Horizontal Models: From Bakers to Cats*

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At the center of quantum chaos research is a particular family of models known as quantum maps. These maps illustrate an important "horizontal" dimension to model construction that has been overlooked in the literature on models. Three ways in which quantum maps are being used to clarify the relationship between classical and quantum mechanics are examined. This study suggests that horizontal models may provide a new and fruitful framework for exploring intertheoretic relations.

1. Introduction. One of the most important insights in physics in the last thirty years is the realization that a wide variety of classical systems exhibit chaotic behavior, that is, they exhibit sensitive dependence on initial conditions. This is perhaps surprising in light of the fact that quantum mechanics recently celebrated its hundredth birthday as the fundamental theory that *replaced* classical mechanics. On the standard interpretation of quantum theory, there does not seem to be genuine chaos in quantum mechanics and furthermore, it is difficult to see how classically chaotic behavior could even emerge. This difficulty has given rise to a new subdiscipline in physics known as quantum chaos.

Quantum chaos provides a particularly rich arena in which to explore the use of models.[1] On the one hand, the difficulty in solving classical nonlinear equations giving rise to chaotic dynamical behavior requires the

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1. A model can be generally understood as an entity or structure used to represent (Frisch 1998) or denote (Hughes 1997) another system, object, or structure. The specific class of models of interest here will be defined more fully in Section 2.

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use of models to bridge what Michael Redhead (1980) has called "the computational gap." A computational gap occurs when the fundamental equations of a theory are too difficult to solve for the sort of systems you are interested in making empirical predictions about. Quantum mechanics also faces a computational gap when it comes to trying to solve the Schrödinger equation for systems more complicated than the usual textbook examples. In these cases, the use of mathematical models and approximation techniques becomes essential.

In classical mechanics the study of chaotic Hamiltonian systems is greatly aided by the use of discrete area-preserving maps. These maps function as geometrical models in which the long-term chaotic behavior of systems can be readily followed. The success of discrete area-preserving maps in providing insight into classical chaos has led researchers to quantize some of them in the hopes that these "quantum maps" will similarly provide new insights into the problem of quantum chaos.

In the following section, I briefly trace the conceptual history of quantum maps. This history plays an important role in understanding the motivation and current justification of these models. Quantum maps can best be understood as a product of what I call horizontal model construction. These horizontally constructed models have proven to be an ideal tool for exploring the relation between classical and quantum mechanics. Sections 3 through 5 provide a more detailed examination of the role that particular quantum maps are playing in quantum chaos research. Specifically, they are being used, first, to develop a new semiclassical approximation; second, as a concrete test of the correspondence principle; and finally, to demonstrate a novel quantum effect used to explain the quantum suppression of classical chaos. In the final section I return to a general discussion of the nature and function of horizontal models.

My aim in this paper is *not* to offer a solution to the problem of quantum chaos. As of yet there is no satisfactory solution to this problem.[2] Rather, my aim is to illustrate the important role that horizontally constructed models play in the theoretical and philosophical enterprise of elucidating the relationship between two different, but partially overlapping, theories.

2. Horizontal Model Construction: From Poincaré Sections to Quantum Maps. Traditionally, model construction is seen as proceeding in one of two ways: either top down from theory or bottom up from empirical data. Both of these approaches emphasize what might be called the *vertical* component to model construction. When it comes to quantum maps, how-

2. See Bokulich 2001 for an overview and evaluation of current attempts to solve the problem of quantum chaos.

ever, neither of these approaches adequately explains how these models are constructed. The introduction of quantum maps did not come by way of quantum theory itself nor from any particular set of quantum phenomena. Instead, quantum maps can best be understood as a product of *horizontal* model construction. To say that the construction of a model is "horizontal" means that the primary guiding principle in the model's construction came, not by way of theory or any particular set of experimental phenomena, but rather, by way of analogy with models belonging to neighboring theories. Quantum maps are interesting not only because of the work that they are doing in quantum chaos research, but also because they illustrate this important horizontal dimension to model construction. In order to better understand how quantum maps are the product of horizontal model construction, it is necessary to briefly trace their development.

The use of geometrical models to solve problems in kinematics and dynamics has a long history. An innovative move by Henri Poincaré at the end of the nineteenth century was to extend this tradition of using geometrical models in mechanics to more abstract state spaces. In classical mechanics the state of a system can be represented as a point in phase space and the evolution of that state as a trajectory. If the number of integrals of motion is less than the number of degrees of freedom, such as in the case of the infamous three-body problem, then the system is said to be nonintegrable and can exhibit chaotic behavior.

In trying to deal with the three-body problem, Poincaré introduced a new method that became known as a surface of section. The central insight of this method is that, rather than trying to follow the entire trajectory, much of the essential information about the behavior of the system can be determined by examining the intersection of the trajectory with a plane in phase space called the surface of section. Given a piercing point A, the differential equations describing the evolution of the trajectory can be used to determine uniquely the next piercing at B, as shown in Figure 1. Thus, the continuous dynamics can be said to induce a discrete mapping, M, on the plane, such that $M(A) = B$. This map, M, can then be iterated to find all subsequent piercings of S.

In general, the maps associated with real dynamical systems (such as a three-body system) can still be extremely difficult to evaluate. Furthermore, when it comes to the study of chaotic systems, one is interested in their long-term behavior, which corresponds to typically several thousand iterations of these maps. For this reason, researchers interested in studying chaos introduced further simplified and idealized mappings that no longer necessarily corresponded to any real physical system or continuous dynamics. The guiding idea, as with most models, is that these simplified maps would capture certain generic features common to many different

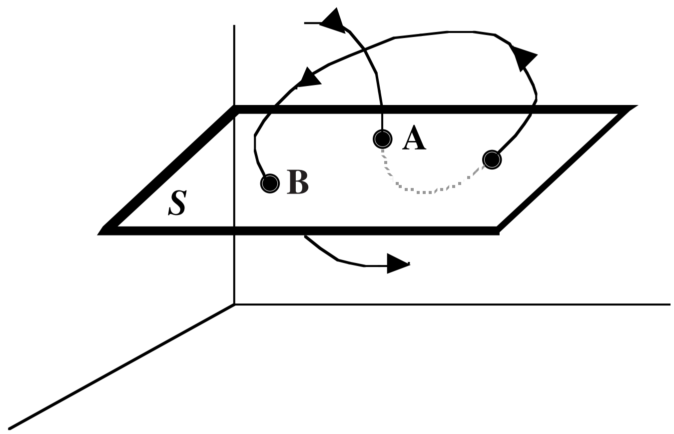


Figure 1. Poincaré surface of section used to reduce continuous dynamics to a discrete map. Only piercings in one direction are considered.

dynamical systems without being encumbered by the idiosyncrasies of particular systems. As it turns out, chaotic behavior can be exhibited even in these simple maps. A particularly important class of maps is the area-preserving transformations of the unit square, used to model Hamiltonian systems. Here the area-preserving property of the map corresponds to the phase-space-volume preserving property of Hamiltonian systems. In this class of maps belong the baker's map, the cat map, and the standard map, which will be discussed in the following sections.

The success of these simple maps in modeling classical chaotic dynamics led researchers in quantum chaos to develop quantum versions of the chaotic area-preserving maps, known as quantum maps.[3] While a classical map, M, is a one-to-one area-preserving transformation of the phase space into itself, a quantum map is a unitary transformation, $\hat{U}$, of the Hilbert space of states. More specifically, quantum maps are unitary $N \times N$ matrices with a particular type of N-dependence in their entries, where $N = 1/2\pi\hbar$.[4] For quantum maps the classical limit is defined as $N \rightarrow \infty$.

Unlike classical maps, quantum maps cannot be obtained directly from the full quantum dynamics by means of a method like the Poincaré surface of section. Rather, quantum maps are obtained horizontally by "quantizing" a particular classical map. As is generally the case in quantizing a

3. The term "quantum map" was first introduced in Berry, Balazs, Tabor and Voros 1979.

4. The basic idea here is to divide the unit phase space square into N minimum uncertainty patches of size h.

classical system, there is no universal prescription for how to quantize maps. For a classical map that can be viewed as a stroboscopic picture of a time-dependent Hamiltonian system, like the standard map to be discussed in Section 5, the quantization procedure is relatively straightforward. For more idealized classical maps, like the baker's map discussed in Section 3 and the cat map discussed in Section 4, ad hoc quantization procedures have to be invoked. As is also generally the case when quantizing a classical system, the quantization of a classical map is not unique. Researchers in quantum chaos identify three constraints that a quantum map should satisfy: it must be unitary (in order to conserve probability), it should respect the same symmetries as the classical map, and finally, it should give the chaotic classical map in the classical limit.

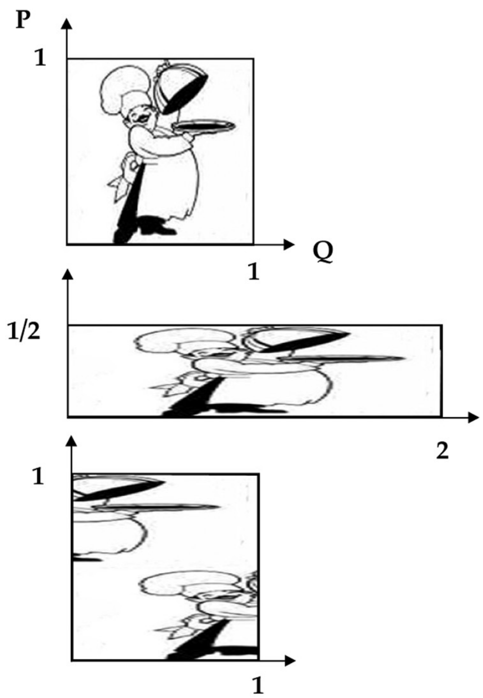
Quantum maps can best be understood as part of a lineage of models with their own internal dynamic and justification. Adam Morton, in his article on mathematical models notes that, "what is new and distinctive in the science of our time is the existence of complex mediating models which themselves have explanatory power and which embody techniques of modeling [that] can be refined and passed down to successor models" (Morton 1993, 664). While maps may not be the sort of model Morton has in mind, his central insight about successor models is a relevant one. As the history of quantum maps shows, the impetus for their introduction does not come from quantum theory nor from any particular set of quantum phenomena. Rather, the motivation—and current justification—of these models comes from the successes of classical maps.

To deny that models are exclusively theory driven does not commit one to the view that theories play no role in the construction of models, nor to the view that models can tell us nothing about theories. Similarly, to deny that a class of models is phenomenologically driven does not commit one to the view that these models have no role to play in explaining phenomena and making contact with experiment. In what follows I will show how these two points are born out by study of classical and quantum maps.

3. The Baker's Maps and Developmental Models. One of the many important functions of models is to aid in the articulation and development of a new theory that has not yet been completely specified. Such models were aptly named "developmental models" by Jarrett Leplin (1980). Developmental models typically have two important functions: first, testing the range and validity of fledgling theories and second, suggesting ways of improving or expanding such theories by indicating certain paths of research to pursue.

When it comes to classically chaotic systems, many of the traditional approximation techniques break down. One of the central projects of quantum chaos is the development a new semiclassical theory[5] that can be applied to classically nonintegrable systems. The essence of semiclassical mechanics is to build up approximations to quantum quantities, such as eigenvalues, through the use of classical objects, such as trajectories. In this context, the so-called baker's map has been important, not only for testing the range and validity of the current semiclassical approximations, but also for suggesting new ways of extending and improving semiclassical methods.

The baker's map is an area-preserving map of the unit phase space square. The action of the map is essentially a repeated halving in the P-direction and doubling in the Q-direction which results in a mixing of the phase space. It is called the baker's map because of the similarity of this action to a baker kneading dough. The motion generated by this map is chaotic. A feature of the baker's map that makes it a particularly useful model is that the dynamics of a point under the action of this map can be



$$Q' = 2Q - [2Q]$$

$$P' = \frac{1}{2}\left(P + [2Q]\right)$$

Figure 2. The baker's map. The square brackets indicate that only the integer part is kept.

5. It should be noted that there is some controversy over whether semiclassical theory should at all be called a 'theory'. Batterman (1995) has argued that semiclassical theory is worthy of the name, but I shall not address this issue here.

given a simple symbolic representation. If one writes the coordinates of a point, X, in binary notation, and puts them back to back separated by a decimal, then the new point, X', that results from one iteration of the baker's map can be found simply by moving the decimal point one place to the right (known as a Bernoulli shift) as shown in the example below.

$$\mathbf{X} = \left(\overbrace{.25}^{Q}, \overbrace{.75}^{P}\right) = \left(\overbrace{\dots 0011}^{P} \bullet \overbrace{0100\dots}^{Q}\right) \Rightarrow \mathbf{X}' = \left(\overbrace{.5}^{Q}, \overbrace{.375}^{P}\right) = \left(\overbrace{\dots 00110}^{P} \bullet \overbrace{100\dots}^{Q}\right)$$

The entire history of the trajectory can be easily followed in this way. Furthermore, in binary notation it is easy to identify the periodic orbits since they will be represented by a finite string of digits repeated an infinite number of times. It is these sorts of features of the baker's map that make it a particularly useful model for understanding chaotic dynamics.

The first quantizations of the baker's map employed ad hoc quantization procedures, guided by intuitive analogies with the classical map (cf. Balazs and Voros 1989 and Saraceno 1989). The action of the quantum map takes one from an initial state Ψ in the N-dimensional Hilbert space to a final state Φ as $\Phi = \hat{B}\Psi$. Just as the classical map divides the phase space square into left and right halves, which get mapped into bottom and top halves respectively, one can decompose the vector space into two orthogonal subspaces L and R whose vectors get mapped into two other orthogonal subspaces B and T. The mapping from Ψ to Φ involves a discrete Fourier transform, F_N, which takes one from the position representation to the momentum representation. What is important here is not the details of the quantum baker's map, but rather, how it is being used to help researchers in quantum chaos learn more about the applicability of semiclassical approximations.

The beginnings of a semiclassical theory that is applicable to nonintegrable as well as integrable systems is the so called Gutzwiller trace formula. This formula gives an expression that allows one to use knowledge about the periodic orbits of the classical system to compute approximate values for the energy eigenvalues of the quantum system. The success of this approach depends on being able to find and classify the actions, periods, and stabilities of all the periodic orbits. Obtaining all of the periodic orbits of a general Hamiltonian system is a formidable task. In general this is not possible except for idealized models. The process of finding the classical periodic orbits is greatly simplified when applied to a discrete map, such as the baker's map.

One of the hallmarks of a developmental model is the role that it plays in testing the range and validity of a fledgling theory. Originally, it was believed that the Gutzwiller trace formula breaks down when the classical chaos has mixed the phase space on a scale smaller than Planck's constant (Sepúlveda et al. 1992). This means that, for a phase space divided up into cells of size *h,* a typical initial state has visited most of these cells. The time at which this breakdown occurs is approximately equal to $\ln(1/\hbar)$, dubbed the "log time." In their study of the baker's map, Patrick O'Connor et al. (1992) were able to show that the semiclassical formula remained valid even after the log time. They cite a number of special features of the baker model that enabled them to first make this discovery. They write, besides the simple encoding of the dynamics,

> the added clarity resides in two additional features of the baker's map which further its appeal as a paradigm. In a typical system, a log time is a somewhat vague concept, varying throughout phase space. The baker's map, though, naturally presents a specific and sharp value for what to call the log time, making it quite clear when we are studying dynamics 'beyond the log time.' The second . . . is that the object which plays the role of a caustic or singularity appears only at two coordinate sites. This means that by confining our study to regions away from these sites, the complications . . . are avoided. (O'Connor et al. 1992, 342)

Thus, it is not simply one feature of the baker's map that makes it a useful model, but rather a variety of simplifying features that, together, make these studies possible.

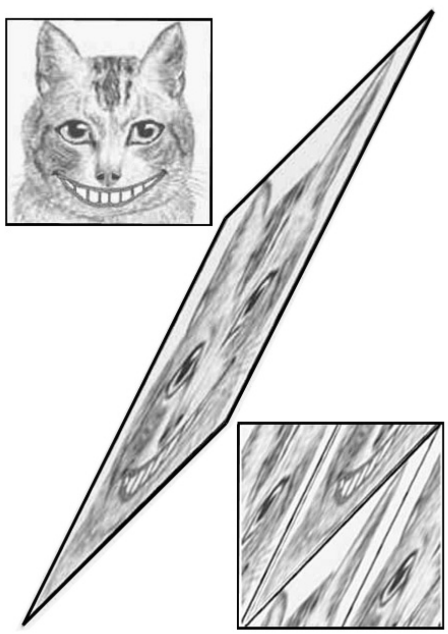
A second hallmark of developmental models is the role they play in improving and extending fledgling theories. The baker's map has been important, not only for testing the range of validity of semiclassical approximations, such as the Gutzwiller formula, but also has played a central role in suggesting ways of extending and developing better semiclassical approximations. One of the fundamental limitations of Gutzwiller's trace formula is that it only gives information about eigenvalues and not wavefunction amplitudes. As such, it is only a partial semiclassical theory. Steven Tomsovic and Eric Heller have used the quantum baker's map to develop a more complete semiclassical theory that goes beyond the information given by the classical periodic orbits used in Gutzwiller's formula. Once again it is the simplifying features of this chaotic model—such as binary coding and linear discrete dynamics—that allowed Tomsovic and Heller (1993) to show that a more complete semiclassical method was, in fact, possible.

4. Cat Maps and the Correspondence Principle. Questions about the relationship between classical and quantum mechanics are often formulated in terms of the correspondence principle.[6] Although many different ver-

6. A discussion of the correspondence principle in the context of quantum chaos can be found in Belot and Earman 1997 and Batterman 1991.

sions of the correspondence principle can be found in the literature, a minimal, though perhaps not historically accurate, version of the correspondence principle asserts that quantum mechanics should be able to reproduce, within experimental error, the empirically well-confirmed successes of classical mechanics. Joseph Ford and his collaborators (1991, 1992) have argued that classical chaos poses an insurmountable threat to the correspondence principle. His controversial argument is based on a particular chaotic model known as the cat map.

The cat map is so-named because Vladimir Arnold and André Avez (1968) first illustrated the action of this map on a drawing of a cat. In the cat map, one begins with a unit phase space square that, when multiplied by a matrix, A, results in a stretched parallelogram. The "modulo one" operation then "folds" the parallelogram back into the unit square. This stretching and folding procedure is repeated for each iteration of the map. The example given above, with $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, is just one possible cat map.



$$\begin{pmatrix} \mathbf{Q}' \\ \mathbf{P}' \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \mathbf{P} \end{pmatrix} \mathbf{mod} 1$$

Figure 3. The cat map. The matrix stretches the figure and the "mod 1" means that only the fractional parts are kept, projecting the image back into the unit square. After just one iteration of this map even the cat's smile is all but gone.

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More generally, the elements of the matrix, A, can be any integers such that the determinant is equal to 1 (to ensure that the map is area-preserving) and the trace of A is greater than two (to ensure the map is hyperbolic, or chaotic).

The quantum version of the cat map is an N $\times$ N unitary matrix, $\hat{U}$. The method by which one derives $\hat{U}$ from the elements of the classical map was first given by John Hannay and Michael Berry (1980). Rather than attempting to quantize the cat map directly, their quantization proceeds by first making an analogy to optics, where classical mechanics is likened to ray optics and quantum mechanics is likened to wave optics. They construct an optical system of lenses and rays of monochromatic light, whose dynamics are described by the action of the classical cat map. They then obtain the quantum cat map by considering the wave optics counterpart of this system. It is important to note, however, that while these contrived optical systems played a central role in constructing the quantum version of this map, they are not the systems that physicists are trying to better understand through this model. Although deriving the expression for the quantum cat map is nontrivial, once one obtains the quantum cat map its behavior can be readily determined and compared with the classical cat map.

Joseph Ford and his collaborators have analyzed the classical and quantum cat maps and drawn the following conclusion:

> We here quantize the Arnol'd cat and examine its quantum motion for signs of chaos using algorithmic complexity as a litmus. Our analysis reveals that the quantum cat is not chaotic in the deep quantum domain nor does it become chaotic in the classical limit as required by the correspondence principle. We therefore conclude that the correspondence principle . . . fails for the quantum Arnol'd cat (Ford et al. 1991, 493).

One of the central motivations for Ford's claim that the correspondence principle fails is the observation that, for any finite N, the quantum dynamics described by the quantum cat map is always periodic. That is to say, for any wavefunction there exists an m such that after m iterations of the map one is back to the original wavefunction, $\hat{U}^m\psi_0 = \psi_0$ (up to a phase factor). By contrast, the classical dynamics, described by the classical cat map, are not periodic (except for those trajectories described by rational coordinates which are of measure zero).

There are several different ways in which one can respond to Ford's polemical claim that the correspondence principle fails for the cat map. For example, Gordon Belot and John Earman (1997) have challenged the adequacy of Ford's notion of algorithmic complexity to characterize chaos. Another possibility is simply to reject the quantum cat map as a legitimate model with which to test the correspondence principle. Another approach, however, which has proven to be fruitful, is to examine whether the model itself is rich enough to provide the resources for responding to Ford's claim that it violates the correspondence principle.

A central question for testing the validity of the correspondence principle is to determine what happens to the periodic quantum motion in the classical limit, which in quantum maps is given by $N \rightarrow \infty$. Jonathan Keating (1991) has studied the orbits of the cat map in great detail and has shown that on average, the periods of the periodic orbits of the quantum cat map tend to infinity as $N \rightarrow \infty$. In other words, the regular periodic quantum behavior can, for all practical purposes, be said to mimic the classical chaotic nonperiodic behavior in the classical limit.[7] Recall that the minimalist correspondence principle requires that *in the classical limit* the behavior of a quantum system should be empirically indistinguishable from the classical behavior. Since one of the defining features of classical chaotic behavior is its non-periodicity, it is significant that, in the cat map, the quantum orbits (which are always periodic) can nonetheless be made empirically indistinguishable from the chaotic orbits by having extremely long periods. Thus, Keating's result suggests that, contrary to Ford, the correspondence principle does not fail for the quantum cat map.

This debate shows how quantum maps can be used as a concrete model situation in which to explore what are otherwise often intractable questions about the validity of the correspondence principle. Again, it is the simplified features of the model that make possible this kind of analysis on a level that would not be technically feasible in a more realistic system. Unlike many more realistic systems, the classical limit of the cat map is both well defined and mathematically solvable. Although the cat maps are caricatures of dynamical systems, they play an essential role in setting out a research program and developing techniques with which to address questions about the validity of the correspondence principle in more realistic models. Before turning to a more general discussion of these points in Section 6, it will be useful to have one more case study in hand.

5. The Standard Maps and the Explanatory Power of Models. Although there are many interesting projects in quantum chaos, ultimately an adequate theory of quantum chaos should explain two things: first, how classically chaotic behavior emerges out of quantum mechanics in the classical limit, and second, how classically chaotic behavior becomes suppressed in the quantum limit. As of yet, there is no complete answer to either of these questions. Furthermore, the answers to these questions may involve very different mechanisms. With regard to the latter question of how classical

7. See Keating 1991 for further details.

chaos becomes suppressed in the quantum limit, quantum maps have played an important role in providing the beginnings of an explanation.

The so-called "standard map" gets it name from the wide variety of dynamical systems that it can be used to model. The standard map is given by the following set of equations,

$$P' = P + K \sin\theta \quad \theta' = \theta + P' \quad \mod 2\pi$$

where the unit square is defined as $0 \leq P, \theta < 2\pi$. This map can be interpreted as a stroboscopic sampling of the continuous dynamics of a kicked rotor at certain intervals of time (rather than a Poincaré section in space).

The Hamiltonian for the continuous kicked rotor system is

$$H = p^2/2I + k \cos\theta \delta_T(t)$$

where p is the angular momentum (or action), θ is the angular displacement (or phase), I is the moment of inertia (which can be set equal to one), k is the kick strength, and $\delta_T(t)$ is a periodic delta function. The standard map can be obtained by integrating the equations of motion over one period. Although the standard map greatly simplifies the theoretical analysis of the kicked rotor model and reduces the continuous dynamics to a discrete dynamics, it brings no new idealization.

The quantum standard map can be obtained by quantizing the classical kicked rotor in the standard way and then considering the time evolution of the quantum kicked rotor over one period of the driving force. Thus the quantum standard map is a mapping of the wavefunction over a period T,

$$\psi(\theta, t + T) = \hat{U}\psi(\theta, t)$$

in which the unitary evolution operator, $\hat{U}$, is the product of three noncommuting unitary operators, the first of which corresponds to free ro-

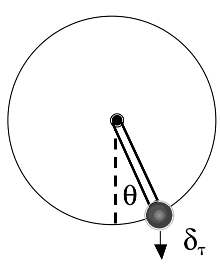

Figure 4. The kicked rotor is essentially a rigid pendulum, free to rotate around a central fixed point, that periodical receives a kick causing it to rotate erratically when the kicking strength is above some critical value.

tation during the first half of a period, the second operator describing the kick, and the third operator describing the free rotation during the second half of the period. The quantum standard map allows the dynamical behavior of the continuous quantum model to be more readily explored.

Classically, one of the manifestations of chaos in the kicked rotor, or standard map, is that the energy and momentum will diffuse without bound as the number of kicks, or iterations of the map, increases.[8] To determine what happens to the energy and momentum of the quantum kicked rotor we can use the quantum standard map to obtain the energy as a function of the number of kicks (or iterations). The result, first obtained by Giulio Casati and his collaborators (1979), is that in the quantum model, the energy growth followed the classical energy growth only up to some critical time, after which it leveled off, as shown in Figure 5.

The difference in predictions between the quantum and classical standard maps is taken as providing part of an explanation for what is called the quantum suppression of classical chaos.[9] There are a number of general theoretical arguments for why there can be no genuine chaos in quan-

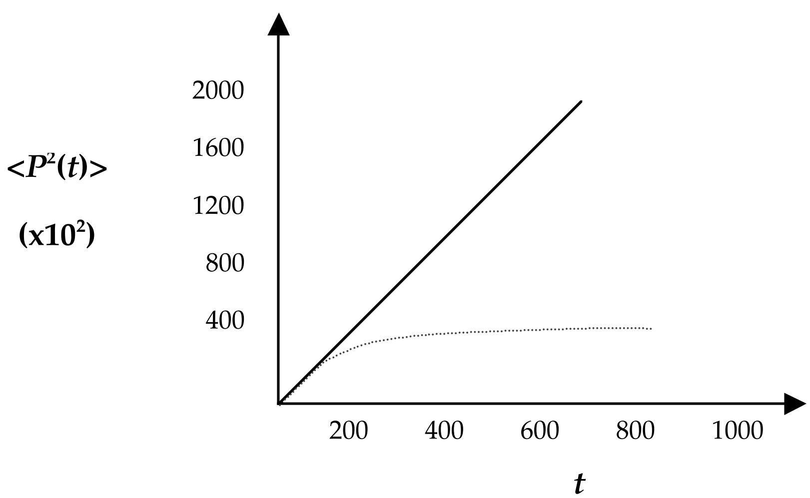

Figure 5. Classical (solid line) and quantum (dashed line) diffusion of energy as a function of number of map iterations in the standard map for K = 5 (Adapted from Casati and Chirikov 1995, 13).

8. Specifically the mean square of the momentum averaged over all the trajectories grows linearly in time: $<P^2(t)> \approx K^2t/2$.

9. A closer examination of the model reveals that the smaller the parameter, *k,* (i.e., the more quantum mechanical it is) the more diffusion is suppressed. See Izrailev 1990 for further details.

tum mechanics. For example, two quantum states that begin "close together" will always remain close together under unitary Schrödinger evolution (that is, there can be no exponential divergence characteristic of chaos).[10] While these arguments may explain the absence of chaotic behavior in quantum systems, they do not explain how the chaotic behavior—evident at the classical level—disappears or becomes suppressed in the quantum limit. It is in this important theoretical task of explaining the quantum suppression of classical chaos that the present model, the quantum standard map, has proven particularly fruitful.

A mechanism for the suppression of chaos was suggested by an analogy drawn between the behavior of the quantum standard map (depicted in Figure 5) and a similar effect in a model in condensed matter physics (known as the Anderson model), which describes the localization of electrons in a disordered solid (Grempel et al. 1984). The analogy suggests that the mechanism for the suppression of diffusion is a novel quantum interference effect that is now called dynamical localization. Dynamical localization is essentially a destructive interference phenomenon that limits the spread of the rotor wave function over the available angular momentum space. What is important for the discussion here is the fact that it was the study of this quantum map and the comparison of it with its classical counterpart that led to the discovery of a new physical effect in quantum mechanics.

With corresponding quantum and classical models, the intractable question of what happens to classical chaos in the quantum limit becomes the more tractable question of what happens to the chaotic behavior of the classical standard map in the transition to the quantum standard map. As was done here, one can look at the behavior of an observable in the classical model and then ask to what extent that behavior is followed in the quantum model. Although the quantum standard map was the product of horizontal model construction, this does not mean that such models cannot ultimately make contact with empirical data. Recently, researchers have been able to construct an experimental realization of this quantum map and found that its behavior conforms with the predictions of this model.[11] What is noteworthy here is that the model was not introduced to account for a set of empirical data; rather, the model was introduced and studied long before anyone knew what would count as a physical instantiation of the model. In the next section I will argue that this sort of independence from empirical phenomena is one of the striking characteristics of horizontal models.

10. See, for example, Berry 1989, 335.

11. See, for example, Moore et al. 1995.

6. Horizontal Models and Intertheoretic Relations. From the preceding three case studies one can begin to see the key features of horizontal models emerge. What makes this class of models philosophically interesting is both the way in which they were constructed and the uses to which they are being put.

Historically most of the philosophical literature on model construction has focused on top-down model construction, in which models are constructed from a full theory through the introduction of various idealizations. More recently this emphasis on the top-down approach has been challenged by several philosophers of science. For example, Matthias Frisch in his discussion of models in electromagnetism notes, "according to a widespread view, mathematical models are derived from the laws of a theory in such a way that the models satisfy the laws" (Frisch 1998, ix). He goes on to argue that this conception of models is inadequate in the context of electromagnetism, where one finds models of that theory that do not necessarily obey the laws of electromagnetism. Nancy Cartwright, Towfic Shomar, and Mauricio Suárez (1995) have also challenged the top-down approach, referring to it as the "theory-driven" view of models. They criticize it on the grounds that "it is rarely the case that models of the phenomena are arrived at as de-idealizations of theoretical models" (Cartwright et al., 142). Instead, they argue for an alternative mode of model construction, which they call phenomenological model building. In phenomenological model building, models are built "bottom up," directly from the empirical data or phenomena, largely independent of any theory.

While this recent philosophical work has expanded our understanding of modeling in many important ways, it is still confined within the framework of what I earlier called the "vertical" approach. This vertical view of model building (either top down from theory or bottom up from data) is reinforced by the traditional view that the function of models is to mediate between theory and data. While this is certainly one of the functions of models, it is emphatically not their only function. The foregoing detailed study of quantum maps points to a number of ways in which our philosophical understanding of the construction and function of models in scientific practice needs to be further expanded.

Recall that what makes a model *horizontal* is the way in which it was constructed. Rather than being derived from either theory or data, these models are developed by way of analogy with models of a neighboring theory. In the case of quantum maps, these models were developed by drawing analogies to the already successful classical maps. In the discussion of the quantum baker's map, for example, it was shown how this model was not derived top down from quantum theory, nor derived from any quantum phenomena or experimental data. Instead the quantum baker's map was developed via the classical model. This same method of horizontal model construction was found to be at work in the case of the quantum cat map and the quantum standard map.

This mode of construction has a number of implications for understanding the nature and function of these models. First, horizontal models are surprisingly independent from both theory and experiment. Instead of mediating between theory and data, these models take on a life of their own. Here we see these models playing a dual role: standing in for the theory on one hand, and themselves becoming the central object of experimental investigation on the other. For example, as was seen in the case of the cat map, one can perform thousands of iterations of this map and analyze the behavior of a "trajectory" produced by these iterations. The results from these manipulations of the model have become the relevant "experimental" data. At the same time, however, the quantum cat map is taken as a representative model from which to draw conclusions about quantum theory in general: If the quantum cat map cannot exhibit true chaos, then quantum theory cannot exhibit true chaos; if the quantum cat map mimics chaotic behavior in the classical limit, then quantum theory also mimics this behavior in the classical limit.

The independence of horizontal models, such as quantum maps, from theory and data suggests that their primary function is not to mediate between theory and data. Instead, the examination of the baker's, cat, and standard maps in the previous sections reveals that the primary function of these horizontal models is to mediate between two theories, in this case classical and quantum mechanics. Recall that the three uses to which these particular models are being put are the following: the development of a new semiclassical approximation, a test of the correspondence principle, and the discovery of a novel quantum effect to explain the discrepancy between the predictions of classical and quantum theory. All three of these uses fall under the rubric of investigating intertheoretic relations.[12]

Traditionally discussions of intertheoretic relations have focused on the question of whether the laws of one theory can be derived from, or shown to be a limiting case of, the laws of the other theory.[13] What is new and progressive about horizontal models is that they provide an alternative framework for discussing intertheoretic relations—a framework other than the usual nomocentric approach. Understanding the relationship between science's various theoretical descriptions of the world is one of the

12. I prefer the term "intertheoretic relations" to the more common "theoretical reductionism" because reductionism is just one of several possible ways that two theories can be related.

13. See, for example, the classic works on reductionism by Nagel (1961) and Nickles (1973).

central projects in the philosophy of science. The present study of quantum maps demonstrates that by focusing on models rather than laws, fruitful headway can be made on this problem.

One of the reasons that horizontal models are so useful for exploring intertheoretic relations is that they provide the researcher with a concrete model pair. For example, the largely impenetrable question of whether classical mechanics is a limiting case of quantum mechanics becomes, with the help of horizontal models, the more tractable question of the relationship between the classical standard map and the quantum standard map. These simplified model pairs, standing in for classical and quantum mechanics respectively, allow a detailed level of analysis that would not be feasible otherwise. As was seen in Section 5, the careful comparison of the behavior of the classical standard map with that of the quantum standard map led researchers to discover a novel physical effect that provided a mechanism for the quantum suppression of classical chaos. Similar progress was made through detailed comparisons of the classical and quantum cat maps, and the classical and quantum baker's maps. To reiterate, the horizontal construction of models provides one with a pair of closely related simplified models, each of which is used to stand in for the full theories whose relationship is under investigation.

Although I have focused on the use of horizontal models in quantum chaos research, these models are not limited to this field. The present study suggests that horizontal models are particularly effective when the two theories involved bear a certain structural similarity to each other, such as is the case with classical and quantum mechanics.[14] It is precisely in these sorts of cases that substantive questions concerning the nature of intertheoretic relations arise.

The present study of quantum maps in quantum chaos research deepens our understanding of the nature of modeling practices in science in a number of ways. One lesson that emerges is that model construction is a more rich and varied practice than many traditional philosophical accounts have allowed. Recognizing that models can be constructed horizontally, by way of analogy with models in neighboring theories, provides an alternative to the theory-driven-versus-phenomenologically-driven dichotomy. A second lesson that emerges is that discussions of intertheoretic relations do not need to be carried out within the context of laws. Models—particularly horizontal models—have proven themselves in the context of quantum chaos research to be a fruitful way in which to come to a deeper understanding of the relationship between two theories.

14. The structural similarity between classical and quantum mechanics can be seen, for example, in the fact that both theories can be given a Hamiltonian formulation. This structural similarity leaves open the question of whether a reductive or non-reductive relationship exists between these two theories.

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