AN EXTREME-VALUE THEORY APPROXIMATION SCHEME IN REINSURANCE AND INSURANCE-LINKED SECURITIES

BY

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ABSTRACT

We establish a "top-down" approximation scheme to approximate loss distributions of reinsurance products and Insurance-Linked Securities based on three input parameters, namely the *Attachment Probability*, *Expected Loss* and *Exhaustion Probability*. Our method is rigorously derived by utilizing a classical result from Extreme-Value Theory, the Pickands–Balkema–de Haan theorem. The robustness of the scheme is demonstrated by proving sharp error-bounds for the approximated curves with respect to the *supremum* and L^2 norms. The practical implications of our findings are examined by applying it to Industry Loss Warranties: the method performs very accurately for each transaction. Our approach can be used in a variety of applications such as vendor model blending, portfolio optimization and premium calculation.

KEYWORDS

Extreme-value theory, loss exceedance probability curve, reinsurance, insurance-linked securities.
2000 Mathematics Subject Classification. Primary, 91B50; Secondary, 91B16

1. INTRODUCTION

Modeling of natural catastrophes and other insurance risks has been substantially advanced in the last few decades, motivating the development of specialized vendor models, which are now intensively used by the (re)insurance industry and asset managers investing in Insurance-Linked Securities (ILS). These models are constantly improving in terms of their ability to incorporate realistic features, new scientific findings and provide multiple views on a risk. Following a "bottom-up" approach, the models' ambition is to estimate risk through computation of a peril's impact on underlying exposures endowed with certain vulnerability profiles and insurance terms. The output is usually a list of events conjugated with their frequency and severity characteristics, the

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so-called *Event Loss Table*. From this output, one can deduce risk parameters[1] such as the *Attachment Probability*, *Expected Loss*, *Exhaustion Probability* and, in essence, the entire *Loss Distribution*.

Due to a commercial rationale, the intellectual property associated with these models is often kept secret or at least described only at a high-level, making the models regarded by some practitioners as "black boxes" (see for instance Global [Re]Insurance, 2013). What we attempt to develop in this note is an elementary approximation framework retrieving *Loss Distributions* by using as input the preceding *risk parameters*[2] extracted from vendor models. That is, rather than relying on a holistic derivation of each point of the loss distribution through a model, we adapt a "top-down" philosophy and reversely recover it (Section 3.1) via a parametric curve fitting process. To accomplish our humble efforts, we make use of a classical theorem from Extreme-Value Theory (EVT), the Pickands–Balkema–de Haan theorem. So, while our method is relatively straightforward, it still benefits from the rigor of a well-established mathematical theory. We further investigate the accuracy of this approach, aiming at answering the question of how far can we deviate from the "correct" distribution extracted from the model? It turns out, not too much, as shown by establishing sharp bounds (Section 3.3) with respect to both the *supremum* and L^2 norms. Thereby proving the robustness of our methodology.

On the practical side, we study in Section 4, the performance of the method for a set of Industry Loss Warranties (ILWs), where we conclude that each fitted distribution is impressively approximating the ones obtained from the model (see Figure 1).

We now outline some related literature: first, the ILS-related ones. Embrechts and Meister (1997) study catastrophe futures and pricing aspects including utility maximization. Lane (2000) sets a pricing paradigm incorporating ideas from economics (The Cobb–Douglas production function). Wang (2004) uses probability transforms to price catastrophe bonds. Zimbidis *et al.* (2007) use maximum-likelihood to model earthquake risk with EVT and price hypothetical catastrophe bonds. In a work incorporating seasonality patterns, Hainaut (2010) uses Esscher transforms to price catastrophe bonds. For a comprehensive exposition of EVT including applications, we refer to Embrechts *et al.* (1997). Some other interesting EVT-related works include those of Rootzen and Tajvidi (1997), Morales (2004) and Vandewalle and Beirlant (2006).

The structure of the paper is as follows. Section 2 deals with preliminaries, introduces notations from reinsurance and ILS and presents a theorem from EVT. Section 3 presents the approximation scheme, shows its well-posedness and proves robustness by establishing sharp error-estimates. Last, Section 4 tests the approximation method and its accuracy for ILWs. We conclude the paper in Section 5.

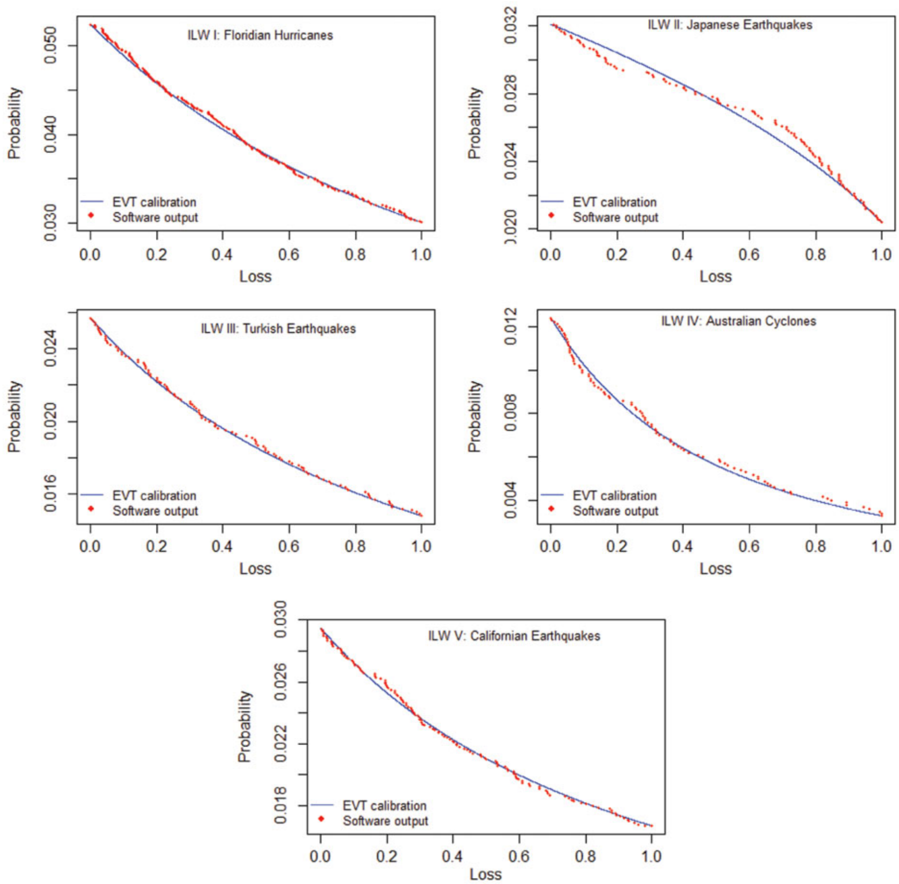

FIGURE 1: Loss Exceedance Probability curves: EVT approximation versus software outputs for ILW I–V.

2. SETUP

Throughout this paper, we will focus on *Excess of Loss* layers. These structures occupy a prominent portion of the non-life reinsurance and ILS product space. Nonetheless, our methods can be employed for more complex mechanisms and other types of contracts such as quota-shares. We first introduce the necessary notations from reinsurance and ILS (Section 2.1), and then present a result from EVT (Section 2.2) that will be used in subsequent sections.

2.1. Notations from reinsurance and ILS

For a given *Excess of Loss* layer within a cedant's reinsurance program, denote by a and b the *attachment* and *exhaustion* level of the layer, respectively, and by $\widetilde{L}$ the monetary ground-up loss of the cedant during 1 year. Here, $\widetilde{L}$ is a non-negative random variable. Assuming the risk associated with the layer has been

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transferred to a reinsurer, a payout will be triggered if $\widetilde{L} > a$, in which case it will be equal to $\min\{\widetilde{L}-a, b-a\}$. Otherwise (if $\widetilde{L} \le a$), there are no payments made by the reinsurer to the cedant. We set below a normalized version[3] ($L \in [0, 1]$) of the random variable $\widetilde{L}$ defined by

$$L = \min\left\{\max\left\{\frac{\widetilde{L}-a}{b-a}, 0\right\}, 1\right\}.$$

There are three important metrics associated with *Excess of Loss* structures: the *probability of attachment* ($P_{\text{attachment}}$), *Expected Loss* (EL) and *probability of exhaustion* ($P_{\text{exhaustion}}$), which are defined as

$$P_{\text{attachment}} := P\left(L > 0\right) = P\left(\widetilde{L} > a\right),$$

$$EL := E\left[L\right] = P\left(\widetilde{L} \ge b\right) + \frac{E\left[\widetilde{L} I_{\{a \le \widetilde{L} < b\}}\right] - aP\left(a \le \widetilde{L} < b\right)}{b-a},$$

where I_A denotes the indicator of the set A, and

$$P_{\text{exhaustion}} := P\left(L = 1\right) = P\left(\widetilde{L} \ge b\right).$$

Obviously, the inequality $P_{\text{exhaustion}} \le EL \le P_{\text{attachment}}$ always holds true. The *Loss Exceedance Probability* curve associated with the normalized loss random variable is defined by

$$S_L(x) = P\left(L > x\right),$$

for all $x \in [0, 1]$. For natural catastrophe risks, these three metrics ($P_{\text{attachment}}$, EL and $P_{\text{exhaustion}}$) as well as the *Loss Exceedance Probability* curve are usually obtained through specialized modeling software programs. For the *Standard Deviation* $SD(L) := \sqrt{E\left[L^2\right] - (EL)^2}$, the following inequality holds true:

$$\sqrt{EL\left(1 - EL\right)} \ge \tag{1}$$

$$SD(L) \ge \sqrt{\max\left\{0, P_{\text{exhaustion}} + \frac{\left(EL - P_{\text{exhaustion}}\right)^2}{P_{\text{attachment}} - P_{\text{exhaustion}}} - \left(EL\right)^2\right\}}. \tag{2}$$

The above inequality is important because it shows that for every normalized loss variable L, the associated risk metrics *(Attachment Probability, Expected Loss and Exhaustion Probability)* form a range of feasible values for its standard deviation. To our knowledge, such a result (though elementary) was not published. We show now the validity of this inequality. The left-hand side of the inequality is evidently satisfied since $EL \ge E\left[L^2\right]$. The right-hand side of the inequality can be proved by considering the change of measure

$$dQ/dP = \frac{I_{\{0<L<1\}}}{P_{\text{attachment}} - P_{\text{exhaustion}}},$$

in case that $P_{\text{attachment}} \neq P_{\text{exhaustion}}$, and then applying the inequality

$$E_Q\left[L^2\right] \geq \left(E_Q[L]\right)^2.$$

Moreover, the above inequality is sharp, i.e. no better lower and upper bounds can be achieved. This is demonstrated in Appendix A.

2.2. Result from EVT: The Pickands–Balkema–de Haan theorem

We present the following theorem from EVT dealing with the limiting behavior of distributions over large thresholds. This result is due to Balkema and De Haan (1974) and Pickands (1975). In Section 3, we will link it to *Excess of Loss* layers.

Theorem 2.1. *(Pickands–Balkema–de Haan theorem) For a large class of distributions, we can find a function* $\sigma(u)$ *such that*

$$\lim_{u \to x_F} \sup_{0 \leq x < x_F - u} \left| F_u(x) - G_{\xi,\sigma(u)(x)}(x) \right| = 0, \tag{3}$$

where

$$F_u(x) = P\left(X \leq u + x \middle| X > u\right) = \frac{F(x+u) - F(u)}{1 - F(u)}$$

is the excess distribution over threshold u *of a random variable* X *with a cumulative distribution function* F, x_F *is the right endpoint of the distribution* F*, and here*

$$G_{\xi,\sigma}(x) = 1 - \left(1 + \frac{\xi}{\sigma}x\right)^{-1/\xi},$$

for $\xi \neq 0$, *where* $\sigma > 0$ *and* $x \geq 0$ *when* $\xi \geq 0$ *and* $0 \leq x \leq -\frac{\sigma}{\xi}$ *when* $\xi < 0$, *and*

$$G_{0,\sigma}(x) = 1 - \exp\left(-x/\sigma\right).$$

It is important to stress that the above class of distributions for which the limit (3) holds true includes all the well-known distributions encountered in actuarial and financial applications. More precisely, a distribution F satisfies the limit (3) if and only if it belongs to the so-called *Domain of Attraction* of a certain non-degenerate distribution function. We refer to Chapters 7.1, 7.2 and 7.3 in McNeil *et al.* (2010) for a comprehensive overview of the topic.

3. EVT Approximation

We concentrate now on developing an approximation method based on the Pickands–Balkema–de Haan theorem for the *Loss Exceedance Probability* curve $S_L(x)$. The approximation is based on a straightforward distribution fitting scheme. We will then prove the well-posedness (Section 3.2) of the method and examine its robustness (Section 3.3).

3.1. The approximation method

In the spirit of the notations introduced in Section 2.1, we let the random variables $\widehat{\widetilde{L}}$ and $\widehat{L}$ stand for the approximated monetary and normalized loss, respectively, and denote accordingly by $\widehat{S}_L$, $\widehat{F}_{\widehat{L}}$ and $\widehat{F}_{\widetilde{L}}$ the associated Loss Exceedance Probability curve and Cumulative Distribution functions. Assuming that the attachment level a is sufficiently large and following Theorem 2.1, we assume that

$$\frac{\widehat{F}_{\widetilde{L}}(x+a) - \widehat{F}_{\widetilde{L}}(a)}{1 - \widehat{F}_{\widetilde{L}}(a)} = G_{\xi,\sigma(a)}(x),$$

for $x \geq 0$. Notice that the above is equivalent to

$$\widehat{S}_L(x) = P_{\text{attachment}} \left(1 - G_{\xi,\sigma(a)}((b-a)x)\right),$$

for $x \in [0, 1]$. As expected, we get $\widehat{S}_L(0) = P_{\text{attachment}}$, since $G_{\xi,\sigma(a)}(0) = 0$. We denote by $\sigma := \frac{\sigma(a)}{(b-a)}$ and obtain

$$\widehat{S}_L(x) = P_{\text{attachment}} \left(1 + \frac{\xi}{\sigma} x\right)^{-1/\xi},$$

for all $x \in [0, 1]$ and $\xi \neq 0$. We proceed by fitting the parameters σ and ξ through the values $P_{\text{exhaustion}}$ and EL by imposing the following equations

$$P_{\text{exhaustion}} = \widehat{S}_L(1),$$

and

$$EL = \int_0^1 \widehat{S}_L(x)dx.$$

This yields the following two equations determining the values of ξ and σ:

$$\sigma = \frac{\xi}{\left(\frac{P_{\text{attachment}}}{P_{\text{exhaustion}}}\right)^{\xi} - 1}, \tag{4}$$

and

$$\frac{EL}{P_{\text{attachment}}} = \frac{\xi}{(\xi - 1)\left((P_{\text{attachment}}/P_{\text{exhaustion}})^{\xi} - 1\right)} \left(\left(\frac{P_{\text{attachment}}}{P_{\text{exhaustion}}}\right)^{\xi-1} - 1\right). \tag{5}$$

We arrived at a non-linear equation, which generally does not have a closed-form solution, but can be tackled numerically. It is interesting to note that the values ξ and σ solely depend on the ratios $\frac{EL}{P_{\text{attachment}}}$ and $\frac{P_{\text{exhaustion}}}{P_{\text{attachment}}}$ that can be interpreted as the *conditional expected loss* and *conditional probability of total loss*

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as seen through the following relations:

$$\frac{EL}{P_{\text{attachment}}} = E\left[L \middle| L > 0\right],$$

and

$$\frac{P_{\text{exhaustion}}}{P_{\text{attachment}}} = P\left[L = 1 \middle| L > 0\right].$$

For the case $\xi = 0$, we have

$$S_{\tilde{L}}(x) = P_{\text{attachment}} \cdot \exp\left(-\frac{x}{\sigma}\right).$$

By imposing $S_{\tilde{L}}(1) = P_{\text{exhaustion}}$ as above, we obtain

$$\sigma = \frac{1}{\log\left(P_{\text{attachment}}/P_{\text{exhaustion}}\right)}, \tag{6}$$

and by requiring $\int_0^1 S_{\tilde{L}}(x)dx = EL$, it is easy to check that the following relation must hold as well

$$\frac{EL}{P_{\text{attachment}}} = \frac{1}{\log\left(\frac{P_{\text{attachment}}}{P_{\text{exhaustion}}}\right)}\left(1 - \frac{P_{\text{exhaustion}}}{P_{\text{attachment}}}\right).$$

Example 3.1. *For $\xi = 1$, a closed-form expression is feasible. Note that Equation (5) implies that*

$$\sigma = \frac{1}{\frac{P_{\text{attachment}}}{P_{\text{exhaustion}}} - 1}.$$

Moreover, by letting ξ tend to 1 in Equation (5), it is easy to check that $\xi = 1$ if and only if the following holds true

$$\frac{EL}{P_{\text{attachment}}} = \frac{P_{\text{exhaustion}}}{P_{\text{attachment}} - P_{\text{exhaustion}}} \log\left(P_{\text{attachment}}/P_{\text{exhaustion}}\right).$$

3.2. Well-posedness: Existence and uniqueness

This section is concerned with proving that the approximation method is well-posed, that is, the retrieved parameters σ and ξ are uniquely determined. This boils down to proving that Equation (5) attains a unique solution, and, to this end, we prove the following simple lemma.

Lemma 3.1. *Denote*

$$\alpha = \frac{EL}{P_{\text{attachment}}},$$

and

$$\beta = \frac{P_{\text{exhaustion}}}{P_{\text{attachment}}}.$$

The following equation

$$\alpha = \frac{x}{(x-1)(\beta^{-x}-1)}\left(\beta^{1-x}-1\right)$$

has a unique solution $x \in (-\infty, \infty)$.

Proof. See Appendix B. ■

3.3. Robustness and error bounds

The objective of the current subsection is to derive error estimates for the fitted result in terms of the input parameters (*probability of attachment, expected loss* and *probability of exhaustion*) with respect to the *supremum* and L^2 norms. That is, we provide bounds for the difference between the approximated result and the one derived from the vendor models with respect to the preceding norms. We complement then those findings by exploring specific examples.

3.3.1. *Error bounds with respect to the supremum norm.* Let $\mathcal{F}_C$ denote the set of all non-increasing functions $f : [0, 1] \to [0, 1]$ satisfying:

$$(i)\quad f(0) = 1,$$

$$(ii)\quad f(1) = 0,$$

and

$$(iii)\quad \int_0^1 f(x)dx = C,$$

for some $C \in (0, 1]$. By $||\cdot||_\infty$, we denote the supremum norm defined as $||f||_\infty := \max_{x\in[0,1]} |f(x)|$ for every $f \in \mathcal{F}_C$. We begin with presenting the following result formulated as a variational problem.

Lemma 3.2. *For an arbitrary* $g_0 \in \mathcal{F}_C$, *consider the following variational problem*

$$V(g_o) = \sup_{f\in\mathcal{F}_C} ||f - g_0||_\infty.$$

Then, the solution of this problem is given by

$$V(g_0) = \max_{x\in[C,1]} \{C/x - g_0(x)\}.$$

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Moreover, a maximizer to the problem (not belonging to the set $\mathcal{F}_C$) is of the form

$$f_{\max}(x) = 1_{\{x \le x_0\}} \frac{C}{x_0},$$

for some $x_0 \in [C, 1]$.

Proof. See Appendix C. ■

We turn now to examining the implications of Lemma 3.2 on our approximation scheme. Recall that

$$\widehat{S}_L(x) = P_{\text{attachment}} \left(1 + \frac{\xi}{\sigma} x\right)^{-1/\xi}.$$

Consider the function

$$g_0(x) = \frac{\widehat{S}_L(x) - P_{\text{exhaustion}}}{P_{\text{attachment}} - P_{\text{exhaustion}}},$$

and note that $g_0 \in \mathcal{F}_C$ for

$$C := \int_0^1 g_0(x)dx = \frac{EL - P_{\text{exhaustion}}}{P_{\text{attachment}} - P_{\text{exhaustion}}}.$$

Assuming that S_L is a continuous function and applying Lemma 3.2 on the function g_0, we get

$$\left\|\widehat{S}_L - S_L\right\|_\infty \le (P_{\text{attachment}} - P_{\text{exhaustion}}) \max_{x\in[C,1]} \left(\frac{C}{x} - g_0(x)\right)$$

$$= P_{\text{exhaustion}} + \max_{x\in[C,1]} \left(\frac{EL - P_{\text{exhaustion}}}{x} - \widehat{S}_L(x)\right)$$

$$= P_{\text{exhaustion}} + \max_{x\in[C,1]} \left(\frac{EL - P_{\text{exhaustion}}}{x} - P_{\text{attachment}} \left(1 + \frac{\xi}{\sigma} x\right)^{-1/\xi}\right).$$

The above maxima cannot be calculated explicitly for all values of ξ and σ but can be computed numerically. This provides a sharp error estimate for the sup-distance between the fitted curve $\widehat{S}_L$ and an arbitrary software extracted *Loss Exceedance Probability* curve S_L.

Example 3.2. *For the case $\xi = \sigma = 1$, a closed-form solution is feasible. In this case, one easily checks that*

$$P_{\text{exhaustion}} = 1/2 \cdot P_{\text{attachment}},$$

and

$$EL = log2 \cdot P_{\text{attachment}}.$$

Using the above notations, we obtain that

$$C = \log 4 - 1,$$

and it follows from the above inequality that

$$\left\| \widehat{S}_L - S_L \right\|_\infty \leq 1/2 \cdot P_{\text{attachment}} + P_{\text{attachment}} \max_{x \in [C,1]} \left(\frac{\log 4 - 1}{2x} - \frac{1}{1+x} \right).$$

Some arithmetics yields

$$\left\| \widehat{S}_L - S_L \right\|_\infty \leq \left(2 - \frac{2}{\log 4} \right) \left(P_{\text{attachment}} - P_{\text{exhaustion}} \right).$$

Note that this bound is better than the a-priori *trivial estimate* $\left\| \widehat{S}_L - S_L \right\|_\infty \leq P_{\text{attachment}} - P_{\text{exhaustion}}$.

3.3.2. *Error bounds with respect to the L^2-norm.* By changing variables, note that the second moment of the random variable $\widehat{L}$ is given by

$$E\left[\left(\widehat{L}\right)^2 \right] = \int_0^1 P\left[\left(\widehat{L}\right)^2 > x \right] dx = 2 \int_0^1 x P\left[\left(\widehat{L}\right)^2 > x^2 \right] dx$$

$$= 2 \int_0^1 x \cdot \widehat{S}_L(x)\, dx = 2 P_{\text{attachment}} \int_0^1 x \left(1 + \frac{\xi}{\sigma} x \right)^{-1/\xi} dx,$$

and cannot be calculated explicitly for arbitrary values of ξ and σ. By recalling inequality (2), we arrive at the following error estimate

$$\left| E\left[\left(\widehat{L}\right)^2 - L^2 \right] \right| \leq \max \left\{ \left| 2 P_{\text{attachment}} \int_0^1 x \left(1 + \frac{\xi}{\sigma} x \right)^{-1/\xi} dx - EL \right|, \right.$$

$$\left. \left| 2 P_{\text{attachment}} \int_0^1 x \left(1 + \frac{\xi}{\sigma} x \right)^{-1/\xi} dx - P_{\text{exhaustion}} - \frac{\left(EL - P_{\text{exhaustion}} \right)^2}{P_{\text{attachment}} - P_{\text{exhaustion}}} \right| \right\}.$$

Example 3.3. *For the special case $\xi = \sigma = 1$, we obtain*

$$E\left[(\widehat{L})^2 \right] = 2\left(1 - \log 2 \right) P_{\text{attachment}}.$$

Recall that according to Example 3.2, we have $P_{\text{exhaustion}} = 1/2 \cdot P_{\text{attachment}}$ and $EL = log2 \cdot P_{\text{attachment}}$. Therefore, we get that

$$\left| E\left[\left(\widehat{L}\right)^2 - L^2 \right] \right| \leq 2\left(3 \log 2 - 2 \right) \left(P_{\text{attachment}} - P_{\text{exhaustion}} \right),$$

which is again better than the a-priori *bound* $\left| E\left[\left(\widehat{L}\right)^2 - L^2 \right] \right| \leq P_{\text{attachment}} - P_{\text{exhaustion}}$.

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4. Case Study: Industry Loss Warranties (ILWs)

An ILW is a reinsurance or derivative contract providing protection to a cedant seeking to make a recovery in the event of a pre-defined type and level of loss caused to the insurance industry, rather than due to a loss caused to the cedant's own portfolio. Unlike a risk transfer based on indemnification, in the ILW case, the protection buyer usually assumes a certain level of basis risk, whereas the seller of the protection benefits from transparency and a lower level of moral hazard. Prevalently traded both in ILS and traditional reinsurance markets, we use our methodology to fit the loss distribution of various ILWs (Section 4.1). We examine the accuracy of our approximation method by comparing the obtained results with the output from the modeling software. Throughout this section, all the model extracted data and outputs are derived from AIR's CATRADER software.

4.1. EVT approximation for loss distribution of ILWs

We consider a set of five *independent* ILW transactions covering different territories and perils:

ILW I: *Hurricanes in Florida*, USD 20 bn xs USD 30 bn.
ILW II: *Earthquakes in Japan*, USD 10 bn xs USD 20 bn.
ILW III: *Earthquakes in Turkey*, USD 3 bn xs USD 3 bn.
ILW IV: *Cyclones in Australia*, USD 4 bn xs USD 4 bn.
ILW V: *Earthquakes in California*, USD 15 bn xs USD 15 bn.

To illustrate the mechanics of an ILW, assume an investment of USD 5 mn in ILW I. Assume further that during the risk period of the transaction, a hurricane has caused a loss to the insurance industry in Florida amounting to USD 45 bn. Then, the associated loss is equal to the investment size (USD 5 mn) multiplied by the respective percentage of the eroded layer $\left(\frac{45-30}{50-30}\right)$, which is USD 3.75 mn. The risk figures for the above ILWs as modeled through the software are outlined in Table 1. Based on our technique, we provide a summary of the fitted parameters and the fitted Loss Exceedance Probability curves for each ILW contract in Table 3. A comparison between the approximated curves versus the modeling output for each ILW is found in Table 2: the *Standard Deviation* as well as the *supremum-distance* between the two is computed. Further, we provide a visualization of the Loss Exceedance Probability curves in Figure 1. As one can note, the EVT approximation scheme proves to be a very accurate approximation of the modeled outputs.

5. Conclusion and Further Research

We developed a robust approximation scheme postulated on EVT to approximate loss distributions of reinsurance and ILS products. The technique is

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TABLE 1

ILW CONTRACTS: RISK METRICS AS DERIVED VIA A MODELING SOFTWARE.

Transaction	Attachment Probability	Expected Loss	Exhaustion Probability
ILW I	0.0524	0.0393	0.0301
ILW II	0.0321	0.0271	0.0204
ILW III	0.0257	0.0191	0.0148
ILW IV	0.0124	0.0063	0.0033
ILW V	0.0295	0.0217	0.0167

TABLE 2

ILW CONTRACTS: EVT APPROXIMATION VERSUS SOFTWARE OUTPUTS.

	EVT Approximation	Software Output
ILW I: Standard Deviation	0.1843	0.1848
ILW I: Real-max Error	8.7×10^{-4}	
ILW I: Sup-norm Distance	1.2×10^{-2}	
ILW II: Standard Deviation	0.1564	0.1570
ILW II: Real-max Error	10^{-3}	
ILW II: Sup-norm Distance	6×10^{-3}	
ILW III: Standard Deviation	0.1302	0.1305
ILW III: Real-max Error	3.9×10^{-4}	
ILW III: Sup-norm Distance	6×10^{-3}	
ILW IV: Standard Deviation	0.0699	0.0706
ILW IV: Real-max Error	5.1×10^{-4}	
ILW IV: Sup-norm Distance	5×10^{-3}	
ILW V: Standard Deviation	0.1383	0.1384
ILW V: Real-max Error	5.9×10^{-4}	
ILW V: Sup-norm Distance	7×10^{-3}	

elementary, provides a closed-form parameterization and uses only three input parameters (*Attachment Probability, Expected Loss* and *Exhaustion Probability*). We applied the method on ILWs showing very content results for each transaction.

There are a number of applications and topics of further research that would be interesting to investigate as highlighted below:

1. Model Blending and Risk Aggregation: For independent transactions, this paper offers a simple method to aggregate risk and blend models. In case of correlation, developing a "top-down" method to retrieve dependence structures from various vendor models (e.g. by using Copulas) is of high interest.

2. The Dynamic Angle: This paper assumes a static risk-period. However, especially for ILS (and in particular, catastrophe bonds which are OTC traded on

TABLE 3

ILW CONTRACTS: FITTED PARAMETERS.

Transaction	ξ	σ
ILW I	0.9105	1.3867
ILW II	−2.9879	4.0272
ILW III	1.3262	1.2291
ILW IV	0.6052	0.4927
ILW V	1.3415	1.1713

daily basis), a setting of a rich time-dependent "top-down" framework would be of value.

3. Portfolio Optimization: Developing portfolio optimization techniques with tail constraints by assuming the EVT-approximated curves (and accounting for correlation) in the spirit of Rockafellar and Uryasev (2000) and Chekhlov *et al.* (2005) would be of high practical interest.

4. Premium Calculation: Some Premium Calculation principles make use of the entire distribution (e.g. *Utility Indifference Pricing* or *Probability Transform*), in which case our method offers a clear perspective and advantage compared to a discrete distribution function.

5. Sub-layers: Investigating how the approximated EVT-based loss curve could be further "disassembled" in order to approximate smaller sub-layers associated with the same underlying risk, would be a highly regarded application for practitioners.

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NOTES

1. The mathematical definition of these metrics is given in Section 2.1.
2. *Attachment Probability*, *Expected Loss* and *Exhaustion Probability.*
3. This allows us for comparing various loss metrics of reinsurance contracts and ILS based on different monetary units and structures.

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APPENDIX A. SHARPNESS OF THE STANDARD DEVIATION BOUNDS

We will prove now the sharpness of the upper and lower bounds in inequality (2). First, consider a random variable $L(\delta, \varepsilon)$ satisfying $P(L(\delta, \varepsilon) > x) = P_{\text{attachment}}(1 - x)$, for $x \in [0, \delta)$, $P(L(\delta, \varepsilon) > x) = P_{\text{exhaustion}} + (1 - x)$, for $x \in [\varepsilon, 1)$, and $P(L(\delta, \varepsilon) > x) = EL$, for $x \in [\delta, \varepsilon]$ for appropriate small $\delta, \varepsilon > 0$ such that $E[L(\delta, \varepsilon)] = EL$. One checks that in this case

$$\lim_{\delta, \varepsilon \to 0} SD(L(\delta, \varepsilon)) = \sqrt{EL(1 - EL)}.$$

Next, assume that

$$P_{\text{exhaustion}} + \frac{(EL - P_{\text{exhaustion}})^2}{P_{\text{attachment}} - P_{\text{exhaustion}}} \leq (EL)^2 .$$

As in the previous case, we consider a random variable $L(\delta, \varepsilon, \varepsilon')$ such that $P(L(\delta, \varepsilon, \varepsilon') > x) = P_{\text{attachment}}(1-x)$, for $x \in [0, \delta)$, $P(L(\delta, \varepsilon, \varepsilon') > x) = P_{\text{exhaustion}} + (1-x)$, for $x \in [\varepsilon, 1)$, and

$$P\left(L(\delta, \varepsilon, \varepsilon') > x\right) = P_{\text{exhaustion}} + \frac{(EL - P_{\text{exhaustion}})^2}{(EL)^2 - P_{\text{exhaustion}}} - \varepsilon',$$

for $x \in (\delta, \varepsilon)$. One can check that

$$\lim_{\delta,\varepsilon,\varepsilon' \to 0} SD\left(L(\delta, \varepsilon, \varepsilon')\right) = 0.$$

Last, assume that

$$P_{\text{exhaustion}} + \frac{(EL - P_{\text{exhaustion}})^2}{P_{\text{attachment}} - P_{\text{exhaustion}}} > (EL)^2 .$$

Define a random variable L with $P(L = 1) = P_{\text{exhaustion}}$ and

$$P\left(L = \frac{EL - P_{\text{exhaustion}}}{P_{\text{attachment}} - P_{\text{exhaustion}}}\right) = P_{\text{attachment}} - P_{\text{exhaustion}}.$$

It is easy to check that in this case

$$E\left[L^2\right] = P_{\text{exhaustion}} + \frac{(EL - P_{\text{exhaustion}})^2}{(EL)^2 - P_{\text{exhaustion}}}.$$

This accomplishes the proof of the sharpness of the bounds.

APPENDIX B. PROOF OF LEMMA 3.1

Denote

$$g(x) = \frac{x}{(x-1)(\beta^{-x}-1)}\left(\beta^{1-x} - 1\right).$$

Notice that

$$\lim_{x \to 0} g(x) = \frac{1-\beta}{\log(1/\beta)}, \tag{7}$$

and

$$\lim_{x \to 1} g(x) = \frac{\beta}{1-\beta}\log(1/\beta). \tag{8}$$

It is not hard to check that $\frac{1-\beta}{\log(1/\beta)} > \frac{\beta}{1-\beta}\log(1/\beta)$, for all $\beta \in (0, 1)$. Moreover, it is easy to note that

$$\lim_{x \to -\infty} g(x) = 1,$$

and

$$\lim_{x \to \infty} g(x) = \beta.$$

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Now, since $\alpha \in (\beta, 1)$ and $g(x)$ is a continuous function, it follows that the equation admits a solution. It is left to check now that this solution is unique. Consider the function $f(x) = \frac{x}{\beta^{-x}-1}$, and note that $g(x) = \frac{f(x)}{f(x-1)}$. We will prove that $g'(x) < 0$, for all $x \in (-\infty, \infty)$. First, we will prove that g' never vanishes. Assume toward a contrary that there exists $x_0 \in (-\infty, \infty)$ such that $g'(x_0) = 0$, or equivalently

$$\left.\frac{d}{dx}\right|_{x=x_0} (\log f(x) - \log f(x-1)) = 0. \tag{9}$$

By definition, we have

$$(\log f(x))' = \frac{1}{x} + \frac{\beta^{-x}\log\beta}{\beta^{-x}-1},$$

yielding

$$\frac{\beta^{1-x_0}-1}{x_0-1} = \log(\beta)\,(1-\beta)\,\beta^{-x_0}\frac{x_0}{(\beta^{-x_0}-1)}.$$

This implies that

$$g(x_0) = \log(\beta)\,(1-\beta)\,\beta^{-x_0}\frac{(x_0)^2}{(\beta^{-x_0}-1)^2} < 0,$$

which is a contradiction, as $g(x) > 0$ for all $x \in (-\infty, \infty)$. Note that g' is continuously differentiable, and due to (7), (8) and the inequality thereafter, it follows that $g'(x) < 0$. This accomplishes the proof of the Lemma. ■

APPENDIX C. PROOF OF LEMMA 3.2

First, for each $x_0 \in [C, 1)$, consider a suitable function $\widehat{f}_{x_0} \in \mathcal{F}$, which is close (with respect to the sup-norm) to the step-function $f_{x_0}(x) = 1_{\{x \le x_0\}}\frac{C}{x_0}$. One can check that

$$V(g_0) \ge \max_{x_0\in(C,1)}\max_{x\in[0,1]}|\widehat{f}_{x_0}(x) - g_0(x)| \ge \max_{x\in[C,1]}\{C/x - g_0(x)\}.$$

We turn now to proving the second part of the inequality. For an arbitrary $\varepsilon > 0$, let $f_0 \in \mathcal{F}$ be such that $\max_{x\in[0,1]}|f_0(x) - g_0(x)| = V(g_0) - \varepsilon$, and assume that (the alternative case can be treated similarly to the below approach) $\max_{x\in[0,1]}(f_0(x) - g_0(x)) = V(g_0) - \varepsilon$. Denote

$$y := max_{z\in[0,1]}\{f_0(z) = V(g_0) - \varepsilon + g_0(z)\}.$$

Assume that $y \le C$. Then, for arbitrary small $\delta > 0$, one can find a function $\widehat{h} \in \mathcal{F}$ close (in the sup-norm sense) to the step-function $h(x) = 1_{\{x\le C\}}$ so that

$$\widehat{h}(C) - g_0(C) \ge f_0(y) - g_o(y) - \delta = V(g_0) - \varepsilon - \delta.$$

This shows that in this case

$$V(g_0) = \sup_{x\in[0,C]}\left(1_{\{x\le C\}} - g_0(x)\right) = 1 - g_0(C).$$

If $y > C$, then

$$C = \int_0^1 f_0(x)dx \geq y \cdot f_0(y) = y \cdot (g_0(y) + V(g_0) - \varepsilon)\,,$$

or equivalently

$$\frac{C}{y} - g_0(y) \geq V(g_0) - \varepsilon,$$

which in particular yields

$$\max_{x\in(C,1]} \{C/x - g_0(x)\} \geq V(g_0) - \varepsilon.$$

This proves the result. ■