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ARTICLE

On Ramsey numbers of hedgehogs

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Abstract
The hedgehog H_t is a 3-uniform hypergraph on vertices $1, \ldots, t + \binom{t}{2}$ such that, for any pair (i, j) with $1 \leqslant i < j \leqslant t$, there exists a unique vertex $k > t$ such that $\{i, j, k\}$ is an edge. Conlon, Fox and Rödl proved that the two-colour Ramsey number of the hedgehog grows polynomially in the number of its vertices, while the four-colour Ramsey number grows exponentially in the square root of the number of vertices. They asked whether the two-colour Ramsey number of the hedgehog H_t is nearly linear in the number of its vertices. We answer this question affirmatively, proving that $r(H_t) = O(t^2 \ln t)$.

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1. Introduction

For a k-uniform hypergraph H, the Ramsey number $r(H)$ is the smallest n such that any two-colouring of $K_n^{(k)}$, the complete k-uniform hypergraph on n vertices, contains a monochromatic copy of H. Let $r(H; q)$ denote the analogous Ramsey number for q-colourings, so that $r(H) = r(H; 2)$.

It is a major open problem to determine the growth of $r(K_t^{(3)})$, the Ramsey number of the complete 3-uniform hypergraph on t vertices. It is known [6, 7] that there are constants $c, c' > 0$ such that

$$2^{ct^2} \leqslant r(K_t^{(3)}) \leqslant 2^{2^{c't}}.$$

Erdős conjectured that $r(K_t^{(3)}) = 2^{2^{\Theta(t)}}$, that is, the upper bound is closer to the truth. Erdős and Hajnal gave some evidence that this conjecture is true by showing that $r_3(K_t^{(3)}; 4) \geqslant 2^{2^{ct}}$, that is, the four-colour Ramsey number of $K_t^{(3)}$ is double-exponential in t (see *e.g.* [9]).

Definition. The *hedgehog* H_t is a 3-uniform hypergraph on $t + \binom{t}{2}$ vertices $1, \ldots, t + \binom{t}{2}$ such that, for each $1 \leqslant i < j \leqslant t$, there exists a unique vertex $k > t$ such that $\{i, j, k\}$ is an edge, and there are no additional edges.

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We sometimes refer to the first t vertices as the *body* of the hedgehog. For any $k \geqslant 4$, one can also define a k-uniform hedgehog $H_t^{(k)}$ on $t + \binom{t}{k-1}$, with a body of size t and a unique hyperedge for every $k-1$-sized subset of the body. In this notation, we have $H_t = H_t^{(3)}$.

Hedgehogs are interesting because their two-colour Ramsey number $r(H_t; 2)$ is polynomial in t, while their four-colour Ramsey number $r(H_t; 4)$ is exponentially large in t [3, 10]. This suggests that the bound $r(K_t^{(3)}; 4) \geqslant 2^{2^{ct}}$ by Erdős and Hajnal may not be such strong evidence that $r(K_t^{(3)}) = 2^{2^{\Theta(t)}}$.

Hedgehogs are also interesting because they are a natural family of hypergraphs with *degeneracy* 1. Degeneracy is a notion of sparseness for graphs and hypergraphs. For graphs, the degeneracy is defined as the minimum d such that every subgraph induced by a set of vertices has a vertex of degree at most d. The Burr–Erdős conjecture [2] states that there exists a constant $c(d)$ depending only on d such that the Ramsey number of any d-degenerate graph G on n vertices satisfies $r(G) \leqslant c(d) \cdot n$. Building on the work of Kostochka and Sudakov [11] and Fox and Sudakov [8], Lee [12] recently proved this conjecture. We can similarly define the degeneracy of a hypergraph as the minimum d such that every sub-hypergraph induced by a subset of vertices has a vertex of degree at most d. Under this definition, Conlon, Fox and Rödl [3] observe that the 4-uniform analogue of the Burr–Erdős conjecture is false: the 4-uniform hedgehog $H_t^{(4)}$, which is 1-degenerate, satisfies $r(H_t^{(4)}) \geqslant 2^{ct}$. They also observe that the 3-uniform analogue of the Burr–Erdős conjecture is false for three or more colours: the 3-uniform hedgehog, which is 1-degenerate, satisfies $r(H_t; 3) \geqslant \Omega(t^3 / \log^6 t)$.

However, the analogue of the Burr–Erdős conjecture for 3-uniform hypergraphs and two colours remains open. In particular, it was not known whether the Ramsey number of the hedgehog H_t is linear, or even near-linear, in the number of vertices, $t + \binom{t}{2}$. Conlon, Fox and Rödl [3] show $r(H_t; 2) \leqslant 4t^3$, and, with the above in mind, ask if $r(H_t; 2) = t^{2+o(1)}$. We answer this question affirmatively.

Theorem 1.1. *If $t \geqslant 10$ and $n \geqslant 200t^2 \ln t + 400t^2$, then every two-colouring of the complete 3-uniform hypergraph on vertices contains a monochromatic copy of the hedgehog H_t. That is,*

$$r(H_t) < 200t^2 \ln t + 400t^2 + 1.$$

We make no attempt to optimize the absolute constants here.

2. Ramsey number of hedgehogs

Throughout this section we assume $t \geqslant 10$, and that we have a fixed two-colouring of the edges of a complete 3-uniform hypergraph $\mathcal{H}$ on vertex set V with $n \geqslant 200t^2 \ln t + 400t^2$ vertices. Let

$$m_{\max} := 2t + \binom{t}{2}.$$

Let $\binom{S}{2}$ denote the set of pairs of elements of S. For integer a, let $[a] = \{1, 2, \ldots, a\}$. For vertices u and v of $\mathcal{H}$, we write uv as an abbreviation for the unordered pair $\{u, v\}$.

For $u, v \in V$, let

$$d_{uv}^{(r)} := |\{w : \{u, v, w\} \text{ red}\}|,$$
$$d_{uv}^{(b)} := |\{w : \{u, v, w\} \text{ blue}\}|.$$

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For a set of pairs $F \subset \binom{V}{2}$, let

$$N^{(b)}(F) := \{w : \exists uv \in F \text{ s.t. } \{u, v, w\} \text{ blue}\},$$
$$N^{(r)}(F) := \{w : \exists uv \in F \text{ s.t. } \{u, v, w\} \text{ red}\}.$$

Here, and throughout, we use b and r to refer to the colours blue and red, respectively. For a vertex v and set X, let

$$U^{(b)}_{\leqslant m}(v, X) = \{u \in X : d^{(r)}_{uv} \leqslant m\},$$
$$U^{(r)}_{\leqslant m}(v, X) = \{u \in X : d^{(b)}_{uv} \leqslant m\}.$$

If X is omitted, take $X = V$. We define $U^{(b)}_{\leqslant m}(v, X)$ to be sets of u such that $d^{(r)}_{uv}$ is small, rather than those such that $d^{(b)}_{uv}$ is small, because we wish to think of the $U^{(b)}$ as sets helpful in finding a blue hedgehog. Similarly, we think of the $U^{(r)}$ as sets helpful in finding a red hedgehog.

Lemma 2.1. *For any* $0 \leqslant m < |V|/2 - 1$, *and* $v \in V$,

$$\min\left(|U^{(b)}_{\leqslant m}(v)|, |U^{(r)}_{\leqslant m}(v)|\right) \leqslant 2m.$$

Proof. Fix m and v. For convenience, let $A = U^{(b)}_{\leqslant m}(v)$ and $B = U^{(r)}_{\leqslant m}(v)$. Assume for contradiction that $|A|, |B| \geqslant 2m + 1$. For every u, we have $d^{(r)}_{uv} + d^{(b)}_{uv} = |V| - 2 > 2m$, so A and B are disjoint. Consider the set E' of edges of $\mathcal{H}$ containing v, one element of A, and one element of B. On one hand, $|E'| = |A| \cdot |B|$. On the other hand, for every $u \in A$, the pair uv is in at most m such red triples, so the number of red triples of E' is at most $|A| \cdot m$. Additionally, for every $u \in B$, the pair uv is in at most m such blue triples, so the number of blue triples of E' is at most $|B| \cdot m$. Hence, $(|A| + |B|) \cdot m \leqslant |E'| = |A| \cdot |B|$, a contradiction to $|A|, |B| \geqslant 2m + 1$. □

The following 'matching condition' for hedgehogs is useful.

Lemma 2.2. *Let* $S \subset V$ *be a set of* t *vertices. If, for all non-empty sets* $F \subset \binom{S}{2}$, *we have* $|N^{(b)}(F)| \geqslant |F| + t$, *then there exists a blue hedgehog with body* S. *Similarly, if, for all non-empty sets* $F \subset \binom{S}{2}$, *we have* $|N^{(r)}(F)| \geqslant |F| + t$, *then there exists a red hedgehog with body* S.

Proof. By symmetry, it suffices to prove the first part. Consider the bipartite graph G between pairs in $\binom{S}{2}$ and vertices of $V \setminus S$, where $uv \in \binom{S}{2}$ is connected with $w \in V \setminus S$ if and only if triple $\{u, v, w\}$ is blue. If, for all non-empty $F \subset \binom{S}{2}$, we have $|N^{(b)}(F)| \geqslant |F| + t$, then any such F has at least $|F| + t - |S| = |F|$ neighbours in G. By Hall's marriage lemma on G, there exists a matching in G using every element of $\binom{S}{2}$. Taking triples $\{u, v, w\}$, where $uv \in \binom{S}{2}$ and $w \in V \setminus S$ is the vertex matched with pair uv, gives a blue hedgehog with body S. □

2.1 Special cases

We start by finding monochromatic hedgehogs in two specific classes of colourings on $\mathcal{H}$. We base our proof of Theorem 1.1 on the argument for the first class of colourings, which we call *simple colourings*. We use the result for the second class of colourings, which we call *balanced colourings*, as a specific case in the general argument.

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2.1.1 Simple colourings

Consider hypergraphs that are coloured in the following way.

(1) Start with a graph G on $[n]$.
(2) Colour a complete hypergraph $\mathcal{H}$ on $[n]$ by colouring the triple $\{u, v, w\}$ blue if at least one of uv, uw, vw is in G, and red otherwise.

Lemma 2.3. *If $n \geqslant t^2 + t$, any hypergraph coloured as above has a monochromatic H_t.*

Proof. Set $X = V(G)$. For $i = t-1, t-2, \dots, 0$, pick a vertex $v_i \in X$ whose degree in G is at least i and let $\hat{U}(v_i) \subset X$ be an arbitrary set of i neighbours of v_i. Remove $v_i \cup \hat{U}(v_i)$ from X. We call this the *peeling step* of v_i. Figure 1 shows the first three peeling steps of this process for $t = 5$. If this process succeeds, we have found a set $S = \{v_{t-1}, \dots, v_0\}$ of t vertices and disjoint sets of vertices $\hat{U}(v_0), \dots, \hat{U}(v_{t-1})$ also disjoint from S, from which we can greedily embed a blue hedgehog in $\mathcal{H}$ with body $\{v_0, \dots, v_{t-1}\}$: for each $v_i v_j$ with $i < j$, pick an arbitrary unused element of $\hat{U}(v_j)$ for the third vertex of the hedgehog's edge containing $v_i v_j$.

Now suppose this process finds vertices $v_{t-1}, v_{t-2}, \dots, v_{i+1}$ but fails to find v_i for some $i \leqslant t-1$. After picking v_j, we remove v_j and j of its neighbours from X, for a total of $j+1$ vertices. Then we have removed exactly

$$t + (t-1) + \cdots + (i+2) = \binom{t+1}{2} - \binom{i+2}{2}$$

vertices from X. Hence,

$$|X| \geqslant (t^2 + t) - \binom{t+1}{2} + \binom{i+2}{2} = \binom{t+1}{2} + \binom{i+2}{2} > \frac{t^2 + i^2}{2} \geqslant ti,$$

and every vertex has degree at most $i-1$ in the subgraph of G induced X. Thus, there exists an independent set $S \subset X$ in G of size at least $|X|/i \geqslant t$. Furthermore, any vertex has at most $i-1$ neighbours in X, so any two vertices $u, v \in S$ share at least

$$|X| - 2i \geqslant t + \binom{t}{2} + \binom{i+2}{2} - 2i > t + \binom{t}{2}$$

red triples in the sub-hypergraph of $\mathcal{H}$ induced by X, so we can greedily find a red hedgehog with body S. □

2.1.2 Balanced colourings

In this section we consider the case where our colouring is 'balanced'. Lemma 2.1 tells us that, for every vertex v and every non-negative integer m less than $|V|/2 - 1$, one of

$$|U^{(b)}_{\leqslant m}(v)| = \#\{u : d^{(r)}_{uv} \leqslant m\} \quad \text{and} \quad |U^{(r)}_{\leqslant m}(v)| = \#\{u : d^{(b)}_{uv} \leqslant m\}$$

is at most $2m$. In 'balanced' colourings, we assume, for all $v \in V$ and all $2t \leqslant m \leqslant m_{\max} := 2t + \binom{t}{2}$, *both* of $|U^{(b)}_{\leqslant m}(v)|$ and $|U^{(r)}_{\leqslant m}(v)|$ are $O(m)$. We show, in this case, there is a monochromatic hedgehog. The proof is by choosing a random subset of approximately $4t$ vertices, and showing that, with positive probability, we can remove vertices so that the remaining set of t vertices is the body of some red hedgehog.

Lemma 2.4. *Let $c \geqslant 1$. Consider a two-coloured hypergraph $\mathcal{H} = (V, E)$ on $n \geqslant 40ct^2$ vertices. Suppose that, for all $2t \leqslant m \leqslant m_{\max}$ and all $v \in V$, we have*

$$|U^{(b)}_{\leqslant m}(v)| \leqslant cm. \tag{2.1}$$

Then $\mathcal{H}$ has a red hedgehog H_t.

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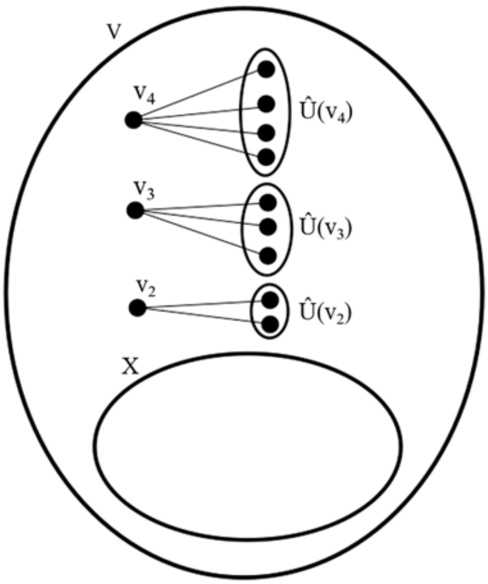

Figure 1. Peeling v_4, v_3, v_2 in Lemma 2.3.

Proof. It suffices to prove for $n = 40ct^2$, so assume without loss of generality that $n = 40ct^2$. Pick a random set S by including each vertex of V in S independently with probability $4t/n$. By the Chernoff bound, $\mathbb{P}[|S| \leqslant 3t] \leqslant e^{-t/8}$.

Fix m such that $2t \leqslant m \leqslant m_{\max}$ and m is a multiple of t. Let $e_1, \ldots, e_p$ be the pairs such that $d^{(r)}_{e_\ell} \leqslant m$ for all $\ell \in [p]$, and let $X_1, \ldots, X_p$ the indicator random variables for these pairs being in $\binom{S}{2}$. Let $X = X_1 + \cdots + X_p$. By (2.1), we have $p \leqslant cmn/2$. Each X_ℓ for $\ell \in [p]$ is a Bernoulli$(16t^2/n^2)$ random variable. Consider a graph on $[p]$ where ℓ and ℓ' are adjacent (written $\ell \sim \ell'$) if e_ℓ and $e_{\ell'}$ share a vertex. This is a valid dependency graph for $\{X_\ell\}$ as X_ℓ is independent of all $X_{\ell'}$ such that $e_{\ell'}$ is vertex-disjoint from e_ℓ. Furthermore, by the condition (2.1), each endpoint of any pair e_ℓ is in at most cm pairs, so each $\ell \in [p]$ has degree at most $2cm$ in the dependency graph, and the total number of pairs (ℓ, ℓ') such that $\ell \sim \ell'$ is at most $2cmp$. We have

$$\mathbb{E}[X] = \frac{16t^2 p}{n^2} = \frac{2p}{5cn} \leqslant \frac{m}{5} < \frac{3m}{4} - t, \tag{2.2}$$

$$\begin{aligned}
\mathrm{Var}[X] &= \sum_{\ell,\ell' \in [p]} \mathbb{E}[X_\ell X_{\ell'}] - \mathbb{E}[X_\ell]\mathbb{E}[X_{\ell'}] \\
&= \sum_{\ell \sim \ell'} \mathbb{E}[X_\ell X_{\ell'}] - \mathbb{E}[X_\ell]\mathbb{E}[X_{\ell'}] \\
&\leqslant 2cmp \cdot \left(\left(\frac{4t}{n}\right)^3 - \left(\frac{4t}{n}\right)^4 \right) \\
&< \frac{128t^3 cmp}{n^3} \\
&\leqslant \frac{64t^3 c^2 m^2}{n^2} \\
&= \frac{m^2}{25t}.
\end{aligned}$$

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Hence,

$$\mathbb{P}\left[\#\left\{uv \in \binom{S}{2} : d_{uv}^{(r)} \leqslant m\right\} > m - t\right] = \mathbb{P}[X > m - t]$$
$$= \mathbb{P}[X - \mathbb{E}[X] \geqslant m - t - \mathbb{E}[X]]$$
$$\leqslant \mathbb{P}[X - \mathbb{E}[X] \geqslant m/4]$$
$$\leqslant \frac{\mathrm{Var}[X]}{(m/4)^2}$$
$$< \frac{16}{25t}.$$

The first inequality is by (2.2) and the second is by Chebyshev's inequality. By the union bound over the multiples of t in $[2t, m_{\max}]$, of which there are less than t, the probability that there exists some $m \in [2t, m_{\max}]$ a multiple of t with

$$\#\left\{uv \in \binom{S}{2} : d_{uv}^{(r)} \leqslant m\right\} \leqslant m - t \tag{2.3}$$

is less than $t \cdot 16/(25t) = 16/25$. Again by the union bound, with probability more than $1 - (16/25 + e^{-t/8}) > 0$ over the randomness of S, we have (i) $|S| \geqslant 3t$, and (ii) for all m a multiple of t in $[2t, m_{\max}]$, (2.3) holds. Hence, there exists an S such that (i) and (ii) hold, so consider such an S. Remove $|S| - t \geqslant 2t$ vertices from S, at least one from each of the $2t$ pairs with smallest $d_{uv}^{(r)}$, to obtain a set of t vertices T such that, for all m a multiple of t in $[2t, m_{\max}]$, we have

$$\#\left\{uv \in \binom{T}{2} : d_{uv}^{(r)} \leqslant m\right\} \leqslant \max\left(0, \#\left\{uv \in \binom{S}{2} : d_{uv}^{(r)} \leqslant m\right\} - 2t\right) \leqslant \max(0, m - 3t).$$

Then, for all m with $2t \leqslant m \leqslant m_{\max} - t$, set m' to be the smallest multiple of t larger than m, so that

$$\#\left\{uv \in \binom{T}{2} : d_{uv}^{(r)} \leqslant m\right\} \leqslant \#\left\{uv \in \binom{T}{2} : d_{uv}^{(r)} \leqslant m'\right\} \leqslant \max(0, m' - 3t) \leqslant m - 2t. \tag{2.4}$$

Now, we show that our matching condition holds. Setting $m = 2t$ in (2.4), we have $d_{uv}^{(r)} > 2t$ for all $uv \in \binom{T}{2}$. Hence, for any non-empty subset $F \subset \binom{T}{2}$ of size at most t, any $uv \in F$ satisfies $d_{uv}^{(r)} > t + |F|$. If $F \subset \binom{T}{2}$ has size greater than t, then, by setting $m = t + |F|$ in (2.4), we know that there are at most $m - 2t = |F| - t$ pairs $uv \in F$ such that $d_{uv}^{(r)} \leqslant t + |F|$, so again there exists $uv \in F$ such that $d_{uv}^{(r)} > t + |F|$. We conclude that, for all non-empty subsets of pairs $F \subset \binom{T}{2}$, there exists $uv \in F$ such that $|N^{(r)}(F)| \geqslant d_{uv}^{(r)} \geqslant t + |F|$. By Lemma 2.2, there exists a red hedgehog with body T. □

2.2 Proof of Theorem 1.1

2.2.1 Proof outline

To prove Theorem 1.1, we follow the proof of Lemma 2.3. First, 'peel off' vertices v into a set S to try to find a blue or red hedgehog.[1] If we succeed, we are done. If we fail, we end up with an induced two-coloured hypergraph that is 'balanced' in the sense of Lemma 2.4. In this case, we simply apply Lemma 2.4.

In the proof of Lemma 2.3, we started with $X = V$ and iteratively removed from X a vertex v and a set $\hat{U}(v)$ of size t such that, for all $u \in \hat{U}(v)$, vertices u and v share many blue triples.

[1] For technical reasons, we peel vertices to find both blue and red hedgehogs, as opposed to Lemma 2.3 where we only peeled vertices to find a blue hedgehog.

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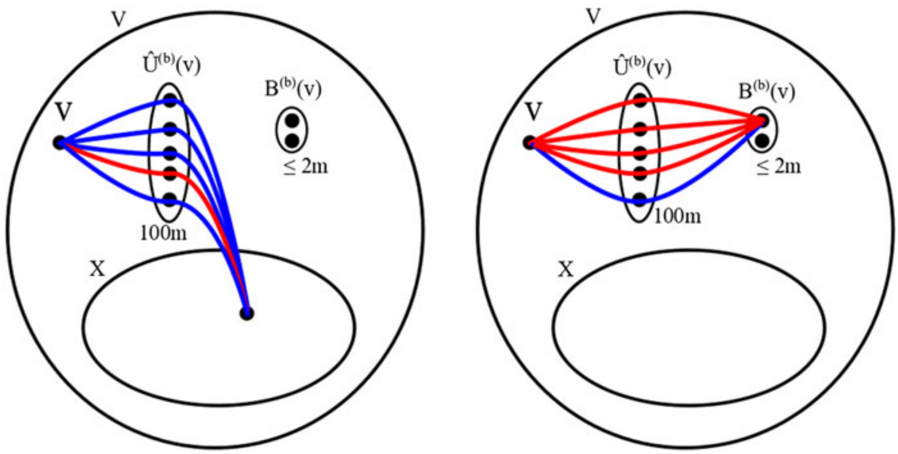

Figure 2. Peeling v with many blue-heavy neighbours. For every $w \in X$, edge $\{u, v, w\}$ is blue for many $u \in \hat{U}^{(b)}(v)$. Vertices $w \in B^{(b)}(v)$ are the exception. Ideally we simply delete vertex v, set $\hat{U}^{(b)}(v)$, and set $B^{(b)}(v)$ from X (depicted), but instead we maintain fractional penalties $\alpha^{(x)}(\cdot)$ and $\beta^{(x)}(\cdot)$. We have $|\hat{U}^{(b)}(v)| = 10m$ by definition, and $|B^{(b)}(v)| \leqslant 2m$ by Lemma 2.6.

This deletes $O(t)$ vertices per round, which is small enough for the argument to succeed. For general hypergraphs, we peel off vertices v with many 'blue-heavy neighbours', meaning there exists some m such that $|U^{(b)}_{\leqslant m}(v, X)| \geqslant 10m$.[2] However, m can be $\Theta(t^2)$, so if we simply deleted v along with $10m$ of its blue-heavy neighbours $\hat{U}^{(b)}(v) \subset U^{(b)}_{\leqslant m}(v, X)$, we could delete $\Theta(t^2)$ vertices for every v, which is too many. Instead, when we peel off v, we delete v from X, add a *penalty* of t/m to each $u \in \hat{U}^{(b)}(v)$, accumulated as $\alpha^{(b)}(u)$, and delete from X every vertex u with $\alpha^{(b)}(u) \geqslant 1/2$. With these penalties, we guarantee that, on average, we delete $O(t)$ vertices from X per peeled vertex v.

However, we need more care. In Lemma 2.3, we can find a hedgehog with body S because, for any peeled vertices $v, v' \in S$, the edges $\{u, v, v'\}$ are blue for *every* $u \in \hat{U}(v)$. However, in our procedure, for a v chosen with corresponding $\hat{U}^{(b)}(v)$ of size $10m$, there are some vertices w such that $\{u, v, w\}$ is blue for few (at most $4m$) vertices $u \in \hat{U}^{(b)}(v)$. We denote this set of 'bad' vertices by $B^{(b)}(v)$. As much as possible, we wish to avoid choosing both v and, at some later step, $w \in B^{(b)}(v)$ for the body $S^{(b)}$ of our blue hedgehog. Ideally, we simply delete all vertices $u \in B^{(b)}(v)$ in the step we peel off v. However, $B^{(b)}(v)$ can have $\Omega(m)$ vertices, which again could be too many if $m = \Theta(t^2)$. Instead, for each $w \in B^{(b)}(v)$ we add a *penalty* of $t/d^{(b)}_{wv}$, accumulated as $\beta^{(b)}(w)$, and delete from X every vertex w with $\beta^{(b)}(w) \geqslant 1/4$. We guarantee that, on average, we delete $O(t \ln t)$ vertices from X per peeled vertex v (Lemma 2.9). See Figure 2.

To finish the proof, we show that if our peeling produces a set $S^{(b)} = \{v_1, \ldots, v_t\}$ (where v_i is chosen before v_{i+1}), then, because we track the penalties $\alpha^{(b)}(u)$ and $\beta^{(b)}(w)$ carefully, the matching condition of Lemma 2.2 holds. On the other hand, if the peeling procedure fails, the sub-hypergraph induced by X is large and balanced, in which case we apply Lemma 2.4.

2.2.2 The peeling procedure

We now describe the procedure formally. Start with $S^{(b)} = S^{(r)} = \emptyset$, and $X = V$. For all $u \in V$, initialize $\alpha^{(r)}(u) = \alpha^{(b)}(u) = \beta^{(r)}(u) = \beta^{(b)}(u) = 0$. If, at any point, $S^{(b)}$ or $S^{(r)}$ has t vertices, stop.

[2] For technical reasons, we peel vertices v in increasing order of the corresponding m.

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Recall that $m_{\max} = 2t + \binom{t}{2}$. For $m = 2t, 2t+1, \ldots, m_{\max}$, do the following, which we refer to as Stage(m).

While there exists a vertex $v \in X$ and a colour $\chi \in \{b, r\}$ such that $|U^{(\chi)}_{\leqslant m}(v, X)| \geqslant 10m$:

(a) let $\hat{U}^{(\chi)}(v)$ be the set $U^{(\chi)}_{\leqslant m}(v, X)$ truncated to $10m$ vertices arbitrarily,
(b) let $B^{(\chi)}(v) = \{w : |u \in \hat{U}^{(\chi)}(v) : \{u, v, w\} \text{ is colour } \chi| \leqslant 4m\}$,
(c) add v to $S^{(\chi)}$,
(d) for all $u \in \hat{U}^{(\chi)}(v)$, add t/m to $\alpha^{(\chi)}(u)$,
(e) for all $w \in B^{(\chi)}(v)$, add $\min(1/4, t/d^{(\chi)}_{vw})$ to $\beta^{(\chi)}(w)$,
(f) delete from X all vertices u with $\alpha^{(\chi)}(u) \geqslant 1/2$ or $\beta^{(\chi)}(u) \geqslant 1/4$,
(g) delete v from X.

Note that $B^{(\chi)}(v)$ and $\hat{U}^{(\chi)}(v)$ are only defined for $v \in S^{(\chi)}$. We refer to steps (a)–(g) as the *peeling step* for v, denoted Peel(v). We let m_v denote the value such that the peeling step for v occurred during Stage(m_v), and call m_v the *peeling parameter* of v. Throughout the analysis, let X_v denote the set X immediately before Peel(v). For any $m \in [2t, m_{\max}]$, let X_m denote the set X immediately after Stage(m), so that $X_{m_{\max}}$ is the set X at the end of the peeling procedure.

The above process terminates in one of two ways. Either we 'get stuck', that is, we complete Stage$(m_{\max})$ and $|S^{(b)}| < t$ and $|S^{(r)}| < t$, or we 'finish', that is, we terminate earlier with $|S^{(b)}| = t$ or $|S^{(r)}| = t$. We show there is a monochromatic hedgehog in each case. In Section 2.2.5 we handle the case where we 'get stuck'. In Section 2.2.6 we handle the case where we 'finish'.

2.2.3 Basic facts about peeling

We first establish the following facts about the procedure.

Lemma 2.5. *For any m such that $2t \leqslant m \leqslant m_{\max}$, for any time in the procedure after* Stage(m)*, the following holds: for all colours $\chi \in \{b, r\}$, for all m' with $2t \leqslant m' \leqslant m$, and for all vertices $v \in X$, we have $|U^{(\chi)}_{\leqslant m'}(v, X)| < 10m'$.*

Proof. Fix m with $2t \leqslant m \leqslant m_{\max}$. We have $|U^{(\chi)}_{\leqslant m}(v, X_m)| < 10m$ for all $v \in X_m$. If not, then there exists a vertex $v \in X_m$ with $|U^{(\chi)}_{\leqslant m}(v, X_m)| \geqslant 10m$, in which case we would have peeled vertex v during Stage(m), and we would have deleted v from X_m during Peel(v), which is a contradiction. Throughout the procedure, X is non-increasing. Thus, at any point in the procedure after Stage(m), we have $X \subset X_m$, so for all $v \in X$, we have $v \in X_m$ and

$$|U^{(\chi)}_{\leqslant m}(v, X)| \leqslant |U^{(\chi)}_{\leqslant m}(v, X_m)| < 10m. \qquad \square$$

Lemma 2.6. *For all colours $\chi \in \{b, r\}$ and all vertices $v \in S^{(\chi)}$, we have $|B^{(\chi)}(v)| \leqslant 2m_v$.*

Proof. We prove this for $\chi = b$, and the case $\chi = r$ follows from symmetry. We double-count the number Z of red triples $\{u, v, w\}$ such that $u \in \hat{U}^{(b)}(v)$ and $w \in B^{(b)}(v)$. On one hand, every $u \in \hat{U}^{(b)}(v)$ is in at most m_v red triples because we chose $\hat{U}^{(b)}(v)$ as a subset of $U^{(b)}_{\leqslant m_v}(v, X_v)$, so the total number of red triples is at most $m_v \cdot |\hat{U}^{(b)}(v)| = 10m_v^2$. On the other hand, by definition of $B^{(b)}(v)$, each $w \in B^{(b)}(v)$ is in at least $|\hat{U}^{(b)}(v)| - 4m_v = 6m_v$ such red triples. Thus, the number of such triples is at least $|B^{(b)}(v)| \cdot 6m_v$. Hence, $10m_v^2 \geqslant Z \geqslant 6m_v|B^{(b)}(v)|$, so $|B^{(b)}(v)| \leqslant 2m_v$ as desired. $\square$

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Lemma 2.7. *For all colours $\chi \in \{b, r\}$ and all vertices $v, v' \in S^{(\chi)}$, we have $d^{(\chi)}_{vv'} \geqslant 4t$.*

Proof. Assume for the sake of contradiction that $d^{(\chi)}_{vv'} < 4t$. Without loss of generality, v was added to $S^{(\chi)}$ before v'. We have $d^{(\chi)}_{vv'} < 4t < 4m_v$, so during Peel($v$), vertex v' is included in $B^{(\chi)}(v)$. Hence, $\min(1/4, t/d^{(\chi)}_{vv'}) = 1/4$ is added to $\beta^{(\chi)}(v')$ during step (e) of Peel(v), so during step (f) of Peel(v), vertex v' is deleted from X if it has not been deleted already. Thus, we could not have added v' to $S^{(\chi)}$ after Peel(v), which is a contradiction, so $d^{(\chi)}_{vv'} \geqslant 4t$, as desired. □

2.2.4 Bounding the number of deleted vertices

Lemma 2.8. *For all colours $\chi \in \{b, r\}$ and all vertices $v \in S^{(\chi)}$, during* Peel(v), *the total increase in $\alpha^{(\chi)}(u)$ over all $u \in V$ is exactly* $10t$.

Proof. Fix $v \in S^{(\chi)}$. We have $|\hat{U}^{(\chi)}(v)| = 10m_v$ by definition, and, for $u \in \hat{U}^{(\chi)}(v)$, each $\alpha^{(\chi)}(u)$ increases by exactly t/m_v, for a total increase of $10m_v \cdot (t/m_v) = 10t$. □

Lemma 2.9. *For all colours $\chi \in \{b, r\}$ and all vertices $v \in S^{(\chi)}$, during* Peel(v), *the total increase in $\beta^{(\chi)}(w)$ over all $w \in V$ is at most* $20t \ln t$.

Proof. By symmetry, it suffices to prove the lemma for $\chi = b$. Let $v \in S^{(b)}$. For $m = 0, \dots, 4m_v$, let

$$a_m := \#\{w \in X_v : d^{(b)}_{vw} = m\},$$
$$a_{\leqslant m} := a_0 + a_1 + \cdots + a_m = |U^{(r)}_{\leqslant m}(v, X_v)|.$$

Peel(v) is after Stage($m_v - 1$). Hence, by Lemma 2.5, for $2t \leqslant m \leqslant m_v - 1$, we have $a_{\leqslant m} \leqslant 10m$. We know

$$|U^{(b)}_{\leqslant 4m_v}(v, X_v)| \geqslant |U^{(b)}_{\leqslant m_v}(v, X_v)| \geqslant 10m_v > 8m_v,$$

where the second inequality holds because v was chosen to be peeled in Stage(m_v). Hence, by Lemma 2.1,

$$a_{\leqslant 4m_v} = |U^{(r)}_{\leqslant 4m_v}(v, X_v)| \leqslant |U^{(r)}_{\leqslant 4m_v}(v)| \leqslant 8m_v.$$

As $a_{\leqslant m}$ is non-decreasing in m, we conclude $a_{\leqslant m} \leqslant 10m$ for $2t \leqslant m \leqslant 4m_v$.

For $m = 0, \dots, 4m_v$, for any w with $d^{(b)}_{vw} = m$, the peeling of v increases $\beta^{(b)}(w)$ by exactly $\min(1/4, t/m)$. Thus, for a_m many w, the penalty $\beta^{(b)}(w)$ increases by $\min(1/4, t/m)$. Furthermore, $\beta^{(b)}(w)$ increases only for $w \in B^{(b)}(v)$, which has at most $2m_v$ vertices by Lemma 2.6. For $2m_v - a_{\leqslant 4m_v}$ vertices w, $\beta^{(b)}(w)$ increases by less than $t/4m_v$, giving a total increase in $\beta^{(b)}(w)$ of less than t from those vertices. The total increase in $\beta^{(b)}(w)$ is thus less than

$$\frac{1}{4}(a_0 + a_1 + \cdots + a_{4t}) + \frac{a_{4t+1}t}{4t+1} + \cdots + \frac{a_{4m_v}t}{4m_v} + t. \tag{2.5}$$

The coefficients of $a_0, \dots, a_{4m_v}$ in (2.5) are non-increasing, so (2.5) is t plus a positive linear combination of $a_{\leqslant 4t}, a_{\leqslant 4t+1}, \dots, a_{\leqslant 4m_v}$. Subject to $a_{\leqslant m} \leqslant 10m$ for $2t \leqslant m \leqslant 4m_v$, all of $a_{\leqslant 4t}, a_{\leqslant 4t+1}, \dots, a_{\leqslant 4m_v}$ are simultaneously maximized if $a_0 = 0$ and $a_m = 10$ for $m = 1, \dots, 4m_v$, so (2.5) is maximized there as well. Hence,

$$\begin{aligned}\text{total increase in } \beta^{(b)}(w) &< \frac{1}{4}(a_0 + a_1 + \cdots + a_{4t}) + \frac{a_{4t+1}t}{4t+1} + \cdots + \frac{a_{4m_v}t}{4m_v} + t \\ &\leqslant t + \frac{1}{4} \cdot 40t + \frac{10t}{4t+1} + \frac{10t}{4t+2} + \cdots + \frac{10t}{4m_v}\end{aligned}$$

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$$\leqslant 11t + 10t \ln(4m_v/4t)$$
$$< 20t \ln t,$$

where, for the last inequality, we used $m_v \leqslant t^2$ and $t \geqslant 10$. This is what we wanted to show. □

Lemma 2.10. *The total number of vertices deleted from X in the peeling procedure is at most $200t^2 \ln t$.*

Proof. A vertex is deleted either for being added to $S^{(b)}$ or $S^{(r)}$, having $\alpha^{(b)}(\cdot)$ or $\alpha^{(r)}(\cdot)$ at least 1/2, or having $\beta^{(b)}(\cdot)$ or $\beta^{(r)}(\cdot)$ at least 1/4. At the end of the procedure, we have the following inequalities. For all $\chi \in \{b, r\}$ and all $u \in V$, we have that $\alpha^{(\chi)}(u)$ and $b^{(\chi)}(u)$ are initially 0 and increase only during the peeling step of some vertex $v \in S^{(\chi)}$. Hence, by Lemma 2.8, for $\chi \in \{b, r\}$,

$$\sum_{u \in V} \alpha^{(\chi)}(u) = 10t \cdot |S^{(\chi)}| \leqslant 10t^2.$$

Furthermore, by Lemma 2.9, for $\chi \in \{b, r\}$,

$$\sum_{u \in V} \beta^{(\chi)}(u) \leqslant 20t \ln t \cdot |S^{(\chi)}| \leqslant 20t^2 \ln t.$$

We conclude that, at the end of the procedure,

$$\begin{aligned}
\#\{\text{deleted } u\} &\leqslant |S^{(b)}| + |S^{(r)}| + \#\{u : \alpha^{(b)}(u) \geqslant 1/2\} + \#\{u : \alpha^{(r)}(u) \geqslant 1/2\} \\
&\quad + \#\{u : \beta^{(b)}(u) \geqslant 1/4\} + \#\{u : \beta^{(r)}(u) \geqslant 1/4\} \\
&< 2t + \sum_{u \in V} (2\alpha^{(b)}(u) + 2\alpha^{(r)}(u) + 4\beta^{(b)}(u) + 4\beta^{(r)}(u)) \\
&\leqslant 2t + 2 \cdot 10t^2 + 2 \cdot 10t^2 + 4 \cdot 20t^2 \ln t + 4 \cdot 20t^2 \ln t \\
&< 200t^2 \ln t.
\end{aligned}$$

□

2.2.5 Case 1: Peeling procedure gets stuck

By Lemma 2.10, the number of vertices deleted in the peeling process is at most $200t^2 \ln t$, so, at the end of the peeling procedure, $|X| \geqslant (200t^2 \ln t + 400t^2) - 200t^2 \ln t = 400t^2$.

Consider the complete two-coloured sub-hypergraph $\mathcal{H}'$ of $\mathcal{H}$ induced by the vertex set X. By Lemma 2.5, at the end of the procedure, for all $m = 2t, 2t+1, \ldots, m_{\max}$ and all $v \in X$,

$$|U^{(b)}_{\leqslant m}(v, X)| < 10m, \quad |U^{(r)}_{\leqslant m}(v, X)| < 10m.$$

Applying Lemma 2.4 to $\mathcal{H}'$ with $c = 10$, we conclude $\mathcal{H}'$ (and hence $\mathcal{H}$) has a red hedgehog H_t.[3]

2.2.6 Case 2: Peeling procedure finishes

Suppose we finish with $|S^{(b)}| = t$. The analysis for $|S^{(r)}| = t$ is symmetrical. We try to find a blue hedgehog. For brevity, in the rest of this section, let $S = S^{(b)}$. Let $S = \{v_1, \ldots, v_t\}$, where the v_i were chosen in the order $v_1, \ldots, v_t$. For $i = 1, \ldots, t$, let $m_i = m_{v_i}$ be the peeling parameter for v_i, so that $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_t$.

Definition. Call a pair $v_i v_j \in \binom{S}{2}$ with $i < j$ *bad* if $v_j \in B^{(b)}(v_i)$. Otherwise, call $v_i v_j \in \binom{S}{2}$ *good*. Let $E_{\text{bad}} \subset \binom{S}{2}$ be the set of all bad pairs and let $E_{\text{good}} \subset \binom{S}{2}$ be the set of all good pairs, so that $\binom{S}{2} = E_{\text{bad}} \cup E_{\text{good}}$ is a partition.

[3] By the same reasoning, $\mathcal{H}'$ also has a blue hedgehog.

Lemma 2.11.

$$\sum_{v_iv_j\in E_{\text{bad}}} \frac{1}{d^{(b)}_{v_iv_j}} < \frac{1}{4}.$$

Proof. Fix $2 \leqslant j \leqslant t$. Consider all bad pairs v_iv_j with $i < j$. At the peeling of v_j, $\beta(v_j) < 1/4$, otherwise v_j would have been deleted from X and we could not have peeled v_j. Hence, at the peeling of v_j,

$$\frac{1}{4} > \beta^{(b)}(v_j) = \sum_{\substack{i\,:\,i<j,\\ v_j\in B^{(b)}(v_i)}} \min\left(\frac{1}{4}, \frac{t}{d^{(b)}_{v_iv_j}}\right) = \sum_{\substack{i\,:\,i<j,\\ v_iv_j\in E_{\text{bad}}}} \min\left(\frac{1}{4}, \frac{t}{d^{(b)}_{v_iv_j}}\right) = \sum_{\substack{i\,:\,i<j,\\ v_iv_j\in E_{\text{bad}}}} \frac{t}{d^{(b)}_{v_iv_j}}.$$

The first equality is by definition of $\beta^{(b)}(v_j)$, the second is by definition of E_{bad}, and the last is because $d^{(b)}_{v_iv_j} \geqslant 4t$ for all $i < j$ by Lemma 2.7. Thus,

$$\sum_{v_iv_j\in E_{\text{bad}}} \frac{1}{d^{(b)}_{v_iv_j}} = \sum_{j=2}^{t} \sum_{\substack{i\,:\,i<j,\\ v_iv_j\in E_{\text{bad}}}} \frac{1}{d^{(b)}_{v_iv_j}} \leqslant \sum_{j=2}^{t} \frac{1}{4t} < \frac{1}{4}.$$ □

We prove that there is a blue hedgehog with body S, by showing that the matching condition of Lemma 2.2 holds. Consider an arbitrary $F \subset \binom{S}{2}$. Partition $F = F_{\text{bad}} \cup F_{\text{good}}$, where $F_{\text{bad}} = F \cap E_{\text{bad}}$ and $F_{\text{good}} = F \cap E_{\text{good}}$. We wish to show that $N^{(b)}(F) \geqslant |F| + t$.

Subcase 1.

$|F_{\text{bad}}| \geqslant |F_{\text{good}}|$. By Lemma 2.11,

$$\frac{|F_{\text{bad}}|}{\max_{v_iv_j\in F_{\text{bad}}} d^{(b)}_{v_iv_j}} \leqslant \sum_{v_iv_j\in F_{\text{bad}}} \frac{1}{d^{(b)}_{v_iv_j}} \leqslant \sum_{v_iv_j\in E_{\text{bad}}} \frac{1}{d^{(b)}_{v_iv_j}} < \frac{1}{4}.$$

Thus, there exists some $v_iv_j \in F_{\text{bad}}$ such that $d^{(b)}_{v_iv_j} > 4|F_{\text{bad}}|$. Furthermore, this v_iv_j satisfies $d^{(b)}_{v_iv_j} \geqslant 4t$ by Lemma 2.7, so $d^{(b)}_{v_iv_j} \geqslant 2|F_{\text{bad}}| + 2t$. Hence,

$$|N^{(b)}(F)| \geqslant d^{(b)}_{v_iv_j} \geqslant 2|F_{\text{bad}}| + 2t \geqslant |F_{\text{bad}}| + |F_{\text{good}}| + 2t > |F| + t,$$

as desired. The first inequality is because the blue edges containing v_iv_j are all elements of $N^{(b)}(F)$. The second inequality is because $d^{(b)}_{v_iv_j}$ is at least $4|F_{\text{bad}}|$ and at least $4t$ by above. The third inequality is by the assumption $|F_{\text{bad}}| \geqslant |F_{\text{good}}|$. The fourth inequality is because $|F| = |F_{\text{bad}}| + |F_{\text{good}}|$ and $2t > t$.

Subcase 2.

$|F_{\text{bad}}| < |F_{\text{good}}|$. In particular, $|F_{\text{good}}| > 0$, so $|F|$ has some good pair v_iv_j with $i < j$. This pair is in at least $4m_i \geqslant 8t$ blue triples, so $|N^{(b)}(F)| \geqslant 8t$.

Let I be the set of all indices i such that there exists j with $i < j \leqslant t$ with $v_iv_j \in F_{\text{good}}$. For each i, there are less than t indices j such that $i < j \leqslant t$, so

$$|I| \cdot t > |F_{\text{good}}|. \tag{2.6}$$

For each $i \in I$, arbitrarily fix $j_i > i$ such that $v_iv_{j_i}$ is good. For $i \in I$, define

$$U^*_i := N^{(b)}(\{v_iv_{j_i}\}) \cap \hat{U}^{(b)}(v_i), \quad U^*_I := \bigcup_{i\in I} U^*_i,$$

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so that $U_I^* \subset N^{(b)}(F)$. For all $i \in I$, the pair $v_i v_{j_i}$ is good, so $v_{j_i} \notin B^{(b)}(v_i)$. Hence, by the definition of $B^{(b)}(v_i)$, there are more than $4m_i$ vertices $u \in \hat{U}^{(b)}(v_i)$ such that $\{u, v_i, v_{j_i}\}$ is blue. Thus, for all $i \in I$, the set U_i^* has at least $4m_i$ vertices. In the peeling of v_i, the penalty $\alpha^{(b)}(u)$ increases by t/m_i for each $u \in U_i^*$. Hence, in peeling v_i, the sum of penalties $\sum_{u \in U_I^*} \alpha^{(b)}(u)$, increases by at least $4m_i \cdot t/m_i = 4t$. Thus,

$$4t \cdot |I| \leqslant \sum_{u \in U_I^*} \alpha^{(b)}(u). \tag{2.7}$$

On the other hand, the vertex u is deleted from X whenever $\alpha^{(b)}(u) \geqslant 1/2$, the penalty $\alpha^{(b)}(u)$ increases by at most $t/2t = 1/2$ in any peeling step, and the penalty $\alpha^{(b)}(u)$ never changes after u is deleted from X. Thus, for all vertices $u \in V$, we have

$$\alpha^{(b)}(u) \leqslant 1. \tag{2.8}$$

We conclude that

$$2|F| \leqslant 4|F_{\text{good}}| \leqslant 4t|I| \leqslant \sum_{u \in U_I^*} \alpha^{(b)}(u) \leqslant \sum_{u \in U_I^*} 1 = |U_I^*| \leqslant |N^{(b)}(F_{\text{good}})| \leqslant |N^{(b)}(F)|.$$

The first inequality is by the assumption $|F_{\text{bad}}| < |F_{\text{good}}|$, the second is by (2.6), the third is by (2.7), the fourth is by (2.8), the fifth is by $U_I^* \subset N^{(b)}(F_{\text{good}})$, and the sixth is by $F_{\text{good}} \subset F$. Combining with $|N^{(b)}(F)| \geqslant 8t$, we conclude $|N^{(b)}(F)| \geqslant |F| + t$, as desired.

This covers all subcases, so we have proved that, for any non-empty subset $F \subset \binom{S}{2}$, we have $N^{(b)}(F) \geqslant |F| + t$. Hence, the matching condition of Lemma 2.2 holds, so there is a blue hedgehog with body S, as desired. This completes the proof of Theorem 1.1. □

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