16}$$

for every $X \in \mathfrak{X}(\mathbb{S}^n)$. Thus, tracing (5.16) with respect to $\langle , \rangle_0$ we have

$$\theta_\eta = \frac{2f\Delta_0 f + (n-4)\|\nabla^0 f\|_0^2 - nf^2}{2nf^4}. \tag{5.17}$$

In the next result, and as a consequence of proposition 5.2, we observe that every codimension two compact spacelike submanifold contained in $\Lambda^+$ is, up to a conformal diffeomorphism, as in example 5.3.

Corollary 5.4. *Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two compact spacelike submanifold contained in $\Lambda^+$. Then there exists a conformal diffeomorphism $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ such that*

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle$$

*with $u = -\langle \psi, \mathbf{e}_1\rangle = \psi_1 > 0$, and $\psi = \psi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$ and $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ is the embedding*

$$\psi_f(p) = (f(p), f(p)p).$$


*In particular, the immersion $\psi$ is an embedding.*

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For the proof simply consider $u$ and $\Psi$ as in the proof of proposition 5.2, and recall that in this situation $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$ is a conformal diffeomorphism with

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle.$$

Let $\Phi : \mathbb{S}^n \to \Sigma^n$ be the inverse of $\Psi$. Then taking $f = u \circ \Phi$ one has $f \circ \Psi = u$ and $\psi = \psi_f \circ \Psi$, since

$$\psi_f \circ \Psi(p) = (f(\Psi(p)), f(\Psi(p))\Psi(p)) = (u(p), \psi_2(p), \ldots, \psi_{n+2}(p)) = \psi(p).$$

## 6. Trapped submanifolds into the light cone. First results

The original formulation of trapped surfaces was given by Penrose [**14**] in terms of the signs or the vanishing of the null expansions. Following this approach, for a codimension two spacelike submanifold in $\mathbb{L}^{n+2}$ we have the following:

(a) $\Sigma$ is a trapped submanifold if and only if either both $\theta_\xi < 0$ and $\theta_\eta < 0$ (future trapped), or both $\theta_\xi > 0$ and $\theta_\eta > 0$ (past trapped).

(b) $\Sigma$ is a marginally trapped submanifold if and only if either $\theta_\xi = 0$ and $\theta_\eta \neq 0$ (future marginally trapped if $\theta_\eta < 0$ and past marginally trapped if $\theta_\eta > 0$), or $\theta_\xi \neq 0$ and $\theta_\eta = 0$ (future marginally trapped if $\theta_\xi < 0$ and past marginally trapped if $\theta_\xi > 0$).

(c) $\Sigma$ is a weakly trapped submanifold if and only if either both $\theta_\xi \leqslant 0$ and $\theta_\eta \leqslant 0$ with $\theta_\xi^2 + \theta_\eta^2 > 0$ (future weakly trapped), or both $\theta_\xi \geqslant 0$ and $\theta_\eta \geqslant 0$ with $\theta_\xi^2 + \theta_\eta^2 > 0$ (past weakly trapped).

Using (2.7) and (2.8), this is equivalent to the following

(a) $\Sigma$ is trapped if and only if either $\mathbf{H}$ is timelike and future-pointing (future trapped) or $\mathbf{H}$ is timelike and past-pointing (past trapped).

(b) $\Sigma$ is marginally trapped if and only if either $\mathbf{H}$ is null and future-pointing (future marginally trapped) or $\mathbf{H}$ is null and past-pointing (past marginally trapped).

(c) $\Sigma$ is weakly trapped if and only if either $\mathbf{H}$ is causal and future-pointing (future weakly trapped) or $\mathbf{H}$ is causal and past-pointing (past weakly trapped).

The following known result [**1**, remark 4.2] establishes the non-existence of compact weakly trapped submanifolds into $\mathbb{L}^{n+2}$ (more generally, see [**9**, theorem 2] for the non-existence of compact weakly trapped submanifolds into strictly stationary spacetimes). For the convenience of the reader, we give here a proof of it using our approach.

Proposition 6.1. *There exists no codimension two compact weakly trapped submanifold in* $\mathbb{L}^{n+2}$.

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*Proof.* Let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ be an $n$-dimensional compact weakly trapped submanifold and consider the function $u = -\langle \psi, \mathbf{e}_1\rangle$, whose gradient is given by $\nabla u = -\mathbf{e}_1^\top = \mathbf{e}_1^\perp - \mathbf{e}_1$. Taking derivatives here, as we did in the previous section for the case where the submanifold is contained in the light cone, we obtain again

$$A_{\mathbf{e}_1^\perp} X = \nabla_X \nabla u$$

and from here we compute

$$\Delta u = \mathrm{tr}(A_{\mathbf{e}_1^\perp}) = n\langle \mathbf{H}, \mathbf{e}_1^\perp\rangle = n\langle \mathbf{H}, \mathbf{e}_1\rangle.$$

On the other hand, the mean curvature vector field $\mathbf{H}$ satisfies $\langle \mathbf{H}, \mathbf{e}_1\rangle < 0$ or $\langle \mathbf{H}, \mathbf{e}_1\rangle > 0$ since $\mathbf{H}$ is not spacelike. Suppose that $\langle \mathbf{H}, \mathbf{e}_1\rangle < 0$, then

$$\Delta u = n\langle \mathbf{H}, \mathbf{e}_1\rangle < 0.$$

Now, from the divergence theorem we have

$$\int_\Sigma \Delta u \mathrm{d}\Sigma = 0$$

what implies $\Delta u \equiv 0$ and gives us a contradiction. The proof for the case $\langle \mathbf{H}, \mathbf{e}_1\rangle > 0$ ends in a similiar way. □

On the other hand, from proposition 3.2 and (2.7), the mean curvature vector field of a codimension two spacelike submanifold $\Sigma$ which is contained in $\Lambda^+$ is given by

$$\mathbf{H} = -\frac{1}{2nu^2}(2u\Delta u - n(1 + \|\nabla u\|^2))\xi - \eta.$$

In particular, the null expansion $\theta_\xi$ is always $\theta_\xi = 1 > 0$ and

$$\langle \mathbf{H}, \mathbf{H}\rangle = -\frac{1}{nu^2}(2u\Delta u - n(1 + \|\nabla u\|^2)). \tag{6.1}$$

Recall also that, by (3.8), the scalar curvature of $\Sigma$ is $\mathrm{Scal} = n(n-1)\langle \mathbf{H}, \mathbf{H}\rangle$. As a consequence of these computations we have the following.

Corollary 6.2. *Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the future component of the light cone $\Lambda^+$ of the Lorentz-Minkowski spacetime. Let $u$ be the positive function $u = -\langle \psi, \mathbf{e}_1\rangle$. They are satisfied:*

(i) *$\Sigma$ is (necessarily past) trapped if and only if $u$ satisfies the differential inequality*

$$2u\Delta u - n(1 + \|\nabla u\|^2) > 0 \quad \text{on } \Sigma. \tag{6.2}$$

(ii) *$\Sigma$ is (necessarily past) marginally trapped if and only $u$ satisfies the differential equation*

$$2u\Delta u - n(1 + \|\nabla u\|^2) = 0 \quad \text{on } \Sigma. \tag{6.3}$$

(iii) $\Sigma$ *is (necessarily past) weakly trapped if and only if* $u$ *satisfies the differential inequality*

$$2u\Delta u - n(1 + \|\nabla u\|^2) \geqslant 0 \quad on\ \Sigma. \tag{6.4}$$

Corollary 6.3. *Let* $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ *be a codimension two spacelike submanifold which is contained in the future component of the light cone* $\Lambda^+$ *of the Lorentz-Minkowski spacetime. Let* $u$ *be the positive function* $u = -\langle \psi, \mathbf{e}_1 \rangle$. *The following are equivalent:*

(i) $\Sigma$ *is (necessarily past) marginally trapped.*

(ii) $u$ *satisfies the differential equation* $2u\Delta u - n(1 + \|\nabla u\|^2) = 0$ *on* $\Sigma$.

(iii) $\Sigma$ *has zero scalar curvature,* $\mathrm{Scal} = 0$.

## 6.1. Examples

In this subsection, we present explicit examples of weakly trapped and marginally trapped submanifolds contained in the future component of the light cone $\Lambda^+$. In the first example, we construct a marginally trapped submanifold.

Example 6.4. Let $\psi : \mathbb{R}^n \to \mathbb{L}^{n+2}$ be a map given by

$$\psi(p) = \left( \frac{\|p\|^2 + 1}{2}, \frac{\|p\|^2 - 1}{2}, p \right).$$

We compute

$$\langle \psi(p), \psi(p) \rangle = -\frac{(\|p\|^2 + 1)^2 + (\|p\|^2 - 1)^2}{4} + \|p\|^2 = 0$$

and we also have

$$u(p) = -\langle \psi(p), \mathbf{e}_1 \rangle = \frac{\|p\|^2 + 1}{2} > 0.$$

Therefore $\psi(\mathbb{R}^n)$ is contained in $\Lambda^+$.

On the other hand, for $\mathbf{v}, \mathbf{w} \in T_p\mathbb{R}^n$, $p \in \mathbb{R}^n$, we obtain

$$\mathrm{d}\psi_p(\mathbf{v}) = (\|p\|\mathbf{v}, \|p\|\mathbf{v}, \mathbf{v})$$

and hence,

$$\psi^*(\langle \mathbf{v}, \mathbf{w} \rangle) = \langle \mathrm{d}\psi_p(\mathbf{v}), \mathrm{d}\psi_p(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{R}^n}.$$

Thus, $\psi$ is an isometric immersion of $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$ into $\Lambda^+ \subset \mathbb{L}^{n+2}$. In particular, the gradient and Laplacian operators of $u$ are respectively

$$\nabla u(p) = \nabla^{\mathbb{R}^n} u(p) = p \quad \text{and} \quad \Delta u(p) = \Delta_{\mathbb{R}^n} u(p) = n.$$

Therefore, the function $u$ satisfies the differential equation (6.3),

$$2u\Delta u - n(1 + \|\nabla u\|^2) = n(\|p\|^2 + 1) - n(1 + \|p\|^2) = 0$$

and $\psi$ is a marginally trapped immersion of $\mathbb{R}^n$.

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In the next example, we construct a weakly trapped submanifold which is marginally trapped if, and only if, the dimension is $n = 2$.

Example 6.5. Let $\phi : (0, +\infty) \times \mathbb{H}^{n-1} \to \mathbb{L}^{n+2}$ be the map given by

$$\phi(t, p) = (p, \cos(t), \sin(t)),$$

where we are denoting by $\mathbb{H}^{n-1}$ the $(n-1)$-dimensional hyperbolic space,

$$\mathbb{H}^{n-1} = \{p = (p_1, \dots, p_n) \in \mathbb{L}^n : \langle p, p\rangle = -1, p_1 > 0\}.$$

We have that

$$\langle \phi(t, p), \phi(t, p)\rangle = \langle p, p\rangle_{\mathbb{H}^{n-1}} + \cos(t)^2 + \sin(t)^2 = -1 + 1 = 0,$$

and

$$u(t, p) = p_1 > 0,$$

so that $\phi((0, +\infty) \times \mathbb{H}^{n-1})$ is contained in $\Lambda^+$.

We have that for every $z = (t, p) \in (0, +\infty) \times \mathbb{H}^{n-1}$ the tangent space at $z$ is generated by two types of vectors, $\mathbf{v}_1 = (1, \mathbf{0})$ with $\mathbf{0} \in T_p\mathbb{H}^{n-1}$ and $\mathbf{v}_2 = (0, \mathbf{w})$ with $\mathbf{w} \in T_p\mathbb{H}^{n-1}$. Thus, we compute

$$\mathrm{d}\phi_z(1, \mathbf{0}) = (0, -\sin(t), \cos(t))$$

and

$$\mathrm{d}\phi_z(0, \mathbf{w}) = (\mathbf{w}, 0, 0) \quad \text{for every } \mathbf{w} \in T_p\mathbb{H}^{n-1}.$$

From here, we can easily see that

$$\phi^*(\langle , \rangle) = \mathrm{d}t^2 + \langle , \rangle_{\mathbb{H}^{n-1}},$$

that is, the induced metric is nothing but the product metric in $(0, +\infty) \times \mathbb{H}^{n-1}$. In other words, $\phi$ gives an isometric immersion of the Riemannian product manifold $(0, +\infty) \times \mathbb{H}^{n-1}$ into $\Lambda^+ \subset \mathbb{L}^{n+2}$.

Therefore, in this case $u(t, p) = v(p)$ for every $(t, p) \in (0, +\infty) \times \mathbb{H}^{n-1}$, where the function $v : \mathbb{H}^{n-1} \to (0, +\infty)$ is given by $v(p) = -\langle p, \mathbf{e}_1\rangle_{\mathbb{L}^n}$ for every $p \in \mathbb{H}^{n-1}$, with $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{L}^n$. In particular,

$$\nabla u(t, p) = (0, Dv(p))$$

where $D$ denotes the gradient operator on $\mathbb{H}^{n-1}$. Since $v(p) = -\langle p, \mathbf{e}_1\rangle_{\mathbb{L}^n}$, it is clear that $Dv(p) = -\mathbf{e}_1^\top$, where $^\top$ denotes here the component which is tangent to $\mathbb{H}^{n-1}$

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as a spacelike hypersurface of $\mathbb{L}^n$. Hence, the vector $\mathbf{e}_1$ decomposes along $\mathbb{H}^{n-1}$ as

$$\mathbf{e}_1 = -Dv(p) + v(p)p$$

for every $p \in \mathbb{H}^{n-1}$ and

$$\|Dv\|^2 = -1 + v^2.$$

Since $\nabla u = (0, Dv)$ this is equivalent to

$$\|\nabla u\|^2 = -1 + u^2.$$

On the other hand, for every $\mathbf{v} \in T_p\mathbb{H}^{n-1}$ we have

$$\nabla^0_{\mathbf{v}}\mathbf{e}_1 = 0 = -\nabla^0_{\mathbf{v}}Dv + \mathbf{v}(v(p)p) = -D_{\mathbf{v}}Dv + v(p)\mathbf{v} + \mathbf{v}(v)p$$

where $\nabla^0$ and $D$ denote, respectively, the Levi-Civita connections of $\mathbb{L}^n$ and $\mathbb{H}^{n-1}$. From here we have that $D_{\mathbf{v}}Dv = v(p)\mathbf{v}$ for every $\mathbf{v} \in T_p\mathbb{H}^{n-1}$, and then

$$\Delta_{\mathbb{H}^{n-1}}v = (n-1)v.$$

Hence

$$\Delta u(t,p) = \Delta_{\mathbb{H}^{n-1}}v(p) = (n-1)u(t,p).$$

From these computations we finally get

$$2u\Delta u - n(1 + \|\nabla u\|^2) = (n-2)u^2$$

and, from equation (6.3), we have that $\Sigma$ is a weakly trapped submanifold, and it is marginally trapped if, and only if $n = 2$.

## 7. Non-existence of weakly trapped submanifolds into the light cone

We know from proposition 6.1 that, in particular, there is no compact weakly trapped submanifold contained in the light cone of $\mathbb{L}^{n+2}$. The following corollary is a direct consequence of proposition 5.2 and this fact.

Corollary 7.1. *There exists no codimension two complete weakly trapped immersed submanifold $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ satisfies*

$$u \leqslant Cr\log r, \quad r >> 1.$$

*In particular, there is no codimension two complete weakly trapped immersed submanifold in $\Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function $u$ is bounded from above.*

In this section, we will extend this non-existence result to the more general case of stochastically complete submanifolds. Recall that a (non necessarily complete) Riemannian manifold $\Sigma$ is said to be *stochastically complete* if its Brownian motion is stochastically complete, that is, the probability of a particle to be found in the state space is constantly equal to 1. In particular, every parabolic Riemannian manifold is stochastically complete.

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As proved by Pigola, Rigoli and Setti [15], the stochastic completeness of a Riemannian manifold $\Sigma$ is equivalent to the fact that the weak maximum principle for the Laplacian holds on $\Sigma$ (see also [16, theorem 3.1] and, more generally, [2]). Following the terminology introduced by Pigola, Rigoli and Setti in [16], the weak maximum principle for the Laplacian is said to hold on a (non necessarily complete) Riemannian manifold $\Sigma^n$ if, for any smooth function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in $\Sigma$ with the properties

$$(i) \quad u(p_k) > u^* - \frac{1}{k}, \quad \text{and} \quad (ii) \quad \Delta u(p_k) < \frac{1}{k}. \tag{7.1}$$

We are now ready to prove the following

THEOREM 7.2. *There exists no codimension two stochastically complete weakly trapped immersed submanifold contained in $\Lambda^+ \subset \mathbb{L}^{n+2}$ for which the positive function $u = -\langle \psi, \mathbf{e}_1 \rangle$ is bounded from above.*

*Proof.* Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be an $n$-dimensional stochastically complete weakly trapped submanifold such as $\psi(\Sigma) \subset \Lambda^+$. If we define $u = -\langle \psi, \mathbf{e}_1 \rangle$ as usual, it satisfies (6.4) in corollary 6.2

$$2u\Delta u - n(1 + \|\nabla u\|^2) \geqslant 0.$$

Supose that $u$ is bounded from above, that is, $u^* = \sup_\Sigma u < +\infty$. Since $\Sigma$ is stochastically complete, by the weak maximum principle there exists a sequence $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma$ with

$$\Delta u(p_k) < \frac{1}{k} \quad \text{for every} \quad k \in \mathbb{N}$$

and putting this into (6.4) we obtain

$$n \leqslant n(1 + \|\nabla u(p_k)\|^2) \leqslant 2u(p_k)\Delta u(p_k) < 2\frac{u(p_k)}{k}.$$

Making $k \to +\infty$ we get

$$n \leqslant 0$$

which is not possible. □

For the proof of the next theorem we need the following analytical result whose proof can be derived from that of theorem 3.3 in [10].

THEOREM 7.3. *Let $(\Sigma, \langle , \rangle)$ be a complete Riemannian manifold and let $v \geqslant 0$ be a solution of*

$$v\Delta v + av^2 - bv \geqslant -A\|\nabla v\|^2 \tag{7.2}$$

*on $\Sigma$, with $a \leqslant 0$, $b > 0$ and $A \in \mathbb{R}$. Suppose that for some $\alpha > 1$, $\beta > -1$, $\beta \geqslant A$*

$$v \in L^{\alpha(\beta+1)}(\Sigma). \tag{7.3}$$

*Then $v \equiv 0$.*

*Proof.* Let $v \geqslant 0$ be a solution of (7.2). Fix $\varepsilon > 0$ and set

$$w_\varepsilon = (v^2 + \varepsilon)^{(\beta+1)/2}.$$

We compute

$$w_\varepsilon \Delta w_\varepsilon = (\beta+1)(v^2+\varepsilon)^\beta v\Delta v + \\ (\beta+1)(v^2+\varepsilon)^\beta \left(1 + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2.$$

If follows from (7.2) and the assumptions $a \leqslant 0$ and $b > 0$ that

$$\begin{aligned} w_\varepsilon \Delta w_\varepsilon &\geqslant (\beta+1)(v^2+\varepsilon)^\beta \left(\left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2 + bv - av^2\right) \\ &\geqslant (\beta+1)(v^2+\varepsilon)^\beta \left(\left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2 + a\left(1 - \frac{v^2}{v^2+\varepsilon}\right)\right) \\ &= (\beta+1)(v^2+\varepsilon)^\beta \left(a\varepsilon + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\right). \end{aligned}$$

Let $\tilde{r}(t) \in C^1(\mathbb{R})$ and $s(t) \in C^0(\mathbb{R})$ defined as

$$\tilde{r}(t) = t^{\alpha-2} \quad \text{and} \quad s(t) = c_\alpha t^{\alpha-2},$$

where $\alpha > 1$ and $c_\alpha = \min\{\alpha - 1, 1\}$. They satisfy the conditions

$$\tilde{r}(w_\varepsilon) \geqslant 0 \tag{7.4}$$

and

$$\tilde{r}(w_\varepsilon) + w_\varepsilon \tilde{r}'(w_\varepsilon) = (\alpha-1)w_\varepsilon^{\alpha-2} \geqslant c_\alpha w_\varepsilon^{\alpha-2} = s(w_\varepsilon) > 0. \tag{7.5}$$

Consider the vector field

$$Z = w_\varepsilon \tilde{r}(w_\varepsilon)\nabla w_\varepsilon = w_\varepsilon^{\alpha-1}\nabla w_\varepsilon$$

and, for fixed $t$ and $\delta > 0$, let $\psi_\delta$ be the Lipschitz function defined by

$$\psi_\delta(p) = \begin{cases} 1 & \text{if} \quad r(p) \leqslant t \\ \frac{t+\delta-r(p)}{\delta} & \text{if} \quad t < r(p) < t+\delta \\ 0 & \text{if} \quad r(p) \geqslant t+\delta. \end{cases}$$

Using conditions (7.4), (7.5) and the definition of $\psi_\delta$ we compute

$$\begin{aligned} \operatorname{div}(\psi_\delta Z) &= \psi_\delta \operatorname{div}(Z) + \langle \nabla\psi_\delta, Z\rangle \\ &= \left(w_\varepsilon^{\alpha-1}\Delta w_\varepsilon + (\alpha-1)w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2\right)\psi_\delta - \frac{1}{\delta}\langle \nabla r, w_\varepsilon^{\alpha-1}\nabla w_\varepsilon\rangle \\ &= \left(w_\varepsilon^{\alpha-2} w_\varepsilon \Delta w_\varepsilon + (\alpha-1)w^{\alpha-2}\|\nabla w\|^2\right)\psi_\delta - \frac{1}{\delta}\langle \nabla r, w^{\alpha-1}\nabla w\rangle \\ &\geqslant w_\varepsilon^{\alpha-2}(\beta+1)(v^2+\varepsilon)^\beta \left(a\varepsilon + \left(1 - A + (\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right)\psi_\delta \\ &\quad + c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2\psi_\delta - \frac{1}{\delta}\langle \nabla r, w_\varepsilon^{\alpha-2}\nabla w_\varepsilon\rangle \chi_{B_{t+\delta}\setminus B_t}, \end{aligned}$$

where we have used $\nabla\psi_\delta = -1/\delta\nabla r\chi_{B_{t+\delta}\setminus B_t}$.

Then, integrating and using the divergence theorem and Cauchy-Schwarz inequality we obtain

$$\int_{B_t} w_\varepsilon^{\alpha-2}(\beta+1)(v^2+\varepsilon)^\beta\left(a\varepsilon+\left(1-A+(\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right)\psi_\delta + \int_{B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2 \leqslant \frac{1}{\delta}\int_{B_{t+\delta}\setminus B_t} w_\varepsilon^{\alpha-1}\|\nabla w_\varepsilon\|. \tag{7.6}$$

By Hölder inequality the integral on the right-hand side is bounded above as follows

$$\begin{aligned}\int_{\bar{B}_{t+\delta}\setminus B_t} w_\varepsilon^{\alpha-1}\|\nabla w_\varepsilon\| &= \int_{\bar{B}_{t+\delta}\setminus B_t}\left(\frac{1}{\sqrt{\delta c_\alpha}}w_\varepsilon^{\alpha/2}\right)\left(\frac{\sqrt{c_\alpha}}{\sqrt{\delta}}w_\varepsilon^{\alpha/2-1}\|\nabla w_\varepsilon\|\right)\\ &\leqslant \left(\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t}\frac{w_\varepsilon^\alpha}{c_\alpha}\right)^{\frac{1}{2}}\left(\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|\right)^{1/2}.\end{aligned}$$

Inserting this into inequality (7.6) and letting $\delta \to 0^+$ we obtain that

$$\int_{B_t} w_\varepsilon^{\alpha-2}(\beta+1)(v^2+\varepsilon)^\beta\left(a\varepsilon+\left(1-A+(\beta-1)\frac{v^2}{v^2+\varepsilon}\right)\|\nabla v\|^2\right)+ + \int_{B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2 \leqslant \left(\int_{\partial B_t}\frac{w_\varepsilon^\alpha}{c_\alpha}\right)^{1/2}\left(\int_{\partial B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2\right)^{1/2}, \tag{7.7}$$

where we have used the co-area formula, that is,

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t}\frac{w_\varepsilon^\alpha}{c_\alpha} = \int_{\partial B_t}\frac{w_\varepsilon^\alpha}{c_\alpha}$$

and

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\int_{\bar{B}_{t+\delta}\setminus B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2 = \int_{\partial B_t} c_\alpha w_\varepsilon^{\alpha-2}\|\nabla w_\varepsilon\|^2.$$

As $\varepsilon \to 0$, then $w_\varepsilon \to w_0 = v^{\beta+1}$. Therefore, using the dominated convergence theorem in (7.6) we get

$$(\beta+1)(\beta-A)\int_{B_t} v^{2\beta}w_0^{\alpha-2}\|\nabla v\|^2 + \int_{B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2 \leqslant \left(\int_{\partial B_t}\frac{w_0^\alpha}{c_\alpha}\right)^{1/2}\left(\int_{\partial B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2\right)^{1/2}. \tag{7.8}$$

We define now

$$h(t) = \int_{B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2,$$

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and then, by the co-area formula, $h$ is Lipschitz and

$$h'(t) = \int_{\partial B_t} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2.$$

From our assumptions on $\beta$ and $A$, we know that

$$(\beta+1)(\beta-A)\int_{B_t} v^{2\beta} w_0^{\alpha-2}\|\nabla v\|^2 \geqslant 0$$

so, from (7.8), it is satisfied

$$h(t) \leqslant \left(\int_{\partial B_t} \frac{w_0^\alpha}{c_\alpha}\right)^{1/2} h'(t)^{1/2}. \tag{7.9}$$

Our aim now is to show that $w_0 = v^{\beta+1}$ is constant. Let us suppose it is not and reason by contradiction.Then, if $w_0$ is not constant, there exists $R_0 >> 1$ such that $h(t) > 0$ for every $t \geqslant R_0$. Then, dividing in (7.9) by $h(t)$ we have

$$1 \leqslant \frac{h'(t)}{h(t)^2}\int_{\partial B_t} \frac{w_0^\alpha}{c_\alpha}$$

or, equivalently,

$$\frac{h'(t)}{h(t)^2} \geqslant \left(\int_{\partial B_t} \frac{w_0^\alpha}{c_\alpha}\right)^{-1}.$$

Taking $R_0 \leqslant r < R$ and integrating the previous inequality, we obtain

$$\begin{aligned}\left(\int_{B_r} c_\alpha w_0^{\alpha-2}\|\nabla w_0\|^2\right)^{-1} &= \frac{1}{h(r)} \geqslant \frac{1}{h(r)} - \frac{1}{h(R)} \\ &= \int_r^R \frac{h'(t)}{h(t)^2} \geqslant \int_r^R \left(\int_{\partial B_t} \frac{w_0^\alpha}{c_\alpha}\right)^{-1}.\end{aligned}$$

Since $w_0 = v^{\beta+1}$, from here it follows

$$\int_r^R \left(\int_{\partial B_t} v^{\alpha(\beta+1)}\right)^{-1} \leqslant C\left(\int_{B_r} w_0^{\alpha-2}\|\nabla w_0\|^2\right)^{-1} < +\infty,$$

where $C \in \mathbb{R}$. Then,

$$\left(\int_{\partial B_t} v^{\alpha(\beta+1)}\right)^{-1} \in L^1(+\infty). \tag{7.10}$$

If we define now

$$\phi(t) = \int_{B_t} v^{\alpha(\beta+1)}$$

we obtain

$$\phi'(t) = \int_{\partial B_t} v^{\alpha(\beta+1)} \geqslant 0.$$

Taking into account (7.10), $\phi'(t)$ has to satisfy

$$\lim_{t\to+\infty} \frac{1}{\phi'(t)} = 0,$$

or equivalently

$$\lim_{t\to+\infty} \phi(t) = +\infty.$$

As $\phi$ is a non-decreasing function, it implies that $\phi$ tends to infinity, that is,

$$\lim_{t\to+\infty} \int_{B_t} v^{\alpha(\beta+1)} = \int_{\Sigma} v^{\alpha(\beta+1)} = +\infty.$$

However, this contradicts the assumption (7.3), and we get that $w_0$ has to be constant and so does $v$. Taking into account that $v$ satisfies equation (7.2),

$$av^2 - bv \geqslant 0$$

with $a \leqslant 0$ and $b > 0$, then we conclude that $v \equiv 0$. □

As a consequence of theorem 7.3 we have the following

THEOREM 7.4. *There is no codimension two complete weakly trapped immersed submanifold* $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ *for which the positive function* $u = -\langle \psi, \mathbf{e}_1 \rangle$ *satisfies*

$$u \in L^q(\Sigma) \tag{7.11}$$

*for any* $q > 0$.

*Proof.* Let $\Sigma$ be a codimension two complete weakly trapped immersed submanifold into the light cone $\Lambda^+$ and assume that $u \in L^q(\Sigma)$ for some $q > 0$. Define $v = u^2 > 0$. From (6.3) we have

$$v\Delta v - nv \geqslant \frac{n+2}{4} \|\nabla v\|^2. \tag{7.12}$$

We now apply theorem 7.3 with the choices $a = 0$, $b = n$, $A = -(n+2)/4$. Note that, since $n \geqslant 2$, then $A \leqslant -1$ and the only condition on $\beta$ required in theorem 7.3 is now $\beta > -1$. Choose then $\beta = -1 + q/4 > -1$ and take $\alpha = 2$, so that

$$\alpha(\beta+1) = \frac{q}{2}.$$

This implies that

$$v \in L^{\alpha(\beta+1)}(\Sigma) \tag{7.13}$$

and, by theorem 7.3, $v \equiv 0$, which is a contradiction, completing the proof of the theorem. □

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## 8. Codimension two spacelike submanifolds into a null hyperplane

In this last section, we briefly study the case of codimension two spacelike submanifolds contained in a different null hypersurface of the Lorentz-Minkowski spacetime, a null hyperplane. Let us start by giving the definition of such a hypersurface.

DEFINITION 8.1. Let $\mathbf{a} \in \mathbb{L}^{n+2}$ be a null vector. The subset

$$\mathcal{L}_{\mathbf{a}} = \{x \in \mathbb{L}^{n+2} : \langle x, \mathbf{a} \rangle = 0, x \neq \mathbf{a}\}$$

is a null hyperplane on the Lorentz-Minkowski spacetime $\mathbb{L}^{n+2}$.

Let us suppose $\mathbf{a}$ future-pointing and let $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the null hyperplane $\mathcal{L}_{\mathbf{a}}$. In this case

$$\xi = \mathbf{a}$$

is a future-pointing null vector field which is normal to the submanifold and hence, it can be chosen as the first vector field of our globally defined future-pointing normal null frame. We consider the function $u : \Sigma \to \mathbb{R}$ by $u = -\langle \psi, \mathbf{e}_1 \rangle = \psi_1$ and following the expressions obtained in (2.2) and (2.3) we have

$$\nu = \frac{\mathbf{e}_1 + \nabla u}{\sqrt{1 + \|\nabla u\|^2}},$$

and

$$\eta = -\frac{1 + \|\nabla u\|^2}{2\langle \mathbf{e}_1, \mathbf{a} \rangle^2}\mathbf{a} - \frac{1}{\langle \mathbf{e}_1, \mathbf{a} \rangle}(\mathbf{e}_1 + \nabla u).$$

Therefore, we have that $\eta$ is a globally defined normal null vector field which is future-pointing and satisfies $\langle \xi, \eta \rangle = -1$. In this setting, we can state the following.

PROPOSITION 8.2. *Let $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the null hyperplane $\mathcal{L}_{\mathbf{a}}$. Then,*

$$\xi = \mathbf{a} \quad \textit{and} \quad \eta = -\frac{1 + \|\nabla u\|^2}{2\langle \mathbf{e}_1, \mathbf{a} \rangle^2}\mathbf{a} - \frac{1}{\langle \mathbf{e}_1, \mathbf{a} \rangle}(\mathbf{e}_1 + \nabla u)$$

*are two globally defined normal null vector fields along the submanifold which are future-pointing and satisfy $\langle \xi, \eta \rangle = -1$.*

Similarly as in proposition 3.2 we can compute the Weingarten endomorphisms and null mean curvatures of $\Sigma$ with respect to $\{\xi, \eta\}$.

PROPOSITION 8.3. *Let $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ be a codimension two spacelike submanifold which is contained in the null hyperplane $\mathcal{L}_{\mathbf{a}}$. Then, the Weingarten* *endomorphisms associated with* $\xi$ *and* $\eta$ *are, respectively,*

$$A_\xi = 0 \quad \text{and} \quad A_\eta = -\frac{1}{\langle \mathbf{e}_1, \mathbf{a}\rangle} \nabla^2 u$$

*In particular,*

$$\theta_\xi = 0 \qquad \text{and} \qquad \theta_\eta = -\frac{1}{n\langle \mathbf{e}_1, \mathbf{a}\rangle} \Delta u. \tag{8.1}$$

From the previous proposition we have the expression for the mean curvature vector field,

$$\mathbf{H} = \frac{\Delta u}{n\langle \mathbf{e}_1, \mathbf{a}\rangle} \mathbf{a} \tag{8.2}$$

and then, $\langle \mathbf{H}, \mathbf{H}\rangle = 0$. Thus, we have the following.

Proposition 8.4. *Let* $\psi : \Sigma^n \to \mathcal{L}_{\mathbf{a}} \subset \mathbb{L}^{n+2}$ *be a codimension two spacelike submanifold which is contained in the null hyperplane* $\mathcal{L}_{\mathbf{a}}$. *Then* $\Sigma$ *is marginally trapped except at points where* $\Delta u = 0$ *on* $\Sigma$.

In what follows, and without loss of generality, we may assume that the future-pointing null vector is $\mathbf{a} = (1, 0, \ldots, 0, 1)$. Now we denote the null hyperplane $\mathcal{L}_{\mathbf{a}}$ simply by $\mathcal{L}$. Our next result corresponds to proposition 5.2.

Proposition 8.5. *Let* $\psi : \Sigma^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ *be a codimension two spacelike submanifold which is contained in the null hyperplane* $\mathcal{L}$. *Assume that* $\Sigma$ *is complete. Then* $\Sigma$ *is isometric to the Euclidean space* ($\mathbb{R}^n$, $\langle , \rangle_{\mathbb{R}^n}$).

*Proof.* Since $\psi(\Sigma) \subset \mathcal{L}$, for every $p \in \Sigma$ we can write

$$\psi(p) = (u(p), \psi_2(p), \ldots, \psi_{n+1}(p), u(p)).$$

We define the function

$$\begin{aligned} \Psi \colon \Sigma^n &\to \mathbb{R}^n \\ p &\mapsto (\psi_2(p), \ldots \psi_{n+1}(p)). \end{aligned}$$

For every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we compute

$$d\Psi_p(\mathbf{v}) = (\mathbf{v}(\psi_2), \ldots, \mathbf{v}(\psi_{n+1}))$$

and

$$\begin{aligned} \langle d\Psi_p(\mathbf{v}), d\Psi_p(\mathbf{w})\rangle_{\mathbb{R}^n} &= \sum_{i=2}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) \\ &= -\mathbf{v}(u)\mathbf{w}(u) + \sum_{i=2}^{n+1} \mathbf{v}(\psi_i)\mathbf{w}(\psi_i) + \mathbf{v}(u)\mathbf{w}(u) \\ &= \langle d\psi_p(\mathbf{v}), d\psi_p(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle. \end{aligned}$$

In other words, $\Psi^*(\langle , \rangle_{\mathbb{R}^n}) = \langle , \rangle$, which means that $\Psi$ is a local isometry. After this point, since $\Sigma$ is complete and $\mathbb{R}^n$ is simply connected, we obtain that $\Psi$ is in fact a global isometry. $\square$

In the next example, we show that for each smooth function on $\mathbb{R}^n$ we can construct an embedding of $\mathbb{R}^n$ to $\mathbb{L}^{n+2}$ through $\mathcal{L}$.

EXAMPLE 8.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth real function. We define $\phi_f : \mathbb{R}^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ given by

$$\phi_f(p) = (f(p), p, f(p)).$$

For every $\mathbf{v}, \mathbf{w} \in T_p\Sigma$ we have

$$\mathrm{d}(\phi_f)_p(\mathbf{v}) = (\mathbf{v}(f), \mathbf{v}, \mathbf{v}(f))$$

and

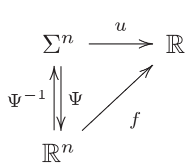
$$\langle \mathrm{d}(\phi_f)_p(\mathbf{v}), d(\phi_f)_p(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{R}^n}.$$

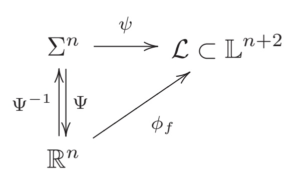
That is, $\phi_f^*(\langle , \rangle) = \langle , \rangle_{\mathbb{R}^n}$ and $\phi_f$ determines a spacelike isometric immersion of the Euclidean space through $\mathcal{L}$. Moreover, the immersion is marginally trapped except at points where $\Delta_{\mathbb{R}^n} f = 0$ on $\mathbb{R}^n$.

At this point, from proposition 8.5 we know that every codimension two complete spacelike submanifold factorizing through $\mathcal{L}$ is, up to an isometry, as in example 8.6.

COROLLARY 8.7. *Let $\psi : \Sigma^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ be a codimension two complete spacelike submanifold which is contained in $\mathcal{L}$ . Then there exists an isometry $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$ such that $\psi = \phi_f \circ \Psi$, where $f = u \circ \Psi^{-1}$ with $u = -\langle \psi, \mathbf{e}_1 \rangle = \psi_1$ and $\phi_f : \mathbb{R}^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ is the embedding*

$$\phi_f(p) = (f(p), p, f(p)).$$




*In particular, the immersion $\psi$ is an embedding and it is marginally trapped except at points where $\Delta u = 0$ on $\Sigma$.*

As a consequence, we can characterize codimension two spacelike submanifolds which are contained in $\mathcal{L}$ and that have parallel mean curvature vector as follows.

COROLLARY 8.8. *Let $\psi : \Sigma^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ be a codimension two complete spacelike submanifold which is contained in $\mathcal{L}$ and that has parallel mean curvature vector. Then there exists an isometry $\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$ such that $\psi = \phi_{\varrho,c} \circ \Psi$, where $\phi_{\varrho,c} : \mathbb{R}^n \to \mathcal{L} \subset \mathbb{L}^{n+2}$ is the embedding*

$$\phi_{\varrho,c}(p) = (\varrho(p) + c\|p\|^2, p, \varrho(p) + c\|p\|^2)$$

*for some harmonic function $\varrho$ on $\mathbb{R}^n$ and $c \in \mathbb{R}$. Moreover:*

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(i) $\Sigma$ *is minimal if, and only if,* $c = 0$.

(ii) $\Sigma$ *is future marginally trapped if, and only if,* $c < 0$.

(iii) $\Sigma$ *is past marginally trapped if, and only if,* $c > 0$.

*Proof.* Since $\langle \mathbf{a}, \mathbf{e}_1 \rangle = -1$, it follows from (8.2) that

$$\mathbf{H} = -\frac{\Delta u}{n}\mathbf{a}. \tag{8.3}$$

From (8.3), $\mathbf{H}$ is parallel if, and only if, $\Delta u =$ constant on $(\Sigma, \langle , \rangle)$. Equivalently, since $u = f \circ \Psi$ with $\Psi$ an isometry between $(\Sigma, \langle , \rangle)$ and $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$, $\mathbf{H}$ is parallel if, and only if, $\Delta_{\mathbb{R}^n} f =$ constant on $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$.

Consider the function

$$g(p) = \frac{\Delta_{\mathbb{R}^n} f}{2n} \|p\|^2$$

for every $p \in \mathbb{R}^n$. It is satisfied

$$\nabla^{\mathbb{R}^n} g = \frac{\Delta_{\mathbb{R}^n} f}{n} p \quad \text{and} \quad \Delta_{\mathbb{R}^n} g = \Delta_{\mathbb{R}^n} f.$$

Then, defining $\varrho(p) = f(p) - g(p)$ we have $\Delta_{\mathbb{R}^n} \varrho = 0$, that is, $\varrho$ is an harmonic function on $\mathbb{R}^n$ and $f(p) = \varrho(p) + c\|p\|^2$ where $c = \frac{\Delta_{\mathbb{R}^n} f}{2n} \in \mathbb{R}$. The last assertions follow from (8.3) since $\mathbf{H} = -c/n\mathbf{a}$, with $\mathbf{a}$ future-pointing. □

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