The Journal of Symbolic Logic
Volume 85, Number 1, March 2020

CHAITIN'S Ω AS A CONTINUOUS FUNCTION

RUPERT HÖLZL, WOLFGANG MERKLE, JOSEPH MILLER,
FRANK STEPHAN, AND LIANG YU

Abstract. We prove that the continuous function $\widehat{\Omega}\colon 2^\omega \to \mathbb{R}$ that is defined via $X \mapsto \sum_n 2^{-K(X\upharpoonright n)}$ for all $X \in 2^\omega$ is differentiable exactly at the Martin-Löf random reals with the derivative having value 0; that it is nowhere monotonic; and that $\int_0^1 \widehat{\Omega}(X)\,\mathrm{d}X$ is a left-c.e. wtt-complete real having effective Hausdorff dimension 1/2.

We further investigate the algorithmic properties of $\widehat{\Omega}$. For example, we show that the maximal value of $\widehat{\Omega}$ must be random, the minimal value must be Turing complete, and that $\widehat{\Omega}(X) \oplus X \geq_T \emptyset'$ for every X. We also obtain some machine-dependent results, including that for every $\varepsilon > 0$, there is a universal machine V such that $\widehat{\Omega}_V$ maps every real X having effective Hausdorff dimension greater than ε to a real of effective Hausdorff dimension 0 with the property that $X \leq_{tt} \widehat{\Omega}_V(X)$; and that there is a real X and a universal machine V such that $\Omega_V(X)$ is rational.

§1. Introduction.

In 1975, Chaitin [12] introduced a celebrated number as

$$\Omega = \sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)}.$$

Ω is an example of a naturally occuring Martin-Löf random number. It can be seen as an analogue of the halting problem in the theory of algorithmic randomness. An overview over the research into Chaitin's Ω can be found in Barmpalias [2].

Subsequently, other authors studied variants of Ω: Downey et al. [14] investigated Ω when relativized to oracles; and Becher and Grigorieff [7], as well as Becher et al. [6] studied it as a function from subsets of natural numbers to the real numbers. More recently, Barmpalias et al. [5] studied analogues of Ω in the c.e. sets, and in two articles Barmpalias, Cenzer, and Porter [3, 4] studied more generally the probabilities that, given random oracles, universal Turing machines display certain behaviors (other than halting).

In this article, we study a version of Ω as a function from Cantor space to the reals.

Definition 1.1. Let a prefix-free Turing machine M be given. Then for a real $X \in 2^\omega$, let

$$\widehat{\Omega}_M(X) = \sum_n 2^{-K_M(X\upharpoonright n)}$$

be the INITIAL SEGMENT $\widehat{\Omega}$ NUMBER of X.

Received May 19, 2019.
2010 *Mathematics Subject Classification.* 03D32, 68Q30, 26A27, 26A30.
Key words and phrases. Kolmogorov complexity, Chaitin's number, computable analysis.

0022-4812/20/8501-0022
DOI:10.1017/jsl.2019.60

Furthermore, for every finite string σ, define

$$\widehat{\Omega}_M(\sigma) = \sum_{\tau \succeq \sigma} 2^{-K_M(\tau)}.$$

In this article, we will be mostly interested in optimal machines. In cases where the respective statements are independent of the choice of optimal machine, the subscript M will be omitted.

The following statement is immediate.

Fact 1.2. *$\widehat{\Omega}\colon 2^\omega \to \mathbb{R}$ is computable in $\emptyset'$ and consequently continuous.*

In this article we will analyse this natural function both from the point of view of computable analysis and from that of computability theory. The article is organized as follows: inSection 2, we provide essential definitions and preliminaries. In Section 3, we investigate the function $n \mapsto 2^n\,\widehat{\Omega}(X \restriction n)$, which serves as preparation for the study of the analysis aspects of $\widehat{\Omega}$ in Section 4. In Section 5, we investigate the algorithmic aspects of $\widehat{\Omega}$.

§**2. Preliminaries.** We assume that that reader has a general background in computability theory and algorithmic randomness, as provided by the textbooks of Downey and Hirschfeldt [13] and Nies [20]. When we talk about a "real" X we mean either $X \in 2^\omega$ or $X \in [0,1]$ depending on the context; we identify these two interpretations with each other in the canonical way. For finite strings $\sigma \in 2^{<\omega}$ we use $[\sigma]$ to denote the basic open set $\{X \in 2^\omega\colon X \succ \sigma\}$.

Definition 2.1. A real X is left-c.e. if there is a computable nondecreasing sequence $(X_s)_s$ such that $\lim_s X_s = X$. Similarly, a real X is right-c.e. if there is a computable nonincreasing sequence $(X_s)_s$ such that $\lim_s X_s = X$.

We fix a standard universal prefix-free machine U and use $K(\sigma)$ to denote $K_U(\sigma)$. A prefix-free machine V is *optimal* if there is a constant c such that, for every σ, $K_V(\sigma) \leq K(\sigma) + c$. Often we will simply refer to a Martin-Löf random real as a "random".

Definition 2.2. Given $X \in 2^\omega$, $Y \in [0,1]$, and a set $\mathcal{P} \subseteq 2^{<\omega}$, we say that X has density Y in $\mathcal{P}$ if $\lim_n 2^n \mu([X \restriction n] \cap \mathcal{P}) = Y$.

The following notion was studied intensively in the literature (see, for example, Bienvenu et al. [10] and Bienvenu et al. [8]).

Definition 2.3. A real $X \in 2^\omega$ is density random if X is Martin-Löf random and has density 1 in every Π^0_1 subset of 2^ω that contains X.

Lemma 2.4 (Ample Excess Lemma; Miller and Yu [18]). *If X is random, then $\sum_n 2^{n-K(X \restriction n)}$ is finite.*

Definition 2.5. A real A is weakly low for K if and only if there are a constant c and infinitely many $\sigma \in 2^{<\omega}$ such that

$$K(\sigma) \leq K^A(\sigma) + c.$$

A is WEAKLY LOW ALONG X if and only if there is some constant c and infinitely many n such that

$$K(X \upharpoonright n) \leq K^A(X \upharpoonright n) + c.$$

A is WEAKLY LOW ALONG ITSELF if A is weakly low along A.

DEFINITION 2.6. A function $f : \mathbb{N} \to \mathbb{N}$ is right-c.e. if there exists a computable function $\widehat{f} : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that

$$(\forall n)(\forall s)(\widehat{f}(n, s+1) \leq \widehat{f}(n, s)) \text{ and } (\forall n)(\lim_s \widehat{f}(n, s) = f(n)).$$

DEFINITION 2.7. A function $f : \mathbb{N} \to \mathbb{N}$ is a SOLOVAY FUNCTION RELATIVE TO A, if

- — f is right-c.e. relative to A,
- — f is an upper bound for $n \mapsto K^A(n)$ up to an additive constant, and
- — this upper bound is tight up to an additive constant infinitely often.

f is a SOLOVAY FUNCTION if it is a Solovay function relative to $\emptyset$.

THEOREM 2.8 (Bienvenu and Downey [9]; Hölzl, Kräling, and Merkle [16]). *A right-c.e. function f is a Solovay function relative to X if and only if $\sum_n 2^{-f(n)}$ is Martin-Löf random relative to X.*

DEFINITION 2.9. Given two reals $A \in 2^\omega$ and $X \in [0, 1]$, we say that

(1) A has EFFECTIVE HAUSDORFF DIMENSION X if $\underline{\lim}_n \frac{K(A\upharpoonright n)}{n} = X$,
(2) A has EFFECTIVE PACKING DIMENSION X if $\overline{\lim}_n \frac{K(A\upharpoonright n)}{n} = X$.

THEOREM 2.10 (Levin and Gács [15]; Chaitin [12]). *There is a constant c such that, for all strings σ and τ,*

$$|K(\sigma\tau) - K(\sigma) - K(\tau|\sigma^*)| \leq c,$$

where for a string ρ we let ρ^* denote the shortest τ such that $U(\tau) = \rho$.

§3. On $2^n\,\widehat{\Omega}(X\upharpoonright n)$.

As we will show, the function $n \mapsto 2^n\,\widehat{\Omega}(X\upharpoonright n)$ plays an important role in the investigation of $\widehat{\Omega}$. First observe that the mapping

$$\sigma \mapsto \sum_{\tau \succ \sigma} 2^{-K(\tau)+|\sigma|}$$

is a left-c.e. supermartingale. Thus, the following proposition is immediate.

PROPOSITION 3.1. *If $R \in 2^\omega$ is random, then $2^n\,\widehat{\Omega}(R\upharpoonright n)$ is bounded.*

In fact, the following stronger statement holds.

LEMMA 3.2. *For every random real R, $\underline{\lim}_n 2^n\,\widehat{\Omega}(R\upharpoonright n) = 0$.*

PROOF. We prove that, for every rational p, the set

$$\mathcal{X}_p = \{X \in 2^\omega : (\forall n)(2^n\,\widehat{\Omega}(X\upharpoonright n) > p)\}$$

can be covered by a Martin-Löf test $(U_n)_{n\in\omega}$. To see this, for every n, we inductively define $U_{n,s}$ over stages s as follows:

At stage 0, search for the length-lexicographically smallest string τ such that a sequence $l_1^\tau, \dots, l_{n+1}^\tau$ exists with

$$0 = l_1^\tau < l_2^\tau < \dots < l_n^\tau = |\tau| < l_{n+1}^\tau \leq s$$

and such that for all $1 \leq i \leq n$,

$$\sum_{\nu \succeq \tau \restriction l_i^\tau \wedge |\nu| \leq l_{i+1}^\tau} 2^{-K_{l_{i+1}^\tau}(\nu)} > p2^{-l_i^\tau}.$$

Once τ is found, let $U_{n,0} = \{\tau\}$ and call $l_1^\tau, \dots, l_{n+1}^\tau$ the *section* of τ. It will be important that we take $l_1^\tau, \dots, l_{n+1}^\tau$ to be the least possible sequence satisfying the condition, which mostly follows from the minimality of τ.

At the beginning of stage $s+1$, finitely many strings σ are in $U_{n,s}$, and we assume that for each of them a sequence of numbers $l_1^\sigma, \dots, l_{n+1}^\sigma$ is defined such that

$$0 = l_1^\sigma < l_2^\sigma < \dots < l_n^\sigma = |\sigma| < l_{n+1}^\sigma \leq s.$$

As before, we call this sequence the *section* of σ.

Let T be the set of strings τ such that τ is incomparable with every $\sigma \in U_{n,s}$. For a string $\tau \in T$, let i_τ be the largest number such that there is $\sigma_\tau \in U_{n,s}$ such that $\sigma_\tau \restriction l_{i_\tau}^{\sigma_\tau} = \tau \restriction l_{i_\tau}^{\sigma_\tau}$. Note that for each $\tau \in T$ such an i_τ and σ_τ exist, since all $l_1^{\sigma_\tau}$ equal 0.

Now, in length-lexicographically ascending order, we search for a string $\tau \in T$ such that there is a finite sequence $l_{i_\tau+1}^\tau, \dots, l_{n+1}^\tau$ with

$$l_{i_\tau+1}^\tau = l_{i_\tau+1}^{\sigma_\tau} < l_{i_\tau+2}^\tau < \dots < l_n^\tau = |\tau| < l_{n+1}^\tau$$

such that for every $i_\tau + 1 \leq i \leq n$,

$$\sum_{\nu \succeq \tau \restriction l_i^\tau \wedge |\nu| \leq l_{i+1}^\tau} 2^{-K_{l_{i+1}^\tau}(\nu)} > p2^{-l_i^\tau}.$$

Once such a τ is found, let $U_{n,s+1} = U_{n,s} \cup \{\tau\}$ and define τ's section as

$$l_1^{\sigma_\tau}, \dots, l_{i_\tau}^{\sigma_\tau}, l_{i_\tau+1}^\tau, \dots, l_{n+1}^\tau.$$

As before, we assume that $l_{i_\tau+1}^\tau, \dots, l_{n+1}^\tau$ is as small as possible. This completes stage $s+1$.

Finally, for every n, define $U_n = \bigcup_s U_{n,s}$. By construction, U_n is a prefix-free set. Moreover, by construction, for every $i \leq n$,

(a) $\{\sigma \restriction l_i^\sigma : \sigma \in U_n\}$ is a prefix-free set, and

(b) for every $\sigma \in U_n$, $\sum_{\nu \succeq \sigma \restriction l_i^\sigma \wedge |\nu| \leq l_{i+1}^\sigma} 2^{-K_{l_{i+1}^\sigma}(\nu)} > p2^{-l_i^\sigma} \geq p2^{-|\sigma|}$.

For every $i \leq n$, let

$$q_i = \sum_{\sigma \in U_n} \sum_{\nu \succeq \sigma \restriction l_i^\sigma \wedge |\nu| \leq l_{i+1}^\sigma} 2^{-K_{l_{i+1}^\sigma}(\nu)}.$$

By (b), $q_i > p\mu(U_n)$ for every $i \leq n$.

The following claim follows by an easy inductive argument.

CLAIM 1. *If for any $\sigma_0, \sigma_1 \in U_n$ with $\sigma_0 \neq \sigma_1$ we have that there is some $i < n$ with $\sigma_0 \restriction l_i^{\sigma_0} = \sigma_1 \restriction l_i^{\sigma_0}$, then $l_{i+1}^{\sigma_0} = l_{i+1}^{\sigma_1}$.*

Let $T_i = \{\nu : (\exists \sigma \in U_n)(\nu \succeq \sigma \restriction l_i^\sigma \wedge |\nu| \leq l_{i+1}^\sigma)\}$.

Claim 2. *$T_i \cap T_j = \emptyset$ for all $i \neq j \leq n$.*

Proof of the Claim. Without loss of generality, assume that $i < j$. Assume for a contradiction that $T_i \cap T_j \neq \emptyset$, so that there is a string $\nu \in T_i \cap T_j$ such that there are two distinct strings $\sigma_0, \sigma_1 \in U_n$ with the property that

$$\nu \succeq \sigma_0 \upharpoonright l_i^{\sigma_0} \;\wedge\; |\nu| \leq l_{i+1}^{\sigma_0} \;\wedge\; \nu \succeq \sigma_1 \upharpoonright l_j^{\sigma_1} \;\wedge\; |\nu| \leq l_{j+1}^{\sigma_1}.$$

Then by (a) above, we have that $\sigma_0 \upharpoonright l_i^{\sigma_0} = \sigma_1 \upharpoonright l_i^{\sigma_0}$, and hence by Claim 1 that $l_{i+1}^{\sigma_0} = l_{i+1}^{\sigma_1}$. Then $\nu \succ \sigma_1 \upharpoonright l_j^{\sigma_1} \succeq \sigma_1 \upharpoonright l_{i+1}^{\sigma_1}$ implies $|\nu| > l_{i+1}^{\sigma_1} = l_{i+1}^{\sigma_0}$. But then $\nu \notin T_i$, by definition, which is a contradiction. $\Diamond$

Consequently,

$$1 \geq \sum_{\nu} 2^{-K(\nu)} \geq \sum_{i \leq n} \sum_{\tau \in T_i} 2^{-K(\tau)} \geq \sum_{i \leq n} q_i \geq np\mu(U_n),$$

and thus $\mu(U_n) < 1/np$.

Now for a contradiction, suppose that there is a real $X \in \mathcal{X}_p \setminus U_n$ for some n. Then there is no $\sigma \in U_n$ such that $\sigma \prec X$. Find the largest $j < n$ such that there is some $\sigma \in U_n$ for which $\sigma \upharpoonright l_j^{\sigma} \prec X$; among all such σ let τ denote the length-lexicographically smallest. For every $\sigma \in U_n$, we have that $\sigma \upharpoonright l_{j+1}^{\sigma} \not\prec X$. By the definition of $\mathcal{X}_p$, there must be some least s and a finite sequence $l_{j+1}^{X \upharpoonright s}, \dots, l_{n+1}^{X \upharpoonright s}$ with

$$l_{j+1}^{X \upharpoonright s} = l_{j+1}^{\tau} < \cdots < l_n^{X \upharpoonright s} = s < l_{n+1}^{X \upharpoonright s}$$

such that

$$\sum_{\nu \succeq \tau \upharpoonright l_j^{\tau} \wedge |\nu| \leq l_{j+1}^{X \upharpoonright s}} 2^{-K_{l_{j+1}^{X \upharpoonright s}}(\nu)} > p2^{-l_j^{\tau}}$$

and, for every $j < i < n+1$,

$$\sum_{\nu \succeq X \upharpoonright l_i^{X \upharpoonright s} \wedge |\nu| \leq l_{i+1}^{X \upharpoonright s}} 2^{-K_{l_{i+1}^{X \upharpoonright s}}(\nu)} > p2^{-l_i^{X \upharpoonright s}}.$$

As there are at most finitely many strings lexicographically smaller than $X \upharpoonright s$, by construction, there must be a stage t such that $X \upharpoonright s \in U_{n,t}$.

Now let $V_n = \bigcap_{i \leq \lceil 2^n/p \rceil} U_i$ for each $n \in \omega$. Then $(V_n)_{n \in \omega}$ is a Martin-Löf test covering $\mathcal{X}_p$; thus $R \notin \mathcal{X}_p$. As $p > 0$ was arbitrary, it follows that

$$\underline{\lim}_n 2^n\, \widehat{\Omega}(R \upharpoonright n) = 0. \qquad \dashv$$

By Lemma 3.2, it is easy to see that, for every 2-generic real G,

$$\underline{\lim}_n 2^n\, \widehat{\Omega}(G \upharpoonright n) = 0.$$

Theorem 3.3 (Andrews, Cai, Diamondstone, Lempp, and Miller; for a proof see Miyabe, Nies, and Zhang [19]). *A real R is density random if and only if for every left-c.e. martingale G, $\lim_n G(R \upharpoonright n)$ exists.*

Lemma 3.4. *If R is density random, then $\lim_n 2^n\, \widehat{\Omega}(R \upharpoonright n) = 0$.*

Proof. For all $\sigma \in 2^{<\omega}$, let $L(\sigma) = \sum_{\tau \prec \sigma} 2^{|\tau| - K(\tau)}$ and

$$F(\sigma) = L(\sigma) + 2^{|\sigma|}\, \widehat{\Omega}(\sigma).$$

F is clearly left-c.e.; furthermore, it is a martingale, as

$$\begin{aligned} F(\sigma) &= L(\sigma) + 2^{|\sigma|}\,\widehat{\Omega}(\sigma) \\ &= \frac{L(\sigma 0) + L(\sigma 1) - 2\cdot 2^{|\sigma| - K(\sigma)}}{2} + 2^{|\sigma|}(\widehat{\Omega}(\sigma 0) + \widehat{\Omega}(\sigma 1) + 2^{-K(\sigma)}) \\ &= \frac{L(\sigma 0) + 2^{|\sigma 0|}\widehat{\Omega}(\sigma 0) + L(\sigma 1) + 2^{|\sigma 1|}\widehat{\Omega}(\sigma 1)}{2} \\ &= \frac{F(\sigma 0) + F(\sigma 1)}{2}. \end{aligned}$$

So by Theorem 3.3, $\lim_n F(R\restriction n)$ exists, and by Ample Excess Lemma 2.4

$$\lim_n 2^n\,\widehat{\Omega}(R\restriction n) = \lim_n \left(F(R\restriction n) - \sum_{m<n} 2^{m-K(R\restriction m)} \right)$$

also exists. Then by Lemma 3.2, $\lim_n 2^n\,\widehat{\Omega}(R\restriction n) = \underline{\lim}_n 2^n\,\widehat{\Omega}(R\restriction n) = 0$. ⊣

The following result gives a Kolmogorov complexity characterization of density randomness.

Theorem 3.5. *R is density random if and only if* $\lim_n 2^n\,\widehat{\Omega}(R\restriction n) = 0$.

Proof. The direction from left to right follows from Lemma 3.4.

For the right to left direction, let R be such that $\lim_n 2^n\,\widehat{\Omega}(R\restriction n) = 0$. Then R is random. Now suppose that there is a computable tree $T \subseteq 2^{<\omega}$ such that $R \in [T]$ but $\underline{\lim}_n 2^n\,\mu([R\restriction n]\cap[T]) < 1-\varepsilon$ for some $\varepsilon > 0$. So there is a constant c independent of n and a c.e. prefix-free set $W \subseteq 2^{<\omega}$ such that $[T] = 2^\omega \setminus [W]$ and $(\forall \sigma \in W)(K(\sigma) \leq |\sigma| + c)$.

Now fix an n such that $\mu([R\restriction n]\cap[T]) < 2^{-n}(1-\varepsilon)$. Then

$$\begin{aligned} & 2^n \textstyle\sum_{\tau \succ R\restriction n} 2^{-K(\tau)} \\ \geq\ & 2^n \textstyle\sum_{\tau \succ R\restriction n \,\wedge\, \tau \in W} 2^{-K(\tau)} \\ \geq\ & 2^n \textstyle\sum_{\tau \succ R\restriction n \,\wedge\, \tau \in W} 2^{-|\tau| - c} \\ \geq\ & 2^n \cdot 2^{-n-c}\cdot\varepsilon = 2^{-c}\cdot\varepsilon. \end{aligned}$$

This contradicts $\lim_n 2^n\,\widehat{\Omega}(R\restriction n) = 0$. ⊣

Remark 3.6. The result of Miyabe, Nies, and Zhang [19], that every K-trivial real is low for density randomness, is an immediate corollary of the above theorem.

The following result implies that $2^n\,\widehat{\Omega}(R\restriction n)$ converges to zero slowly.

Proposition 3.7. *There is no real X such that* $\sum_n 2^n\,\widehat{\Omega}(X\restriction n) < \infty$.

Proof. There is a constant c such that for every σ,

$$\begin{aligned} & 2^{|\sigma|} \textstyle\sum_{\tau\succeq\sigma} 2^{-K(\tau)} \\ \geq\ & 2^{|\sigma|} \textstyle\sum_{\tau\succeq\sigma} 2^{-K(|\tau|) - |\tau| - c} \\ =\ & 2^{|\sigma|} \textstyle\sum_{n\geq|\sigma|} \left(2^{n-|\sigma|}\cdot 2^{-K(n)-n-c}\right) \\ =\ & \textstyle\sum_{n\geq|\sigma|} 2^{-K(n)-c}. \end{aligned}$$

Therefore, for every real X,

$$\sum_m 2^m \sum_{\tau \succeq X\restriction m} 2^{-K(\tau)} \geq \sum_m \sum_{n \geq m} 2^{-K(n)-c} = \infty. \qquad \dashv$$

§4. Analytic aspects of $\widehat{\Omega}$.

4.1. Differentiability and monotonicity. We first give a characterization of the points where $\widehat{\Omega}$ is differentiable.

Theorem 4.1. *The following are equivalent*:

(1) *R is random*;
(2) *$\widehat{\Omega}$ is differentiable at R*;
(3) $\widehat{\Omega}'(R) = 0$.

Proof. First note that the implication from (3) to (2) is trivial, while the implication from (2) to (3) follows from the equivalence of (1) and (2) by Lemma 3.2. We now show the remaining implications.

(*1*) *implies* (*2*). Consider the left-c.e. function

$$G(\sigma) = 2^{|\sigma|} \max_{Y \succ \sigma} \sum_{m > |\sigma|} 2^{-K(Y\restriction m)}.$$

We claim that $\lim_n G(R\restriction n) = 0$ for random R. Assume otherwise; then there is a rational $\varepsilon > 0$ such that there are infinitely many n with $G(R\restriction n) > \varepsilon$. For every n, let

$$S_n = \sum_{\{\sigma\colon |\sigma| \leq n \wedge G(\sigma) > \varepsilon\}} 2^{-|\sigma|}\,\varepsilon.$$

For each σ, let $Z_\sigma \succ \sigma$ be a real such that

$$\sum_{m > |\sigma|} 2^{-K(Z_\sigma\restriction m)} = \max_{Y \succ \sigma} \sum_{m > |\sigma|} 2^{-K(Y\restriction m)}.$$

Note that such a Z_σ exists by compactness because

$$Y \mapsto \sum_{m > |\sigma|} 2^{-K(Y\restriction m)}$$

is a continuous function from Cantor space to $\mathbb{R}$.

For every n, inductively partition $\{\sigma \in 2^{<n+1}\colon G(\sigma) > \varepsilon\}$ into $\{A_i\}_{i \leq k_n}$ as follows: Let $A_0 = \emptyset$. To define A_{i+1}, let σ_{i+1} be the lexicographically least element of $\{\sigma \in 2^{<n+1}\colon G(\sigma) > \varepsilon\} \setminus \bigcup_{j \leq i} A_j$ and let

$$A_{i+1} = \{\tau \in 2^{<n+1}\colon G(\tau) > \varepsilon \wedge \tau \succeq \sigma_{i+1} \wedge \tau \prec Z_{\sigma_{i+1}}\}.$$

Continue this process until the first stage $k_n < 2^{n+1}$ at which no σ_{i+1} can be found. Then it is clear that $\{A_i\}_{i \leq k_n}$ partitions $\{\sigma \in 2^{<n+1}\colon G(\sigma) > \varepsilon\}$.

Note that for every $i \leq k_n$ and $\tau \in 2^{<n+1}$ with $G(\tau) > \varepsilon$, $\tau \in A_i$ implies that $\tau \succeq \sigma_i$ and $\tau \prec Z_{\sigma_i}$. Moreover, no σ_i is on Z_{σ_j} for $j \neq i$, so the $Z_{\sigma_i} \restriction [\sigma_i, \infty)$ are disjoint. Finally, if $\tau \succeq \sigma_i$ is in $2^{<n+1}$, then, by the definition of Z_{σ_i},

$$2^{-|\tau|}\,\varepsilon < 2^{-|\tau|}\,G(\sigma_i) \leq 2^{-|\tau|+|\sigma_i|} \sum_{j \geq |\sigma_i|} 2^{-K(Z_{\sigma_i}\restriction j)}.$$

Consequently, for every n,

$$\begin{aligned} S_n &= \textstyle\sum_{\{\sigma\in 2^{<n+1}:\ G(\sigma)>\varepsilon\}} 2^{-|\sigma|}\,\varepsilon \\ &= \textstyle\sum_{i\leq k_n} \sum_{\sigma\in A_i} 2^{-|\sigma|}\,\varepsilon \\ &\leq \textstyle\sum_{i\leq k_n} \sum_{|\sigma_i|\leq m<n+1} 2^{-m+|\sigma_i|} \sum_{|\sigma_i|\leq j} 2^{-K(Z_{\sigma_i}\restriction j)} \\ &\leq \textstyle\sum_{i\leq k_n} 2\cdot \sum_{|\sigma_i|\leq j} 2^{-K(Z_{\sigma_i}\restriction j)} \\ &\leq 2. \end{aligned}$$

Thus,

$$\sum_{\{\sigma:\ G(\sigma)>\varepsilon\}} 2^{-|\sigma|}\varepsilon \leq 2 \text{ and so } \sum_{\{\sigma:\ G(\sigma)>\varepsilon\}} 2^{-|\sigma|} \leq 2\varepsilon^{-1}.$$

Hence, there exists a constant c such that for all σ we have that $G(\sigma) > \varepsilon$ implies that $|\sigma| \geq K(\sigma) - c$. Thus, for every n and every random R with $G(R\restriction n) > \varepsilon$ we have $K(R\restriction n) < n + c$, contradiction.

So, if R is random, then

$$\begin{aligned} 0 &\leq \lim_{Y\to R} \tfrac{|\widehat{\Omega}(Y)-\widehat{\Omega}(R)|}{d(Y,R)} \\ &\leq \lim_{Y\to R} \tfrac{|\widehat{\Omega}(Y)|+|\widehat{\Omega}(R)|}{d(Y,R)} \\ &\leq \lim_n 2^n \textstyle\sum_{m>n} 2^{-K(R\restriction m)} + 2^n \max_{Y\succ R\restriction n} \sum_{m>n} 2^{-K(Y\restriction m)} \\ &= \lim_n 2^n \textstyle\sum_{m>n} 2^{-K(R\restriction m)} + G(R\restriction n) \\ &= 0. \end{aligned}$$

Thus, $\widehat{\Omega}$ is differentiable at R and $\widehat{\Omega}'(R) = 0$.

(*2*) *implies* (*1*). Assume that X is not random and that $\widehat{\Omega}$ is differentiable at X. Then there exists an M such that for all Y we have $\left|\frac{\widehat{\Omega}(Y)-\widehat{\Omega}(X)}{d(Y,X)}\right| \leq M$. Note that there is a constant c such that for every σ and d, if $K(\sigma) \leq |\sigma| - d$, then $K\left(\sigma 0^{2^{2^{|\sigma|}}}\right) \leq |\sigma| - d + c$. To simplify notation, we will assume $c = 0$.

Since X is not random, for every d, there is some n such that $K(X\restriction n) \leq n - d$. We distinguish two cases:

Case 1. $X \succ (X\restriction n)0^i 1$ for some $i > 2^{2^n-1}$. Fix any real $Z_0 \succ (X\restriction n)0^{2^{2^n}}$; then $d(Z_0, X) < 2^{-2^n+1}$. So, $M \geq \left|\frac{\widehat{\Omega}(Z_0)-\widehat{\Omega}(X)}{d(Z_0,X)}\right| > 2^{2^n-1}\,|\widehat{\Omega}(Z_0) - \widehat{\Omega}(X)|$ and therefore $|\widehat{\Omega}(Z_0) - \widehat{\Omega}(X)| < 2^{-2^n+1} M$.

Note that since $2^{2^n-1} - 2^{n^2} > 2^{n^2}$ some $j \in [2^{n^2}, 2^{2^n-1}]$ must have the property that $\sum_{\tau\succeq(X\restriction n)0^j 1} 2^{-K(\tau)} < 2^{-n^2}$. Fix such a j and a real $Z_1 \succ (X\restriction n)0^j 1$. Then $d(Z_1, X) \leq 2^{-n^2}$ and

$$M \geq \left|\frac{\widehat{\Omega}(Z_1) - \widehat{\Omega}(X)}{d(Z_1, X)}\right| \geq 2^{n^2}\,|\widehat{\Omega}(Z_1) - \widehat{\Omega}(X)|.$$

So, $|\widehat{\Omega}(Z_1) - \widehat{\Omega}(X)| < 2^{-n^2} M$ and thus

$$\begin{aligned} 2^{-n} - 2^{-n^2} &\leq 2^{-K((X\upharpoonright n)0^{2^{2^n}})} - \textstyle\sum_{\tau \succeq (X\upharpoonright n)0^j 1} 2^{-K(\tau)} \\ &\leq |\widehat{\Omega}(Z_0) - \widehat{\Omega}(Z_1)| \\ &\leq |\widehat{\Omega}(Z_0) - \widehat{\Omega}(X)| + |\widehat{\Omega}(X) - \widehat{\Omega}(Z_1)| \\ &\leq 2^{-n^2} M + 2^{-2^n+1} M. \end{aligned}$$

For large enough n this is a contradiction.

Case 2. Otherwise. Fix any real $Z_0 \succ X\upharpoonright n0^{2^{2^n}}$ and choose $i \in (2^{2^n-1}, 2^{2^n})$ and a real $Z_1 \succ (X\upharpoonright n)0^i 1$ such that $\sum_{\tau \succeq (X\upharpoonright n)0^i 1} 2^{-K(\tau)} < 2^{-n^2}$. By the assumption, there is a number $0 < j \leq i$ such that $X \succ (X\upharpoonright n)0^j 1$. Since

$$M \geq \left| \frac{\widehat{\Omega}(Z_1) - \widehat{\Omega}(X))}{d(Z_1, X)} \right| \geq 2^n\, |\widehat{\Omega}(Z_1) - \widehat{\Omega}(X)|,$$

we have that $|\widehat{\Omega}(Z_1) - \widehat{\Omega}(X)| \leq 2^{-n} M$. Note that

$$\begin{aligned} &|\widehat{\Omega}(Z_1) - \widehat{\Omega}(X)| \\ = &\left| \textstyle\sum_{j<k\leq i} 2^{(X\upharpoonright n)0^k} + \sum_{m \geq n+i} 2^{-K(Z_1\upharpoonright m)} - \sum_{m \geq j+n} 2^{-K(X\upharpoonright m)} \right| \\ \geq &\left| \textstyle\sum_{j<k\leq i} 2^{(X\upharpoonright n)0^k} - \sum_{m \geq j+n} 2^{-K(X\upharpoonright m)} \right| - \textstyle\sum_{m \geq n+i} 2^{-K(Z_1\upharpoonright m)} \\ \geq &\left| \textstyle\sum_{j<k\leq i} 2^{(X\upharpoonright n)0^k} - \sum_{m \geq j+n} 2^{-K(X\upharpoonright m)} \right| - 2^{-n^2}, \end{aligned}$$

which yields

$$\begin{aligned} &\left| \textstyle\sum_{j<k\leq i} 2^{(X\upharpoonright n)0^k} - \sum_{m \geq j+n} 2^{-K(X\upharpoonright m)} \right| \\ \leq\ & 2^{-n^2} + |\widehat{\Omega}(Z_1) - \widehat{\Omega}(X)| \\ \leq\ & 2^{-n^2} + 2^{-n} M. \end{aligned}$$

Further note that

$$\begin{aligned} 2^n &\left| \textstyle\sum_{j<k\leq i} 2^{(X\upharpoonright n)0^k} + \sum_{m > i+n} 2^{-K(Z_0\upharpoonright m)} - \sum_{m \geq j+n} 2^{-K(X\upharpoonright m)} \right| \\ &\leq \left| \tfrac{\widehat{\Omega}(Z_0) - \widehat{\Omega}(X)}{d(Z_0, X)} \right| \leq M, \end{aligned}$$

and consequently

$$\left| \sum_{j<k\leq i} 2^{(X\upharpoonright n)0^k} + \sum_{m > i+n} 2^{-K(Z_0\upharpoonright m)} - \sum_{m \geq j+n} 2^{-K(X\upharpoonright m)} \right| \leq 2^{-n} M.$$

Thus,

$$\begin{aligned} &\textstyle\sum_{m > i+n} 2^{-K(Z_0\upharpoonright m)} \\ \leq\ &\left| \textstyle\sum_{j<k\leq i} 2^{(X\upharpoonright n)0^k} - \sum_{m \geq j+n} 2^{-K(X\upharpoonright m)} \right| + 2^{-n} M \\ \leq\ & 2^{-n^2} + 2^{-n+1} M \end{aligned}$$

which, for large enough n and d, contradicts

$$\sum_{m>i+n} 2^{-K(Z_0\upharpoonright m)} > 2^{-K\left((X\upharpoonright n)0^{2^{2^n}}\right)} > 2^{-n+d}. \qquad \dashv$$

Next we show that $\widehat{\Omega}$ is nowhere monotone.

Proposition 4.2. *For every computable increasing function g and every universal machine U, there is a prefix-free machine M such that for every weakly 1-generic set G there are infinitely many n such that*

$$(\forall m \in [g(n), g(n+1)])(\forall \sigma \in 2^m)(K_M(G\upharpoonright m) \leq K_U(\sigma) \ \wedge \ G(m) = 0).$$

The same statement holds with 1 in place of 0.

Proof. We define a prefix-free machine M as follows:

First, define $(l_n)_n$ via $l_0 = 0$ and $l_{n+1} = g(l_n + 1 + 2^{g(l_n+1)})$ for $n > 0$.

Next, at stage $n+1$, fix an enumeration $(\sigma_i)_{i<2^{g(l_n+1)}}$ of finite strings with length $g(l_n+1)$. For each $i < 2^{g(l_n+1)}$, define

$$\tau_i = \sigma_i 0^{l_{n+1}-g(l_n+1)} = \sigma_i 0^{g(l_n+1+2^{g(l_n+1)})-g(l_n+1)}.$$

Now for each such i and $g(l_n+1+i) \leq k < g(l_n+2+i)$, we let

$$K_M(\tau_i\upharpoonright k) = \min\{K_U(\nu) \colon |\nu| = k\}.$$

Without loss of generality, we may assume that M is a prefix-free machine. By construction, for every σ, we can effectively find a string $\tau \succ \sigma$ and a number n such that

$$(\forall m \in [g(n), g(n+1)])(\forall \nu \in 2^m)(K_M(\tau\upharpoonright m) \leq K_U(\nu) \ \wedge \ \tau(m) = 0).$$

Then, for every n, the Σ^0_1 set

$$\{Y \colon (\forall m \in [g(n), g(n+1)])(K_M(Y\upharpoonright m) \text{ is defined } \wedge \ Y(m) = 0)\} =$$
$$\{Y \colon (\forall m \in [g(n), g(n+1)])(\forall \nu \in 2^m)(K_M(Y\upharpoonright m) \leq K_U(\nu) \ \wedge \ Y(m) = 0)\}$$

is dense. The proposition follows immediately. $\dashv$

Lemma 4.3. *There is a constant c such that for every weakly 1-generic real G and $i \in \{0, 1\}$, there are infinitely many n such that*

$$(\forall m \in [n, 2^{2^{2^n}}])(K(G\upharpoonright m) \leq K(m) + c \ \wedge \ G(m) = i).$$

Proof. By Proposition 4.2, there is a machine M such that for every weakly 1-generic real G, there are infinitely many n such that

$$(\forall m \in [n, 2^{2^{2^n}}])(\forall \sigma \in 2^m)(K_M(G\upharpoonright m) \leq K(\sigma)).$$

Then there must be a constant c such that

$$(\forall m \in [n, 2^{2^{2^n}}])(\forall \sigma \in 2^m)(K(G\upharpoonright m) \leq K(m) + c). \qquad \dashv$$

Lemma 4.4. *If G is weakly 1-generic, then*

$$\overline{\lim}_{X\to G^+} \frac{\widehat{\Omega}(G) - \widehat{\Omega}(X)}{d(X, G)} = \infty \quad \text{and} \quad \overline{\lim}_{X\to G^-} \frac{\widehat{\Omega}(G) - \widehat{\Omega}(X)}{d(X, G)} = \infty.$$

Proof. Let G be weakly 1-generic. By Lemma 4.3, fix a number l such that $(\forall m \in [l, 2^{2^{2^l}}])(K(G\upharpoonright m) \le K(m) + c)$. By Theorem 2.10, there are constants c_1 and c_2 such that for every $n \in [l, 2^l]$, $m \in [0, 2^{2^n} - n]$, and every $\sigma \in 2^m$,

$$\begin{aligned} & K((G\upharpoonright n)\sigma) - K(G\upharpoonright (n+m)) \\ &= K(G\upharpoonright n) + K(\sigma\upharpoonright (G\upharpoonright n)^*) - K(G\upharpoonright n) - K(0^m\upharpoonright (G\upharpoonright n)^*) + c_1 \\ &\ge K(\sigma\upharpoonright (G\upharpoonright n)^*) - K(m) - c_2. \end{aligned}$$

It is clear that there is a real R with $(\forall m)(K(R\upharpoonright m|(G\upharpoonright n)^*) \ge m)$. If we let $\sigma = R\upharpoonright m$ in the above inequality, then, for some constant d',

$$K((G\upharpoonright n)\sigma) - K(G\upharpoonright n + m) \ge m - K(m) - c_2 \ge m - 2\log m - d'.$$

Note that there is a constant d'' such that $m - 2\log m - d' > 0$ holds for every $m \ge d''$. Let $e = \max\{|m - 2\log m - d'| \colon m < d''\}$. Then, for some constant c_3, we have

$$\begin{aligned} & 2^n \textstyle\sum_m (2^{-K(G\upharpoonright (n+m))} - 2^{-K((G\upharpoonright n)(R\upharpoonright m))}) \\ =\ & 2^n (\textstyle\sum_{m \le 2^{2^n} - n} (2^{-K(G\upharpoonright (n+m))} - 2^{-K((G\upharpoonright n)(R\upharpoonright m))}) \\ & + \textstyle\sum_{m > 2^{2^n} - n} (2^{-K(G\upharpoonright (n+m))} - 2^{-K((G\upharpoonright n)(R\upharpoonright m))})) \\ \ge\ & 2^n (\textstyle\sum_{m \le 2^{2^n} - n} (2^{-K(G\upharpoonright (n+m))} - 2^{-K((G\upharpoonright n)(R\upharpoonright m))}) \\ & - \textstyle\sum_{m > 2^{2^n} - n} 2^{-K((G\upharpoonright n)(R\upharpoonright m))}) \\ \ge\ & 2^n (\textstyle\sum_{m \le 2^{2^n} - n} (2^{-K(G\upharpoonright (n+m))} - 2^{-K((G\upharpoonright n)(R\upharpoonright m))}) \\ & - \textstyle\sum_{m > 2^{2^n} - n} 2^{-K(R\upharpoonright m|(G\upharpoonright n)^*) - c_3}) \\ \ge\ & 2^n \textstyle\sum_{m \le 2^{2^n} - n} (2^{-K(G\upharpoonright (n+m))} - 2^{-K((G\upharpoonright n)(R\upharpoonright m))}) - 2^{n - 2^n - c_3 - 1} \\ =\ & 2^n \textstyle\sum_{m \le 2^{2^n} - n} 2^{-K(G\upharpoonright (n+m))} (1 - 2^{-K((G\upharpoonright n)(R\upharpoonright m)) + K(G\upharpoonright (n+m))}) \\ & - 2^{n - 2^n - c_3 - 1} \\ \ge\ & 2^n \textstyle\sum_{m \le 2^{2^n} - n} 2^{-K(G\upharpoonright (n+m))} (1 - 2^{-m + 2\log m + d'}) - 2^{2n - 2^n - c_3 - 1} \\ =\ & 2^n (\textstyle\sum_{m < d''} 2^{-K(G\upharpoonright (n+m))} (1 - 2^{-m + 2\log m + d'}) \\ & + \textstyle\sum_{d'' \le m \le 2^{2^n} - n} 2^{-K(G\upharpoonright (n+m))} (1 - 2^{-m + 2\log m + d'})) \\ & - 2^{2n - 2^n - c_3 - 1} \\ \ge\ & 2^n (\textstyle\sum_{d'' \le m \le 2^{2^n} - n} 2^{-K(G\upharpoonright (n+m)) - 1} - \textstyle\sum_{m < d''} 2^{-K(G\upharpoonright (n+m)) + e}) \\ & - 2^{2n - 2^n - c_3 - 1}. \end{aligned}$$

It is clear that we may fix numbers $n_0, n_1 \in [l, 2^l]$ such that $n_1 > n_0 + d''$, $K(n_0) \ge \log n_0$, and $K(n_1) \le \log\log n_0 - 1$. Then, by the inequality above, there is a constant e' such that

$$\begin{aligned} & 2^{n_0} \textstyle\sum_m (2^{-K(G\upharpoonright (n_0+m))} - 2^{-K((G\upharpoonright n_0)(R\upharpoonright m))}) \\ \ge\ & 2^{n_0} (\tfrac{1}{\log n_0} - \textstyle\sum_{m < d''} 2^{-K(G\upharpoonright n_0) + m + 1 + e'}) - 2^{2n_0 - 2^{n_0} - c_3 - 1} \\ \ge\ & 2^{n_0} (\tfrac{1}{\log n_0} - \tfrac{2^{1+e'+d''}}{n_0}) - 2^{2n_0 - 2^{n_0} - c_3 - 1}. \end{aligned}$$

So, if $n_0 \to \infty$, then $2^{n_0} \sum_m (2^{-K(G \upharpoonright (n_0+m))} - 2^{-K((G \upharpoonright n_0)(r \upharpoonright m))}) \to \infty$. By the properties of R, there must be some $i, j \in \left[l, 2^{2^{2^l}}\right)$ such that $R(i) = 1$ and $R(j) = 0$. Then by Lemma 4.3, both the case $(G \upharpoonright n_0)R > G$ and the case $(G \upharpoonright n_0)R < G$ are possible. Thus,

$$\overline{\lim}_{X \to G^+} \frac{\widehat{\Omega}(G) - \widehat{\Omega}(X)}{d(X, G)} = \infty \quad \text{and} \quad \overline{\lim}_{X \to G^-} \frac{\widehat{\Omega}(G) - \widehat{\Omega}(X)}{d(X, G)} = \infty. \qquad \dashv$$

As a consequence, we obtain the following corollary.

Corollary 4.5. *If G is weakly 1-generic, then for every n there are random reals $R_0 > G > R_1$ such that $R_0 \upharpoonright n = G \upharpoonright n = R_1 \upharpoonright n$ and*

$$\max\{\widehat{\Omega}(R_0), \widehat{\Omega}(R_1)\} < \widehat{\Omega}(G).$$

The following theorem summarizes the results obtained in this subsection.

Theorem 4.6. *$\widehat{\Omega}$ is a continuous, nowhere monotone, almost everywhere differentiable function.*

4.2. Integral. In this section, we identify finite binary strings with rational numbers. Similarly, we identify reals $X \in [0, 1]$ with their binary expansions; then every $X \upharpoonright n$ is a finite binary string of length n.

Lemma 4.7.

(1) *If we let $p_n = \sum_{|\sigma|=n} 2^{-K(\sigma)}$ for all n, then there is some constant c such that for all n we have $K(p_n) \geq n - c$.*

(2) *For every set A, let $p_n^A = \sum_{|\sigma|=n \,\wedge\, \sigma \leq A} 2^{-K(\sigma)}$, where $\sigma \leq A$ means that σ is either an initial segment of A or to the left of A. Then*

$$A \neq 0 \implies (\exists c)(\forall n)(K(p_n^A) \geq n - K(A \upharpoonright n) - c).$$

Proof. (1): Let f be a partial computable function that maps every pair of the form (p_n, n) to a string σ_n that has maximal complexity among all strings of length n. Then, for some constants c, c_1, c_2, and c_3,

$$\begin{aligned} K(p_n) &\geq K(p_n, n) - K(n) - c_1 \\ &\geq K(\sigma_n) - K(n) - c_2 \\ &\geq n + K(n) - K(n) - c_3 \\ &\geq n - c. \end{aligned}$$

(2): Since $A \neq 0$, there must be a set $B \leq A$ and a constant d such that $(\forall n)(K(B \upharpoonright n) \geq n - d)$. Let f be a partial computable function such that $f(p_n^A, A \upharpoonright n) = \sigma_n$ for all n, where σ_n is the leftmost string of length n having the property $K(\sigma_n) \geq n - d$. Then

$$\begin{aligned} K(p_n^A) &\geq K(p_n^A, A \upharpoonright n) - K(A \upharpoonright n) - c_1 \\ &\geq K(\sigma_n) - K(A \upharpoonright n) - c_2 \\ &\geq n - d - K(A \upharpoonright n) - c_3 \\ &\geq n - K(A \upharpoonright n) - c, \end{aligned}$$

for some constants c, c_1, c_2, and c_3. $\dashv$

Theorem 4.8.

(1) *$\mathcal{E} = \sum_n 2^{-n} \sum_{|\sigma|=n} 2^{-K(\sigma)}$ is a left-c.e., wtt-complete real of effective Hausdorff dimension $1/2$.*

(2) *Let $\mathcal{E}(A) = \sum_n 2^{-n} \sum_{|\sigma|=n \,\wedge\, \sigma \preceq A} 2^{-K(\sigma)}$. Then for every A, $\mathcal{E}(A)$ is an A-left-c.e. real that is not random relative to A. If $A \neq 0$ and A is of effective packing dimension 0, then $\mathcal{E}(A)$ is of effective Hausdorff dimension at least $1/2$.*

Proof. (1): It is obvious that $\mathcal{E}$ is left-c.e. Note that, for every n and $\sigma \in 2^n$, there is a constant c such that $K(\sigma) \leq n + 2\log n + c$. W.ithout loss of generality, we may even assume that, for every stage s, $K_s(\sigma) \leq n + 2\log n + c$.

For every stage s, let $\mathcal{E}_s = \sum_{n \leq s} 2^{-n} \sum_{|\sigma|=n} 2^{-K_s(\sigma)}$. Then, for every $\varepsilon > 0$, there is a constant c_ε such that for every s with $\mathcal{E}\restriction n = \mathcal{E}_s\restriction n$ we have that $p_{\lceil n/2 - \varepsilon n\rceil - c_\varepsilon} = p_{\lceil n/2 - \varepsilon n\rceil - c_\varepsilon, s}$ and $p_{n,s} = \sum_{|\sigma|=n} 2^{-K_s(\sigma)}$ (where p_n is as in Lemma 4.7). In other words, there is a computable function f which maps $\mathcal{E}\restriction n$ to $p_{\lceil n/2 - \varepsilon n\rceil - c_\varepsilon}$. Then, by Lemma 4.7, there is a constant d with

$$K(\mathcal{E}\restriction n) \geq n/2 - \varepsilon n - d,$$

and thus $\mathcal{E}$ is of effective Hausdorff dimension at least $1/2$.

For every m, by the assumption above, the approximation to $\sum_{|\sigma|=m} 2^{-K(\sigma)}$ changes at most $2^m(m + 2\log m + c) \leq 2^{m+2\log m+c}$ times; that is,

$$\left|\left\{ s : \sum_{|\sigma|=m} 2^{-K_s(\sigma)} \neq \sum_{|\sigma|=m} 2^{-K_{s+1}(\sigma)} \right\}\right| \leq 2^{m+2\log m+c}.$$

For arbitrary r, write

$$\begin{aligned} C(r) &= \textstyle\sum_{n \geq m > n/2 + 2\log n + c} \sum_{|\sigma|=m} 2^{-m-K_r(\sigma)} \\ D(r) &= \textstyle\sum_{m \leq n/2 + 2\log n + c} \sum_{|\sigma|=m} 2^{-m-K_r(\sigma)}, \end{aligned}$$

and write $C(\omega)$ and $D(\omega)$ for the same expressions where "K_r" has been replaced with "K". Then, using the arguments above, for every $\varepsilon > 0$ and large enough n, the approximation to $D(\omega)$ changes at most

$$\sum_{m \leq n/2 + 2\log n + c} 2^{m+2\log m+c} \leq 2^{(1+\varepsilon)(n/2 + 2\log n + c)}$$

many times. Note that

$$C(\omega) \leq \sum_{n \geq m > n/2} \sum_{|\sigma|=m} 2^{-m-K(\sigma)} < 2^{-\lceil n/2\rceil - 1},$$

and thus $C(\omega)\restriction n/2 = 0^{\lceil n/2\rceil}$. Then $|\{s : C(s)\restriction n \neq C(s+1)\restriction n\}| \leq 2^{n/2}$.

By using that $\sum_{m>n} \sum_{|\sigma|=m} 2^{-m-K(\sigma)} < \sum_{m>n} 2^{-m} \leq 2^{-n}$ we obtain for some $j \leq 3$ that

$$\mathcal{E}\restriction n = j2^{-n+1} + \sum_{i<n \wedge (C(\omega))(i)=1} 2^{-i} + \sum_{i<n \wedge (D(\omega))(i)=1} 2^{-i}.$$

Thus, $\mathcal{E}\restriction n$ can be effectively approximated by letting, for each stage s and every $j \leq 3$,

$$\mathcal{E}_s^j\restriction n = j2^{-n+1} + \sum_{i<n \wedge (C(s))(i)=1} 2^{-i} + \sum_{i<n \wedge (D(s))(i)=1} 2^{-i}.$$

By the discussion above,

$$|\{s : \mathcal{E}^j_s \neq \mathcal{E}^j_{s+1}\}| \leq 2^{(1+\varepsilon)(n/2+2\log n+1)} + 2^{n/2} \leq 2^{(1+\varepsilon)(n/2+2\log n+3)}$$

for each $j \leq 3$ and $\mathcal{E}\restriction n = \lim_s \mathcal{E}^j_s$ for some $j \leq 3$. To know $\mathcal{E}\restriction n$, it is therefore enough to know the correct j and the last time the above approximation changes. This means there are at most

$$4(2^{(1+\varepsilon)(n/2+2\log n+1)} + 2^{n/2}) \leq 2^{(1+\varepsilon)(n/2+2\log n+4)}$$

possible values for $\mathcal{E}\restriction n$ and that, for some constant d,

$$K(\mathcal{E}\restriction n) \leq (1+\varepsilon)(n/2 + 2\log n + 4) + K(n) + d.$$

Hence, the effective Hausdorff dimension of $\mathcal{E}$ is no more than $1/2$.

Finally, since $\mathcal{E}$ is left-c.e. and of d.n.c. degree, we can apply Arslanov's [1] completeness criterion (see Soare [22, Theorem 5.1 and Exercise 5.8]) to a c.e. set which is wtt-equivalent to $\mathcal{E}$; this way, we obtain that $\mathcal{E}$ is wtt-complete.

(2): First, $\mathcal{E}(A)$ is clearly A-left-c.e. Second, there is a constant c such that for every n,

$$2^{-K(n)-c} \leq \sum_{|\sigma|=n \,\wedge\, \sigma\leq A} 2^{-K(\sigma)} \leq \sum_{|\sigma|=n} 2^{-K(\sigma)} \leq 2^{-K(n)+c}.$$

So, $G(n) = -\log 2^{-n}\sum_{|\sigma|=n \,\wedge\, \sigma\leq A} 2^{-K(\sigma)}$ is not an A-Solovay function, and $\mathcal{E}(A)$ cannot be A-random. As in the proof of (1), for every n and every $\varepsilon > 0$, there is a constant c_ε such that if for every s,

$$\mathcal{E}(A) - \sum_{\tau\leq A\restriction n}\sum_{m\leq n} 2^{-K_s(\tau\restriction m)-m} \leq 2^{-n},$$

then $p^A_{n/2-\varepsilon n-c_\varepsilon} = p^A_{n/2-\varepsilon n-c_\varepsilon,s}$, where p^A_n is as in Lemma 4.7. Then, by Lemma 4.7, there is a constant d such that

$$\begin{aligned} K(\mathcal{E}(A)\restriction n) &\geq K((\mathcal{E}(A)\restriction n, A\restriction n)) - K(A\restriction n) \\ &\geq n/2 - \varepsilon n - K(A\restriction n) - K(A\restriction (n/2 - \varepsilon - c_\varepsilon)) - d. \end{aligned}$$

Consequently, $\mathcal{E}(A)$ has effective Hausdorff dimension at least $1/2$ if A has effective packing dimension 0. ⊣

THEOREM 4.9.

(1) $\mathcal{O} = \int_0^1 \widehat{\Omega}(X)dX = \lim_n 2^{-n}\sum_{|\sigma|=n}\sum_{m\leq n} 2^{-K(\sigma\restriction m)}$ *is a left-c.e., wtt-complete real of effective Hausdorff dimension* $1/2$.

(2) *Let* $\mathcal{O}(A) = \int_0^A \widehat{\Omega}(X)dX = \sum_n 2^{-n}\sum_{|\sigma|=n \,\wedge\, \sigma\leq A}\sum_{m\leq n} 2^{-K(\sigma\restriction m)}$. *Then* $\mathcal{O}(A)\oplus\emptyset' \equiv_T A\oplus\emptyset'$. *If* $A \neq 0$ *has effective packing dimension* 0, *then* $\mathcal{O}(A)$ *has effective Hausdorff dimension at least* $1/2$.

PROOF. (1): Note that for every n,

$$\begin{aligned} &2^{-n}\textstyle\sum_{|\sigma|=n}\sum_{m\leq n} 2^{-K(\sigma\restriction m)} \\ = \;&2^{-n}\textstyle\sum_{m\leq n} 2^{n-m}\sum_{|\sigma|=m} 2^{-K(\sigma)} \\ = \;&\textstyle\sum_{m\leq n} 2^{-m}\sum_{|\sigma|=m} 2^{-K(m)}. \end{aligned}$$

Therefore, $\mathcal{O} = \mathcal{E}$, and the claim follows from Theorem 4.8(1).

(2): That $\mathcal{O}(A) \leq_T A \oplus \emptyset'$ is immediate. Note that, when restricted to sets that are both infinite and coinfinite, $A \mapsto \mathcal{O}(A)$ is a $\emptyset'$-computable, increasing, and therefore injective function; thus, $A \leq_T \mathcal{O}(A) \oplus \emptyset'$.

The second part of the claim can be shown with a method similar to that used in the proof of Theorem 4.8(2). ⊣

It is obvious that both $A \mapsto \mathcal{E}(A)$ and $A \mapsto \mathcal{O}(A)$ satisfy the premises of the following fact; thus, we obtain the corollary below.

FACT 4.10. *Suppose that f is a continuous function from Cantor space to $\mathbb{R}$ such that for every $x \neq y$, if $(x, y) = \emptyset$, then $f(x) = f(y)$. Then the range of f must be an interval.*

COROLLARY 4.11. *The ranges of $A \mapsto \mathcal{E}(A)$ and of $A \mapsto \mathcal{O}(A)$ are intervals.*

§5. Algorithmic aspects of $\widehat{\Omega}$-operators.

In this section, we investigate the algorithmic properties of $\widehat{\Omega}$, some of which will be dependent on the machine used to define $\widehat{\Omega}$.

5.1. Machine-independent results.

PROPOSITION 5.1. *A real $X \in 2^\omega$ is weakly low along itself if and only if $\widehat{\Omega}(X)$ is X-random.*

PROOF. By Theorem 2.8, a function f is a Solovay function relative to X if and only if $\sum_n 2^{-f(n)}$ is X-random. For some constant c and all X, we have that $K(X \upharpoonright n) \geq K^X(n) - c$; therefore, the function $n \mapsto K(X \upharpoonright n)$ is right-c.e. relative to X and is an upper bound of K^X up to an additive constant. So, it suffices to observe that this upper bound is infinitely often tight up to an additive constant if and only if X is weakly low along itself. ⊣

A real X is *d.c.e.* if it is a difference of two left-c.e. reals. An oracle A is called *low for d.c.e. reals* if every d.c.e. real relative to A is a d.c.e. real.

THEOREM 5.2 (Miller [13]). *A is K-trivial if and only if A is low for d.c.e. reals.*

PROPOSITION 5.3. *If X is K-trivial, then $\widehat{\Omega}(X)$ is left-c.e.*

PROOF. If X is low for K, then, by Theorem 5.2, X is low for d.c.e. reals. Now $\widehat{\Omega}(X)$ is left-c.e. relative to X, hence it is d.c.e. relative to X, hence it is d.c.e.

On the other hand, by Proposition 5.1, $\widehat{\Omega}(X)$ is X-random. So, by a result of Rettinger and Zheng [21, Theorem 2.5], $\widehat{\Omega}(X)$ is left-c.e. or right-c.e. Since $\widehat{\Omega}(X)$ is left-c.e. relative to X, if it were also right-c.e., then it would be computable from X, which contradicts the fact that $\widehat{\Omega}(X)$ is X-random. Hence, $\widehat{\Omega}(X)$ must be left-c.e. ⊣

PROPOSITION 5.4. *If X is left-c.e., then $\widehat{\Omega}(X)$ is d.c.e.*

PROOF. Fix an approximation $X_0, X_1, \dots$ to X that witnesses that X is left-c.e. Consider the approximation $\widehat{\Omega}(X)[s]$ to $\widehat{\Omega}(X)$ given by

$$\widehat{\Omega}(X)[s] = \sum_{j<s} 2^{-K_s(X_s \upharpoonright j)}.$$

In this approximation, the values $\widehat{\Omega}(X)[s]$ are fluctuating up and down over the stages s, but we will argue that the total sum of all increases and the total sum of all decreases can each be bounded by 1; then $\widehat{\Omega}(X)$ is d.c.e. To see this, we look at these two sums separately.

For every given string σ let I_σ be the set of all numbers s such that $|\sigma| < s$ and σ is an initial segment of X_s. Since X is left-c.e., each set I_σ is an interval of natural numbers that could possibly be empty, finite, or cofinite. If I_σ is empty, then σ never contributes to any increases or any decreases of the values $\widehat{\Omega}(X)[s]$.

If I_σ is cofinite, then σ is an initial segment of X and whenever the approximation to the true value of $K(\sigma)$ improves, the value of $\widehat{\Omega}(X)[s]$ increases. But the total sum of such increases is clearly bounded by $2^{-K(\sigma)}$. The same argument also allows bounding the increases caused by σ's with finite I_σ at stages $s \in I_\sigma$. In total, the total sum of all increases over all $\sigma \in 2^{<\omega}$ is bounded by 1.

It remains to look at the decreases of the values $\widehat{\Omega}(X)[s]$ over stages s. The only σ's that contribute any decreases are the ones where I_σ is finite but nonempty; a decrease occurs when the positive contribution at stages $s \in I_\sigma$ of one such σ falls away at stages $s > \max I_\sigma$. But for every such σ its positive contribution was at most $2^{-K(\sigma)}$; therefore, the sum of all decreases over all $\sigma \in 2^{<\omega}$ is again bounded by 1. ⊣

PROPOSITION 5.5. *Let X be Δ^0_2 and let $\widehat{\Omega}(X)$ be X-random. Then X is K-trivial.*

PROOF. By the assumption on X and Theorem 2.8, $K(X\restriction n)$ is an X-Solovay function. Hence, X is weakly low for K. Since $X \leq_T \Omega$, we have that X is K-trivial by a result of Miller [17]. ⊣

LEMMA 5.6. $\widehat{\Omega}(X) \oplus X \geq_T \emptyset'$ *for every X.*

PROOF. We build a prefix-free machine V by enumerating a bounded request set as follows: For every n, if $n \in \emptyset'_{s_n+1} \setminus \emptyset'_{s_n}$ for some s_n, then pick the least $m_n \leq 4^n$ such that

$$\sum_{\sigma\in 2^{m_n}} 2^{-K_{U,s_n+1}(\sigma)} \leq 4^{-n}$$

and enumerate $(K_{U,s_n+1}(\sigma) - n, \sigma)$ for every $\sigma \in 2^{m_n}$. Note that an m_n as above must exist since $\sum_{|\sigma|\leq 4^n} 2^{-K_{U,s_n+1}(\sigma)} < 1$.

Then a prefix-free machine V as required exists, since

$$\begin{aligned} &\textstyle\sum_n \sum_{\sigma\in 2^{m_n}} 2^{-K_{U,s_n+1}(\sigma)+n} \\ &= \textstyle\sum_n 2^n \sum_{\sigma\in 2^{m_n}} 2^{-K_{U,s_n+1}(\sigma)} \\ &\leq \textstyle\sum_n 2^n \cdot 4^{-n} \leq \sum_n 2^{-n} \leq 1. \end{aligned}$$

We claim that for almost every n, if $\widehat{\Omega}_U(X) - \sum_{m\leq s} 2^{K_{U,s}(X\restriction m)} < 2^{-8^n}$ at stage $s > 4^n$, then $n \in \emptyset'$ if and only if $n \in \emptyset'_{s+1}$. Assume otherwise and fix n and s such that $\widehat{\Omega}_U(X) - \sum_{m\leq s} 2^{K_{U,s}(X\restriction m)} < 2^{-8^n}$ but $n \in \emptyset' \setminus \emptyset'_{s+1}$. Then $n \in \emptyset'_{s_n+1} \setminus \emptyset'_{s_n}$ for some $s_n \geq s+1$. So

$$K_V(X\restriction m_n) \leq K_{U,s_{n+1}}(X\restriction m_n) - n. \tag{†}$$

But we have $K_U(X\restriction m_n) \leq 2m_n \leq 2\cdot 4^n < 8^n$ when m_n is large enough, and we also have $\widehat{\Omega}_U(X) - \sum_{m\leq s} 2^{K_{U,s}(X\restriction m)} < 2^{-8^n}$. This implies

$$K_{U,s_n+1}(X\restriction m_n) = K_U(X\restriction m_n).$$

Therefore, if there are infinitely many n as in (†), then V is a prefix-free machine with $\overline{\lim}_n K_U(n) - K_V(n) = \infty$, which is a contradiction. ⊣

Definition 5.7. We let

$$\widehat{\Omega}(2^\omega) = \{X : (\exists A)(\widehat{\Omega}(A) = X)\}$$

denote the image of $\widehat{\Omega}$.

Proposition 5.8. *Both* $\min \widehat{\Omega}(2^\omega)$ *and* $\max \widehat{\Omega}(2^\omega)$ *are left-c.e. In addition,* $\max \widehat{\Omega}(2^\omega)$ *is Martin-Löf random.*

Proof. Let $X = \max \widehat{\Omega}(2^\omega)$. For all s, define

$$X_s = \sup_{\tau \in 2^s} \sum_{n \leq s} 2^{-K_s(\tau \upharpoonright n)}.$$

Then $(X_s)_s$ is a nondecreasing sequence such that $X = \lim_s X_s$. Now for every n, search for a stage s such that there is a real A_n such that

$$\sum_{l<s} 2^{-K_s(A_n \upharpoonright l)} \in [X - 2^{-n}, X).$$

Then for every such A_n, we have that $K(A_n \upharpoonright s) > n$. It is clear that there is a partial computable function f mapping $X \upharpoonright n+1$ to such an $A_n \upharpoonright s$. Then f witnesses that X is Martin-Löf random.

Let $Y = \min \widehat{\Omega}(2^\omega)$. Define $Y_s = \inf_{\tau \in 2^s} \sum_{n \leq s} 2^{-K_s(\tau \upharpoonright n)}$. Then $(Y_s)_s$ is a nondecreasing sequence such that $Y = \lim_s Y_s$. ⊣

It is natural to ask which reals can be preimages of $\widehat{\Omega}$'s maximal value. It is easy to see that for the right choice of optimal machine 0^ω can be such a preimage.

Proposition 5.9. *If A is 2-random, then $\widehat{\Omega}(A)$ is not a left-c.e. random real. Thus,* $\widehat{\Omega}(A) \neq \max \widehat{\Omega}(2^\omega)$.

Proof. Assume otherwise, then A is $\widehat{\Omega}(A)$-random, and consequently $\widehat{\Omega}(A)$ is A-random. Then by Theorem 2.8, $K(A \upharpoonright n)$ is an A-Solovay function, which implies that for infinitely many n we have $K(A \upharpoonright n) \leq K^A(n) + c$ for some constant c. This contradicts A's being random. ⊣

Proposition 5.10. $\widehat{\Omega}(2^\omega)$ *is a perfect set, and in particular uncountable.*

Proof. It is clear that $\widehat{\Omega}(2^\omega)$ is a closed set by the compactness of Cantor space. To see that it is a perfect set, it is sufficient to show that it has no isolated points. For every number n, let m_n be the least number such that $\sum_{|\tau| \geq m_n} 2^{-K(\tau)} < 2^{-n}$.

Case 1. $\widehat{\Omega}(A)$ is left-c.e. and random. Then let $R \succ A \upharpoonright m_n$ be a 2-random real. Then $\widehat{\Omega}(R) \in (\widehat{\Omega}(A) - 2^{-n}, \widehat{\Omega}(A) + 2^{-n})$ and, by Proposition 5.9, $\widehat{\Omega}(R)$ is not a left-c.e. random real. Then $\widehat{\Omega}(R) \neq \widehat{\Omega}(A)$.

Case 2. Otherwise. Then let $R = (A \upharpoonright m_n)0^\omega$. Then $\widehat{\Omega}(R)$ is a left-c.e. random real. Thus, $\widehat{\Omega}(R) \in (\widehat{\Omega}(A) - 2^{-n}, \widehat{\Omega}(A) + 2^{-n})$ but $\widehat{\Omega}(R) \neq \widehat{\Omega}(A)$.

In summary, there is a set R with $\widehat{\Omega}(R) \in (\widehat{\Omega}(A) - 2^{-n}, \widehat{\Omega}(A) + 2^{-n})$ but $\widehat{\Omega}(R) \neq \widehat{\Omega}(A)$. Since n was arbitrary, $\widehat{\Omega}(A)$ is not isolated. ⊣

For every set X, write $\widehat{\Omega}^{-1}(X)$ for $\{A : X = \widehat{\Omega}(A)\}$.

Lemma 5.11. *For every set X, $\widehat{\Omega}^{-1}(X)$ is $\Pi^0_1(X \oplus \emptyset')$.*

PROOF. We construct a binary tree T that is computable in $X \oplus \emptyset'$ as follows: Let $T_0 = \{\lambda\}$. At stage $s + 1 > 0$, let t_s be least such that

$$\sum_{|v| \geq t_s} 2^{-K(v)} < 2^{-s-1}.$$

Then for every σ, put σ into T_{s+1} if there is some $\tau \in T_s$ such that $\sigma \succ \tau$, $|\sigma| = t_s$, and $\sum_{l \leq t_s} 2^{-K(\sigma \upharpoonright l)} \in [X - 2^{-s}, X)$. Close T_{s+1} under initial segments.

Let $T = \bigcup_s T_s$. It is obvious that $\widehat{\Omega}(A) = X$ if and only if $A \in [T]$. ⊣

COROLLARY 5.12. *For every set X, $\widehat{\Omega}^{-1}(X)$ is meager.*

PROOF. Otherwise, by Lemma 5.11, $\widehat{\Omega}^{-1}(X)$ must contain an interval. Then by Proposition 5.8, X must be left-c.e. and random and there must be a 2-random A such that $\widehat{\Omega}(A) = X$, contradicting Proposition 5.9. ⊣

PROPOSITION 5.13. *For every left-c.e. real X, $\widehat{\Omega}^{-1}(X)$ has positive measure if and only if there is a 2-random set A such that $\widehat{\Omega}(A) = X$.*

PROOF. The left to right direction is obvious. For the other direction, if X is left-c.e., then by Fact 5.11, the set $\widehat{\Omega}^{-1}(X)$ is $\Pi^0_1(\widehat{\Omega})$. Since there is a 2-random set A such that $\widehat{\Omega}(A) = X$, the set $\widehat{\Omega}^{-1}(X)$ is not null. ⊣

It is unknown whether there is a real X such that $\widehat{\Omega}^{-1}(X)$ has positive measure, but the following result excludes many possible candidates for such an X.

PROPOSITION 5.14. *If $\widehat{\Omega}^{-1}(X)$ has positive measure, then X is left-c.e., Turing complete, and nonrandom.*

PROOF. We first prove that X is left-c.e. By the Lebesgue density theorem, we may assume, without loss of generality, that $\mu(\widehat{\Omega}^{-1}(X)) > 3/4$. Let $X_0 = 0$. For every $s > 0$, let $X_s \geq X_{s-1}$ be a rational number such that there is a stage $t > s$ such that

$$\mu(\{Y \colon \sum_{l \leq t} 2^{-K_t(Y \upharpoonright l)} \in [X_s, X_s + 2^{-s})\}) > 3/4,$$

if such a number exists. By induction over s, X_s exists for every s. Moreover, for any $\varepsilon > 0$, there is some s_ε so that

$$\mu(\{Y \colon \sum_{l \leq s_\varepsilon} 2^{-K_{s_\varepsilon}(Y \upharpoonright l)} \in [X - \varepsilon, X]\}) > 3/4.$$

Then it is clear that $X - \varepsilon \leq X_{s_\varepsilon} + 2^{-s_\varepsilon}$. So, $X_{s_\varepsilon} \leq X \leq X_{s_\varepsilon} + \varepsilon + 2^{-s_\varepsilon}$. Hence, $\lim_s X_s = X$.

It remains to show that X is Turing complete. By Lemma 5.6, for every set $A \in \widehat{\Omega}^{-1}(X)$ we have that $X \oplus A$ is Turing complete. But as $\widehat{\Omega}^{-1}(X)$ has positive measure, a set of reals of positive measure is cupped above $\emptyset'$ by X. This implies that X is Turing complete.

That X is nonrandom follows from Proposition 5.9. ⊣

LEMMA 5.15. *For every real X,*

(1) $\{A \colon \widehat{\Omega}(A) \leq X\}$ *is* $\Pi^0_1(X)$;

(2) *If* $X = \min \widehat{\Omega}(2^\omega)$, *then* $\Omega^{-1}(X)$ *is* $\Pi^0_1(X)$.

(3) *If* $\mathcal{P}$ *is* Π^0_1 *and* $X = \min \widehat{\Omega}(\mathcal{P})$, *then* $\Omega^{-1}(X) \cap \mathcal{P}$ *is* $\Pi^0_1(X)$.

Proof. The first statement follows from the fact that $\widehat{\Omega}(A) \leq X$ if and only if $(\forall s)(\sum_{l \leq s} 2^{K_s(A \upharpoonright l)} \leq X)$. The second and third statements are immediate consequences. ⊣

Corollary 5.16.

(1) *If $\mathcal{P}$ is a nonempty Π^0_1 set, then* $\min \widehat{\Omega}(\mathcal{P})$ *is left-c.e.*
(2) *The sequence* $\min \widehat{\Omega}(2^\omega)$ *is Turing complete.*
(3) *For every nonempty Π^0_1 set $\mathcal{P}$,* $\min \widehat{\Omega}(\mathcal{P})$ *is Turing complete.*

Proof. (1): Let T be a computable tree such that $[T] = \mathcal{P}$. Define

$$X_s = \min \Big\{ \sum_{l \leq s} 2^{-K_s(\sigma \upharpoonright l)} \colon |\sigma| = s \ \wedge\ \sigma \in T \Big\}.$$

Then $(X_s)_s$ is nondecreasing and computable and $X = \lim_s X_s$ as required.
(2): Let $X = \min \widehat{\Omega}(2^\omega)$. By Lemma 5.15, $\widehat{\Omega}^{-1}(X) = \{A \colon \widehat{\Omega}(A) = X\}$ is $\Pi^0_1(X)$. Then there are $A, B \in \widehat{\Omega}^{-1}(X)$ such that all sets that are computable in both $A \oplus X$ and $B \oplus X$ are computable in X; this can be seen, for example, by the Hyperimmune-Free and Low Basis Theorems relative to X. By Lemma 5.6, $A \oplus X \geq_T \emptyset'$ and $B \oplus X \geq_T \emptyset'$, and thus $X \geq_T \emptyset'$.
(3): This is left to the reader. ⊣

5.2. Machine-dependent results. We first study questions related to effective Hausdorff dimension.

Theorem 5.17. *For every $\varepsilon > 0$, there is a universal machine V such that for all $X \in 2^\omega$ having effective Hausdorff dimension greater than ε, we have $X \leq_{tt} \widehat{\Omega}_V(X)$. Moreover, $\widehat{\Omega}_V(X)$ has effective Hausdorff dimension* 0.

Proof. Fix $\varepsilon > 0$ and a constant c_0 such that

$$(\forall \sigma)(|K(\sigma) - K(\sigma \upharpoonright (|\sigma| - 1))| < c_0).$$

We also fix numbers a and $\delta > c_0$ such that $\varepsilon > 2^{-a}$ and $2^\delta/\delta > 2^{a+4}$.

Let U be a universal machine; define another machine V_0 as follows: If

$$(U(p) = \sigma) \ \wedge\ (\exists k)(\exists n)(\exists m < 2\delta)(|p| = 2^n + 2\delta k + m < 2^{n+1} \wedge |\sigma| > n)$$

then let $V_0(1^{2\delta + \delta\sigma(n) - m - 1} 0 p) = \sigma$. Thus, $V_0(q)\downarrow$ only if there exists some n such that $|q| \in [2^n + 2\delta, 2^{n+1} + 3\delta)$ and δ divides $|q| - 2^n$. Then clearly $K_{V_0}(\sigma) \leq K_U(\sigma) + 3\delta$ for every σ with $K(\sigma) \leq 2^{|\sigma|}$. Without loss of generality, we may assume that the inequality holds for every σ. Fix d such that for every σ, $K_{V_0}(\sigma) \leq K_U(\sigma) + d$. Define

$$V(p) = \begin{cases} V_0(q), & \text{if there is a } q \text{ such that } p = 01q, \\ U(q), & \text{if there is a } q \text{ such that } p = 0^{d+1}1q, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then V is a universal machine and, for every σ, we have $K_V(\sigma) = K_{V_0}(\sigma) + 2$. To save the notations, we simply assume that $K_V(\sigma) = K_{V_0}(\sigma)$ for every σ.

Fix a real X having effective Hausdorff dimension greater than $\varepsilon > 2^{-a}$ and assume without loss of generality that a is large enough so that, for all l,

$$\min\{K_V(X \upharpoonright l)), K_U(X \upharpoonright l)\} > 2^{-a} \cdot l.$$

Then for every n, we have that if

$$B_n := \{l\colon K_V(X\restriction l) \in [2^n, 2^{n+1})\}, \text{ then } |B_n| \leq 2^{n+a+1}. \tag{†}$$

Claim 1. *Fix n and let k be such that $2^n + (2k+3)\delta < 2^{n+1}$. Then*

(1) *if $X(n) = 0$ there is some l such that $K_V(X\restriction l) = 2^n + (2k+2)\delta$;*
(2) *if $X(n) = 1$ there is some l such that $K_V(X\restriction l) = 2^n + (2k+1)\delta$.*

Proof of the Claim. Suppose that $X(n) = 0$ but that there is no l such that $K_V(X\restriction l) = 2^n + 2k\delta$. Then by construction, there is no l such that $K(X\restriction l) \in [2^n + 2\delta k, 2^n + (2k+2)\delta)$. Let l_0 be the largest number such that $K(X\restriction l_0) < 2^n + 2\delta k$; then $K(X\restriction l_0 + 1) \geq 2^n + (2k+2)\delta$. In other words, $c_0 > |K(X\restriction l_0+1) - K(X\restriction l_0)| > 2\delta$, a contradiction to the choice of δ.

The proof for the case $X(n) = 1$ is analogous. ◊

Now let n be a number. Set

$$A_{0,n} = \left\{k \geq 0\colon \begin{matrix} (2^n + (2k+1)\delta \leq 2^{n+1}) \ \wedge \\ (\exists i \in (2^n + 2k\delta, 2^n + (2k+1)\delta])(\widehat{\Omega}_V(X)(i) = 1) \end{matrix}\right\}$$

and

$$A_{1,n} = \left\{k > 0\colon \begin{matrix} (2^n + 2k\delta \leq 2^{n+1}) \ \wedge \\ (\exists i \in (2^n + (2k-1)\delta, 2^n + 2k\delta])(\widehat{\Omega}_V(X)(i) = 1) \end{matrix}\right\}.$$

Case 1. $X(n) = 1$. Then for every l with $K_V(X\restriction l) \in [2^n, 2^{n+1})$, there must be some k such that $K_V(X\restriction l) = 2^n + (2k+1)\delta$. By (†), we have that

$$\begin{aligned} &\textstyle\sum_{l\colon K_V(X\restriction l) \geq 2^{n+1}} 2^{-K_V(X\restriction l)} \\ &= \textstyle\sum_{m \geq n+1} \sum_{l\colon K_V(X\restriction l) \in [2^m, 2^{m+1})} 2^{-K_V(X\restriction l)} \\ &\leq \textstyle\sum_{m \geq n+1} 2^{-2^m + m + a + 1} \\ &< 2^{-2^{n+1} + n + a + 3} \\ &< 2^{-2^{n+1} + \delta n}. \end{aligned} \tag{‡}$$

For every k with $(2k+1)\delta < 2^n$, let

$$B_{n,k} = \{l\colon K_V(X\restriction l) = 2^n + (2k+1)\delta\}.$$

Then $\{B_{n,k}\}_{(2k+1)\delta < 2^n}$ is a collection of mutually disjoint sets such that $\bigcup_{(2k+1)\delta<2^n} B_{n,k} = B_n$. Enumerate $A_{1,n} \cap \{k\colon (2k+3)\delta + n\delta < 2^n\}$ as $\{k_1 < k_2 < \cdots < k_{d'}\}$ for some $d' > 0$. Note that by (‡) and the assumption that $X(n) = 1$, the set $\{l\colon K_V(X\restriction l) \in [2^n + (2k_{d'}+1)\delta, 2^{n+1})\}$ must contain at least 2^δ many elements; and this lower bound can only be attained when $|B_{n,k_{d'}}| \geq 2^\delta$ and $B_{n,k} = \emptyset$ for every $k > k_{d'}$. So, $|\bigcup_{k \geq k_{d'}} B_{n,k}| \geq 2^\delta$. This observation can be generalized as follows.

Claim 2. *For every $i \in [1, d']$ and every set*

$$C_i \subseteq \{\sigma\colon (\exists k)(k_i \leq k \wedge (2k+1)\delta < 2^n \wedge |\sigma| = 2^n + (2k+1)\delta)\}$$

with the property that for every $i' \in [i, d']$,

$$\sum_{\substack{\sigma \in C_i \ \wedge \\ |\sigma| \geq 2^n + (2k_{i'}+1)\delta}} 2^{-|\sigma|} \geq \sum_{\substack{(\exists j \in [i', d'])(\widehat{\Omega}_V(X)(k) = 1 \ \wedge \\ k \in (2^n + (2k_j - 1)\delta, 2^n + 2k_j\delta])}} 2^{-k},$$

we have that $|C_i| \geq 2^\delta(d' - i + 1)$.

Proof of the Claim. We proceed by reverse induction. By the assumption that $X(n) = 1$, the fact that $(2k_{d'} + 3)\delta + n\delta < 2^n$, and by (‡), the claim is clear for $i = d'$.

Now suppose that it holds for $i + 1 \leq d'$. For each $j \in [i + 1, d']$, there must be a set $D_j \subseteq C_i \cap \{\sigma\colon |\sigma| \geq 2^n + (2k_j + 1)\delta\}$ with

$$\sum_{\sigma \in D_j} 2^{-|\sigma|} \in [e_j, e_j + 2^{-2^n - (2k_j+1)\delta}]$$

where $e_j = \sum_{(\exists j' \in [j,d'])(\widehat{\Omega}_V(X)(k)=1 \,\wedge\, k \in (2^n + (2k_{j'}-1)\delta, 2^n + 2k_{j'}\delta])} 2^{-k}$.

Let $\widetilde{C}_{i+1} = \bigcup_{j \in [i,d'-1]} D_{j+1}$. Then, by the induction hypothesis, we have that $|\widetilde{C}_{i+1}| \geq 2^\delta(d' - i)$ and that

$$\sum_{\sigma \in \widetilde{C}_{i+1}} 2^{-\sigma} \leq \sum_{j \in [i+1,d']} (e_j + 2^{-2^n - (2k_j+1)\delta}) < 2^{-2^n - (2k_{i+1}+1)\delta + 1} + \sum_{j \in [i+1,d']} e_j.$$

So, if $|C_i| < 2^\delta(d' - i + 1)$, then

$$|C_i \setminus \widetilde{C}_{i+1}| < 2^\delta(d' - i + 1) - 2^\delta(d' - i) = 2^\delta;$$

that is, $|C_i \setminus \widetilde{C}_{i+1}| \leq 2^\delta - 1$. Thus,

$$\begin{aligned}
&\textstyle\sum_{\sigma \in C_i} 2^{-|\sigma|} \\
&= \textstyle\sum_{\sigma \in C_i \setminus \widetilde{C}_{i+1}} 2^{-|\sigma|} + \sum_{\sigma \in \widetilde{C}_{i+1}} 2^{-|\sigma|} \\
&\leq \textstyle\sum_{\sigma \in C_i \setminus \widetilde{C}_{i+1}} 2^{-2^n - (2k_i+1)\delta} + \sum_{\sigma \in \widetilde{C}_{i+1}} 2^{-|\sigma|} \\
&\leq (2^\delta - 1) 2^{-2^n - (2k_i+1)\delta} + 2^{-2^n - (2k_{i+1}+1)\delta + 1} + \textstyle\sum_{j \in [i+1,d']} e_j \\
&< 2^{-2^n - 2k_i\delta} + \textstyle\sum_{j \in [i+1,d']} e_j.
\end{aligned}$$

Since $k_i \in A_{1,n}$, we have that

$$\sum_{\sigma \in C_i} 2^{-|\sigma|} < 2^{-2^n - 2k_i\delta} + \sum_{j \in [i+1,d']} e_j \leq \sum_{\substack{(\exists i' \in [i,d'])(\widehat{\Omega}_V(X)(k)=1 \,\wedge \\ k \in (2^n + (2k_{i'}-1)\delta, 2^n + 2k_{i'}\delta])}} 2^{-k},$$

which is in contradiction with the assumptions on C_i. ◊

For each l, let σ_l be the shortest binary string such that $V(\sigma_l) = X\restriction l$. Now for each $i \in [1, d']$, let $C_i = \{\sigma_l\colon |\sigma_l| \in [2^n + (2k_i + 1)\delta, 2^{n+1})\}$. Then $|C_i| = |\bigcup_{k \geq k_i \wedge (2k+1)\delta < 2^n} B_{n,k}|$. By (‡), it is clear that C_i satisfies the premises of Claim 2 and therefore $|C_i| \geq 2^\delta(d' - i + 1)$. Combining this with (†) we obtain

$$2^{n+a+1} \geq |B_n| \geq |\bigcup_{(2k+1)\delta < 2^n \wedge k \geq k_1} B_{n,k}| \geq C_1 \geq 2^\delta d'.$$

Thus, $2^\delta |A_{1,n} \cap \{k\colon (2k + 3)\delta + n\delta < 2^n\}| \leq 2^{n+a+1}$ and if n is large enough,

$$|A_{1,n}| \leq \frac{2^{n+a+1}}{2^\delta} + (n + 3)\delta \leq \frac{2^{n+a+2}}{2^\delta} = 2^{n+a-\delta+2}.$$

Then by the choice of δ,

$$|A_{1,n}| < 2^{-\delta+n+a+2} < {}^{2^{n-2}}\!/_{\delta^2}.$$

Define

$$\widetilde{A}_{1,n} = \left\{ k : \begin{array}{c} (2^n + (2k+1)\delta \leq 2^{n+1}) \ \wedge \\ (\forall i \in (2^n + 2k\delta, 2^n + (2k+1)\delta])(\widehat{\Omega}_V(x)(i) = 0) \end{array} \right\}.$$

By item (2) in Claim 1, it must be that for every k with $2^n + (2k+1)\delta < 2^{n+1}$, there is some l such that $K_V(X \restriction l) = 2^n + (2k+1)\delta$. Note that if $k \in \widetilde{A}_{1,n}$, then $(\forall i \in (2^n + 2k\delta, 2^n + (2k+1)\delta])(\widehat{\Omega}_V(x)(i) = 0)$. So, if $k \in \widetilde{A}_{1,n}$ with $2^n + (2k+1)\delta < 2^{n+1} - (n+2)\delta n$, then, by (‡), there must be at least 2^δ many elements in $\{l \colon K_V(X \restriction l) \in [2^n + (2k+1)\delta, 2^{n+1})\}$. Then, by the same proof as above, we have for large enough n that

$$|\widetilde{A}_{1,n}| \leq {}^{2^{n-2}}/_{\delta^2}$$

and therefore

$$|A_{0,n}| \geq \frac{2^{n+1} - 2^n}{2\delta} - |\widetilde{A}_{1,n}| \geq {}^{2^{n-1}}/_{\delta} - {}^{2^{n-2}}/_{\delta^2} > {}^{2^{n-2}}/_{\delta^2}.$$

CASE 2. $X(n) = 0$. Then, by the same proof as for Case 1, we have that for large enough n,

$$|A_{0,n}| < {}^{2^{n-2}}/_{\delta^2} \text{ and } |A_{1,n}| > {}^{2^{n-2}}/_{\delta^2}.$$

So, for large enough n, to decide whether $X(n) = 0$ or $X(n) = 1$, we use $\widehat{\Omega}_V(X)$ to compute the cardinality of $|A_{0,n}|$ and $|A_{1,n}|$. If $|A_{0,n}| < |A_{1,n}|$, then $X(n) = 0$; and if $|A_{1,n}| < |A_{0,n}|$, then $X(n) = 1$. It follows that $X \leq_{tt} \widehat{\Omega}_V(X)$.

Finally, since either $A_{0,n}$ or $A_{1,n}$ must have cardinality less than ${}^{2^{n-2}}/_{\delta^2}$, $\Omega_V(X)$ has effective Hausdorff dimension 0. ⊣

COROLLARY 5.18. *Let V be a machine constructed as in the proof of Theorem* 5.17. *Then for every X, $\widehat{\Omega}_V^{-1}(X)$ is null.*

PROOF. Assume otherwise and fix an X such that $\widehat{\Omega}_V^{-1}(X)$ is not null. Then $\widehat{\Omega}_V^{-1}(X)$ contains a set of random reals of positive measure and, by Theorem 5.17, for every such random real $R \in \widehat{\Omega}_V^{-1}(X)$, we have that $R \leq_T X$. But there can be at most countably many reals Turing-below X, contradiction. ⊣

Next, we apply a known result to prove that at least for some universal machines V it is possible that for some sets A we have that $\widehat{\Omega}_V(A)$ is strictly below A in the Turing degrees.

THEOREM 5.19 (Calude, Hay, and Stephan [11]). *For every computable real $\varepsilon \in (0, 1)$ there is a set A and a constant c such that for all n*

$$\varepsilon n - c \leq K(A \restriction n) \leq \varepsilon n + c.$$

THEOREM 5.20. *Let be A be as in Theorem* 5.19 *when letting $\varepsilon = {}^1/_2$. Then there is a universal machine V such that $\widehat{\Omega}_V(A)$ is rational. In particular, there is a universal machine V such that $A >_T \widehat{\Omega}_V(A)$.*

PROOF. Let c be the constant that appears in the statement of Theorem 5.19 and let U be the standard universal machine used for defining prefix-free Kolmogorov complexity K, as it is used there.

We define V as follows: If it holds that

$$U(p) = x \ \wedge \ (|p| < {}^{|x|}/_2 - c \ \vee \ |p| > {}^{|x|}/_2 + c + 1)$$

then let $V(p) = x$; else let $V(q) = x$ for some $q \succ p$ with $|q| = \lceil {}^{|x|}/_2 + {}^3/_2 + c \rceil$.

For this V we have that $K_V(A\restriction 2n) = K_V(A\restriction 2n+1) = n+2+c$ for every n. It follows that $\widehat{\Omega}_V(A) = \sum_m 2^{-m-2-c+1} = 2^{-c}$ and thus $\widehat{\Omega}_V(A)$ is computable. In particular, since A is of d.n.c. degree, $\widehat{\Omega}_V(A) <_T A$.

The machine V can be made universal using the same trick as in the first part of the proof of Theorem 5.17. ⊣

Corollary 5.21. *There is a universal machine V and a Π^0_1 set $\mathcal{P}$ such that $\max \widehat{\Omega}_V(\mathcal{P})$ is a rational number.*

Proof. Let V, c, and A be as in the proof of Theorem 5.20. Define

$$\mathcal{P} = \{X : (\forall n)(K(X_V\restriction 2n) \geq n+2+c \ \wedge \ K_V(X\restriction 2n+1) \geq n+2+c)\}.$$

Then $A \in \mathcal{P}$ and $\widehat{\Omega}_V(A) = \max \widehat{\Omega}_V(\mathcal{P})$. ⊣

To conclude this section, we give an example of a set A that is always mapped to nonrandom reals by $\widehat{\Omega}$, independently of the optimal machine.

Proposition 5.22. *There is a real A such that $\widehat{\Omega}_V(A)$ is not random for any optimal machine V.*

Proof. Let A be a set such that $K(A\restriction n) \in (pn-c, pn+c)$ for some constant c and some rational number $p \in (0,1)$. Then, for every optimal machine V, there is some d such that $K_V(A\restriction n) \in (pn-d, pn+d)$.

For every n, let $s_n = \min\{s : K_{V,s}(A\restriction n) = K_V(A\restriction n)\}$ and define $(n_k)_k$ as an increasing sequence with the property that, for every k,

$$s_{n_k} = \max\{s_m : m \leq n_k\}.$$

Then, for some constant d' and for each k,

$$\begin{aligned} &\widehat{\Omega}_V(A) - \textstyle\sum_{m\leq n_k} 2^{-K_{V,s_{n_k}}(A\restriction m)} \\ =\ &\widehat{\Omega}_V(A) - \textstyle\sum_{m\leq n_k} 2^{-K_V(A\restriction m)} \\ =\ &\textstyle\sum_{m>n_k} 2^{-K_V(A\restriction m)} \leq \sum_{m>n_k} 2^{-pm+d} \leq 2^{-pn_k+d'}. \end{aligned}$$

Thus, there is some constant d'' such that, for each k,

$$K(\widehat{\Omega}_V(A)\restriction pn_k + d') \leq K_V(A\restriction n_k) + d'' \leq pn_k + d + d'';$$

thus $\widehat{\Omega}_V(A)$ is not random. ⊣

§6. Open questions.

Question 6.1. *If $\widehat{\Omega}(X)$ is X-random, must X be K-trivial?*

Note that by Proposition 5.1, $\widehat{\Omega}(X)$ is X-random if and only if there is a constant c such that $(\exists^\infty n)(K(X\restriction n) \leq K^X(n) + c)$. Thus, the answer to this question must be machine-independent. Further note that every Turing degree containing a 2-random real contains a weakly 1-generic real; and all such reals are weakly low for K and infinitely often K-trivial.

The following further open questions are inspired by the machine-dependent results obtained in Section 5.2.

Question 6.2.

1. *Is $\widehat{\Omega}_V^{-1}(X)$ null for every optimal machine V and every real X?*

2. *Is it true that for every optimal machine V there is a real X with*

$$X >_T \widehat{\Omega}_V(X)?$$

3. *Is there a universal machine V such that for every X we have that if $\widehat{\Omega}_V(X) \geq_T X$, then X must be K-trivial?*
4. *How can the elements of $\widehat{\Omega}^{-1}(\max \widehat{\Omega}(2^\omega))$ be characterized?*

Acknowledgments. Stephan was supported in part by the Singapore Ministry of Education Academic Research Fund Tier 2 grant MOE2016-T2-1-019 / R146-000-234-112. Part of the work was done while F. Stephan was on Sabbatical Leave to the University of Heidelberg and the Bundeswehr University in Munich. Liang Yu was partially supported through grant 11671196 by the National Science Foundation of China, by a Fellowship for Experienced Researchers by the Alexander von Humboldt Foundation, and by the Fundamental Research Funds for the Central Universities No. 14380028. Miller was partially supported by grant #358043 from the Simons Foundation.

REFERENCES

[1] M. M. Arslanov, *Some generalizations of a fixed-point theorem*. ***Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika***, vol. 5 (1981), pp. 9–16.

[2] G. Barmpalias, *Aspects of Chaitin's Omega*, preprint, 2017, arXiv:1707.08109.

[3] G. Barmpalias, D. Cenzer, and C. P. Porter, *The probability of a computable output from a random oracle*. ***ACM Transactions on Computational Logic***, vol. 18 (2017), no. 3, pp. 18:1–18:15.

[4] ———, *Random numbers as probabilities of machine behavior*. ***Theoretical Computer Science***, vol. 673 (2017), pp. 1–18.

[5] G. Barmpalias, R. Hölzl, A. E. M. Lewis, and W. Merkle, *Analogues of Chaitin's Omega in the computably enumerable sets*. ***Information Processing Letters***, vol. 113 (2013), no. (5–6), pp. 171–178.

[6] V. Becher, S. Figueira, S. Grigorieff, and J. S. Miller, *Randomness and halting probabilities*, this Journal, vol. 71 (2006), no. 4, pp. 1411–1430.

[7] V. Becher and S. Grigorieff, *Random reals and possibly infinite computations. I. Randomness in* $\emptyset'$, this Journal, vol. 70 (2005), no. 3, pp. 891–913.

[8] L. Bienvenu, A. R. Day, N. Greenberg, A. Kučera, J. S. Miller, A. Nies, and D. Turetsky, *Computing K-trivial sets by incomplete random sets*. ***Bulletin of Symbolic Logic***, vol. 20 (2014), no. 1, pp. 80–90.

[9] L. Bienvenu and R. Downey, *Kolmogorov complexity and Solovay functions*, ***STACS 2009: 26th International Symposium on Theoretical Aspects of Computer Science*** (S. Albers and J.-Y. Marion, editors), LIPICS - Leibniz International Proceedings in Informatics, vol. 3, Schloss Dagstuhl. Leibniz-Zentrum für Informatik, Wadern, 2009, pp. 147–158.

[10] L. Bienvenu, R. Hölzl, J. S. Miller, and A. Nies, *Denjoy, Demuth and density*. ***Journal of Mathematical Logic***, vol. 14 (2014), no. 1-1450004, pp. 1–35.

[11] C. S. Calude, N. J. Hay, and F. Stephan, *Representation of left-computable ε-random reals*. ***Journal of Computer and System Sciences***, vol. 77 (2011), no. 4, pp. 812–819.

[12] G. J. Chaitin, *A theory of program size formally identical to information theory*. ***Journal of the ACM***, vol. 22 (1975), pp. 329–340.

[13] R. G. Downey and D. R. Hirschfeldt, ***Algorithmic Randomness and Complexity***, Theory and Applications of Computability, Springer, New York, 2010.

[14] R. Downey, D. R. Hirschfeldt, J. S. Miller, and A. Nies, *Relativizing Chaitin's halting probability*. ***Journal of Mathematical Logic***, vol. 5 (2005), no. 2, pp. 167–192.

[15] P. Gács, *On the symmetry of algorithmic information*. ***Soviet Mathematics Doklady***, vol. 150 (1974), pp. 1477–1480.

[16] R. Hölzl, T. Kräling, and W. Merkle, *Time-bounded Kolmogorov complexity and Solovay functions*. ***Theory of Computing Systems***, vol. 52 (2013), no. 1, pp. 80–94.

[17] J. S. Miller, *The K-degrees, low for K-degrees, and weakly low for K sets*. ***Notre Dame Journal of Formal Logic***, vol. 50 (2009), no. 3, pp. 381–391.

[18] J. S. Miller and L. Yu, *On initial segment complexity and degrees of randomness*. ***Transactions of the American Mathematical Society***, vol. 360 (2008), no. 6, pp. 3193–3210.

[19] K. Miyabe, A. Nies, and J. Zhang, *Using almost-everywhere theorems from analysis to study randomness*. ***Bulletin of Symbolic Logic***, vol. 22 (2016), no. 3, pp. 305–331.

[20] A. Nies, ***Computability and Randomness***, Oxford Logic Guides, vol. 51, Oxford University Press, Oxford, 2009.

[21] R. Rettinger and X. Zheng, *Solovay reducibility on d-c.e. real numbers*, ***Computing and Combinatorics*** (L. Wang, editor), Lecture Notes in Computer Science, vol. 3595, Springer, Berlin, 2005, pp. 359–368.

[22] R. I. Soare, ***Recursively Enumerable Sets and Degrees***, Springer-Verlag, Berlin, 1987.

INSTITUTE 1, FACULTY OF COMPUTER SCIENCE
BUNDESWEHR UNIVERSITY MUNICH
WERNER-HEISENBERG-WEG 39
85579, NEUBIBERG
GERMANY
E-mail: r@hoelzl.fr

INSTITUT FÜR INFORMATIK
RUPRECHT-KARLS-UNIVERSITÄT
HEIDELBERG 69120, GERMANY
E-mail: merkle@math.uni-heidelberg.de

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WI 53706, USA
E-mail: jmiller@math.wisc.edu

DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
BLOCK S17, 10 LOWER KENT RIDGE ROAD
SINGAPORE 119076, SINGAPORE
E-mail: fstephan@comp.nus.edu.sg

MATHEMATICAL DEPARTMENT
NANJING UNIVERSITY
NANJING, JIANGSU PROVINCE 210093
CHINA
E-mail: yuliang.nju@gmail.com