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Application of the Abel integral equation to an inverse problem in thermoelectricity

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This paper deals with a new method to determine the dependence of the electrical conductivity of metals or semiconductors on temperature. It is based on the fact that the current–voltage relationship is easily measurable. This inverse problem is solved by the classical Abel integral equation.

1 Introduction

The ratio between the electric conductivity σ and the thermal conductivity κ is the same for all metals and depends linearly on the absolute temperature u:

$$\frac{\kappa}{\sigma} = Lu, \tag{1.1}$$

where L is the Lorenz number. This law, found experimentally by G. Wiedemann and R. Franz [12], is only approximately true for semiconductors, ceramics and polymers for which, however, a 'generalised Wiedemann–Franz law' holds:

$$\frac{\kappa}{\sigma} = L(u)u, \tag{1.2}$$

where $L(u)$ is the Lorenz function (see e.g. [7]). On the other hand, the dependence on the temperature of the electric and thermal conductivities is quite simple in metals; more precisely we have [6]

$$\sigma(u) = \frac{C_\sigma}{u},\ \kappa(u) = C_\kappa, \tag{1.3}$$

where C_σ and C_κ are positive constants which depend on the metal, whereas in semiconductors the electric and thermal conductivities laws are more complex.

If we assume $\sigma(u)$ and $\kappa(u)$ to be given functions of the temperature, the electric potential $\varphi(\mathbf{x})$, $\mathbf{x} = (x, y, z)$ and the temperature $u(\mathbf{x})$ inside a three-dimensional conductor to which a difference of potential V is applied are determined by the following boundary value problem (P) [6]:

$$\nabla \cdot (\sigma(u)\nabla v) = 0 \text{ in } \Omega, \tag{1.4}$$

$$\nabla \cdot (\kappa(u)\nabla u) + \sigma(u)\beta(u)\nabla u \cdot \nabla v + \sigma(u)|\nabla v|^2 = 0 \text{ in } \Omega, \tag{1.5}$$

$$v = 0 \text{ on } \Gamma_1,\ v = V \text{ on } \Gamma_2,\ \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_0, \tag{1.6}$$

$$u = \bar{u} \text{ on } \Gamma_1 \cup \Gamma_2,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_0,\ \bar{u} > 0. \tag{1.7}$$

v is the effective potential, given by

$$v = \varphi + \int_{\bar{u}}^{u} \alpha(t)\, dt, \tag{1.8}$$

where $\alpha(u)$ (also a given function of the temperature) accounts for the Thomson effect and

$$\beta(u) = u\alpha'(u). \tag{1.9}$$

In (1.5) and (1.5) Ω is an open and bounded subset of $\mathbf{R}^3$ with a regular boundary Γ consisting of three parts Γ_i, $i = 0, 1, 2$. Γ_1 and Γ_2 are disjoint and represent the electrodes to which the difference of potential V is applied, and Γ_0 is the electrically and thermally insulated part of the body. We note that, in view of the maximum principle, we have $u(\mathbf{x}) \geqslant \bar{u}$ for any regular solution of problem (P). Moreover, in problem (P) the temperature is assumed to be the constant $\bar{u}$ on $\Gamma_1 \cup \Gamma_2$. Thus the boundary conditions are quite special; however, with this assumption, a nearly complete integration of this boundary value problem is possible.

In Section 2, after explaining how problem (P) can be integrated, we address the following inverse problem: reconstruct $\sigma(u)$ from the easily measured current–voltage relation $I = f(V)$ and from the knowledge of the Lorenz function $L(u)$ entering in (1.2). I is the total current crossing the device and is given by

$$I = \int_{\Gamma_2} \mathbf{J} \cdot \mathbf{n}\, dS, \tag{1.10}$$

where $\mathbf{n}$ is the exterior pointing unit vector normal to $\Gamma_1 \cup \Gamma_2$ and

$$\mathbf{J} = -\sigma(u)\nabla v \tag{1.11}$$

is the current density. We prove that $\sigma(u)$ is the solution of a generalised Abel integral equation [11]

$$\mathscr{F}(x) = \int_0^x \frac{G(x,z)\sigma(z)\, dz}{\sqrt{x - z}}. \tag{1.12}$$

If we neglect the Thompson effect, (1.12) reduces to the classical Abel equation [1]

$$\mathscr{F}(x) = \int_0^x \frac{\sigma(z)\, dz}{\sqrt{x - z}}, \tag{1.13}$$

which has a simple explicit solution [9, 10]. In Sections 3 and 4 we deal with two special cases. As pointed out by a referee, the integral solution of this paper could be useful in the so-called ‘spot-welding problem’.

2 Solution of the direct and of the inverse problems

Let us assume $\sigma(u)$, $\kappa(u)$ and $\beta(u)$ to be given $C^1([\bar{u},\infty))$ functions satisfying $\sigma(u) > 0$, $\kappa(u) > 0$ for $u \geqslant \bar{u}$ and

$$\int_{\bar{u}}^{\infty} \frac{\kappa(t)}{\sigma(t)}\,dt = \infty, \quad \left|\frac{\beta(u)\sigma(u)}{\kappa(u)}\right| \leqslant C_1 \text{ for all } u \geqslant \bar{u}. \tag{2.1}$$

If $(v(\mathbf{x}), u(\mathbf{x}))$ is a solution of problem (P), by (1.5) we have

$$\nabla \cdot (v\sigma(u)\nabla v) = \sigma(u)|\nabla v|^2$$

and

$$\nabla \cdot \left[\sigma(u)\left(\int_{\bar{u}}^{u} \beta(t)\,dt\right)\nabla v\right] = \sigma(u)\beta(u)\nabla u \cdot \nabla v.$$

Therefore, (1.5) can be rewritten in divergence form as follows:

$$\nabla \cdot \left\{\sigma(u)\left[v\nabla v + \frac{\kappa(u)}{\sigma(u)}\nabla u + \int_{\bar{u}}^{u} \beta(t)\,dt\nabla v\right]\right\} = 0. \tag{2.2}$$

In view of the special boundary conditions of problem (P), we make the 'ansatz' of the existence between u and v of a functional relation $u = U(v)$ so that $u(\mathbf{x}) = U(v(\mathbf{x}))$. Define

$$\theta = \frac{v^2}{2} + \int_{\bar{u}}^{u} \frac{\kappa(t)}{\sigma(t)}\,dt + \int_0^v \left[\int_{\bar{u}}^{U(\xi)} \beta(t)\,dt\right] d\xi. \tag{2.3}$$

Thus (2.2) becomes

$$\nabla \cdot (\sigma(u)\nabla\theta) = 0. \tag{2.4}$$

Moreover, $\theta(\mathbf{x})$ satisfies the boundary conditions

$$\theta = 0 \text{ on } \Gamma_1, \ \theta = \tilde{C} \text{ on } \Gamma_2, \ \frac{\partial\theta}{\partial n} = 0 \text{ on } \Gamma_0, \tag{2.5}$$

where

$$\tilde{C} = \frac{V^2}{2} + \int_0^V \left[\int_{\bar{u}}^{U(\xi)} \beta(t)\,dt\right] d\xi.$$

Since the functional relation $U(v)$ is not known, $\tilde{C}$ is an unknown constant. We claim that θ and v are related by the functional relation

$$\theta = \frac{\tilde{C}}{V} v. \tag{2.6}$$

Indeed, we have

$$\nabla \cdot (\sigma(U(v))\nabla\theta) = 0 \text{ in } \Omega, \ \theta = 0 \text{ on } \Gamma_1, \ \theta = \tilde{C} \text{ on } \Gamma_2, \ \frac{\partial\theta}{\partial n} = 0 \text{ on } \Gamma_0 \tag{2.7}$$

and

$$\nabla \cdot (\sigma(U(v))\nabla v) = 0 \text{ in } \Omega, \ v = 0 \text{ on } \Gamma_1 \ v = V \text{ on } \Gamma_2, \ \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_0. \tag{2.8}$$

This system is uncoupled, and it is easy to see that the solution $v(\mathbf{x})$ of (2.8) is unique as follows. Define the Kirchhoff transformation

$$w = \mathcal{Q}(v), \ \mathcal{Q}(v) = \int_0^v \sigma(U(t))\,dt.$$

$\mathcal{Q}$ maps $[0, V]$ one-to-one onto $[0, \mathcal{Q}(V)]$. If $w(\mathbf{x})$ is the solution of the mixed problem

$$\Delta w = 0 \text{ in } \Omega, \ w = 0 \text{ on } \Gamma_1, \ w = \mathcal{Q}(V) \text{ on } \Gamma_2, \ \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_0,$$

then $v(\mathbf{x}) = \mathcal{Q}^{-1}(w(\mathbf{x}))$ is the only solution of the problem (2.8). Thus we obtain (2.6) since $\theta(\mathbf{x}) = \frac{\tilde{C}}{V} v(\mathbf{x})$ solves (2.7). From (2.3) we have, setting $\gamma = \frac{\tilde{C}}{V}$,

$$\gamma v = \frac{v^2}{2} + \int_{\bar{u}}^{u} \frac{\kappa(t)}{\sigma(t)}\,dt + \int_0^v \left[\int_{\bar{u}}^{U(\xi)} \beta(t)\,dt \right] d\xi. \tag{2.9}$$

Taking the derivative with respect to v of (2.9) and defining

$$B(u) = \int_{\bar{u}}^{u} \beta(t)\,dt,$$

we obtain for $U(v)$ the ordinary differential equation

$$\frac{\kappa(U)}{\sigma(U)} \frac{dU}{dv} = \gamma - v - B(U), \tag{2.10}$$

which must be supplemented with the conditions

$$U(0) = \bar{u}, \ U(V) = \bar{u}. \tag{2.11}$$

In view of (2.1), the two-point boundary value problem (2.10), (2.11) has one and only one solution $U(v)$. We refer to [2] for the proof. Once $U(v)$ is known, problem (P) is easily solved. Indeed, let us consider again the problem

$$\nabla \cdot (\sigma(U(v)) \nabla v) = 0 \text{ in } \Omega, \ v = 0 \text{ on } \Gamma_1, \ v = V \text{ on } \Gamma_2, \ \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_0. \tag{2.12}$$

Define

$$\psi = G(v), \ G(v) = \int_0^v \sigma(U(\xi))\,d\xi, \ \psi_2 = G(V). \tag{2.13}$$

G maps $[0, V]$ one-to-one onto $[0, \psi_2]$. If $\psi(\mathbf{x})$ solves the problem

$$\Delta \psi = 0 \text{ in } \Omega, \ \psi = 0 \text{ on } \Gamma_1, \ \psi = \psi_2 \text{ on } \Gamma_2, \ \frac{\partial \psi}{\partial n} = 0 \text{ on } \Gamma_0, \tag{2.14}$$

by the maximum principle we have

$$0 \leqslant \psi(\mathbf{x}) \leqslant \psi_2 \text{ in } \bar{\Omega}. \tag{2.15}$$

Defining

$$v(\mathbf{x}) = G^{-1}(\psi(\mathbf{x})) \text{ and } u(\mathbf{x}) = U(v(\mathbf{x})), \tag{2.16}$$

we obtain a solution to problem (P). Let $w(\mathbf{x})$ be the solution of the problem

$$\Delta w = 0 \text{ in } \Omega,\ w = 0 \text{ on } \Gamma_1,\ w = 1 \text{ on } \Gamma_2,\ \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_0, \tag{2.17}$$

then

$$\psi(\mathbf{x}) = \psi_2 w(\mathbf{x}). \tag{2.18}$$

Define

$$k = \int_{\Gamma_2} \frac{\partial w}{\partial n}\, dS. \tag{2.19}$$

By (2.13) and (2.18), we have

$$\nabla\psi = \psi_2 \nabla w = \sigma(u(\mathbf{x}))\nabla v. \tag{2.20}$$

Hence, the total current crossing the conductor is

$$I = k \int_0^V \sigma(U(\xi))\, d\xi. \tag{2.21}$$

As an example we apply the theory to the simple, but important case of metals in which the Wiedemann–Franz law (1.1) holds, the Thomson effect is neglected, $\beta(u) = 0$ and

$$\sigma(u) = \frac{C_\sigma}{u},\ \kappa(u) = C_\kappa,\ L = \frac{C_\kappa}{C_\sigma}, \tag{2.22}$$

where C_σ and C_κ are constants typical of the metal. Problem (2.10), (2.11) becomes

$$LU\frac{dU}{dv} = \gamma - v,\ U(0) = \bar{u},\ U(V) = \bar{u},$$

which is easily solved by separation of variables and has as the solution

$$U(v) = \sqrt{\frac{L\bar{u}^2 + Vv - v^2}{L}}. \tag{2.23}$$

Computing the total current with (2.21), we find

$$I = 2k\sqrt{L}C_\sigma \arctan\left[\frac{V}{2\sqrt{L\bar{u}^2}}\right]. \tag{2.24}$$

Thus the total current remains bounded as $V \to \infty$ in sharp contrast with the total current $I = k\sigma V$, which we obtain if σ is a constant.

To treat the inverse problem of reconstructing $\sigma(u)$ we shall use the two following elementary lemmas.

Lemma 1 *Let $y = f(x) \in C^3((-\delta, \delta))$ be such that*

$$f(0) = 0,\ f'(0) = 0,\ f''(0) > 0. \tag{2.25}$$

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Let $\mathscr{H}_1(y)$ *and* $\mathscr{H}_2(y)$ *be the two branches of the local inverse function of* $y = f(x)$ *for* $0 \leqslant y \leqslant \mu$, $0 \leqslant x \leqslant \eta$, *and* $0 \leqslant y \leqslant \mu$, $-\eta \leqslant x \leqslant 0$, *respectively,* $\mu > 0$, $\eta > 0$. *Then*

$$\lim_{y\to 0^+} \mathscr{H}_1'(y)\sqrt{y} = (2f''(0))^{-1/2}, \quad \lim_{y\to 0^+} \mathscr{H}_2'(y)\sqrt{y} = -(2f''(0))^{-1/2}. \tag{2.26}$$

Proof We have

$$\lim_{y\to 0^+} \frac{\sqrt{y}}{\mathscr{H}_1(y)} = \lim_{x\to 0} \sqrt{\frac{f(x)}{x^2}} = \sqrt{\frac{f''(0)}{2}}. \tag{2.27}$$

On the other hand,

$$\sqrt{y}\mathscr{H}_1'(y) = \frac{\sqrt{y}}{f''(0)\mathscr{H}_1(y) + \lambda(\mathscr{H}_1(y))}, \tag{2.28}$$

for some function $\lambda(x)$ satisfying

$$\lim_{y\to 0^+} \frac{\lambda(\mathscr{H}_1(y))}{\mathscr{H}_1(y)} = 0. \tag{2.29}$$

Hence $(2.26)_1$ follows from (2.27), (2.28) and (2.29). Similarly we obtain $(2.26)_2$. □

Lemma 2 *Let* $U(v)$ *be the solution of problem (2.10), (2.11). Then*

$$U(v) \ \textit{has only one point of maximum}\ v_{\mathrm{M}} \in (0, V), \tag{2.30}$$

$$U'(v) > 0 \ \textit{for}\ v \in [0, v_{\mathrm{M}}),\ \ U'(v) < 0 \ \textit{for}\ v \in (v_{\mathrm{M}}, V]. \tag{2.31}$$

Proof From (2.10) it follows that

$$\text{if } v^* \in (0, V) \text{ and } U'(v^*) = 0 \text{ then } U''(v^*) < 0. \tag{2.32}$$

Let v_{m} and v_{M} be, respectively, the points of absolute minimum and maximum of $U(v)$ in $[0, V]$; $v_{\mathrm{m}} \notin (0, V)$, since, if $v_{\mathrm{m}} \in (0, V)$ we have $U'(v_{\mathrm{m}}) = 0$ and $U''(v_{\mathrm{m}}) \geqslant 0$ contradicting (2.32). Hence $v_{\mathrm{m}} \in \{0, V\}$, $U(v) > \bar{u}$ in $(0, V)$ and $v_{\mathrm{M}} \in (0, V)$. On the other hand, $U(v)$ cannot have points of relative maximum or minimum in $(0, V)$. Relative minima are excluded by (2.32) and relative maxima v_{rM} are also not possible since they would imply the existence of a relative minimum between v_{rM} and v_{M}. □

Theorem 1 *Let* $U(v, V)$ *be the solution to problem (2.10), (2.11). Then there exists a continuous function* $G(x, z)$, $0 \leqslant z < x$, *such that*

$$\int_0^V \sigma(U(v, V))\, dv = \int_0^{M(V)} \frac{G(M(V), z)\sigma(z)}{\sqrt{M(V) - z}}\, dz \tag{2.33}$$

with

$$M(V) = U(v_{\mathrm{M}}(V), V). \tag{2.34}$$

Moreover,

$$G(z, z) \neq 0. \tag{2.35}$$

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Proof Let $v = \mathscr{H}_1(z, V)$, $0 \leqslant z < M(V)$, $0 \leqslant v < v_M$ and $v = \mathscr{H}_2(z, V)$, $0 \leqslant z < M(V)$, $v_M < v \leqslant V$, be the two branches of the inverse function of $U(v, V)$. By (2.26)

$$\lim_{z \to M(V)^-} \mathscr{H}_i'(z, V)\sqrt{M(V) - z} \neq 0, \; i = 1, 2. \tag{2.36}$$

We have

$$\int_0^V \sigma(U(v, V))\, dv = \int_0^{v_M} \sigma(U(v, V))\, dv + \int_{v_M}^V \sigma(U(v, V))\, dv. \tag{2.37}$$

By (2.26) the functions $\mathscr{H}_1(z, V)$ and $\mathscr{H}_2(z, V)$ are absolutely continuous. Therefore, the substitution $z = U(v, V)$ in (2.37) is admissible. This gives

$$\int_0^V \sigma(U(v, V))\, dv = \int_0^{M(V)} \sigma(z)\left[\mathscr{H}_1'(z, V) - \mathscr{H}_2'(z, V)\right] dz. \tag{2.38}$$

By Lemmas 1 and 2, there exists a continuous function $G(x, z)$ such that

$$\left(\mathscr{H}_1'(z, V) - \mathscr{H}_2'(z, V)\right)\sqrt{M(V) - z} = G(M(V), z). \tag{2.39}$$

Thus (2.33) and (2.35) follow. □

Let us assume that

$$x = M(V) \text{ is globally invertible} \tag{2.40}$$

and define the function

$$\mathscr{F}(x) = \frac{1}{k} f(M^{-1}(x)). \tag{2.41}$$

Using (2.33) the inverse problem of reconstructing $\sigma(u)$ from the current–voltage characteristic $I = f(V)$ is reduced to the search of a solution of the generalised Abel integral equation

$$\mathscr{F}(x) = \int_0^x \frac{G(x, z)\sigma(z)}{\sqrt{x - z}}\, dz, \tag{2.42}$$

which in turn can be transformed into a Volterra integral equation of the first kind (see [11] p. 60) for which a fully developed theory exists [10]. In particular, in view of (2.35), the integral equation (2.42) has one and only one solution. The condition (2.40), probably true in general, is certainly satisfied in the following two important cases in which (2.10) has an integrating factor.

3 Determination of the electric conductivity in the thermistor problem

In this section the generalised Wiedemann–Franz law (1.2) is assumed and the Thomson effect neglected; thus we set $\beta(u) = 0$. In this case problem (P) is known as the 'thermistor problem' and has been thoroughly investigated in recent years. We refer in this connection to the books [5] and [8]. In this case problem (2.10), (2.11) is reduced to

$$\frac{\kappa(U)}{\sigma(U)}\frac{dU}{dv} = \gamma - v, \;\; U(0) = \bar{u}, \;\; U(V) = \bar{u}. \tag{3.1}$$

If we define a new scale for the temperature setting

$$\xi = F(U) = \int_{\bar{u}}^{U} \frac{\kappa(t)}{\sigma(t)}\, dt, \;\; U \geqslant \bar{u}, \tag{3.2}$$

problem (3.1) is easily solved and we find as solution

$$U(v, V) = F^{-1}\left(\frac{Vv}{2} - \frac{v^2}{2}\right). \tag{3.3}$$

We note that the constant γ disappears in (3.3) since it has been used to satisfy, together with the constant of integration, the two boundary conditions of problem (3.1). To solve the integral equation

$$\frac{1}{k} f(V) = \int_0^V \sigma\left(F^{-1}\left(\frac{Vv}{2} - \frac{v^2}{2}\right)\right) dv \tag{3.4}$$

in the unknown $\sigma(u)$ it is more convenient to proceed in a slightly different way from that of Theorem 1. So, let us define

$$\tilde{\sigma}(u) = \sigma(F^{-1}(u)). \tag{3.5}$$

Thus (3.4) can be rewritten as

$$\frac{1}{k} f(V) = \int_0^V \tilde{\sigma}\left(\frac{Vv}{2} - \frac{v^2}{2}\right) dv.$$

With the change of variable of integration $\zeta = v - \frac{V}{2}$ we obtain

$$\frac{1}{k} f(V) = 2 \int_0^{V/2} \tilde{\sigma}\left(\frac{V^2}{8} - \frac{\zeta^2}{2}\right) d\zeta. \tag{3.6}$$

With the further change of variable

$$z = \frac{V^2}{8} - \frac{\zeta^2}{2}, \tag{3.7}$$

we have

$$\frac{1}{k} f(V) = \sqrt{2} \int_0^{\frac{V^2}{8}} \frac{\tilde{\sigma}(z)}{\sqrt{\frac{V^2}{8} - z}}\, dz. \tag{3.8}$$

Setting

$$x = \frac{V^2}{8} \text{ and } \mathscr{F}(x) = \frac{1}{k\sqrt{2}} f(2\sqrt{2x}), \tag{3.9}$$

(3.8) transforms into

$$\mathscr{F}(x) = \int_0^x \frac{\tilde{\sigma}(z)}{\sqrt{x - z}}\, dz. \tag{3.10}$$

In this case we arrive at the classical Abel equation. Assuming $\mathscr{F}(x)$ to be of class C^1, we have from [9], since $\mathscr{F}(0) = 0$,

$$\tilde{\sigma}(x) = \frac{1}{\pi} \int_0^x \frac{\mathscr{F}'(t)}{\sqrt{x - t}}\, dt \tag{3.11}$$

or, in terms of the original scale of temperature,

$$\sigma(u) = \frac{1}{\pi} \int_0^{F(u)} \frac{\mathscr{F}'(t)}{\sqrt{F(u) - t}}\, dt. \tag{3.12}$$

Remark. In (3.12) σ seems to depend on the constants $\bar{u}$ and k. However, one has to remember that $f(u)$ is an experimental datum which in turn varies with $\bar{u}$ and k. But if, for example, we take $I = f(V)$ with f given by (2.24), from (3.12) we obtain $\sigma(u) = C_\sigma/u$, which depends neither on $\bar{u}$ nor on k.

4 The inverse problem with the Thomson effect

A second case of integrability of (2.10) occurs when

$$\frac{\kappa(u)}{\sigma(u)} = Lu \text{ and } \beta(u) = Cu. \tag{4.1}$$

Setting $\tau = C/L$, we have

$$\beta(u) = \tau \frac{\kappa(u)}{\sigma(u)}. \tag{4.2}$$

Under these assumptions, (2.10) has the integrating factor $e^{\tau v}$ and the first integral

$$F(u) + \frac{1}{\tau} v - \frac{1}{\tau^2} - \frac{\gamma}{\tau} = C_1\, e^{-\tau v}, \tag{4.3}$$

where, by $(4.1)_1$,

$$F(u) = \int_{\bar{u}}^{u} \frac{\kappa(t)}{\sigma(t)}\, dt = \frac{L}{2}(u^2 - \bar{u}^2),\ u \geqslant \bar{u}. \tag{4.4}$$

Computing in (4.3) γ and C_1, with the help of (2.11) we obtain, as solution of problem (2.10), (2.11),

$$U(v, V) = F^{-1}(H(v, V)), \tag{4.5}$$

where

$$H(v, V) = \frac{V}{\tau} \frac{1 - e^{-\tau v}}{1 - e^{-\tau V}} - \frac{v}{\tau}. \tag{4.6}$$

Set

$$\tilde{\sigma}(\xi) = \sigma(F^{-1}(\xi)),\ \xi \geqslant 0. \tag{4.7}$$

The integral equation to be solved is now

$$\frac{1}{k} f(V) = \int_0^V \tilde{\sigma}(H(v, V))\, dv \tag{4.8}$$

in the unknown $\tilde{\sigma}(\xi)$. Let $v_{\mathscr{M}}$ be the unique point of maximum of $H(v, V)$ in $(0, V)$ and $\mathscr{M}(V) = H(v_{\mathscr{M}}, V)$. Let

$$v = \mathscr{H}_1(z, V),\ 0 \leqslant z \leqslant \mathscr{M}(V),\ 0 \leqslant v \leqslant v_{\mathscr{M}}, \tag{4.9}$$

$$v = \mathscr{H}_2(z, V),\ 0 \leqslant z \leqslant \mathscr{M}(V),\ v_{\mathscr{M}} \leqslant v \leqslant V \tag{4.10}$$

be the two branches of the inverse function of $H(v,V)$. $\mathscr{H}_1(z,V)$ and $\mathscr{H}_2(z,V)$ can be explicitly computed in terms of the Lambert functions $W_0(z)$, $W_{-1}(z)$. For the properties of the Lambert function we refer to [3]. For a short but clearly written description we suggest visiting Wikipedia. Setting

$$A(z,V)=\frac{\tau V\, e^{\frac{\tau(V-\tau z+z\tau e^{-\tau v})}{e^{-\tau v}-1}}}{e^{-\tau V}-1}$$

we have

$$\mathscr{H}_1(z,V)=\frac{\left[W_0(A(z,V))\,e^{-\tau V}-W_0(A(z,V))+\tau V-z\tau^2+z\tau^2\,e^{-\tau V}\right]}{\tau(e^{-\tau V}-1)}$$

and

$$\mathscr{H}_2(z,V)=\frac{\left[W_{-1}(A(z,V))\,e^{-\tau V}-W_{-1}(A(z,V))+\tau V-z\tau^2+z\tau^2\,e^{-\tau V}\right]}{\tau(e^{-\tau V}-1)}.$$

Using (4.9) and (4.10) we obtain

$$\begin{aligned}\int_0^V \tilde{\sigma}(H(v,V))\,dv &= \int_0^{v_{\mathscr{M}}} \tilde{\sigma}(H(v,V))\,dv+\int_{v_{\mathscr{M}}}^V \tilde{\sigma}(H(v,V))\,dv \\ &= \int_0^{M(V)} \tilde{\sigma}(z)\left[\mathscr{H}_1'(z,V)-\mathscr{H}_2'(z,V)\right]dz.\end{aligned} \tag{4.11}$$

On the other hand,

$$\lim_{z\to\mathscr{M}(V)}\left[\mathscr{H}_1'(z,V)-\mathscr{H}_2'(z,V)\right]\sqrt{\mathscr{M}(V)-z}=\frac{1}{\sqrt{2}}.$$

Thus there exists a continuous function $G(x,z)$ in $0\leqslant z\leqslant x$ such that we have

$$\mathscr{H}_1'(z,V)-\mathscr{H}_2'(z,V)=\frac{G(\mathscr{M}(v),z)}{\sqrt{\mathscr{M}(V)-z}}.$$

In this way the integral equation (4.8) becomes

$$\frac{1}{k}f(V)=\int_0^{\mathscr{M}(V)}\frac{G(\mathscr{M}(V),z)\tilde{\sigma}(z)}{\sqrt{\mathscr{M}(V)-z}}\,dz. \tag{4.12}$$

Since the function $x=\mathscr{M}(V)$ is globally invertible, we can define

$$\mathscr{F}(x)=\frac{1}{k}f(\mathscr{M}^{-1}(x)),$$

obtaining finally from (4.12) the Abel equation

$$\mathscr{F}(x)=\int_0^x\frac{G(x,z)\tilde{\sigma}(z)}{\sqrt{x-z}}\,dz.$$

5 Conclusion

The solution given by N. H. Abel to the integral equation which bears his name dates back to the beginning of the 19th century; however, this result is still capable of applications in fields as diverse as stereology and spectroscopy, to quote only a few (see [4]). The possibility of reconstructing the temperature–conductivity dependence in a thermistor from the current–voltage characteristic is yet another confirmation of the permanency of Abel's result.

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