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Pfaffian Formulas for Spanning Tree Probabilities

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We show that certain topologically defined uniform spanning tree probabilities for graphs embedded in an annulus can be computed as linear combinations of Pfaffians of matrices involving the line-bundle Green's function, where the coefficients count cover-inclusive Dyck tilings of skew Young diagrams.

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1. Introduction

Consider a graph $\mathcal{G}$ with edge weights, possibly with multiple edges connecting pairs of vertices, and possibly with self-loops. The edge weights correspond to conductances in an electrical network. Suppose that $\mathcal{G}$ comes with a set of distinguished vertices, which we call *nodes*. We think of the nodes as being 'boundary vertices', and the other vertices as being internal. Suppose $\mathcal{G}$ is the induced subgraph of some larger graph, which is connected to the larger graph only through the boundary vertices. Then a spanning tree on the larger graph, when restricted to $\mathcal{G}$, is a *grove* of $\mathcal{G}$, which is defined to be spanning forest of $\mathcal{G}$ such that each tree contains at least one node. (Groves were first studied by Carroll and Speyer [1], and then more systematically by Kenyon and Wilson [7], who gave this current definition.) We can study spanning trees of the larger graph by studying groves of $\mathcal{G}$. Each grove of $\mathcal{G}$ defines a partition σ of the nodes, according to which nodes are contained in the same tree. We let $Z[\sigma]$ denote the weighted sum of groves of $\mathcal{G}$ whose induced partition is σ, where the weight of a grove is the product of its edge weights. Suppose that the nodes are labelled $1,\dots,n$. Both $Z[1,\dots,n]$ (where the grove has one tree) and $Z[1|\cdots|n]$ (where the grove has n trees) can be computed using the matrix-tree theorem. We would like to compute $Z[\sigma]$ for other partitions σ, or else the ratios

$$\ddot{Z}[\sigma] := \frac{Z[\sigma]}{Z[1|2|\cdots|n]} \tag{1.1}$$

or

$$\overline{Z}[\sigma] := \frac{Z[\sigma]}{Z[1,\ldots,n]}. \tag{1.2}$$

Of special interest are *circular-planar* graphs $\mathcal{G}$, which are embedded in a disk with every node on the boundary of the disk, arranged in cyclic order $1,\ldots,n$. For circular-planar graphs, Kenyon and Wilson [7] showed how to compute $\overline{Z}[\sigma]$ for any partition σ as a polynomial in the pairwise electrical resistances between the nodes, and how to compute $\ddot{Z}[\sigma]$ as a polynomial in the entries of the 'response matrix', defined in Section 1.1.

Also of special interest are *annular-one* graphs $\mathcal{G}$, which are embedded in an annulus so that the nodes $1,\ldots,n-1$ are arranged in cyclic order on one boundary of the annulus, while node n is on the other boundary of the annulus (see Figure 1). The grove partition function ratios (1.1) and (1.2) for certain annular-one graphs were used to compute the intensity of loop-erased random walk on $\mathbb{Z}^2$ and other lattices [10], and the probabilities of local events in the abelian sandpile model [14]. In contrast to circular-planar graphs, for annular-one graphs the pairwise resistances are not sufficient to compute $\overline{Z}[\sigma]$, and the response matrix is not sufficient to compute $\ddot{Z}[\sigma]$. However, by using additional boundary measurements, in particular the 'line-bundle Green's function' or the 'line-bundle response matrix' defined in Section 1.1, Kenyon and Wilson showed that $\overline{Z}[\sigma]$ and $\ddot{Z}[\sigma]$ can be computed for any partition σ in which node n is not in a singleton part [10].

In this paper we give a new formula for computing $\overline{Z}[\sigma]$ and $\ddot{Z}[\sigma]$ for annular-one graphs, which expresses them as linear combinations of Pfaffians of matrices defined in

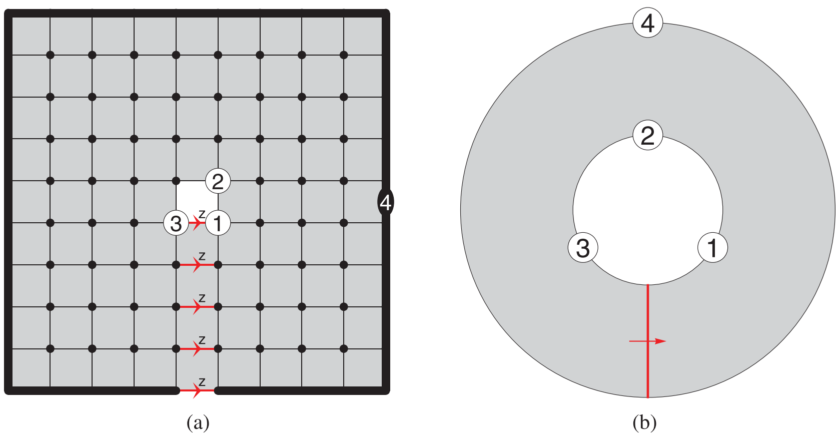

Figure 1. An annular-one graph $\mathcal{G}$ (a) and a schematic diagram (b) showing just the annulus, nodes, and zipper. In this case the outer boundary has just one vertex (node 4) and the bottom edge is a self-loop, and the inner boundary is one of the squares of the grid, which has four vertices, of which three are nodes. There is a 'zipper' (edges crossing a dual path) connecting the inner boundary to the outer boundary of the annulus, and edges crossing the zipper have parallel transport z from their left endpoint to their right endpoint. (This figure first appeared in [10].)

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terms of the boundary measurements, rather than linear combinations of determinants as in [10]. The Pfaffian formulas have fewer terms than the determinant formulas. In addition to being (somewhat) computationally more efficient, the Pfaffian formulas have some consequences, described in Section 1.5, regarding how $\overline{Z}[\sigma]$ and $\ddot{Z}[\sigma]$ depend on the boundary measurements. This in turn simplifies some long-range sandpile height correlation calculations [14].

In Section 1.1 we describe the line-bundle Laplacian together with its associated line-bundle Green's function and line-bundle response matrix, which are the relevant boundary measurements for annular-one graphs. We discuss 'partial pairings' in Section 1.2; for annular-one graphs, $Z[\sigma]$ can be expressed as a linear combination of $Z[\tau]$'s, where each τ is a partial pairing [10]. In Section 1.3 we state our main theorem, which shows how to express each $\overline{Z}[\tau]$ and $\ddot{Z}[\tau]$ as a linear combination of Pfaffians in the boundary measurements. In the interest of clarity, we give a number of examples in Section 1.4. In Section 1.5 we give some corollaries of the main result. In Section 2 we review the relevant determinant formulas from [10], since they are the starting point of the present work. In Section 3 we derive several Pfaffian formulas, including the main theorem. We conclude with some open problems in Section 4.

1.1. Line-bundle Laplacian, Green's function, and response matrix

The graph Laplacian Δ of a graph $\mathcal{G}$ with edge weights c is given by

$$\Delta_{u,v} = \begin{cases} -c_{u,v} & u \neq v, \\ \sum_{w:w\neq u} c_{u,v} & u = v, \end{cases}$$

where $c_{u,v}$ is the weight of edge (u,v), or the sum of such weights in the case of a multigraph. We review here the line-bundle Laplacian, which generalizes this, together with its associated response matrix and Green's function, which is sufficiently general to study annular-one graphs.

Suppose that each edge (u,v), in addition to its weight $c_{u,v}$, has a 'parallel transport' $\phi_{u,v}$ which is a non-zero complex number such that $\phi_{u,v} = \phi_{v,u}^{-1}$. The line-bundle Laplacian is

$$\Delta_{u,v} = \begin{cases} -c_{u,v}\phi_{u,v} & u \neq v, \\ \sum_{w:w\neq u} c_{u,w} & u = v. \end{cases}$$

Note that the parallel transports do not occur on the diagonal. Forman proved a version of the matrix-tree theorem for the line-bundle Laplacian, in which the objects being counted are oriented cycle-rooted spanning forests (OCRSF's) [3]. Each OCRSF is weighted by the product of the edge weights, and also an additional weight for each oriented cycle C, which is $1 - \prod_{(u,v)\in C} \phi_{u,v}$.

(There is a further generalization known as the vector-bundle Laplacian, in which the parallel transports $\phi_{u,v}$ are matrices, usually in $\mathrm{SL}_2(\mathbb{C})$, which can be used to study additional properties of spanning trees [5, 10].)

To compute grove partition functions for annular-one graphs, it suffices to consider the line-bundle Laplacian with the following parallel transports. We draw a path (the 'zipper') connecting the two boundaries of the annulus which may cross edges but not vertices;

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see Figure 1. All directed edges crossing the zipper in one direction get a connection of z; in the opposite direction the connection is $1/z$. For all other edges the connection is 1. We are interested in the limit $z \to 1$.

The Green's function $G^{(s)}_{u,v}$ of a finite weighted graph $\mathcal{G}$ is defined with respect to a 'sink vertex' s, and has the following electrical interpretation: if one unit of current is inserted at u and extracted at s, and s is held at 0 volts, $G^{(s)}_{u,v}$ gives the voltage at v. The function $v \mapsto G^{(s)}_{u,v}$ is harmonic except at u and s. When the graph is connected, the Green's function (when $u, v \neq s$) is the inverse of the Dirichlet Laplacian $\Delta^{\widehat{s}}_{\widehat{s}}$, formed from Δ by removing row and column s:

$$\left(G^{(s)}_{u,v}\right)^{v \neq s}_{u \neq s} = \left(\Delta^{\widehat{s}}_{\widehat{s}}\right)^{-1},$$

so $G^{(s)}_{u,v} = G^{(s)}_{v,u}$. If $u = s$ or $v = s$, then $G^{(s)}_{u,v} = 0$.

The effective electrical resistance $R_{u,v}$ between u and v is (for any s)

$$R_{u,v} = G^{(s)}_{u,u} - G^{(s)}_{u,v} - G^{(s)}_{v,u} + G^{(s)}_{v,v}.$$

The line-bundle Green's function $\mathscr{G}^{(s)}_{u,v}$ generalizes the usual Green's function for the line-bundle Laplacian:

$$\left(\mathscr{G}^{(s)}_{u,v}\right)^{v \neq s}_{u \neq s} = \left(\Delta^{\widehat{s}}_{\widehat{s}}\right)^{-1},$$

and $\mathscr{G}^{(s)}_{u,v} = 0$ if $u = s$ or $v = s$. For annular-one graphs, we take the sink s to be node n (the node on the annulus boundary component with just one node), and write $\mathscr{G}_{u,v}(z)$ to emphasize the dependence on z. Of course when $z = 1$ it specializes to the usual Green's function:

$$G_{u,v} = G_{v,u} = \mathscr{G}^{(s)}_{u,v}(1).$$

The line-bundle Green's function $\mathscr{G}$ has the symmetry $\mathscr{G}_{v,u}(z) = \mathscr{G}_{u,v}(1/z)$. We define

$$G'_{u,v} = \left[\frac{d}{dz}\mathscr{G}^{(s)}_{u,v}(z)\right]_{z=1},$$

which is antisymmetric, and is what we referred to as the derivative of the line-bundle Green's function.

We do not need any further derivatives of $\mathscr{G}_{u,v}(z)$: the boundary measurements $G_{u,v}$ and $G'_{u,v}$, where u and v range over the nodes other than n, are sufficient to compute the grove partition functions for annular-one graphs [10].

There is another set of electrical variables that are useful to work with, the response matrix, or the Dirichlet-to-Neumann matrix $L_{u,v}$. We let $\mathcal{N}$ denote the set of nodes, which we think of as being 'boundary vertices', with the other vertices as being internal. The response matrix gives the linear map from voltages at $\mathcal{N}$ to current flows: when node u is held at 1 volt and the other nodes are at 0 volts, $L_{u,v}$ gives the current leaving the network at node v. When the Laplacian is written in block form,

$$\Delta = \begin{array}{c} \\ u\in\mathcal{N} \\ u\notin\mathcal{N} \end{array} \begin{array}{c} \begin{array}{cc} v\in\mathcal{N} & v\notin\mathcal{N} \end{array} \\ \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \end{array},$$

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the response matrix can be computed:

$$-L = A - BC^{-1}D.$$

Since Δ is symmetric, $L_{v,u} = L_{u,v}$. The response matrix also satisfies

$$\sum_v L_{u,v} = 0$$

for each vertex u.

In the line-bundle setting we denote the response matrix by $\mathscr{L}_{u,v}(z)$, which is computed using the above formula. Here too $\mathscr{L}_{v,u}(z) = \mathscr{L}_{u,v}(1/z)$, and the line-bundle response matrix specializes to the usual response matrix when $z = 1$,

$$L_{u,v} = L_{v,u} = \mathscr{L}_{u,v}(1),$$

which is symmetric, and we define

$$L'_{u,v} = \left[\frac{d}{dz}\mathscr{L}_{u,v}(z)\right]_{z=1},$$

which is antisymmetric. As with the Green's function, we require no further derivatives.

1.2. Partial pairings

For annular-one graphs, for any partition σ of the nodes for which node n is not in a singleton component, $Z[\sigma]$ is a linear combination of $Z[\tau]$'s, where each τ is a 'partial pairing' of the nodes $1,\dots,n$ [10]. (It turns out that for the loop-erased random walk and sandpile applications it suffices to assume that node n is in a doubleton part.) A *partial pairing* is a set of pairs of nodes, singletons, and 'internalized' nodes, which are not listed in the partition, but which may appear in any of the parts (like the other non-node vertices in a grove). Consider for example $Z[2,6,9|3,4,5|7|1,8]$. If we internalize node 6, then

$$\begin{aligned} Z[2,9|3,4,5|7|1,8] = {} & Z[2,6,9|3,4,5|7|1,8] + Z[2,9|3,4,5,6|7|1,8] \\ & + Z[2,9|3,4,5|6,7|1,8] + Z[2,9|3,4,5|7|1,6,8], \end{aligned}$$

which allows us to write

$$\begin{aligned} Z[2,6,9|3,4,5|7|1,8] = {} & Z[2,9|3,4,5|7|1,8] - Z[2,9|3,4,5,6|7|1,8] \\ & - Z[2,9|3,4,5|6,7|1,8] - Z[2,9|3,4,5|7|1,6,8]. \end{aligned}$$

Considering the first term on the right, if we internalize node 4, we obtain

$$Z[2,9|3,4,5|7|1,8] = Z[2,9|3,5|7|1,8],$$

since the other terms in the internalization expansion, such as $Z[2,9|3,5|4,7|1,8]$, are zero for topological reasons, using the fact that the graph is embedded in an annulus. After more transformations of these types, we can obtain a partial pairing expansion

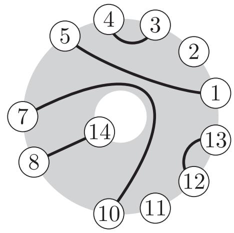
$$\begin{aligned} Z[2,6,9|3,4,5|7|1,8] = {} & Z[2,9|3,5|7|1,8] - Z[2,9|3,6|7|1,8] \\ & - Z[2,9|3,5|6,7|1,8] - Z[2,9|3,5|1,6|7] + Z[2,9|3,5|1,6|7,8]. \end{aligned}$$

Kenyon and Wilson [10] showed that for any partial pairing τ, $Z[\tau]/Z[1|2|\cdots|n]$ can be expressed as a linear combination of determinants involving the $L_{i,j}$'s and $L'_{i,j}$'s, and that $Z[\tau]/Z[1,2,\ldots,n]$ can be expressed as a linear combination of determinants involving the $G_{i,j}$'s and $G'_{i,j}$'s. We shall re-express them as linear combinations of Pfaffians.

1.3. Partial pairings in terms of Pfaffians

For a partial pairing τ in which node n is in a doubleton part, we can encode τ by a string λ of n symbols, where the symbol at position i encodes the role of node i in the partial pairing. For bookkeeping purposes that will soon become apparent, we label each symbol with the label of the node that it represents; when the labels are $1,\ldots,n$ we sometimes omit the labels. For example, for the annular partial pairing

$$\tau = 1,5|2|3,4|7,10|8,14|11|12,13 \quad =$$

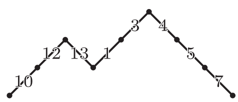


the associated (labelled) encoding string is

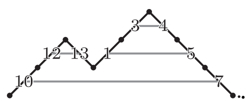
$$\lambda = \lambda(\tau) = \overset{1}{\mathsf{U}}\overset{2}{\mathsf{S}}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{6}{\mathsf{I}}\overset{7}{\mathsf{D}}\overset{8}{\mathsf{F}}\overset{9}{\mathsf{I}}\overset{10}{\mathsf{U}}\overset{11}{\mathsf{S}}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{14}{\odot}. \tag{1.3}$$

Here node n, which is on the other boundary, is given the special symbol $\odot$. The node paired with n is also given a special symbol, F, which stands for 'flat step'. (So each $\lambda(\tau)$ has exactly one F and one $\odot$ symbol.) I in dicates that the node has been internalized, and S indicates a node in a singleton part. The remaining nodes are assigned the symbols U and D, which stand for 'up-step' and 'down-step', so that when the F is cyclically rotated to the end, the substring λ° formed by the U's and D's defines a (labelled) Dyck path whose associated non-crossing matching is the pairing of the nodes in τ. In the above example,

$$\lambda^\circ = \overset{10}{\mathsf{U}}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{1}{\mathsf{U}}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{7}{\mathsf{D}} \quad = \quad$$



$=$



We call the string λ the *augmented cyclic Dyck path* associated with the partial pairing τ – 'augmented' because it contains symbols not in the Dyck path λ°, and 'cyclic' because its start is determined by the location of the F symbol.

Given two labelled augmented cyclic Dyck paths λ and μ, we say that $\lambda \preceq \mu$ if they are the same length, have the same labels, all the letters other than U and D are the same in both λ and μ, and as Dyck paths, λ° lies below μ°.

If λ is a *labelled string*, we let λ_i denote its ith labelled symbol, and we let $\lambda(i)$ denote the label of λ_i.

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For a labelled augmented cyclic Dyck path μ, we define μ^{I} to be the labelled string obtained from μ by deleting all the S letters, and replacing each $\overset{i}{\mathsf{I}}$ with the two letters $\overset{i}{\odot}\overset{i}{\bigcirc}$. We also define μ^{S} to be the labelled string obtained from μ by deleting all the I letters, and replacing each $\overset{i}{\mathsf{S}}$ with the two letters $\overset{i}{\bigcirc}\overset{i}{\odot}$. For example, if μ is the labelled augmented cyclic Dyck path in (1.3), then

$$\mu^{\mathsf{I}} = \overset{1}{\mathsf{U}}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{6}{\odot}\overset{6}{\bigcirc}\overset{7}{\mathsf{D}}\overset{8}{\mathsf{F}}\overset{9}{\odot}\overset{9}{\bigcirc}\overset{10}{\mathsf{U}}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{14}{\odot}$$

and

$$\mu^{\mathsf{S}} = \overset{1}{\mathsf{U}}\overset{2}{\bigcirc}\overset{2}{\odot}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{7}{\mathsf{D}}\overset{8}{\mathsf{F}}\overset{10}{\mathsf{U}}\overset{11}{\bigcirc}\overset{11}{\odot}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{14}{\odot}.$$

Next we define $\ddot{\mu}$ and $\overline{\mu}$. Recall that there is only one letter F in μ; let f be its label, so that $\overset{f}{\mathsf{F}} \in \mu$. For each letter $\overset{i}{\mathsf{U}}$, $\overset{i}{\mathsf{D}}$, $\overset{f}{\mathsf{F}}$ in μ^{S}, we make the substitutions

$$\overset{i}{\mathsf{U}} \mapsto \begin{cases} \overset{i}{\oplus} & i < f \\ \overset{i}{\bigcirc} & i > f \end{cases} \qquad \overset{i}{\mathsf{D}} \mapsto \begin{cases} \overset{i}{\bigcirc} & i < f \\ \overset{i}{\ominus} & i > f \end{cases} \qquad \overset{f}{\mathsf{F}} \mapsto \overset{f}{\bigcirc}$$

to obtain $\overline{\mu}$. We let $\ddot{\mu}$ be the result of these same substitutions applied to μ^{I}. For our example,

$$\ddot{\mu} = \overset{1}{\oplus}\overset{3}{\oplus}\overset{4}{\bigcirc}\overset{5}{\bigcirc}\overset{6}{\odot}\overset{6}{\bigcirc}\overset{7}{\bigcirc}\overset{8}{\bigcirc}\overset{9}{\odot}\overset{9}{\bigcirc}\overset{10}{\bigcirc}\overset{12}{\bigcirc}\overset{13}{\ominus}\overset{14}{\odot}$$

and

$$\overline{\mu} = \overset{1}{\oplus}\overset{2}{\bigcirc}\overset{2}{\odot}\overset{3}{\oplus}\overset{4}{\bigcirc}\overset{5}{\bigcirc}\overset{7}{\bigcirc}\overset{8}{\bigcirc}\overset{10}{\bigcirc}\overset{11}{\bigcirc}\overset{11}{\odot}\overset{12}{\bigcirc}\overset{13}{\ominus}\overset{14}{\odot}.$$

The original string μ can be recovered from either $\overline{\mu}$ or $\ddot{\mu}$.

Given a string σ of m labelled symbols $\oplus$, $\ominus$, $\bigcirc$, $\odot$, such as as the ones above, we define an $m \times m$ matrix $M_\sigma(A, A')$ by

$$M_\sigma(A,A') := \begin{matrix} \\ \sigma_i \neq \odot \\ \sigma_i = \odot \end{matrix} \left[\begin{matrix} \sigma_j \neq \odot & \sigma_j = \odot \\ -A'_{\sigma(i),\sigma(j)} + A_{\sigma(i),\sigma(j)} \begin{pmatrix} +1_{\sigma_i=\oplus} - 1_{\sigma_j=\oplus} \\ -1_{\sigma_i=\ominus} + 1_{\sigma_j=\ominus} \end{pmatrix} & A_{\sigma(i),\sigma(j)} \\ -A_{\sigma(i),\sigma(j)} & 0 \end{matrix} \right]_{\substack{j=1,\ldots,m \\ i=1,\ldots,m}}$$

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The symbols ⊕, ⊖, and ○ are mnemonic for +1, −1, and 0, which go into the coefficient of $A_{\sigma(i),\sigma(j)}$ when $\sigma_i, \sigma_j \neq \odot$. For example, when σ is the above value for $\overline{\mu}$, this matrix is

$$
\begin{array}{c} \\ {}^{+}1 \\ 2 \\ \odot 2 \\ \\ {}^{+}3 \\ 4 \\ \\ 5 \\ 7 \\ \vdots \end{array}
\begin{array}{cccccccccc}
\bar{1} & 2 & \overset{\odot}{2} & \bar{3} & 4 & 5\ 7\ 8\ 10\ 11 & \overset{\odot}{11} & 12 & \overset{+}{13} & \overset{\odot}{14} \\
\hline
0 & A_{1,2}-A'_{1,2} & A_{1,2} & -A'_{1,3} & A_{1,4}-A'_{1,4} & \cdots\cdots\cdots & A_{1,11} & A_{1,12}-A'_{1,12} & 2A_{1,13}-A'_{1,13} & A_{1.14} \\
A'_{1,2}-A_{1,2} & 0 & A_{2,2} & -A_{2,3}-A'_{2,3} & -A'_{2,4} & \cdots\cdots\cdots & A_{2,11} & -A'_{2,12} & A_{2,13}-A'_{2,13} & A_{2,14} \\
-A_{1,2} & -A_{2,2} & 0 & -A_{2,3} & -A_{2,4} & \cdots\cdots\cdots & 0 & -A_{2,12} & -A_{2,13} & 0 \\
\vdots & \vdots & \ddots & & & & & & & \\
 & & & 0 & A_{3,4}-A'_{3,4} & \cdots\cdots\cdots & A_{3,11} & A_{3,12}-A'_{3,12} & 2A_{3,13}-A'_{3,13} & A_{3,14} \\
 & & & & 0 & \cdots\cdots\cdots & A_{4,11} & -A'_{4,12} & A_{4,13}-A'_{4,13} & A_{4,14} \\
 & & & & & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & &
\end{array}.
$$

We define $M_\mu^{(L)} = M_{\dddot{\mu}}(L, L')$, and $M_\mu^{(G)} = M_{\overline{\mu}}(G, G')$, where $G_{i,n}$ is replaced with 1. Both $M_\mu^{(L)}$ and $M_\mu^{(G)}$ are antisymmetric. The new formulas involve Pfaffians of these matrices $M_\mu^{(L)}$ and $M_\mu^{(G)}$.

The new formulas have coefficients that are defined in terms of '*cover-inclusive Dyck tilings*', which were first defined in [8] and independently in [13], and were studied further in [11, 12, 10, 9, 4], and whose definition we now recall. If λ and μ are Dyck paths such that λ is below μ, then the region λ/μ is a skew Young diagram (rotated 45°). A Dyck tile is a ribbon tile which is shaped like a Dyck path, that is, a collection of $\sqrt{2} \times \sqrt{2}$ boxes rotated 45° centred at the points of a Dyck path. A Dyck tiling of λ/μ is a tiling of it by Dyck tiles. We say that one Dyck tile covers another Dyck tile if it contains a box which is directly (not diagonally) above a box of the other tile. A cover-inclusive (c.i.) Dyck tiling is one for which, whenever a Dyck tile T_1 covers another Dyck tile T_2, the range of x-coordinates of T_1 is a subset of the range of x-

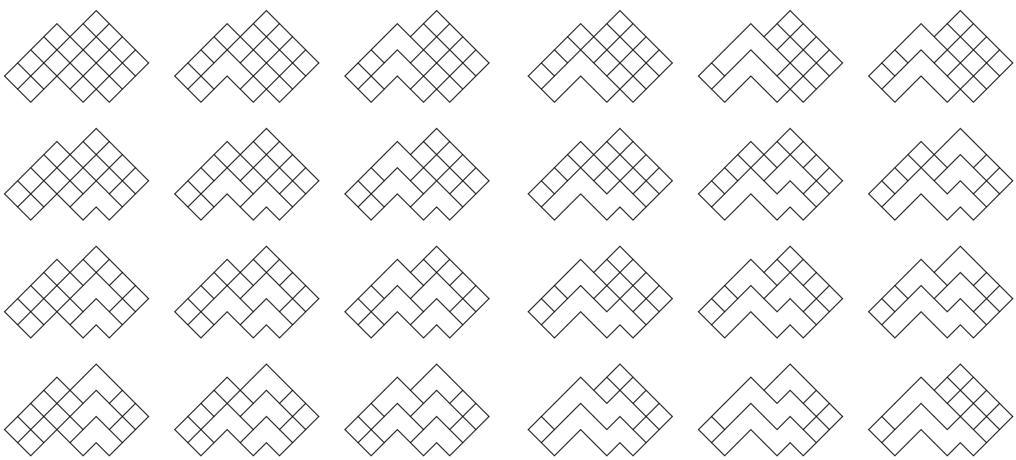

Figure 2. Cover-inclusive Dyck tilings of a skew shape. (This figure first appeared in [8].)

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coordinates of T_2. See Figure 2 for a list of the Dyck tilings of a particular skew shape λ/μ.

Theorem 1.1. *Suppose τ is a partial pairing of the nodes of an annular-one graph with n nodes, where node n is paired in τ. Let λ be the labelled augmented cyclic Dyck path which encodes τ. Then*

$$\frac{Z[\tau]}{Z[1|2|\cdots|n]} = \sum_{\mu \geq \lambda} [\#\text{ of c.i. Dyck tilings of } \lambda^\circ/\mu^\circ] \times \operatorname{Pf} M_\mu^{(L)} \tag{1.4}$$

and

$$\frac{Z[\tau]}{Z[1,2,\ldots,n]} = \sum_{\mu \geq \lambda} [\#\text{ of c.i. Dyck tilings of } \lambda^\circ/\mu^\circ] \times \operatorname{Pf} M_\mu^{(G)}. \tag{1.5}$$

1.4. Examples

We give a couple of examples.

For the partial pairing $1,3|2,4$, the encoding string λ is $\overset{1}{\mathsf{D}}\overset{2}{\mathsf{F}}\overset{3}{\mathsf{U}}\overset{4}{\odot}$; the only μ in the sum is $\mu = \overset{1}{\mathsf{D}}\overset{2}{\mathsf{F}}\overset{3}{\mathsf{U}}\overset{4}{\odot}$, for which the skew Young diagram λ°/μ° has only the empty Dyck tiling, so the coefficient is 1. For this $\mu = \overset{1}{\mathsf{D}}\overset{2}{\mathsf{F}}\overset{3}{\mathsf{U}}\overset{4}{\odot}$, $\ddot{\mu} = \overset{1}{\bigcirc}\overset{2}{\bigcirc}\overset{3}{\bigcirc}\overset{4}{\odot}$, so

$$\frac{Z[1,3|2,4]}{Z[1|2|3|4]} = \operatorname{Pf} \underbrace{\begin{bmatrix} 0 & -L'_{1,2} & -L'_{1,3} & L_{1,4} \\ L'_{1,2} & 0 & -L'_{2,3} & L_{2,4} \\ L'_{1,3} & L'_{2,3} & 0 & L_{3,4} \\ -L_{1,4} & -L_{2,4} & -L_{3,4} & 0 \end{bmatrix}}_{M^{(L)}_{\mathsf{DFU}\odot}} = -L'_{1,2}L_{3,4} - L'_{2,3}L_{1,4} - L'_{3,1}L_{2,4},$$

which matches [10, eqn 5.5b], and $\overline{\mu} = \overset{1}{\bigcirc}\overset{2}{\bigcirc}\overset{3}{\bigcirc}\overset{4}{\odot}$, so

$$\frac{Z[1,3|2,4]}{Z[1,2,3,4]} = \operatorname{Pf} \underbrace{\begin{bmatrix} 0 & -G'_{1,2} & -G'_{1,3} & 1 \\ G'_{1,2} & 0 & -G'_{2,3} & 1 \\ G'_{1,3} & G'_{2,3} & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}}_{M^{(G)}_{\mathsf{DFU}\odot}} = -G'_{1,2} - G'_{2,3} - G'_{3,1},$$

which matches [10, eqn 5.6b].

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For the partial pairing $1,4|2|6,7$ the encoding string is $\lambda = \overset{1\,2\,3\,4\,5\,6\;\,7}{\mathsf{USIDIF}\odot}$, the only $\mu \succeq \lambda$ is $\mu = \overset{1\,2\,3\,4\,5\,6\;\,7}{\mathsf{USIDIF}\odot}$, and $\ddot{\mu} = \overset{1}{\oplus}\overset{3}{\odot}\overset{3}{\bigcirc}\overset{4}{\bigcirc}\overset{5}{\odot}\overset{5}{\bigcirc}\overset{6}{\bigcirc}\overset{7}{\odot}$, so

$$\frac{Z[1,4|2|6,7]}{Z[1|2|3|4|5|6|7]}$$

$$= \operatorname{Pf} \underbrace{\begin{bmatrix} 0 & L_{1,3} & L_{1,3}-L'_{1,3} & L_{1,4}-L'_{1,4} & L_{1,5} & L_{1,5}-L'_{1,5} & L_{1,6}-L'_{1,6} & L_{1,7} \\ -L_{1,3} & 0 & -L_{3,3} & -L_{3,4} & 0 & -L_{3,5} & -L_{3,6} & 0 \\ L'_{1,3}-L_{1,3} & L_{3,3} & 0 & -L'_{3,4} & L_{3,5} & -L'_{3,5} & -L'_{3,6} & L_{3,7} \\ L'_{1,4}-L_{1,4} & L_{3,4} & L'_{3,4} & 0 & L_{4,5} & -L'_{4,5} & -L'_{4,6} & L_{4,7} \\ -L_{1,5} & 0 & -L_{3,5} & -L_{4,5} & 0 & -L_{5,5} & -L_{5,6} & 0 \\ L'_{1,5}-L_{1,5} & L_{3,5} & L'_{3,5} & L'_{4,5} & L_{5,5} & 0 & -L'_{5,6} & L_{5,7} \\ L'_{1,6}-L_{1,6} & L_{3,6} & L'_{3,6} & L'_{4,6} & L_{5,6} & L'_{5,6} & 0 & L_{6,7} \\ -L_{1,7} & 0 & -L_{3,7} & -L_{4,7} & 0 & -L_{5,7} & -L_{6,7} & 0 \end{bmatrix}}_{M^{(L)}_{\mathsf{USIDIF}\odot}},$$

while $\overline{\mu} = \overset{1}{\oplus}\overset{2}{\bigcirc}\overset{2}{\odot}\overset{4}{\bigcirc}\overset{6}{\bigcirc}\overset{7}{\odot}$, so

$$\frac{Z[1,4|2|6,7]}{Z[1,2,3,4,5,6,7]} = \operatorname{Pf} \underbrace{\begin{bmatrix} 0 & G_{1,2}-G'_{1,2} & G_{1,2} & G_{1,4}-G'_{1,4} & G_{1,6}-G'_{1,6} & 1 \\ G'_{1,2}-G_{1,2} & 0 & G_{2,2} & -G'_{2,4} & -G'_{2,6} & 1 \\ -G_{1,2} & -G_{2,2} & 0 & -G_{2,4} & -G_{2,6} & 0 \\ G'_{1,4}-G_{1,4} & G'_{2,4} & G_{2,4} & 0 & -G'_{4,6} & 1 \\ G'_{1,6}-G_{1,6} & G'_{2,6} & G_{2,6} & G'_{4,6} & 0 & 1 \\ -1 & -1 & 0 & -1 & -1 & 0 \end{bmatrix}}_{M^{(G)}_{\mathsf{USIDIF}\odot}}.$$

For $1,2|3,7|4,6$ we have $\lambda = \mathsf{UDFUID}\odot$. There are two μ's such that $\mu \succeq \lambda$:

$$\frac{Z[1,2|3,7|4,6]}{Z[1,2,3,4,5,6,7]} = \operatorname{Pf} M^{(G)}_{\mathsf{UDFUID}\odot} + \operatorname{Pf} M^{(G)}_{\mathsf{DDFUIU}\odot}.$$

For the partial pairing $1,3|2|4,10|5,6|7,9$ we have $\lambda = \mathsf{USDFUDUID}\odot$. There are five μ's such that $\mu \succeq \lambda$, and for one of these μ's the skew Young diagram λ°/μ° has two Dyck tilings:

$$\frac{Z[1,3|2|4,10|5,6|7,9]}{Z[1,2,3,4,5,6,7,8,9,10]} = \operatorname{Pf} M^{(G)}_{\mathsf{USDFUDUID}\odot} + \operatorname{Pf} M^{(G)}_{\mathsf{USDFUUDID}\odot}$$
$$+ \operatorname{Pf} M^{(G)}_{\mathsf{DSDFUDUIU}\odot} + \operatorname{Pf} M^{(G)}_{\mathsf{DSDFUUDIU}\odot} + 2 \times \operatorname{Pf} M^{(G)}_{\mathsf{DSDFUUUID}\odot}.$$

1.5. Corollaries

The formulas in Theorem 1.1 immediately imply the following statement.

Corollary 1.2. *For a partition τ on $\{1,\dots,n\}$ in which n is not in a singleton part, on an annular-one graph with n nodes, the ratio $Z[\tau]/Z[1|2|\cdots|n]$ is a polynomial in the variables*

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L and L' with integer coefficients. Similarly, $Z[\tau]/Z[1,2,\ldots,n]$ is a polynomial in G and G' with integer coefficients.

It was known that these ratios are polynomials in the L and L' variables, or the G and G' variables, and that the coefficients were half-integers [10], but the integrality of the coefficients was previously a mystery.

Recalling

$$\frac{Z[1,3|2,4]}{Z[1,2,3,4]} = -G'_{1,2} - G'_{2,3} - G'_{3,1},$$

observe that this polynomial is invariant under the substitution $G'_{i,j} \to G'_{i,j} + f(i) - f(j)$. The next corollary states that this is a general phenomenon for the G-G'-polynomials of any partition.

Corollary 1.3. *For a partition τ on $\{1,\ldots,n\}$ in which n is not in a singleton part, on an annular-one graph with n nodes, the G-G'-polynomial for $Z[\tau]/Z[1,2,\ldots,n]$ is invariant under replacing each $G'_{i,j}$ with $G'_{i,j} + f(i) - f(j)$.*

Proof of Corollary 1.3. Consider each Pfaffian in the formula from Theorem 1.1. Since the last column (row) is all 1's (−1's), we can add an all-$f(i)$'s row to row i and subtract an all-$f(i)$'s column from column i, without changing the value of the Pfaffian. Since $G'_{i,j}$ occurs only in row i and column j (and row j and column i), with coefficient 1 (and −1), these operations replace each $G'_{i,j}$ with $G'_{i,j} + f(i) - f(j)$ and keep the Pfaffian invariant. □

We remark that it was known [10] that substituting $G'_{i,j} \to G'_{i,j} + f(i) - f(j)$ and then evaluating the polynomial at the values of G and G' that arise from an annular-one graphs will give a result independent of f. Corollary 1.3 is a stronger statement, since it was not known whether the values of G and G' that arise from annular-one graphs are full-dimensional or whether they satisfy algebraic relations which cause the substituted G-G'-polynomials, when evaluated at these values, to be independent of f.

2. Determinant formulas

For an annular partial pairing τ on n nodes, let λ be its encoding string, let T denote the set of internalized nodes, and let Q denote the set of singleton nodes. Let k denote the order of the Dyck path λ°, that is, half its length, so that $n = 2k + 2 + |Q| + |T|$.

Let $S \subset \{1,\ldots,n\} \setminus (Q \cup T)$ be a subset of the paired nodes which has size $k+1$ and includes n, and let $R = \{1,\ldots,n\} \setminus (S \cup Q \cup T)$ be the complementary set of paired nodes. Given λ and S, Kenyon and Wilson [10] defined

$$\mathbf{B}_{\lambda,S}(\zeta) = \sum_{\mu \succeq \lambda} [\#\text{ of c.i. Dyck tilings of } \lambda/\mu] \times \zeta^{\#\text{ indices in } S \text{ at which } \mu \text{ has an up-step}} \times \zeta^{-\#\text{ indices in } S\setminus\{n\} \text{ after } \lambda\text{'s flat step} + \#\text{ down-steps of } \lambda \text{ after } \lambda\text{'s flat step}}, \tag{2.1}$$

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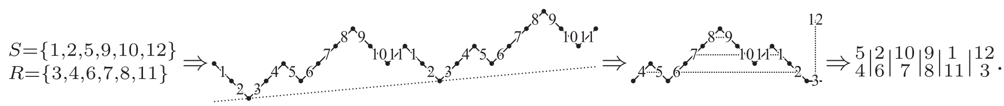
and showed how to use these polynomials $\mathbf{B}_{\lambda,S}$ to compute the ratios of grove partition functions. Specifically,

$$\frac{Z[\tau]}{Z[1|2|\cdots|n]} = (-1)^{|T|} \times \lim_{z\to 1} \sum_{R,S} \frac{\mathbf{B}_{\lambda,S}(z^2)}{(1-z^2)^k} \det \mathscr{L}_{R,T}^{S,T}, \tag{2.2}$$

where $\mathscr{L}_{R,T}^{S,T}$ denotes the submatrix of $\mathscr{L}$ whose rows are indexed by R and T and whose columns are indexed by S and T, and we need to specify a pairing between the indices of R and S to determine the signs of the determinants. We use the Dvoretzky–Motzkin cycle lemma bijection to make this pairing, as indicated below (figure taken from [10]). Essentially we make a path with period $2k+1$ which has an up-step at each index in R and a down-step at each index in $S \setminus \{n\}$. The up- and down-steps are the endpoints of chords underneath the path, and these chords define the pairing, where the extra up-step is paired with n:



Recall that $\mathscr{L}_{i,j} = \mathscr{L}_{i,j}(z)$ is a function of z. We change variables to $z = e^t$ (here we differ slightly from the notation in [10], which used $\zeta = z^2 = e^t$). We expand $\mathscr{L}_{i,j}(e^t) = L_{i,j} + L'_{i,j}t + \cdots$, and let $\hat{\mathscr{L}}_{i,j}$ denote its linearized approximation $\hat{\mathscr{L}}_{i,j} = L_{i,j} + L'_{i,j}t$. In general the series expansion for $\mathscr{L}_{i,j}(e^t)$ will have more terms, but while it is not *a priori* obvious, the limit (2.2) can be evaluated using $\hat{\mathscr{L}}_{i,j}$ in place of $\mathscr{L}_{i,j}(e^t)$:

$$\frac{Z[\tau]}{Z[1|2|\cdots|n]} = (-1)^{|T|} \times \lim_{t\to 0} \frac{1}{(-2t)^k} \sum_{R,S} \mathbf{B}_{\lambda,S}(e^{2t}) \det \hat{\mathscr{L}}_{R,T}^{S,T}, \tag{2.3}$$

and a similar formula

$$\frac{Z[\tau]}{Z[1,2,\ldots,n]} = \lim_{t\to 0} \frac{1}{(-2t)^k} \sum_{R,S} \mathbf{B}_{\lambda,S}(e^{2t}) \det \hat{\mathscr{G}}_{R,Q}^{S,Q} \tag{2.4}$$

holds, where $\hat{\mathscr{G}}_{i,j} = G_{i,j} + G'_{i,j}t$ and each $\hat{\mathscr{G}}_{i,n}$ is replaced with 1 [10]. From these formulas we derive the Pfaffian formulas.

3. Pfaffian formulas

We start in Section 3.1 by showing that a Pfaffian can be expressed as a sum of determinants. In Section 3.2 we give an application of this identity to tripartite pairings. Then we use the Pfaffian identity and equations (2.3) and (2.4) to prove Theorem 1.1 in Section 3.3.

3.1. The Pfaffian as a sum of determinants

For any matching $M = (i_1, j_1), \ldots, (i_k, j_k)$, we define $\operatorname{sign}(M) = (-1)^{\operatorname{cr}(M)}$, where $\operatorname{cr}(M)$ is the number of crossings of arcs from M when M is drawn as k arcs between the points $\{1, \ldots, 2k\}$ on a line. For the left endpoint of each arc we can associate an up-step,

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and for each right endpoint we can associate a down-step, which results in a Dyck path. The down-steps of the matching M are the down-steps of its Dyck path, that is, $\{\max(i_1, j_1), \ldots, \max(i_k, j_k)\}$.

Given a set of positive integers R for which $R \subset \{1, \ldots, 2|R|\}$, we define d_R as follows. We let $n = 2|R|$ and $S = \{1, \ldots, n\} \setminus R$. For an arbitrary matrix A we define

$$d_R(A) := \begin{cases} \det[A_{i,j}]_{i\in R}^{j\in S} & n \in S, \\ \det[-A_{i,j}]_{i\in R}^{j\in S} & n \in R, \end{cases} \tag{3.1}$$

where R and S are ordered according to the Dvoretzky–Motzkin bijection as described above. For example,

$$d_{\{3,4,6,7,8,11\}}(A) = \det[A_{i,j}]_{i=4,6,7,8,11,3}^{j=5,2,10,9,1,12}.$$

Lemma 3.1. *Suppose $n \geqslant 0$ is even, $R \subset \{1, \ldots, n\}$, $|R| = n/2$, and $S = \{1, \ldots, n\} \setminus R$. Let A be an arbitrary $n \times n$ matrix. Then*

$$d_R(A) = \sum_{\substack{\text{directed matchings } M \text{ s.t.} \\ M \text{ matches } R \text{ to } S}} (-1)^{\operatorname{cr}(M)} \prod_{(r,s)\in M} (-1)^{1_{r>s}} A_{r,s}.$$

Proof. Let $k = n/2$. The arrangement of elements of R and S in d_R is given by the Dvoretzky–Motzkin cycle lemma bijection, and in particular corresponds to a matching

$$M_0 = \{(r_1, s_1), \ldots, (r_k, s_k)\}$$

(each $r_\ell \in R$ and $s_\ell \in S$), which has no crossings when drawn in the annulus. By the determinant expansion,

$$\det A_R^S = \sum_{\pi \in \mathfrak{S}_k} \operatorname{sign}(\pi) \prod_{\ell=1}^{k} A_{r_\ell, s_{\pi(\ell)}}. \tag{3.2}$$

Suppose $n \notin R$. When we draw matching M_0 on a line, there may be crossings of the arc (j, n) from arcs (a, b), such that $a > j > b$; these are precisely the arcs whose starting point is larger than its endpoint. When drawn on the line, the number of crossings is $\operatorname{cr}(M_0) = \sum_{(i,j)\in M_0} 1_{i>j}$. If instead $n \in R$, then $\operatorname{cr}(M_0) = \sum_{(i,j)\in M_0} 1_{i<j}$.

For a permutation π let the matching $M(\pi)$ be

$$M(\pi) = \{(r_1, s_{\pi(1)}), \ldots, r_k, s_{\pi(k)}\}.$$

The matching M_0 corresponds to the identity permutation, so at least when the permutation π is the identity, we have

$$\operatorname{sign}(\pi) = (-1)^{\operatorname{cr}(M(\pi))} (-1)^{\sum_{(r,s)\in M(\pi)} 1_{r>s}} (-1)^{(n/2) 1_{n\in R}},$$

a formula which we now verify for the other permutations. Any permutation π can be expressed as a sequence of transpositions, and it is a straightforward case analysis to verify that any transposition changes the parity of the number of crossings in the matching plus the number of arcs directed backwards. □

Theorem 3.2. *Suppose $n \geqslant 0$ is even. If A is an arbitrary $n \times n$ matrix, and $d_R(A)$ is as defined in* (3.1)*, then*

$$\sum_{\substack{R\subset\{1,\dots,n\}\\ |R|=n/2}} d_R(A) = \operatorname{Pf}\left[A - A^T\right], \tag{3.3}$$

where A^T is the transpose of A.

Proof. From Lemma 3.1, we see that the left-hand side of (3.3) equals

$$\sum_{\text{directed matchings } M} (-1)^{\operatorname{cr}(M)} \prod_{(r,s)\in M} (-1)^{1_{r>s}} A_{r,s}.$$

Let $n = 2k$, and let $W_{i,j} = A_{i,j} - A_{j,i}$. We can expand the Pfaffian as

$$\operatorname{Pf}[W] = \sum_{\substack{\text{undirected matchings } M\\ M=\{(i_1,j_1),\dots,(i_k,j_k)\}\\ i_1<j_1,\dots,i_k<j_k\\ j_1<\cdots<j_k}} (-1)^{\operatorname{cr}(M)} \prod_{\ell=1}^{k} W_{i_\ell,j_\ell}.$$

When we make the substitution $W_{i,j} = A_{i,j} - A_{j,i}$, this has the effect of choosing directions for each pairing, converting the sum over undirected matchings into a sum over directed matchings:

$$\operatorname{Pf}\left[A - A^T\right] = \sum_{\text{directed matchings } M} (-1)^{\operatorname{cr}(M)} \prod_{(r,s)\in M} (-1)^{1_{r>s}} A_{r,s}. \qquad \square$$

3.2. Applications of the Pfaffian identity

Before continuing with our main result, we mention an interesting consequence of Theorem 3.2. Curtis, Ingerman and Morrow [2] gave an interpretation of the determinant $\det L_R^S$ when $R = \{r_1,\dots,r_k\}$ and $S = \{s_1,\dots,s_k\}$ are disjoint subsets of $\{1,\dots,n\}$, which, when translated into the language of groves, asserts that

$$\det L_{r_1,\dots,r_k}^{s_1,\dots,s_k} = \sum_{\pi\in\mathfrak{S}_k} \operatorname{sign}(\pi) \frac{Z\left[r_1, s_{\pi(1)}|\cdots|r_k, s_{\pi(k)}|(\text{other nodes singletons})\right]}{Z\left[1|\cdots|n\right]}. \tag{3.4}$$

This formula holds for any graph.

If B and C are two disjoint sets of nodes, and we set $A_{i,j} = 0$ when $i \in C$ or $j \in B$ and otherwise set $A_{i,j} = L_{i,j}$, then Theorem 3.2 with the above interpretation of the minors implies that $\operatorname{Pf}[A - A^T]$ is a sum over directed matchings for which the nodes in B are sources and the nodes in C are destinations, of the sign of the directed matching times the grove ratio associated with that matching. In particular, nodes of B are only paired with nodes not in B, and nodes in C are only paired with nodes not in C. If a matching M contains a pair (i,j) where $i,j \in B$ or $i,j \in C$, then M is not included in the sum. Notice that if $i,j \notin B \cup C$, then the matching $(M \setminus \{(i,j)\}) \cup \{(j,i)\}$, in which the pair (i,j) has

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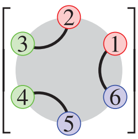
been reversed, has the same weight as M but opposite sign. Thus

$$\operatorname{Pf} \begin{array}{c} \scriptstyle i\in B \\ \scriptstyle i\notin B\cup C \\ \scriptstyle i\in C \end{array} \overset{\begin{array}{ccc} \scriptstyle j\in B & \scriptstyle j\notin B\cup C & \scriptstyle j\in C \end{array}}{\begin{bmatrix} 0 & L_{i,j} & L_{i,j} \\ -L_{i,j} & 0 & L_{i,j} \\ -L_{i,j} & -L_{i,j} & 0 \end{bmatrix}} = \sum_{\substack{\text{directed matchings } M \\ \text{if } (i,j)\in M \text{ then} \\ i\in B \text{ or } j\in C \text{ or both}}} \left((-1)^{\operatorname{cr}(M)} \prod_{(i,j)\in M} (-1)^{1_{j<i}} \right) \ddot{Z}[i_1, j_1|\cdots|i_{n/2}, j_{n/2}]. \tag{3.5}$$

When the graph is circular-planar (*i.e.*, the nodes lie on the outer face of a planar graph), and $B=\{1,\dots,|B|\}$ and $C=\{n+1-|C|,\dots,n\}$, there is only one matching M for which $\ddot{Z}[M]\neq 0$, and the sign is positive, so $\ddot{Z}[M]$ is the Pfaffian. For example,



$$\ddot{Z}\left[\,\cdot\,\right] = \frac{Z[1,6|2,3|4,5]}{Z[1|2|3|4|5|6]} = \operatorname{Pf}\begin{bmatrix} 0 & 0 & L_{1,3} & L_{1,4} & L_{1,5} & L_{1,6} \\ 0 & 0 & L_{2,3} & L_{2,4} & L_{2,5} & L_{2,6} \\ -L_{1,3} & -L_{2,3} & 0 & 0 & L_{3,5} & L_{3,6} \\ -L_{1,4} & -L_{2,4} & 0 & 0 & L_{4,5} & L_{4,6} \\ -L_{1,5} & -L_{2,5} & -L_{3,5} & -L_{4,5} & 0 & 0 \\ -L_{1,6} & -L_{2,6} & -L_{3,6} & -L_{4,6} & 0 & 0 \end{bmatrix}.$$

This is one of several tripartite matching formulas that were derived earlier by Kenyon and Wilson [6] using a different method [7].

The determinant formula (3.4) has been extended in several directions. Kenyon and Wilson [10] showed that if $Q=\{q_1,\dots,q_\ell\}$ and $T=\{t_1,\dots,t_m\}$, and Q,R,S,T partition $\{1,\dots,n\}$, then

$$\det \mathscr{L}^{s_1,\dots,s_k,t_1,\dots,t_m}_{r_1,\dots,r_k,t_1,\dots,t_m} = (-1)^m \sum_{\pi\in\mathfrak{S}_k} \operatorname{sign}(\pi) \frac{\mathscr{Z}\left[{}^{s_{\pi(1)}}_{r_1}|\cdots|{}^{s_{\pi(k)}}_{r_k}|q_1|\cdots|q_\ell\right]}{\mathscr{Z}[1|\cdots|n]}, \tag{3.6}$$

where the $\mathscr{Z}$'s give the weighted sum of 'cycle-rooted groves'. (The cycle weights go to zero and $\mathscr{Z}$ converges to Z when $z\to 1$: see [10] for further explanation.) When we combine Theorem 3.2 with the above formula, we obtain the following theorem.

Theorem 3.3. *Suppose there are n nodes, P,Q,T partition $\{1,\dots,n\}$, and $|P|=2k$ is even. For each $i\in P$ let α_i and β_i be parameters, and for $i\in T$ let $\alpha_i=\beta_i=1$. List the nodes $p_1,\dots,p_{2k}$, $t'_1,t_1,\dots,t'_m,t_m$, where t'_i is a second copy of t_i, and let $T'=\{t'_1,\dots,t'_m\}$. Then*

$$\operatorname{Pf} \begin{array}{c} \scriptstyle i\in P\cup T \\ \scriptstyle i\in T' \end{array} \overset{\begin{array}{cc} \scriptstyle j\in P\cup T & \scriptstyle j\in T' \end{array}}{\begin{bmatrix} \alpha_i\beta_j\mathscr{L}_{i,j} - \alpha_j\beta_i\mathscr{L}_{j,i} & \alpha_i\mathscr{L}_{i,j} \\ -\alpha_j\mathscr{L}_{j,i} & 0 \end{bmatrix}}^{j=p_1,\dots,p_{2k},t'_1,t_1,\dots,t'_m,t_m}_{i=p_1,\dots,p_{2k},t'_1,t_1,\dots,t'_m,t_m} = \sum_{\substack{\text{directed matchings } M \text{ of } P \\ M=\{(r_1,s_1),\dots,(r_k,s_k)\}}} \left((-1)^{\operatorname{cr}(M)} \prod_{(r,s)\in M} (-1)^{1_{s<r}} \alpha_r\beta_s \right) \times \frac{\mathscr{Z}\left[{}^{s_1}_{r_1}|\cdots|{}^{s_k}_{r_k}|q_1|\cdots|q_\ell\right]}{\mathscr{Z}[1|\cdots|n]}. \tag{3.7}$$

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Proof. If $i \in T'$ then take $\alpha_i = 0$ and $\beta_i = 1$. Then apply Theorem 3.2 with $A_{i,j} = \alpha_i \beta_j \mathscr{L}_{i,j}$, and use (3.6) to interpret the determinants. The factor of $(-1)^m$ in (3.6) is absorbed into the Pfaffian because we listed each t_i' before t_i. □

Any of (3.4) or (3.5) or (3.6) can be recovered from (3.7) by choosing the α's and β's suitably and/or setting $z = 1$.

3.3. Proof of main theorem

Our approach to proving the Pfaffian formulas in Theorem 1.1 is to prove that the right-hand sides of (2.3) and (2.4) are equal as polynomials in formal variables to the Pfaffian expressions. We will not, for example, use the fact that $\sum_j L_{i,j} = 0$ or other relations that the electrical network quantities might satisfy, since the $L_{i,j}$'s and the $G_{i,j}$'s satisfy different relations. By working with formal variables that do not satisfy these extra relations, the same proof works for both the L-L' polynomials and the G-G' polynomials.

The roles of the S and I symbols are reversed between the L-L' polynomials for $\ddot{Z}[\tau]$ and the G-G' polynomials for $\overline{Z}[\tau]$. As a matter of convenience, we will give these symbols the roles they have for the G-G' polynomials. To obtain the L-L' polynomials, we will at some point substitute I for S and S for I.

Let λ be a labelled augmented cyclic Dyck path with n symbols. Let λ^* be the substring obtained from λ by excising all S and I symbols that it contains, let n^* be the length of λ^*, and let E_λ be the set of labels of S symbols. For our running example

$$\lambda = \overset{1}{\mathsf{U}}\overset{2}{\mathsf{S}}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{6}{\mathsf{I}}\overset{7}{\mathsf{D}}\overset{8}{\mathsf{F}}\overset{9}{\mathsf{I}}\overset{10}{\mathsf{U}}\overset{11}{\mathsf{S}}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{14}{\odot},$$

we have

$$\lambda^* = \overset{1}{\mathsf{U}}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{7}{\mathsf{D}}\overset{8}{\mathsf{F}}\overset{10}{\mathsf{U}}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{14}{\odot}, \quad n^* = 10, \quad E_\lambda = \{2, 11\}.$$

For an arbitrary $n \times n$ matrix $\mathscr{A}$ we define

$$\overset{\star}{\mathscr{Z}}_\lambda(\mathscr{A}) := \sum_{\substack{R^* \subset \{1,\dots,n^*\} \\ |R^*| = n^*/2 \\ \{n^*\} \cap R = \varnothing \\ S^* = \{1,\dots,n^*\} \setminus R^*}} \frac{\mathbf{B}_{\lambda^*,S^*}(e^{2t})}{(1 - e^{2t})^k} \det \mathscr{A}^{\lambda^*(S^*),E_\lambda}_{\lambda^*(R^*),E_\lambda}. \tag{3.8}$$

Recall from (2.1) that $\mathbf{B}_{\lambda,S}(\zeta)$ is a sum over $\mu \succeq \lambda$ of (# of c.i. Dyck tilings of λ°/μ°) times ζ to the power

$$\begin{aligned} &(\#\text{ up-steps of } \mu \text{ in } S \text{ before flat step}) \\ &\quad - (\#\text{ down-steps of } \mu \text{ in } S \text{ after flat step}) \\ &\qquad + (\#\text{ down-steps of } \lambda \text{ after flat step}). \end{aligned}$$

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We define

$$\overset{\star}{\mathscr{Z}}^{\mu}(\mathscr{A}) := \frac{1}{(1-e^{2t})^k} \sum_{\substack{R^*\subset\{1,\dots,n^*\}\\ |R^*|=n^*/2\\ \{n^*\}\cap R^*=\varnothing\\ S^*=\{1,\dots,n^*\}\setminus R^*}} \begin{array}{l} \det \mathscr{A}^{\mu^*(S^*),E_\mu}_{\mu^*(R^*),E_\mu} \times \\ \exp\big[2t\,|(\text{up-steps of } \mu^* \text{ before flat step}) \cap S^*|\big] \div \\ \exp\big[2t\,|(\text{down-steps of } \mu^* \text{ after flat step}) \cap S^*|\big]. \end{array} \tag{3.9}$$

Observe that if $\mu \succeq \lambda$ then $\mu^*(\cdot) = \lambda^*(\cdot)$ and $E_\mu = E_\lambda$, so

$$\begin{aligned} \overset{\star}{\mathscr{Z}}_{\lambda}(\mathscr{A}) &= \exp\big[2t\,(\#\text{ down-steps of } \lambda \text{ after flat step})\big] \\ &\quad\times \sum_{\mu\succeq\lambda} [\#\text{ of c.i. Dyck tilings of } \lambda^\circ/\mu^\circ]\,\overset{\star}{\mathscr{Z}}^{\mu}(\mathscr{A}). \end{aligned} \tag{3.10}$$

The following lemma will help us evaluate $\overset{\star}{\mathscr{Z}}^{\mu}(\mathscr{A})$.

Lemma 3.4. *Suppose $n \geqslant 0$ is even, $B, C, U, V \subset \{1,\dots,n\}$, $B\cap C=\varnothing$, and $U\cap V=\varnothing$. Let A be an arbitrary $n\times n$ matrix. Then*

$$\begin{aligned} &\sum_{\substack{R\subset\{1,\dots,n\}\\ |R|=n/2\\ B\subset R\\ C\cap R=\varnothing\\ S=\{1,\dots,n\}\setminus R}} \exp\big[2t\big(|S\cap U|-|S\cap V|\big)\big]\, d_R(A) \\ &= \exp\big[t(|U|-|V|)\big] \times \operatorname{Pf} \begin{array}{c} \\ i\in B \\ i\notin B\cup C \\ i\in C \end{array} \begin{bmatrix} \overset{j\in B}{0} & \overset{j\notin B\cup C}{\widetilde{A}_{i,j}} & \overset{j\in C}{\widetilde{A}_{i,j}} \\ -\widetilde{A}_{j,i} & \widetilde{A}_{i,j}-\widetilde{A}_{j,i} & \widetilde{A}_{i,j} \\ -\widetilde{A}_{j,i} & -\widetilde{A}_{j,i} & 0 \end{bmatrix}_{\substack{j=1,\dots,n\\ i=1,\dots,n}} \end{aligned} \tag{3.11}$$

where

$$\widetilde{A}_{i,j} = A_{i,j}\exp\big[t(1_{j\in U} - 1_{i\in U} - 1_{j\in V} + 1_{i\in V})\big].$$

Proof. Observe that $2|S\cap U| = |S\cap U| - |R\cap U| + |U|$, and similarly for $2|S\cap V|$. Since $\widetilde{A}$ is obtained from A by multiplying the ith row by $\exp[t(1_{i\in V} - 1_{i\in U})]$ and jth column by $\exp[t(1_{j\in U} - 1_{j\in V})]$, the determinants $d_R(A)$ and $d_R(\widetilde{A})$ differ by a factor depending on R and S:

$$\exp\big[2t\big(|S\cap U|-|S\cap V|\big)\big] \times d_R(A) = \exp\big[t(|U|-|V|)\big] \times d_R(\widetilde{A}).$$

We can set $A_{i,j}=0$ whenever $j\in B$ or $i\in C$, since these variables do not occur in equation (3.11). We then remove the restrictions $B\subset R$ and $C\cap R=\varnothing$ in the summation on the left-hand side of equation (3.11), since with the above variables zeroed out, $d_R(A)=0$ whenever $B\not\subset R$ or $C\cap R\neq\varnothing$. Without these restrictions on the sum, we can apply Theorem 3.2 to sum up the $d_R(\widetilde{A})$'s to obtain (3.11). □

Lemma 3.5. *Let μ be a labelled augmented cyclic Dyck path with n symbols, and suppose $\overline{\mu}$ has length m. Let U denote the set of labels in $\overline{\mu}$ above $\oplus$ symbols, and let V denote the set of labels in $\overline{\mu}$ above $\ominus$ symbols. Let $\mathscr{A}$ be an arbitrary $n\times n$ matrix, and let*

$$\widetilde{\mathscr{A}}_{i,j} = \mathscr{A}_{i,j}\exp\big[t(1_{j\in U} - 1_{i\in U} - 1_{j\in V} + 1_{i\in V})\big]. \tag{3.12}$$

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Then

$$\overset{\star}{\mathscr{Z}}^{\mu}(\mathscr{A}) = \operatorname{Pf} \begin{array}{c} \\ \overline{\mu}_i \neq \odot \\ \\ \overline{\mu}_i = \odot \end{array} \left[\begin{array}{cc} \overline{\mu}_j \neq \odot & \overline{\mu}_j = \odot \\ \dfrac{\widetilde{\mathscr{A}_{\overline{\mu}(i),\overline{\mu}(j)}} - \widetilde{\mathscr{A}_{\overline{\mu}(j),\overline{\mu}(i)}}}{1-e^{2t}} & \widetilde{\mathscr{A}_{\overline{\mu}(i),\overline{\mu}(j)}} \\ -\widetilde{\mathscr{A}_{\overline{\mu}(j),\overline{\mu}(i)}} & 0 \end{array} \right]_{\substack{j=1,\dots,m \\ i=1,\dots,m}} \tag{3.13}$$

Proof. Recall that μ^{S} is the string obtained from μ by replacing each $\overset{i}{\mathsf{S}}$ symbol with $\overset{i}{\bigcirc}\overset{i}{\odot}$ and omitting each I symbol. Let B_μ denote the positions of these new $\bigcirc$'s (replacing an S) in μ^{S}, let C_μ denote the positions of these new $\odot$'s in μ^{S}. The strings μ^{S} and $\overline{\mu}$ have the same length, which we are calling m. If μ is our earlier example

$$\mu = \overset{1}{\mathsf{U}}\overset{2}{\mathsf{S}}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{6}{\mathsf{I}}\overset{7}{\mathsf{D}}\overset{8}{\mathsf{F}}\overset{9}{\mathsf{I}}\overset{10}{\mathsf{U}}\overset{11}{\mathsf{S}}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{14}{\odot},$$

then

$$\mu^{\mathsf{S}} = \overset{1}{\mathsf{U}}\overset{2}{\bigcirc}\overset{2}{\odot}\overset{3}{\mathsf{U}}\overset{4}{\mathsf{D}}\overset{5}{\mathsf{D}}\overset{7}{\mathsf{D}}\overset{8}{\mathsf{F}}\overset{10}{\mathsf{U}}\overset{11}{\bigcirc}\overset{11}{\odot}\overset{12}{\mathsf{U}}\overset{13}{\mathsf{D}}\overset{14}{\odot}$$

and

$$B_\mu = \text{positions of } \{\overset{2}{\bigcirc},\overset{11}{\bigcirc}\} = \{2,10\} \quad \text{and} \quad C_\mu = \text{positions of } \{\overset{2}{\odot},\overset{11}{\odot}\} = \{3,11\}.$$

Let U_μ denote the set of positions at which μ has a U before its F, and let V_μ denote positions at which μ has a D after its F.

For a given μ, the subsets R^* of $\{1,\dots,n^*\}$ for which $|R^*| = n^*/2$ are in straightforward bijective correspondence with those subsets R of $\{1,\dots,m\}$ for which $|R| = m/2$, $B_\mu \subset R$ and $C_\mu \cap R = \varnothing$, that is, $R = \overline{\mu}^{-1}(\mu^*(R^*)) \cup B_\mu$. Consider the pairing between R^* and $S^* = \{1,\dots,n^*\} \setminus R^*$ given by the cycle lemma bijection. This pairing naturally extends to a pairing between R and $S = \{1,\dots,m\} \setminus R$, where a pair (r^*, s^*) gets mapped to the pair

$$(\overline{\mu}^{\,-1}(\mu^*(r^*)), \overline{\mu}^{\,-1}(\mu^*(s^*))),$$

with the pairing between R and S also containing the pairs $(b, b+1)$ for each $b \in B_\mu$. Provided $n^* \notin R^*$, this extended pairing is precisely the pairing between R and S given by the cycle lemma bijection. Thus

$$\sum_{\substack{R^* \subset \{1,\dots,n^*\} \\ |R^*| = n^*/2 \\ \{n^*\} \cap R^* = \varnothing \\ S^* = \{1,\dots,n^*\} \setminus R^*}} \begin{pmatrix} \det \mathscr{A}_{\mu^*(R^*),E_\mu}^{\mu^*(S^*),E_\mu} \times \\ \exp\left[2t\,|U_{\mu^*} \cap S^*|\right] \div \\ \exp\left[2t\,|V_{\mu^*} \cap S^*|\right] \end{pmatrix} = \sum_{\substack{R \subset \{1,\dots,m\} \\ |R| = m/2 \\ \{m\} \cap R = \varnothing \\ B_\mu \subset R \\ C_\mu \cap R = \varnothing \\ S = \{1,\dots,m\} \setminus R}} \begin{pmatrix} \det \mathscr{A}_{\mu^{\mathsf{S}}(R)}^{\mu^{\mathsf{S}}(S)} \times \\ \exp\left[2t\,|U_{\mu^{\mathsf{S}}} \cap S|\right] \div \\ \exp\left[2t\,|V_{\mu^{\mathsf{S}}} \cap S|\right] \end{pmatrix} \tag{3.14}$$

We could apply Lemma 3.4 with $B = B_\mu$ and $C = C_\mu \cup \{m\}$ to evaluate the right-hand side of (3.14), but it turns out to work better with $B = \varnothing$, $C = C_\mu \cup \{m\}$. So long as $(C_\mu \cup \{m\}) \cap R = \varnothing$, if $B_\mu \not\subset R$, then the determinant $\det \mathscr{A}_{\mu^{\mathsf{S}}(R)}^{\mu^{\mathsf{S}}(S)}$ has at least one repeated column and therefore does not contribute to the sum. Applying Lemma 3.4 with $B = \varnothing$, $C = C_\mu \cup \{m\}$, $U = U_{\mu^{\mathsf{S}}}$, $V = V_{\mu^{\mathsf{S}}}$, and $n = m$, and then using the fact that $\mu^{\mathsf{S}}(\cdot) = \overline{\mu}(\cdot)$, we see that the right-hand side of (3.14) equals

$$\operatorname{Pf}\begin{array}{c} \\ i\notin C \\ i\in C\end{array}\begin{array}{c} \begin{array}{cc} \quad j\notin C \quad & \quad j\in C\end{array} \\ \left[\begin{array}{cc} \widetilde{\mathscr{A}}_{\overline{\mu}(i),\overline{\mu}(j)}-\widetilde{\mathscr{A}}_{\overline{\mu}(j),\overline{\mu}(i)} & \widetilde{\mathscr{A}}_{\overline{\mu}(i),\overline{\mu}(j)} \\ -\widetilde{\mathscr{A}}_{\overline{\mu}(j),\overline{\mu}(i)} & 0 \end{array}\right]_{\substack{j=1,\dots,m\\ i=1,\dots,m}}\end{array}$$

with $\widetilde{\mathscr{A}}$ defined as in (3.12). Observe that $C_\mu\cup\{m\}$, U_μs, and V_μs are the locations of $\odot$, $\oplus$, and $\ominus$ symbols in $\overline{\mu}$ respectively (which is of course the reason we defined $\overline{\mu}$ the way we did).

The definition of $\overset{\star}{\mathscr{Z}}{}^{\mu}(\mathscr{A})$ also contains a factor of $1/(1-e^{2t})^{n^*/2-1}$. If for some x we scale the rows and columns not in C by a factor of $x^{1/2}$, and scale the rows and columns in C by a factor of $x^{-1/2}$, the Pfaffian is scaled by a factor of $x^{[(m-|C|)-|C|]/2}$. Now $m=n^*+2|E|$ and $|C|=|E|+1$, so $[(m-|C|)-|C|]/2=n^*/2-1$. Upon taking $x=1/(1-e^{2t})$, we obtain (3.13). □

So far all these calculations are exact. Next we take the limit $t\to 0$.

Lemma 3.6. *Let μ be a labelled augmented cyclic Dyck path with n symbols, and suppose $\overline{\mu}$ has length m. Let $\mathscr{A}$ be an $n\times n$ matrix of formal power series for which $\mathscr{A}_{i,j}(t)=\mathscr{A}_{j,i}(-t)=A_{i,j}+A'_{i,j}t+O(t^2)$. Then*

$$\overset{\star}{\mathscr{Z}}{}^{\mu}(\mathscr{A})=\operatorname{Pf}\begin{array}{c} \\ \overline{\mu}_i\neq\odot \\ \overline{\mu}_i=\odot\end{array}\underbrace{\begin{array}{c}\begin{array}{cc}\quad\overline{\mu}_j\neq\odot\quad & \quad\overline{\mu}_j=\odot\end{array}\\ \left[\begin{array}{cc}\begin{pmatrix}+1_{\overline{\mu}_i=\oplus}-1_{\overline{\mu}_j=\oplus}\\ -1_{\overline{\mu}_i=\ominus}+1_{\overline{\mu}_j=\ominus}\end{pmatrix}A_{\overline{\mu}(i),\overline{\mu}(j)}-A'_{\overline{\mu}(i),\overline{\mu}(j)} & A_{\overline{\mu}(i),\overline{\mu}(j)}\\ -A_{\overline{\mu}(i),\overline{\mu}(j)} & 0\end{array}\right]_{\substack{j=1,\dots,m\\ i=1,\dots,m}}\end{array}}_{M_{\overline{\mu}}(A,A')}+O(t)$$

Proof. Straightforward series expansion of the expression from Lemma 3.5. □

It is also straightforward to extract the coefficients of higher powers of t in the series expansion $\overset{\star}{\mathscr{Z}}{}^{\mu}(\mathscr{A})$ using Lemma 3.5. As discussed earlier, the constant term is relevant for computing grove probabilities. The term linear in t is relevant for computing expected winding [10], and also depends on just the $A_{i,j}$'s and $A'_{i,j}$'s.

Proof of Theorem 1.1. Immediate from (2.3), (2.4), (3.8), (3.9), (3.10) and Lemma 3.5. For the G-G' polynomials we substitute G for A and G' for A'. For the L-L' polynomials we first substitute I for S and S for I, and then L for A and L' for A', and we absorb the factor of $(-1)^{|T|}$ from (2.3) into the Pfaffian by writing $\overset{i}{\odot}\overset{i}{\bigcirc}$ rather than $\overset{i}{\bigcirc}\overset{i}{\odot}$. □

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4. Open problems

The coefficients in the Pfaffian formulas in Theorem 1.1 count Dyck tilings whose lower path is λ° and whose upper path depends on the summand. It is known that the sum of these coefficients is the number of increasing labellings of the planted plane tree associated with the Dyck path λ° [12]. Is there something more to understand here?

Is there a polynomial-time algorithm for evaluating $Z[\tau]$? For certain τ's there will be few or even just one Pfaffian, though for general τ the number of Pfaffians is exponentially large in the number of nodes. But these Pfaffians are all closely related to one another, which suggests the possibility that some clever linear algebra could be used to evaluate the sum without evaluating each individual Pfaffian.

References

[1] Carroll, G. D. and Speyer, D. (2004) The cube recurrence. *Electron. J. Combin.* **11** R73.

[2] Curtis, E. B., Ingerman, D. and Morrow, J. A. (1998) Circular planar graphs and resistor networks. *Linear Algebra Appl.* **283** 115–150.

[3] Forman, R. (1993) Determinants of Laplacians on graphs. *Topology*, **32** 35–46.

[4] Josuat-Vergés, M. and Kim, J. S. (2014) Generalized Dyck tilings (extended abstract). In *26th Formal Power Series and Algebraic Combinatorics: FPSAC*, *Discrete Math. Theor. Comput. Sci. Proc.* pp. 181–192.

[5] Kenyon, R. (2011) Spanning forests and the vector bundle Laplacian. *Ann. Probab.* **39** 1983–2017.

[6] Kenyon, R. W. and Wilson, D. B. (2009) Combinatorics of tripartite boundary connections for trees and dimers. *Electron. J. Combin.* **16** R112.

[7] Kenyon, R. W. and Wilson, D. B. (2011) Boundary partitions in trees and dimers. *Trans. Amer. Math. Soc.* **363** 1325–1364.

[8] Kenyon, R. W. and Wilson, D. B. (2011) Double-dimer pairings and skew Young diagrams. *Electron. J. Combin.* **18** 130.

[9] Kenyon, R. W. and Wilson, D. B. (2014) The space of circular planar electrical networks. arXiv:1411.7425

[10] Kenyon, R. W. and Wilson, D. B. (2015) Spanning trees of graphs on surfaces and the intensity of loop-erased random walk on planar graphs. *J. Amer. Math. Soc.* **28** 985–1030.

[11] Kim, J. S. (2012) Proofs of two conjectures of Kenyon and Wilson on Dyck tilings. *J. Combin. Theory Ser. A*, **119** 1692–1710.

[12] Kim, J. S., Mészáros, K., Panova, G. and Wilson, D. B. (2014) Dyck tilings, increasing trees, descents, and inversions. *J. Combin. Theory Ser. A*, **122** 9–27.

[13] Shigechi, K. and Zinn-Justin, P. (2012) Path representation of maximal parabolic Kazhdan–Lusztig polynomials. *J. Pure Appl. Algebra* **216** 2533–2548.

[14] Wilson, D. B. (2014) Local statistics of the abelian sandpile model. Manuscript.

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