Canad. J. Math. Vol. 73 (2), 2021 pp. 542–571
http://dx.doi.org/10.4153/S0008414X20000073

Generalized Beilinson Elements and Generalized Soulé Characters

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Abstract. The generalized Soulé character was introduced by H. Nakamura and Z. Wojtkowiak and is a generalization of Soulé's cyclotomic character. In this paper, we prove that certain linear sums of generalized Soulé characters essentially coincide with the image of generalized Beilinson elements in K-groups under Soulé's higher regulator maps. This result generalizes Huber–Wildeshaus' theorem, which is a cyclotomic field case of our results, to an arbitrary number fields.

1 Introduction

Let N be a positive integer and let ζ be an N-th root of unity. For each positive integer m, A. A. Beilinson constructed an element $c_m(\zeta) \in K_{2m-1}(\mathbf{Q}(\mu_N)) \otimes_{\mathbf{Z}} \mathbf{Q}$ called *Beilinson's cyclotomic element* characterized by the equality $\mathrm{reg}_m^{\mathrm{H}}(c_m(\zeta)) = (\mathcal{R}_m(\mathrm{Li}_m(\sigma(\zeta)))_\sigma$. Here, $\mathrm{reg}_m^{\mathrm{H}}$ is the Beilinson regulator map

$$\mathrm{reg}_m^{\mathrm{H}} \colon K_{2m-1}(\mathbf{Q}(\mu_N)) \longrightarrow \bigoplus_{\sigma:\mathbf{Q}(\mu_N)\hookrightarrow\mathbf{C}} \mathbf{R}(m-1),$$

$\mathcal{R}_m(z)$ is defined to be $(z+(-1)^{m-1}\overline{z})/2$ for each $z \in \mathbf{C}$, and $\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} z^n/n^m$ is the m-th classical polylogarithm function. Since $\mathrm{Li}_m(\sigma(\zeta))$ is a linear sum of partial zeta values over $\mathbf{Q}(\mu_N)$, the Beilinson element $c_m(\zeta)$ can be regarded as a *zeta element* in the K-group.

For each prime number ℓ, an ℓ-adic analogue of $c_m(\zeta)$ was constructed by C. Soulé in [25] by *twisting* cyclotomic units, which are also considered as zeta elements. We fix a system $\{\zeta_{\ell^n}\}_{n\geq 1}$ of ℓ-powers roots of unity and regard it as a basis of the Galois module $\mathbf{Z}_\ell(1)$. Soulé defined a continuous group homomorphism

$$\chi_m^{\zeta} \colon \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_{N\ell^\infty})\right) \longrightarrow \mathbf{Z}_\ell(m)$$

called *the Soulé character* by the equations

$$\chi_m^{\zeta}(\sigma) \mod \ell^n = \Big(\prod_{1\leq a\leq \ell^n,\, \ell\nmid a} (1-\zeta_{\ell^n}^a)^{a^{m-1}} \Big)^{\frac{1}{\ell^n}(\sigma-1)} \otimes \zeta_{\ell^n}^{\otimes(m-1)} \text{ in } \mathbf{Z}/\ell^n\mathbf{Z}(m) \tag{1.1}$$

for all positive integers n. A modification $\widetilde{\chi}_m^{\zeta}$ of χ_m^{ζ} is defined by a similar equation to (1.1), but a runs over every integer such that $1 \leq a \leq \ell^n$. If N is prime to ℓ, then there exists a simple relation $\chi_m^{\zeta} = \widetilde{\chi}_m^{\zeta} - \ell^m \widetilde{\chi}_m^{\zeta^{1/\ell}}$ between these two homomorphisms. The homomorphism χ_m^{ζ} can extend to a continuous 1-cocycle on the absolute Galois

Received by the editors February 28, 2019; revised January 3, 2020.
Published online on Cambridge Core February 6, 2020.
AMS subject classification: **11G55**, 14H45.
Keywords: generalized Beilinson element, generalized Soulé character, polylogarithm.

group $\mathscr{G}_{\mathbf{Q}(\mu_N)} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_N))$ of $\mathbf{Q}(\mu_N)$ and denote by $[\chi_m^{\zeta}]$ its cohomology class in $H^1(\mathbf{Q}(\mu_N), \mathbf{Q}_\ell(m))$. Let $\mathrm{reg}_m^{\ell\text{-ét}}$ be the ℓ-adic higher regulator map ([25]):

$$\mathrm{reg}_m^{\ell\text{-ét}}\colon K_{2m-1}(\mathbf{Q}(\mu_N)) \longrightarrow H^1\big(\mathbf{Q}(\mu_N), \mathbf{Q}_\ell(m)\big).$$

The main result of [18] relates these two zeta elements by the regulator.

Theorem 1.1 (Beilinson, Deligne, Huber, Wildeshaus, Kings [16,18]) *For each prime number ℓ, the image of $c_m(\zeta)$ multiplied by $-(m-1)!$ under $\mathrm{reg}_m^{\ell\text{-ét}}$ coincides with the class of $\widetilde{\chi}_m^{\zeta}$:*

$$-(m-1)!\,\mathrm{reg}_m^{\ell\text{-ét}}\big(c_m(\zeta)\big) = [\widetilde{\chi}_m^{\zeta}].$$

This theorem plays an important role in the proof of Bloch–Kato's Tamagawa number conjecture for Dirichlet motives ([17]). The purpose of this paper is to generalize Theorem 1.1 to a general number field K.

Let $\mathbf{Z}[K\backslash\{0,1\}]$ be the free abelian group generated by the symbols $\{z\}$ for $z \in K\backslash\{0,1\}$ and suppose that m is greater than 1. For each positive integer k, the symbol $\mathscr{L}_k^{\mathrm{Cl}}$ denotes the *k-th single-valued classical polylogarithm* (see (3.5) for the precise definition). We call an element $\sum_{i=1}^{n} a_i\{z_i\} \in \mathbf{Z}[K\backslash\{0,1\}]$ satisfies the *m-th Bloch condition* if we have two equalities

$$\sum_{i=1}^{n} a_i\phi^{m-2}(z_i) z_i \wedge (1-z_i) = 0 \quad \text{in } \bigwedge^2 K^\times \otimes_{\mathbf{Z}} \mathbf{Q},$$
$$\sum_{i=1}^{n} a_i\phi^{m-k}(z_i)\mathscr{L}_k^{\mathrm{Cl}}(\sigma(z_i)) = 0$$

for any group homomorphism $\phi\colon K^\times \to \mathbf{Q}$, for any $\sigma\colon K \hookrightarrow \mathbf{C}$, and for any positive integer k such that $2 \le k \le m-1$. Denote by $A_m^{\mathrm{H}}(K)$ the subgroup of $\mathbf{Z}[K\backslash\{0,1\}]$ satisfying the m-th Bloch condition. Note that, by definition, $\{\zeta\}$ is an element of $A_m^{\mathrm{H}}(K)$ for any root of unity ζ of K. In [6], R. de Jeu constructed an element $c_m(\xi) \in K_{2m-1}(K) \otimes_{\mathbf{Z}} \mathbf{Q}$ for each $\xi \in A_m^{\mathrm{H}}(K)$ characterized by $\mathrm{reg}_m^{\mathrm{H}}(c_m(\xi)) = (\mathscr{L}_m^{\mathrm{Cl}}(\sigma(\xi)))_\sigma$. Note that Beilinson and Deligne also gave a conditional construction of the elements $c_m(\xi)$ in [1] independently. In this paper, we call $c_m(\xi)$ a *generalized Beilinson element*. Note that, since $\mathscr{L}_m^{\mathrm{Cl}}(\zeta) = \mathcal{R}_m(\mathrm{Li}_m(\zeta))$ for any root of unity $\zeta \in \mathbf{C}\backslash\{1\}$, we have $c_m(\{\zeta\}) = c_m(\zeta)$.

On the other side, in the paper [20], H. Nakamura and Z. Wojtkowiak defined a continuous group homomorphism

$$\widetilde{\chi}_m^{z}\colon \mathscr{G}_{K(\mu_{\ell^\infty}, z^{1/\ell^\infty})} \to \mathbf{Z}_\ell(m)$$

by the Kummer properties

$$(1.2) \qquad \widetilde{\chi}_m^{z}(\sigma) \mod \ell^n = \Big(\prod_{1\le a\le \ell^n} (1-\zeta_{\ell^n}^a z^{1/\ell^n})^{\frac{a^{m-1}}{\ell^n}}\Big)^{\sigma-1} \otimes \zeta_{\ell^n}^{\otimes(m-1)} \text{ in } \mathbf{Z}/\ell^n\mathbf{Z}(m),$$

which is a generalization of Soulé's cyclotomic character. For each formal linear sum $\xi = \sum_{i=1}^{l} a_i\{z_i\} \in \mathbf{Z}[K\backslash\{0,1\}]$, we define K_ξ to be $K(\{\mu_{\ell^\infty}, z_i^{1/\ell^\infty}\}_{i=1}^{l})$ and define $\widetilde{\chi}_m^{\xi}$

https://doi.org/10.4153/S0008414X20000073 Published online by Cambridge University Press

to be the linear sum of group homomorphisms

$$\widetilde{\chi}_m^{\xi} := \sum_{i=1}^{l} a_i \chi_m^{z_i} \colon \mathscr{G}_{K_\xi} \longrightarrow \mathbf{Z}_\ell(m).$$

We will prove that if ξ is an element of $A_m^{\mathrm{H}}(K)$, then $\widetilde{\chi}_m^{\xi}$ can extend to a continuous 1-cocycle on $\mathscr{G}_K$ (Corollary 4.15, Corollary 6.5) and denote by the same notation this 1-cocycle by abuse of notation. The main theorem of this paper is as follows.

Main Theorem *Let m be a positive integer greater than 1. Let K be a number field and let ξ be an element of $A_m^{\mathrm{H}}(K)$. Then, for each prime number ℓ, we have the equality*

$$-(m-1)!\,\mathrm{reg}_m^{\ell\text{-ét}}(c_m(\xi)) = [\widetilde{\chi}_m^{\xi}].$$

Our approach follows essentially the path laid out by Beilinson and Deligne. The proof of the main theorem is based on analysis of moduli of torsors under fundamental groups, and one of the key ingredients is the motivic fundamental groupoid of $\mathbf{P}^1 \backslash \{0, 1, \infty\}$ whose existence was proved by Deligne and Goncharov [5]. When $K = \mathbf{Q}(\mu_N)$, it seems that the proof of our Main Theorem is a simplification of the original proof of Theorem 1.1 by Huber and Wildeshaus.

Conjecturally, $\{c_m(\xi)\}_{\xi \in A_m^{\mathrm{H}}(K)}$ spans $K_{2m-1}(K) \otimes_{\mathbf{Z}} \mathbf{Q}$ for any number field K and positive integer m greater than 1 ([29, §8, Main Conjecture]). Although this conjecture proved only the case that K is an abelian extension of $\mathbf{Q}$, which is a key of [17], the author hopes that the Main Theorem will be useful for studying the motive $\mathbf{Z}(m)$ over an arbitrary number field.

1.1 Plan

The plan of this paper is as follows. In Section 2, we recall a concept of an abstract modified polylogarithm attached to a series of abstract unipotent Albanese maps in a mixed Tate category. In the following three sections, we see examples of abstract modified polylogarithms. In Section 3, we define the Hodge modified polylogarithms and give a comparison of that polylogarithms with the classical modified polylogarithms (*cf.* Proposition 3.10). In Section 4, we define the ℓ-adic étale modified polylogarithms. We also compare the ℓ-adic étale modified polylogarithms with Wojtkowiak's ℓ-adic polylogarithms and generalized Soulé characters. Then we define the motivic modified polylogarithms in Section 5. In Section 6, we compare the three modified polylogarithms introduced in the previous sections and give a proof of Main Theorem. In Appendix A, we give proofs of technical lemmas that are needed to describe classifying spaces of torsors under algebraic groups in a mixed Tate category.

1.2 Notation

For a field F, we fix its separable closure $\overline{F}$ and denote by $\mathscr{G}_F$ the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F. For any topological group A equipped with a continuous action of $\mathscr{G}_F$, we denote by $H^1(F, A)$ the continuous first Galois cohomology. For a set S, let $\mathbf{Z}[S]$ be the free abelian group generated by symbols $\{s\}$, $s \in S$.

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Let k be a field of characteristic 0 and let V be a finite dimensional k-vector space equipped with an algebraic action of the multiplicative group $\mathbf{G}_{m,k}$. Then we define $V^{(-2n)}$ to be the subspace of V on which $\mathbf{G}_{m,k}$ acts via the n-th power of the standard character $\mathrm{std} := \mathrm{id}_{\mathbf{G}_{m,k}}: \mathbf{G}_{m,k} \to \mathbf{G}_{m,k}$. For each abstract group $\mathcal{G}$, we denote by $\mathcal{G}_k$ the unipotent completion of $\mathcal{G}$ over k in the sense of [12, Appendix A]. Let R be a k-algebra and let X be a k-scheme. We denote by X_R or by $X \otimes_k R$ the base change of X to $\mathrm{Spec}(R)$. For a scheme X, the symbol $\mathcal{O}(X)$ denotes the ring of regular functions on X. We denote by $\mathbf{P}^1_{01\infty}$ the scheme $\mathrm{Spec}(\mathbf{Z}[t, \frac{1}{t(t-1)}]) = \mathbf{P}^1_{\mathbf{Z}} \backslash \{0, 1, \infty\}$.

We mean a left action by an action unless otherwise noted. Let G be a group and let A be a set equipped with an action of G. Then for each $g \in G$ and $a \in A$, we denote by ${}^g a$ the action of g on a. For an object X of a category, the symbol $[X]$ denotes the isomorphism class of X.

2 Abstract Modified Polylogarithms in Mixed Tate Categories

In this section, we recall abstract modified polylogarithms in a mixed Tate category introduced in [22, Section 2]. This notion was referred to as an abstract polylogarithm in that previous work.

2.1 Preliminaries on Mixed Tate Categories

In this and the next subsection, we fix a field k of characteristic 0 and a mixed Tate category $\mathcal{M}$ over k with the invertible object $k(1)$ (*cf.* [9, Appendix 8.1]). Any object M in $\mathcal{M}$ has a natural weight filtration $W_\bullet M$ indexed by even integers such that $\mathrm{gr}^W_{2n} M := W_{2n}M/W_{2n-2}M$ is isomorphic to a direct sum of $k(-2n)$. Let ω be the canonical fiber functor of $\mathcal{M}$ defined by

$$\omega: \mathcal{M} \longrightarrow \mathrm{GrVec}_k \longrightarrow \mathrm{Vec}_k; M \longmapsto \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{M}}\big(k(-n), \mathrm{gr}^W_{2n} M\big),$$

where $GrVec_k$ is the category of finite dimensional graded k-vector spaces. Let $\pi_1(\mathcal{M}, \omega)$ be the Tannakian fundamental group of $\mathcal{M}$ with the base point ω. Since ω factors through GrVec_k, there exists a natural splitting $\pi_1(\mathcal{M}, \omega) = \mathbf{G}_{m,k} \ltimes U(\mathcal{M})$, where $U(\mathcal{M})$ is a pro-unipotent pro-algebraic group over k. Namely, there exists an inverse system $\{U_\alpha\}_\alpha$ of unipotent algebraic groups over k such that $U(\mathcal{M}) = \varprojlim_\alpha U_\alpha$. The *fundamental Lie algebra* $\mathrm{Lie}(\mathcal{M})$ of $\mathcal{M}$ is defined to be the inverse limit $\varprojlim_\alpha \mathrm{Lie}(U_\alpha)$ of the inverse system of Lie algebras $\{\mathrm{Lie}(U_\alpha)\}_\alpha$. The action of $\mathbf{G}_{m,k}$ on $U(\mathcal{M})$ defines the positive grading on the fundamental coLie algebra

$$\mathrm{coLie}(\mathcal{M}) := \varinjlim_\alpha \mathrm{Hom}_k(\mathrm{Lie}(U_\alpha), k) = \bigoplus_{n=1}^{\infty} \mathrm{coLie}(\mathcal{M})^{(2n)}$$

of $\mathcal{M}$ where $\mathrm{coLie}(\mathcal{M})^{(2n)}$ is the subspace of $\mathrm{coLie}(\mathcal{M})$ on which $\mathbf{G}_{m,k}$ acts by the $(-n)$-th power of the standard character $\mathrm{std} := \mathrm{id}_{\mathbf{G}_{m,k}}$. We denote by

$$d_{\mathcal{M}}: \mathrm{coLie}(\mathcal{M}) \longrightarrow \bigwedge^2 \mathrm{coLie}(\mathcal{M})$$

the Lie cobracket of $\mathrm{coLie}(\mathcal{M})$. The following lemma is well known.

Lemma 2.1 (*cf.* [1, Section 2.1]) *We have a canonical isomorphism*

$$\mathrm{coLie}(\mathcal{M})^{(2n),d_{\mathcal{M}}=0} \cong \mathrm{Ext}^1_{\mathcal{M}}(k(0),k(n)) \otimes \mathrm{std}^{-n}$$

as $\mathbf{G}_{m,k}$*-modules. Here, we regard* $\mathrm{Ext}^1_{\mathcal{M}}(k(0),k(n))$ *as a* k*-vector space equipped with the trivial action of* $\mathbf{G}_{m,k}$.

We introduce two conditions for affine group schemes in $\mathcal{M}$ in the sense of Deligne (*cf.* [4, Section 5.4]).

Definition 2.2 Let $G = \mathrm{Sp}(A)$ be an affine group scheme in $\mathcal{M}$. We say that G satisfies (Pos) (resp. (Triv)) if A satisfies the following condition:

$$(\mathbf{Pos}) : W_0 A = k(0) \ \big(\text{resp. } (\mathbf{Triv}) : U(\mathcal{M}) \text{ acts on } \omega(A) \text{ trivially}\big).$$

We say that an affine group scheme $G = \mathrm{Sp}(A)$ in $\mathcal{M}$ is an algebraic group in $\mathcal{M}$ if the k-algebra $\omega(A)$ is finitely generated. The canonical fiber functor ω induces a functor from the category of algebraic groups in $\mathcal{M}$ to the category of algebraic groups over k equipped with an algebraic action of $\pi_1(\mathcal{M},\omega)$. We use the same letter ω for that functor by abuse of notation. The following proposition is a direct consequence of Corollary A.6.

Proposition 2.3 *Let* G *be an algebraic group in* $\mathcal{M}$ *satisfying* (Pos)*. Then the underlying algebraic group of* $\omega(G)$ *is unipotent.*

For each algebraic group G in $\mathcal{M}$, we define the object $\mathrm{Lie}(G)$ in $\mathcal{M}$ by the equation

$$\omega\big(\mathrm{Lie}(G)\big) = \mathrm{Lie}\big(\omega(G)\big).$$

Here, the existence of such an object follows from the Tannakian duality. According to Proposition 2.3, the correspondence $G \mapsto \mathrm{Lie}(G)$ induces an equivalence between the category of algebraic groups in $\mathcal{M}$ satisfying (Pos) and the category of nilpotent Lie algebra objects in $\mathcal{M}$ with negative weights.

For the rest of this subsection, we fix an algebraic group G in $\mathcal{M}$ satisfying (Pos) and (Triv). We denote by $H^1(\mathcal{M},G)$ the set of isomorphism classes of right G-torsors in $\mathcal{M}$. Recall that the pointed set $H^1(\mathcal{M},G)$ is canonically isomorphic to the first rational cohomology $H^1(\pi_1(\mathcal{M},\omega),\omega(G))$ (*cf.* [22, Appendix A6.2]). We recall another description of this pointed set.

Proposition 2.4 ([3, Proposition 5.2]) *Let* G *be an algebraic group in* $\mathcal{M}$ *satisfying* (Pos) *and* (Triv).

(i) *For each* G*-torsor* X*, there exists a unique rational* 1*-cocycle* $c_{[X]}$ *representing the isomorphism class* $[X]$ *of* X *such that* $c_{[X]}|_{\mathbf{G}_{m,k}} = 1$.

(ii) *The correspondence* $[X] \mapsto \log(c_{[X]}|_{U(\mathcal{M})})$ *defines a bijection*

$$\Phi\colon H^1(\mathcal{M},G) \xrightarrow{\sim} \mathrm{Hom}^{\mathbf{G}_{m,k}}_{k\text{-gp}}\big(U(\mathcal{M}),\omega(G)\big) \cong \mathrm{Hom}^{\mathbf{G}_{m,k}}_{k\text{-Lie}}\big(\mathrm{Lie}(\mathcal{M}),\omega(\mathrm{Lie}(G))\big).$$

Here, $\log(c_{[X]}|_{U(\mathcal{M})})$ *is the Lie homomorphism corresponding to the homomorphism* $c_{[X]}|_{U(\mathcal{M})}\colon U(\mathcal{M}) \to \omega(G)$ *of group schemes over* k.

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2.2 Abstract Modified Polylogarithms

For any Lie algebra object L of $\mathcal{M}$ such that $\omega(L)$ is nilpotent, we denote by $\exp(L)$ the associated algebraic group in $\mathcal{M}$.

Definition 2.5 Let m be a non-negative integer. We define *the polylogarithmic quotient* $\mathscr{P}_m^{\mathcal{M}}$ in $\mathcal{M}$, which is an algebraic group in $\mathcal{M}$, by

$$\mathscr{P}_m^{\mathcal{M}} := \exp(\mathfrak{p}_m^{\mathcal{M}}) := \exp\Big(k(1) \ltimes \bigoplus_{n=1}^{m} k(n)\Big).$$

Here, $\mathfrak{p}_m^{\mathcal{M}} := k(1) \ltimes \oplus_{n=1}^{m} k(n)$ is a Lie algebra object in $\mathcal{M}$ such that the abelian Lie algebra $k(1)$ acts on the abelian Lie algebra $\oplus_{n=1}^{m} k(n)$ by $k(1) \otimes k(n) \xrightarrow{\sim} k(n+1)$ for $n < m$ and annihilates $k(m)$. We understand $\mathscr{P}_0^{\mathcal{M}}$ as $\exp(k(1))$.

It is easily checked that the algebraic group $\mathscr{P}_m^{\mathcal{M}}$ in $\mathcal{M}$ satisfies two conditions (Pos) and (Triv). By using Proposition 2.4, we define a natural map r_m that is needed to linearize abstract unipotent Albanese maps.

Lemma 2.6 ([22, Lemma 2.4]) *Let m be a non-negative integer. Then there exists a natural map of pointed sets*

$$r_m\colon H^1(\mathcal{M}, \mathscr{P}_m^{\mathcal{M}}) \longrightarrow \mathrm{coLie}(\mathcal{M})^{(2m+2\epsilon)},$$

where ϵ denotes 0 *or* 1 *when $m > 0$ or $m = 0$, respectively.*

Proof We define the map

$$r_m\colon H^1(\mathcal{M}, \mathscr{P}_m^{\mathcal{M}}) \xrightarrow{\Phi} \mathrm{Hom}_{k\text{-}\mathrm{Lie}}^{\mathbf{G}_{m,k}}\big(\mathrm{Lie}(\mathcal{M}), \omega(\mathfrak{p}_m^{\mathcal{M}})\big) \longrightarrow \mathrm{coLie}(\mathcal{M})^{(2m+2\epsilon)}$$

by

$$r_m([X]) := \Phi([X])|_{\mathrm{Lie}(\mathcal{M})^{(-2m-2\epsilon)}} = \log(c_{[X]})|_{\mathrm{Lie}(\mathcal{M})^{(-2m-2\epsilon)}}.$$

This map coincides with the map defined in [22, Lemma 2.5]. ∎

There exists another description of r_m by using the weight filtration on $U(\mathcal{M})$. The action of $\mathbf{G}_{m,k}$ on $\mathrm{Lie}(\mathcal{M})$ defines a natural filtration $W_{-2n}\,\mathrm{Lie}(\mathcal{M})$ by the equality

$$W_{-2n}\,\mathrm{Lie}(\mathcal{M}) := \text{the closure of } \bigoplus_{i\geq n} \mathrm{Lie}(\mathcal{M})^{(-2i)} \text{ in } \mathrm{Lie}(\mathcal{M}).$$

The graded piece $\mathrm{gr}_{-2n}^{W}\,\mathrm{Lie}(\mathcal{M})$ is canonically isomorphic to $\mathrm{Lie}(\mathcal{M})^{(-2n)}$. Denote by $W_{-2n}U(\mathcal{M})$ the closed sub-pro-algebraic group of $U(\mathcal{M})$ corresponding to $W_{-2n}\,\mathrm{Lie}(\mathcal{M})$, namely,

$$\mathrm{Lie}\big(W_{-2n}U(\mathcal{M})\big) = W_{-2n}\,\mathrm{Lie}(\mathcal{M}).$$

By definition, $\log\colon U(\mathcal{M})(k) \xrightarrow{\sim} \mathrm{Lie}(\mathcal{M})$ induces a natural isomorphism of k-vector spaces

$$\big(\mathrm{gr}_{-2n}^{W}\,U(\mathcal{M})\big)(k) \xrightarrow{\sim} \mathrm{gr}_{-2n}^{W}\,\mathrm{Lie}(\mathcal{M}) \cong \mathrm{Lie}(\mathcal{M})^{(-2n)}. \tag{2.1}$$

Since the following lemma was essentially proved in [22, Lemma 2.4] by using a standard weight argument, we skip its proof here.

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Lemma 2.7 *Let $c\colon \pi_1(\mathcal{M},\omega) \to \omega(\mathscr{P}_m^{\mathcal{M}})$ be a rational 1-cocycle. If m is a positive integer, then the restriction of c to $W_{-2m}U(\mathcal{M})$ induces a group homomorphism*

$$\mathrm{gr}^W_{-2n}(c)\colon \left(\mathrm{gr}^W_{-2n}\, U(\mathcal{M})\right)(k) \longrightarrow \omega(k(m)) = k.$$

Furthermore, this homomorphism depends only on the cohomology class $[c]$ in the first rational cohomology $H^1(\pi_1(\mathcal{M},\omega), \omega(\mathscr{P}_m^{\mathcal{M}}))$.

Let X be a right torsor under $\mathscr{P}_m^{\mathcal{M}}$ and let $c\colon \pi_1(\mathcal{M},\omega) \to \mathscr{P}_m^{\mathcal{M}}$ be any rational 1-cocycle representing X. Then, by the construction of r_m and Lemma 2.7, we have

$$r_m([X]) = \mathrm{gr}^W_{-2n}(c) \tag{2.2}$$

under the isomorphism (2.1). Now, we recall the concept of series of abstract unipotent Albanese maps. For the rest of this section, we fix a field F that will be specified as a number field K in Sections 5 and 6.

Definition 2.8 (*cf.* [22, Definitions 2.3 and 2.5])

(i) *A series of abstract unipotent Albanese maps* $\underline{\mathrm{Alb}} = \{\mathrm{Alb}_n\}_{n\geq 0}$ is an inverse system of maps

$$\mathrm{Alb}_n\colon \mathbf{P}^1_{01\infty}(F) = F\backslash\{0,1\} \longrightarrow H^1(\mathcal{M}, \mathscr{P}_n^{\mathcal{M}})$$

with respect to n satisfying the following two conditions:

(**Hom**) Alb_0 extends to an injective group homomorphism from $F^\times/F^\times_{\mathrm{tor}}$ to $H^1(\mathcal{M}, k(1))$.

(**Ref**) We have $\mathrm{Alb}_1(z) = (\mathrm{Alb}_0(z), \mathrm{Alb}_0(1-z))$ in $H^1(\mathcal{M}, \mathscr{P}_1^{\mathcal{M}}) = H^1(\mathcal{M}, k(1))^{\oplus 2}$ for all $z \in \mathbf{P}^1_{01\infty}(F)$.

(ii) Let m be a positive integer. We define *the m-th abstract modified polylogarithm*

$$\mathscr{L}_m(\underline{\mathrm{Alb}})\colon \mathbf{Z}[\mathbf{P}^1_{01\infty}(F)] \to \mathrm{coLie}(\mathcal{M})^{(2m)}$$

attached to $\underline{\mathrm{Alb}}$ to be the linearization of the composite

$$r_m \circ \mathrm{Alb}_m\colon \mathbf{P}^1_{01\infty}(F) \to \mathrm{coLie}(\mathcal{M})^{(2m)}.$$

Abstract modified polylogarithms satisfy the following differential formula.

Proposition 2.9 ([22, (2.2)], [1, Proposition 2.3]) *For each positive integer m, we have the following differential formula:*

$$d_{\mathcal{M}}\mathscr{L}_m(\underline{\mathrm{Alb}}) = \mathscr{L}_{m-1}(\underline{\mathrm{Alb}}) \wedge \mathscr{L}_0(\underline{\mathrm{Alb}}).$$

The above proposition leads us to define Bloch groups attached to $\underline{\mathrm{Alb}}$.

Definition 2.10 ([22, Definition 2.6]) Let m be a positive integer.

(i) We define $R_m(F, \underline{\mathrm{Alb}})$ the space of functional equations of $\mathscr{L}_m(\underline{\mathrm{Alb}})$ by

$$R_m(F, \underline{\mathrm{Alb}}) := \mathrm{Ker}\left(\mathscr{L}_m(\underline{\mathrm{Alb}})\colon \mathbf{Z}[\mathbf{P}^1_{01\infty}(F)] \longrightarrow \mathrm{coLie}(\mathcal{M})^{(2m)}\right).$$

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(ii) We put

$$T_m := \begin{cases} 0 & \text{if } m = 1, \\ (\bigwedge^2 F^\times) \otimes_{\mathbf{Z}} \mathbf{Q} & \text{if } m = 2, \\ ((\mathbf{Z}[\mathbf{P}^1_{01\infty}(F)]/R_{m-1}(F, \underline{\mathrm{Alb}})) \otimes_{\mathbf{Z}} F^\times) \otimes_{\mathbf{Z}} \mathbf{Q} & \text{if } m > 2, \end{cases}$$

and define the group homomorphism $\delta_m : \mathbf{Z}[\mathbf{P}^1_{01\infty}(F)] \to T_m$ by

$$\delta_m(\{z\}) := \begin{cases} 0 & \text{if } m = 1, \\ (1-z) \wedge z & \text{if } m = 2, \\ \{z\}_{m-1} \otimes z & \text{if } m > 2, \end{cases}$$

where $\{z\}_{m-1}$ is the image of $\{z\}$ in $\mathbf{Z}[\mathbf{P}^1_{01\infty}(F)]/R_{m-1}(F, \underline{\mathrm{Alb}})$. Then we define the subgroup $A_m(F, \underline{\mathrm{Alb}})$ of $\mathbf{Z}[\mathbf{P}^1_{01\infty}(F)]$ to be the kernel of δ_m.

Proposition 2.11 ([22, Lemma 2.7, Proposition 2.9]) *The space of functional equations $R_m(F, \underline{\mathrm{Alb}})$ of $\mathscr{L}_m(\underline{\mathrm{Alb}})$ is a subgroup of $A_m(F, \underline{\mathrm{Alb}})$. We put*

$$B_m(F, \underline{\mathrm{Alb}}) := A_m(F, \underline{\mathrm{Alb}})/R_m(F, \underline{\mathrm{Alb}}),$$

and call this abelian group the m-th Bloch group attached to $\underline{\mathrm{Alb}}$. Then $\mathscr{L}_m(\underline{\mathrm{Alb}})$ induces a well-defined injective group homomorphism

$$\mathscr{L}_m(\underline{\mathrm{Alb}}) : B_m(F, \underline{\mathrm{Alb}}) \hookrightarrow \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(m)).$$

Proof For the reader's convenience, we briefly recall the proof of the second assertion of the proposition. The proof is executed by induction on m. If $m = 1$, there is nothing to prove, because $\mathrm{coLie}(\mathcal{M})^{(2)} = \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(1)) \otimes \mathrm{std}^{-1}$. Thus, we show the case where $m > 1$. By Proposition 2.9 and Lemma 2.1, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} B_m(F, \underline{\mathrm{Alb}}) & \hookrightarrow & \mathbf{Z}[\mathbf{P}^1_{01\infty}(F)]/R_m(F, \underline{\mathrm{Alb}}) & \xrightarrow{\delta_m} & T_m \\ \vdots & & \downarrow{\scriptstyle \mathscr{L}_m(\underline{\mathrm{Alb}})} & & \downarrow{\scriptstyle \mathscr{L}_{m-1}(\underline{\mathrm{Alb}}) \wedge \mathscr{L}_0(\underline{\mathrm{Alb}})} \\ \mathrm{Ext}^1_{\mathcal{M}}(k(0), k(m)) \otimes \mathrm{std}^{-m} & \hookrightarrow & \mathrm{coLie}(\mathcal{M})^{(2m)} & \xrightarrow{d_{\mathcal{M}}} & \bigwedge^2 \mathrm{coLie}(\mathcal{M}). \end{array}$$

Hence, the dotted arrow in the diagram exists, and we have the conclusion. ■

Remark 2.12 Let ξ be an element of $\mathbf{Z}[\mathbf{P}^1_{01\infty}(F)]$. Then, by the diagram in the proof of Proposition 2.11, ξ is contained in $A_m(F, \underline{\mathrm{Alb}})$ if and only if $d_{\mathcal{M}} \mathscr{L}_m(\underline{\mathrm{Alb}})(\xi) = 0$.

3 Hodge Modified Polylogarithms

In the following three sections, we give examples of abstract modified polylogarithms. The first example is the Hodge modified polylogarithm $\mathscr{L}^{\mathrm{H_Q}}_m$. We apply our general theory to the category $\mathcal{H}_{\mathbf{Q}}$ of mixed Hodge–Tate structures over $\mathbf{Q}$ and compare the projections of $\mathscr{L}^{\mathrm{H_Q}}_m$ to the fundamental coLie algebra of the mixed Hodge–Tate structure over $\mathbf{R}$ with the single-valued polylogarithm $\mathscr{L}^{\mathrm{Cl}}_m$.

https://doi.org/10.4153/S0008414X20000073 Published online by Cambridge University Press

3.1 Preliminaries on Mixed Hodge–Tate Structures

Let R be a subring of $\mathbf{C}$. Recall that an R-mixed Hodge structure H is called a *mixed Hodge–Tate structure* if any non-zero Hodge number is of the form $h^{q,q}$ for some integer q. We denote by $\mathcal{H}_R$ the category of mixed Hodge–Tate structures over R. For simplicity of notation, we use the same letter H for the underlying R-module of a mixed Hodge–Tate structure H. We recall some basic facts about $\mathcal{H}_{\mathbf{R}}$ for later use.

According to [1, Section 2.5], giving an object H in $\mathcal{H}_{\mathbf{R}}$ is equivalent to giving a finite dimensional graded $\mathbf{R}$-vector space $V_\bullet = \oplus_{j\in\mathbf{Z}} V_j$ equipped with $\mathbf{R}$-linear homogeneous endomorphisms $N_n(V_\bullet)\colon V_\bullet \to V_\bullet$ of degree n for all positive integers n. Since $V_\bullet$ is finite dimensional, $N_n(V_\bullet)$ is a zero map for sufficiently large n. The mixed Hodge–Tate structure H corresponding to $V_\bullet$ is defined as follows.

- The underlying $\mathbf{R}$-vector space of H is defined by

$$H := \tau(2\pi\sqrt{-1})V_\bullet := \bigoplus_{n\in\mathbf{Z}} V_n \otimes_{\mathbf{R}} \mathbf{R}(n)$$

  and the weight filtration on H is defined by $W_{-2n}H := \oplus_{j\geq n} V_j \otimes_{\mathbf{R}} \mathbf{R}(j)$.
- The Hodge filtration on $H \otimes_{\mathbf{R}} \mathbf{C} = V_\bullet \otimes_{\mathbf{R}} \mathbf{C}$ is defined by

$$F^i(V_\bullet \otimes_{\mathbf{R}} \mathbf{C}) := g\Big(\bigoplus_{j\leq -i} V_j \otimes_{\mathbf{R}} \mathbf{C}\Big),$$

  where g is a unipotent $\mathbf{C}$-linear automorphism on $V_\bullet \otimes_{\mathbf{R}} \mathbf{C}$ satisfying the equality

$$\frac{1}{2}\log(g\overline{g}^{-1}) = \sum_{n=1}^{\infty} N_n(V_\bullet)(2\pi\sqrt{-1})^n \otimes \sqrt{-1}^{-1} \in \mathfrak{gl}(\tau(2\pi\sqrt{-1})V_\bullet) \otimes_{\mathbf{R}} \mathbf{C}. \tag{3.1}$$

  Here we take the complex conjugate $\overline{g}$ of g with respect to the $\mathbf{R}$-structure $\tau(2\pi\sqrt{-1})V_\bullet$ of $V_\bullet \otimes_{\mathbf{R}} \mathbf{C}$.

This implies that the fundamental Lie algebra of $\mathcal{H}_{\mathbf{R}}$ is isomorphic to the nilpotent completion of the free Lie algebra over $\mathbf{R}$ with the set of generators $\{N_n \in \mathrm{Lie}(\mathcal{H}_{\mathbf{R}})^{(-2n)}\}_{n\geq 1}$. Now, we fix such a set of topological generators $\{N_n\}_{n\geq 1}$ of $\mathrm{Lie}(\mathcal{H}_{\mathbf{R}})$.

Example 3.1 Let m be a positive integer and let $b \in \mathbf{R}$. We consider the graded vector space $V_\bullet = \mathbf{R}e_0 \oplus \mathbf{R}e_m$ equipped with the nilpotent endomorphism

$$N_m(V_\bullet)\colon \mathbf{R}e_0 \oplus \mathbf{R}e_m \longrightarrow \mathbf{R}e_0 \oplus \mathbf{R}e_m;\, ae_0 \mapsto abe_m,\, e_m \longmapsto 0$$

of degree m. Then the mixed Hodge–Tate structure H corresponding to $V_\bullet$ is an extension of $\mathbf{R}(0)$ by $\mathbf{R}(m)$ whose isomorphism class is represented by

$$b(2\pi)^m\sqrt{-1}^{m-1} \in \mathbf{R}(m-1) = \mathrm{Ext}^1_{\mathcal{H}_{\mathbf{R}}}\big(\mathbf{R}(0), \mathbf{R}(m)\big).$$

This implies that the image $f_b^{(2m)} \in \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)}$ of $b(2\pi)^m\sqrt{-1}^{m-1}$ under the canonical injection

$$\mathbf{R}(m-1) = \mathrm{Ext}^1_{\mathcal{H}_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(m)) = \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m), d_{\mathcal{H}_{\mathbf{R}}}=0} \hookrightarrow \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})$$

is characterized by $d_{\mathcal{H}_{\mathbf{R}}}(f_b^{(2m)}) = 0$, $f_b^{(2m)}(N_m) = b$, and by $f_b^{(2m)}(N_n) = 0$ for all $n \neq m$.

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3.2 Definition of Hodge Modified Polylogarithms

We fix a positive integer m in this subsection.

Definition 3.2 We define *the m-th Hodge polylogarithmic quotient* $\mathscr{P}_m^{\mathrm{H}}$, which is an algebraic group in $\mathcal{H}_{\mathbf{Q}}$, by $\mathscr{P}_m^{\mathrm{H}} := \mathscr{P}_m^{\mathcal{H}_{\mathbf{Q}}} := \exp(\mathbf{Q}(1) \ltimes \oplus_{n=1}^{m} \mathbf{Q}(n))$.

Let $\pi_1^{\mathrm{top}}(\mathbf{P}^1_{01\infty}(\mathbf{C}), \overrightarrow{01})_{\mathbf{Q}}$ be the unipotent completion of the topological fundamental group of $\mathbf{P}^1_{01\infty}(\mathbf{C}) = \mathbf{C}\backslash\{0,1\}$ with the base point $\overrightarrow{01}$ over $\mathbf{Q}$. Then, by the theory of iterated integrals due to Chen, $\pi_1^{\mathrm{top}}(\mathbf{P}^1_{01\infty}(\mathbf{C}), \overrightarrow{01})_{\mathbf{Q}}$ has a natural structure of a group scheme in $\mathcal{H}_{\mathbf{Q}}$. It is well known that $\mathscr{P}_m^{\mathrm{H}}$ is a quotient of $\pi_1^{\mathrm{top}}(\mathbf{P}^1_{01\infty}(\mathbf{C}), \overrightarrow{01})_{\mathbf{Q}}$ as an affine group scheme in $\mathcal{H}_{\mathbf{Q}}$ (*cf.* [4, Proposition 16.13]). Let

$$u_m^{\mathrm{H}}\colon \pi_1^{\mathrm{top}}(\mathbf{P}^1_{01\infty}(\mathbf{C}), \overrightarrow{01})_{\mathbf{Q}} \twoheadrightarrow \mathscr{P}_m^{\mathrm{H}}$$

be the canonical surjective homomorphism of group schemes in $\mathcal{H}_{\mathbf{Q}}$. For each $z \in \mathbf{P}^1_{01\infty}(\mathbf{C})$, we denote by $\mathscr{P}_m^{\mathrm{H}}(\overrightarrow{01}, z)$ the pushforward by u_m^{H} of the path torsor $\pi_1^{\mathrm{top}}(\mathbf{P}^1_{01\infty}(\mathbf{C}); \overrightarrow{01}, z)_{\mathbf{Q}}$ under $\pi_1^{\mathrm{top}}(\mathbf{P}^1_{01\infty}(\mathbf{C}), \overrightarrow{01})_{\mathbf{Q}}$.

Definition 3.3 *The m-th Hodge-unipotent Albanese map* $\mathrm{Alb}_m^{\mathrm{H}}\colon \mathbf{P}^1_{01\infty}(\mathbf{C}) \to H^1(\mathcal{H}_{\mathbf{Q}}, \mathscr{P}_m^{\mathrm{H}})$ is defined by

$$\mathrm{Alb}_m^{\mathrm{H}_{\mathbf{Q}}}(z) := [\mathscr{P}_m^{\mathrm{H}}(\overrightarrow{01}, z)] \text{ for all } z \in \mathbf{P}^1_{01\infty}(\mathbf{C}).$$

We denote by $\underline{\mathrm{Alb}}^{\mathrm{H}_{\mathbf{Q}}}$ the series of Hodge unipotent Albanese maps $\{\mathrm{Alb}_m^{\mathrm{H}_{\mathbf{Q}}}\}_{m=1}^{\infty}$. We define $\mathrm{Alb}_m^{\mathrm{H}_{\mathbf{R}}}$ to be the composite of $\mathrm{Alb}_m^{\mathrm{H}_{\mathbf{Q}}}$ with the natural map

$$H^1(\mathcal{H}_{\mathbf{Q}}, \mathscr{P}_m^{\mathrm{H}}) \to H^1(\mathcal{H}_{\mathbf{R}}, \mathscr{P}_m^{\mathrm{H}} \times \mathrm{Spec}(\mathbf{R}))$$

induced by the canonical functor $\mathcal{H}_{\mathbf{Q}} \to \mathcal{H}_{\mathbf{R}}$.

It is well known that $\mathscr{P}_1^{\mathrm{H}}(\overrightarrow{01}; z)$ is a direct sum of the Hodge realizations of Kummer torsors $K(z)$ and $K(1-z)$ (*cf.* [4, Proposition 14.2, Proposition 16.26]). Since the Hodge realization of $K(z)$ is represented by $\log(z) \in \mathbf{C}/\mathbf{Q}(1) = \mathrm{Ext}^1_{\mathcal{H}_{\mathbf{Q}}}(\mathbf{Q}(0), \mathbf{Q}(1))$, the series $\underline{\mathrm{Alb}}^{\mathrm{H}}$ of Hodge unipotent Albanese maps the condition of Definition 2.8(i). Note that, however, $\{\mathrm{Alb}_m^{\mathrm{H}_{\mathbf{R}}}\}_{m=0}^{\infty}$ is not a series of abstract unipotent Albanese maps. Indeed, these maps do not satisfy the condition (Hom) because the map

$$\log|\;|\colon \mathbf{C}^{\times}/\mathbf{C}^{\times}_{\mathrm{tor}} \to \mathbf{R}; \quad z \mapsto \log|z|$$

is not injective.

Definition 3.4 We define *the m-th Hodge modified polylogarithm*

$$\mathscr{L}_m^{\mathrm{H}_{\mathbf{Q}}}\colon \mathbf{Z}[\mathbf{P}^1_{01\infty}(\mathbf{C})] \to \mathrm{coLie}(\mathcal{H}_{\mathbf{Q}})^{(2m)}$$

to be $\mathscr{L}_m(\underline{\mathrm{Alb}}^{\mathrm{H}})$ and define $\mathscr{L}_m^{\mathrm{H}_{\mathbf{R}}}$ to be the composite of homomorphisms

$$\mathbf{Z}[\mathbf{P}^1_{01\infty}(\mathbf{C})] \to \mathrm{coLie}(\mathcal{H}_{\mathbf{Q}})^{(2m)} \to \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)},$$

where the last homomorphism is induced by the canonical functor $\mathcal{H}_{\mathbf{Q}} \to \mathcal{H}_{\mathbf{R}}$.

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By construction, $\mathscr{L}_m^{\mathrm{H}_\mathbf{R}}$ is a linearlization of $\mathrm{Alb}_m^{\mathrm{H}_\mathbf{R}}$. We define $R_m^{\mathrm{H}}(\mathbf{C})$, $A_m^{\mathrm{H}}(\mathbf{C})$, and $B_m^{\mathrm{H}}(\mathbf{C})$ by

$$R_m^{\mathrm{H}}(\mathbf{C}) := R_m^{\mathrm{H}}(\mathbf{C}, \underline{\mathrm{Alb}}^{\mathrm{H}}),\ A_m^{\mathrm{H}}(\mathbf{C}) := A_m(\mathbf{C}, \underline{\mathrm{Alb}}^{\mathrm{H}}),\ B_m^{\mathrm{H}}(\mathbf{C}) := B_m(\mathbf{C}, \underline{\mathrm{Alb}}^{\mathrm{H}}).$$

Similarly, for each number field K, we define $R_m^{\mathrm{H}}(K)$ and $A_m^{\mathrm{H}}(K)$ to be the inverse images of $\oplus_{\sigma:K\hookrightarrow\mathbf{C}} R_m^{\mathrm{H}}(\mathbf{C})$ and $\oplus_{\sigma:K\hookrightarrow\mathbf{C}} A_m^{\mathrm{H}}(\mathbf{C})$ under the inclusion

$$\mathbf{Z}[\mathbf{P}^1_{01\infty}(K)] \xrightarrow{\oplus\sigma} \bigoplus_{\sigma:K\hookrightarrow\mathbf{C}} \mathbf{Z}[\mathbf{P}^1_{01\infty}(\mathbf{C})],$$

respectively. Then we define $B_m^{\mathrm{H}}(K) := A_m^{\mathrm{H}}(K)/R_m^{\mathrm{H}}(K)$. By Proposition 2.11, the Hodge modified polylogarithm $\mathscr{L}_m^{\mathrm{H}_\mathbf{Q}}$ induces a well-defined injective group homomorphism

$$\mathscr{L}_m^{\mathrm{H}_\mathbf{Q}} : B_m^{\mathrm{H}}(\mathbf{C}) \hookrightarrow \mathrm{Ext}^1_{\mathcal{H}_\mathbf{Q}}(\mathbf{Q}(0), \mathbf{Q}(m)) = \mathbf{C}/\mathbf{Q}(m)$$

and $\mathscr{L}_m^{\mathrm{H}_\mathbf{R}}$ induces a group homomorphism

$$\mathscr{L}_m^{\mathrm{H}_\mathbf{R}} : B_m^{\mathrm{H}}(\mathbf{C}) \to \mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}}(\mathbf{R}(0), \mathbf{R}(m)) = \mathbf{C}/\mathbf{R}(m) \xleftarrow{\sim} \mathbf{R}(m-1).$$

Note that $\mathscr{L}_m^{\mathrm{H}_\mathbf{R}}$ coincides with the composite of homomorphisms

$$B_m^{\mathrm{H}}(\mathbf{C}) \xrightarrow{\mathscr{L}_m^{\mathrm{H}_\mathbf{Q}}} \mathbf{C}/\mathbf{Q}(m) \longrightarrow \mathbf{C}/\mathbf{R}(m) \xleftarrow{\sim} \mathbf{R}(m-1).$$

3.3 Classifying Spaces of Torsors in $\mathcal{H}_\mathbf{R}$

To calculate $\mathscr{L}_m^{\mathrm{H}_\mathbf{R}}$, we study explicit descriptions of classifying spaces of torsors in $\mathcal{H}_\mathbf{R}$. We fix an algebraic group $G = \mathrm{Sp}(A)$ in $\mathcal{H}_\mathbf{R}$ satisfying the condition (Pos) (*cf.* Definition 2.2). By definition, A is a finitely generated Hopf algebra object in $\mathrm{Ind}(\mathcal{H}_\mathbf{R})$ satisfying the condition $W_0 A = \mathbf{R}$. In particular, all the Hodge weights of A are non-negative and $F^1(A \otimes_\mathbf{R} \mathbf{C})$ is a Hopf ideal of $A \otimes_\mathbf{R} \mathbf{C}$. Let R be $\mathbf{Q}$ or $\mathbf{R}$. For an affine scheme $Y = \mathrm{Sp}(A')$ with $A' \in \mathrm{Ind}(\mathcal{H}_R)$, we sometimes identify Y with the underlying R-*scheme* $\mathrm{Spec}(A'_R)$ where A'_R is the underlying R-algebra of A'.

Lemma 3.5 *Let g be a $\mathbf{C}$-valued point of the underlying $\mathbf{R}$-group scheme G. Let $g^\sharp$ be the $\mathbf{C}$-algebra homomorphism of $A \otimes_\mathbf{R} \mathbf{C}$ induced by the left multiplication of g on G. Then $g^\sharp$ on $A \otimes_\mathbf{R} \mathbf{C}$ preserves the filtration $W_\bullet A \otimes_\mathbf{R} \mathbf{C}$. Moreover, the induced isomorphism on $\mathrm{gr}^W_{2n} A \otimes_\mathbf{R} \mathbf{C}$ is the identity map for each n.*

Proof This proposition is a direct consequence of Corollary A.2 and Lemma A.5. ■

Let $X = \mathrm{Sp}(B)$ be a right G-torsor in $\mathcal{H}_\mathbf{R}$. We say that $x \in X(\mathbf{C})$ is a *Hodge trivialization of X* if the morphism of schemes

$$G_\mathbf{C} \longrightarrow X_\mathbf{C}; g \longmapsto xg$$

preserves Hodge filtrations on the rings of regular functions of both-hand sides. By the exactly same argument as in [24, Lemma 3.3], all Hodge weights of $B = \mathcal{O}(X)$ are also non-negative. Hence, all $F^i(B \otimes_\mathbf{R} \mathbf{C})$ are ideals of $B \otimes_\mathbf{R} \mathbf{C}$. Put $F^0(X_\mathbf{C}) := \mathrm{Spec}(B \otimes_\mathbf{R} \mathbf{C}/F^1(B \otimes_\mathbf{R} \mathbf{C}))$ and put $F^0(G_\mathbf{C}) := \mathrm{Spec}(A \otimes_\mathbf{R} \mathbf{C}/F^1(A \otimes_\mathbf{R} \mathbf{C}))$. Remark that $F^0(X_\mathbf{C})(\mathbf{C})$ coincides with the set of Hodge trivializations of X. Furthermore, $F^0(X_\mathbf{C})$ has a natural structure of a right $F^0(G_\mathbf{C})$-torsor in the usual sense.

Lemma 3.6 *Let X be a right G-torsor in $\mathcal{H}_{\mathbf{R}}$. Then there exists a unique Hodge trivialization of X.*

Proof Since $F^0(G_{\mathbf{C}})$ is the trivial group scheme by the condition (Pos), we have $F^0(X_{\mathbf{C}}) \cong \operatorname{Spec}(\mathbf{C})$. Therefore, the set of Hodge trivializations $F^0(X_{\mathbf{C}})(\mathbf{C})$ is singleton. ∎

Proposition 3.7 *There exists a natural isomorphism of pointed sets*

$$\Psi: G(\mathbf{C})/G(\mathbf{R}) \xrightarrow{\sim} H^1(\mathcal{H}_{\mathbf{R}}, G).$$

Proof Let us take $g \in G(\mathbf{C})$. Then we define the right G-torsor G_g as follows.

- The underlying affine $\mathbf{R}$-scheme of G_g is defined to be G equipped with the right action of G defined by right translations.
- The weight filtration on $\mathcal{O}(G_g)$ is the same as that of $\mathcal{O}(G)$.
- The Hodge filtration on $\mathcal{O}(G_g) \otimes_{\mathbf{R}} \mathbf{C}$ is defined by

$$F^i(\mathcal{O}(G_g) \otimes_{\mathbf{R}} \mathbf{C}) = g^\sharp\big(F^i(\mathcal{O}(G) \otimes_{\mathbf{R}} \mathbf{C})\big).$$

According to Lemma 3.5, $g^\sharp$ is the identity map on each graded piece $\operatorname{gr}^W_{2n} \mathcal{O}(G) \otimes_{\mathbf{R}} \mathbf{C}$. Hence, $\mathcal{O}(G_g)$ is an algebra object in $\operatorname{Ind}(\mathcal{H}_{\mathbf{R}})$, and G_g is actually a right G-torsor in $\mathcal{H}_{\mathbf{R}}$. We put $\Psi(g) := [G_g]$. One can check that $G_g \cong G_{g'}$ if and only if $gG(\mathbf{R}) = g'G(\mathbf{R})$. Hence, Ψ induces an injective map $G(\mathbf{C})/G(\mathbf{R}) \hookrightarrow H^1(\mathcal{H}_{\mathbf{R}}, G)$.

We show the surjectivity of Ψ. Let X be a G-torsor in $\mathcal{H}_{\mathbf{R}}$. According to Lemma 3.6, there exists a unique Hodge trivialization $p_H \in X(\mathbf{C})$ of X. We take $p_w \in X(\mathbf{R})$ an $\mathbf{R}$-valued point of X. Then p_w trivializes the weight filtration of X; that is, the G-equivariant morphism

$$f_{p_w}: G \longrightarrow X; g \longmapsto p_w g$$

preserves their weight filtrations on the rings of regular functions of both-hand sides by Corollary A.3. Let g be an element of $G(\mathbf{C})$ satisfying $p_w = p_H g$. Then f_{p_w} defines an isomorphism of G-torsors between G_g and X. Hence, we have $\Psi(gG(\mathbf{R})) = [X]$, and this completes the proof of the surjectivity of Ψ. ∎

We denote by

$$c^*: G(\mathbf{C}) \xrightarrow{\sim} G(\mathbf{C})$$

the group automorphism induced by the complex conjugate c on $\mathbf{C}$. Then the natural map

$$G(\mathbf{C})/G(\mathbf{R}) \longrightarrow G(\mathbf{C})^{c^*=-1}; gG(\mathbf{R}) \longmapsto gc^*(g^{-1})$$

is bijective. By composing the logarithmic map from $G(\mathbf{C})$ to the Lie algebra of $G_{\mathbf{C}}$, we obtain the canonical isomorphism of pointed sets

$$\Psi': H^1(\mathcal{H}_{\mathbf{R}}, G) \xrightarrow{\sim} (\operatorname{Lie}(G) \otimes_{\mathbf{R}} \mathbf{C})^{c^*=-1} = \operatorname{Lie}(G) \otimes_{\mathbf{R}} \mathbf{R}\sqrt{-1}. \tag{3.2}$$

Remark that c^* acts on $\operatorname{Lie}(G) \otimes_{\mathbf{R}} \mathbf{C}$ by $\operatorname{id} \otimes c$.

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Now, we assume that G satisfies (Triv). By composing Φ^{-1} in Proposition 2.4 and Ψ' in (3.2), we obtain an isomorphism

$$\Psi' \circ \Phi^{-1} \colon \mathrm{Hom}^{\mathbf{G}_{m,\mathbf{R}}}_{\mathbf{R}\text{-}\mathrm{Lie}}(\mathrm{Lie}(\mathcal{H}_{\mathbf{R}}), \omega(\mathrm{Lie}(G))) \xrightarrow{\sim} \mathrm{Lie}(G) \otimes_{\mathbf{R}} \mathbf{R}\sqrt{-1}$$

of pointed sets.

Proposition 3.8 *Let G be an algebraic group in $\mathcal{H}_{\mathbf{R}}$ satisfying* (Pos) *and* (Triv). Then the composite of canonical isomorphisms

$$\begin{aligned}\Psi' \circ \Phi^{-1} \colon \mathrm{Hom}^{\mathbf{G}_{m,\mathbf{R}}}_{\mathbf{R}\text{-}\mathrm{Lie}}\left(\mathrm{Lie}(\mathcal{H}_{\mathbf{R}}), \omega(\mathrm{Lie}(G))\right) &\xrightarrow{\sim} H^1(\mathcal{H}_{\mathbf{R}}, \mathrm{Lie}(G)) \\ &\xrightarrow{\sim} \mathrm{Lie}(G) \otimes_{\mathbf{R}} \mathbf{R}\sqrt{-1}\end{aligned}$$

sends $f \in \mathrm{Hom}^{\mathbf{G}_{m,\mathbf{R}}}_{\mathbf{R}\text{-}\mathrm{Lie}}(\mathrm{Lie}(\mathcal{H}_{\mathbf{R}}), \omega(\mathrm{Lie}(G)))$ to

$$2 \sum_n f(N_n)(2\pi\sqrt{-1})^n \otimes \sqrt{-1}^{-1} \in \mathrm{Lie}(G) \otimes_{\mathbf{R}} \mathbf{R}\sqrt{-1}.$$

Proof According to Lemma A.7, the Lie homomorphism

$$\iota_N \colon \mathrm{Lie}(G) \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow \mathrm{End}_{\mathbf{C}}(W_{2N}A \otimes_{\mathbf{R}} \mathbf{C}); l \longmapsto \log\left(\exp(l)^{\sharp}|_{W_{2N}(A \otimes_{\mathbf{R}} \mathbf{C})}\right)$$

is injective for sufficiently large N. Let us denote by ϕ_N the composite of $\Psi' \circ \Phi^{-1}$ and ι_N. Then, to prove this proposition, it is sufficient to show the equality

$$\phi_N(f) = 2 \sum_n \log\left(\exp(f(N_n))^{\sharp}|_{\omega(W_{2N}A)}\right)(2\pi\sqrt{-1})^n \otimes \sqrt{-1}^{-1} \tag{3.3}$$

in $\mathrm{End}_{\mathbf{C}}(W_{2N}A \otimes_{\mathbf{R}} \mathbf{C})$ for sufficiently large N.

Let f be an element of $\mathrm{Hom}^{\mathbf{G}_{m,\mathbf{R}}}_{\mathbf{R}\text{-}\mathrm{Lie}}\left(\mathrm{Lie}(\mathcal{H}_{\mathbf{R}}), \omega(\mathrm{Lie}(G))\right)$ and let

$$a_f \colon \pi_1(\mathcal{H}_{\mathbf{R}}, \omega) \longrightarrow \mathrm{GL}(\omega(A)); \sigma \longmapsto \exp(f)(\sigma)^{\sharp}$$

be the action of $\pi_1(\mathcal{H}_{\mathbf{R}}, \omega)$ on $\omega(A)$ defined by f. Here, $\exp(f) \colon \pi_1(\mathcal{H}_{\mathbf{R}}, \omega) \to \omega(G)$ is the group homomorphism corresponding to f. According to Lemma 3.5, a_f preserves $\omega(W_{2N}A)$. We denote by $W_{2N}A_f$ the mixed Hodge–Tate structure on $\tau(2\pi\sqrt{-1})\omega(W_{2N}A) = W_{2N}A$ defined by the action a_f (see (3.1)). Then, by definition, $A_f := \varinjlim_N W_{2N}A_f$ has a natural ring structure and $\mathrm{Sp}(A_f)$ is a G-torsor in $\mathcal{H}_{\mathbf{R}}$ representing $\Phi^{-1}(f) \in H^1(\mathcal{H}_{\mathbf{R}}, G)$. Let h be an element of $\mathrm{Aut}_{\mathbf{C}}(W_{2N}A \otimes_{\mathbf{R}} \mathbf{C})$ such that $F^i(W_{2N}A_f \otimes_{\mathbf{R}} \mathbf{C}) = h(F^i(W_{2N}A \otimes_{\mathbf{R}} \mathbf{C}))$. Then, by (3.1), we have

$$\frac{1}{2}\log(h\overline{h}^{-1}) = \sum_n \log\left(\exp(f(N_n))^{\sharp}|_{\omega(W_{2N}A)}\right)(2\pi\sqrt{-1})^n \otimes \sqrt{-1}^{-1}. \tag{3.4}$$

We remark that the left-hand side of equation (3.4) coincides with $\frac{1}{2}\phi_N(f)$. Indeed, we can take h as $g^{\sharp}|_{W_{2N}A \otimes_{\mathbf{R}} \mathbf{C}}$ where $g \in G(\mathbf{C})$ is a representative of $\Psi^{-1} \circ \Phi^{-1}(f) \in G(\mathbf{C})/G(\mathbf{R})$. Therefore, the equality (3.3) holds for all positive integers N, and this completes the proof of the proposition. ∎

3.4 Calculation of $\mathscr{L}_m^{\mathrm{H}_{\mathbf{R}}}$

Now we consider the case $G = \mathscr{P}_m^{\mathrm{H}_{\mathbf{R}}} := \mathscr{P}_m^{\mathrm{H}} \times \mathrm{Spec}(\mathbf{R})$. Define the Lie algebra $\mathfrak{p}_m^{\mathrm{H}}$ over $\mathbf{Q}$ to be the Lie algebra of $\mathscr{P}_m^{\mathrm{H}}$.

Corollary 3.9 *We define the evaluation map* $\mathrm{ev}_m\colon \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)} \to \mathbf{R}(m-1)$ *by* $f \mapsto f(N_m)(2\pi)^m\sqrt{-1}^{m-1}$. *Then the following assertions hold.*

(i) *The evaluation map* ev_m *is the left inverse of the canonical inclusion*

$$\mathbf{R}(m-1) \cong \mathrm{Ext}^1_{\mathcal{H}_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(m)) \hookrightarrow \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)}.$$

(ii) *The diagram*

$$\begin{array}{ccc} H^1(\mathcal{H}_{\mathbf{R}}, \mathscr{P}_m^{\mathrm{H}_{\mathbf{R}}}) & \longrightarrow & \mathfrak{p}_m^{\mathrm{H}} \otimes_{\mathbf{Q}} \mathbf{R}\sqrt{-1} \\ \downarrow & & \downarrow \mathrm{pr}_m \\ \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)} & \xrightarrow{2\,\mathrm{ev}_m} & \mathbf{R}(m-1) \end{array}$$

commutes, where pr_m *is the projection to the last component*

$$\mathfrak{p}_m^{\mathrm{H}} \otimes_{\mathbf{Q}} \mathbf{R}\sqrt{-1} = \mathbf{R}(0) \times \prod_{n=1}^{m} \mathbf{R}(n-1) \twoheadrightarrow \mathbf{R}(m-1).$$

Proof Let b be a real number. Recall that the element $f_b^{(2m)} \in \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)}$ denotes the image of $b(2\pi)^m\sqrt{-1}^{m-1}$ under $\mathbf{R}(m-1) \hookrightarrow \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)}$. Then, by the calculation in Example 3.1, we have equalities

$$\mathrm{ev}_m(f_b^{(2m)}) = f_b^{(2m)}(N_m)(2\pi)^m\sqrt{-1}^{m-1} = b(2\pi)^m\sqrt{-1}^{m-1}.$$

Thus assertion (i) of the corollary holds.

Assertion (ii) is easily checked by the definition of ev_m and Proposition 3.8. ■

Let us recall *the classical modified polylogarithm* $\mathscr{L}_m^{\mathrm{Cl}}\colon \mathbf{P}^1_{01\infty}(\mathbf{C}) \to \mathbf{R}(m-1)$, which is a real analytic function defined by

$$(3.5)\qquad \mathscr{L}_m^{\mathrm{Cl}}(z) := \begin{cases} \sqrt{-1}\,\mathrm{Im}\Big(\sum_{k=0}^{m-1} \frac{B_k}{k!}\log^k(z\bar{z})\,\mathrm{Li}_{m-k}(z)\Big) & \text{if } m \text{ is even,} \\ \mathrm{Re}\Big(\sum_{k=0}^{m-1} \frac{B_k}{k!}\log^k(z\bar{z})\,\mathrm{Li}_{m-k}(z)\Big) & \text{if } m \text{ is odd} \end{cases}$$

(*cf.* [1, 1.5], [29, p. 413 (33)]). Here, B_m is the m-th Bernoulli number defined by

$$\sum_{m=0}^{\infty} \frac{B_m}{m!} t^m := \frac{t}{e^t - 1}.$$

Proposition 3.10 *Let m be a positive integer. Then, for each* $\sum_i a_i\{z_i\} \in A_m^{\mathrm{H}}(\mathbf{C})$, *the equality*

$$\sum_i a_i \mathscr{L}_m^{\mathrm{H}_{\mathbf{R}}}(z_i) = -\sum_i a_i \mathscr{L}_m^{\mathrm{Cl}}(z_i)$$

holds. In other words, $\mathscr{L}_m^{\mathrm{H}_{\mathbf{R}}}\colon B_m^{\mathrm{H}}(\mathbf{C}) \to \mathbf{R}(m-1)$ *coincides with the classical modified polylogarithm multiplied by* -1.

For the proof of Proposition 3.10, we recall the calculation of Beilinson and Deligne in [1] computing the composite of $\Psi'\colon H^1(\mathcal{H}_{\mathbf{R}}, \mathscr{P}_m^{\mathrm{H}_{\mathbf{R}}}) \to \mathfrak{p}_m^{\mathrm{H}} \otimes_{\mathbf{Q}} \mathbf{R}\sqrt{-1}$ and the m-th Hodge unipotent Albanese map.

Lemma 3.11 ([1, Section 1.5]) *For any* $z \in \mathbf{P}^1_{01\infty}(\mathbf{C})$*, the equality*

$$\mathrm{pr}_m \circ \Psi' \circ \mathrm{Alb}^{\mathrm{H}}_m(z) = -2\mathscr{L}^{\mathrm{Cl}}_m(z) \tag{3.6}$$

holds in $\mathbf{R}(m-1)$.

Proof Let z be an element of $\mathbf{P}^1_{01\infty}(\mathbf{C})$. According to [8, (2.20)], the image of z under

$$\mathbf{P}^1_{01\infty}(\mathbf{C}) \to H^1(\mathcal{H}_{\mathbf{R}}, \mathscr{P}^{\mathrm{H}_{\mathbf{R}}}_m) \cong (\mathscr{P}^{\mathrm{H}_{\mathbf{R}}}_m(\mathbf{C}))^{c^*=-1} \subset \mathscr{P}^{\mathrm{H}_{\mathbf{R}}}_m(\mathbf{C}) = \mathbf{C}(1) \times \prod_{n=1}^{m} \mathbf{C}(n)$$

is calculated as

$$\big(\log(z\bar{z}); -\mathrm{Li}^-_1(z), \dots, -\mathrm{Li}^-_m(z)\big).$$

Here, $\mathrm{Li}^-_n \colon \mathbf{P}^1(\mathbf{C}) \to \mathbf{C}$ is a single-valued and real analytic polylogarithm (see [8, Theorem 2.27]). By the Baker–Campbell–Hausdorff formula, we have the equality

$$\Psi' \circ \mathrm{Alb}^{\mathrm{H}}_m(z) = \Big(\log(z\bar{z}); -\mathrm{Li}^-_1(z), \dots, -\sum_{k=1}^{m} \frac{B_k}{k!} \log^k(z\bar{z})\, \mathrm{Li}^-_{m-k}(z)\Big). \tag{3.7}$$

Then the last component of the right-hand side of (3.7) coincides with the right-hand side of the equation (3.6) by [8, (2.21)]. ∎

Proof of Proposition 3.10 Let z be an element of $\mathbf{P}^1_{01\infty}(\mathbf{C})$. Then, by Lemma 3.11 and Corollary 3.9(ii), we have the equalities:

$$2\,\mathrm{ev}_m \circ \mathscr{L}^{\mathrm{H}_{\mathbf{R}}}_m(z) = 2\mathrm{pr}_m \circ \Psi' \circ \mathrm{Alb}^{\mathrm{H}}_m(z) = -2\mathscr{L}^{\mathrm{Cl}}_m(z).$$

Now we take an element $\xi = \sum_i a_i\{z_i\}$ of $A^{\mathrm{H}}_m(\mathbf{C})$. According to Corollary 3.9(i), $\mathscr{L}^{\mathrm{H}_{\mathbf{R}}}_m(\xi)$ coincides with $\mathrm{ev}_m(\mathscr{L}^{\mathrm{H}_{\mathbf{R}}}_m(\xi))$ in $\mathbf{R}(m-1) \subset \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)}$. Thus, we have

$$\mathscr{L}^{\mathrm{H}_{\mathbf{R}}}_m(\xi) = \mathrm{ev}_m(\mathscr{L}^{\mathrm{H}_{\mathbf{R}}}_m(\xi)) = \sum_i a_i\ \mathrm{ev}_m(\mathscr{L}^{\mathrm{H}_{\mathbf{R}}}_m(z_i)) = -\sum_i a_i \mathscr{L}^{\mathrm{Cl}}_m(z_i).$$

This completes the proof of Proposition 3.10. ∎

4 ℓ-adic Étale Modified Polylogarithms

In this section, we fix a rational prime ℓ and a field F of characteristic $p \geq 0$ satisfying the following conditions:

$(\mathbf{cyc})_\ell$ The characteristic p of F does not divide ℓ and the ℓ-adic cyclotomic character on $\mathscr{G}_F$ has an infinite image in $\mathbf{Z}_\ell^\times$.

$(\mathbf{nd})_\ell$ There exists no non-torsion ℓ-divisible element in $F^\times$.

Example 4.1 If F is finitely generated over the prime field k_p of characteristic $p \nmid \ell$, then F satisfies $(\mathrm{cyc})_\ell$ and $(\mathrm{nd})_\ell$.

We fix a coherent system $\zeta_{\ell^\infty} := (\zeta_{\ell^n})_{n\geq 1}$ of ℓ-power roots of unity in $\overline{F}$ and regard ζ_{ℓ^∞} as a $\mathbf{Z}_\ell$-basis of $\mathbf{Z}_\ell(1) = \varprojlim_n \boldsymbol{\mu}_{\ell^n}(\overline{F})$ where $\boldsymbol{\mu}_{\ell^n} := \mathrm{Spec}(\mathbf{Z}[t]/(t^{\ell^n} - 1))$. By condition $(\mathrm{cyc})_\ell$, two $\mathscr{G}_F$-modules $\mathbf{Q}_\ell(m)$ and $\mathbf{Q}_\ell(m')$ are not isomorphic for any two distinct integers m and m'.

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Remark 4.2 Condition $(\mathrm{cyc})_\ell$ is not equivalent to the condition $\cup_{n\geq 1}\boldsymbol{\mu}_{\ell^n}(\overline{F}) \not\subset F$. For example, the maximal totally real subfield $\mathbf{Q}(\mu_{\ell^\infty})^+$ of $\mathbf{Q}(\mu_{\ell^\infty})$ does not contain $\mu_{\ell^\infty} := \cup_{n\geq 1}\boldsymbol{\mu}_{\ell^n}(\overline{\mathbf{Q}})$, although the order of the ℓ-adic cyclotomic character on $\mathscr{G}_{\mathbf{Q}(\mu_{\ell^\infty})^+}$ is two.

We denote by $\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F)$ the category of continuous representations of $\mathscr{G}_F$ on finite dimensional $\mathbf{Q}_\ell$-vector spaces. *An ℓ-adic mixed Tate $\mathscr{G}_F$-module* is an object V in $\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F)$ equipped with an increasing, saturated, and separated filtration $\{W_{2n}V\}_{n\in\mathbf{Z}}$ indexed by even integers such that $\mathrm{gr}^W_{2n}V$ is a direct sum of $\mathbf{Q}_\ell(-n)$ as a $\mathscr{G}_F$-module (*cf.* [11, Section 6, Section 7]). We denote by $\mathcal{MT}_\ell(F)$ the category of ℓ-adic mixed Tate $\mathscr{G}_F$-modules. Then $\mathcal{MT}_\ell(F)$ is a mixed Tate category over $\mathbf{Q}_\ell$. The second example of an abstract modified polylogarithm is the ℓ-adic étale modified polylogarithm. We apply our abstract formalism to $\mathcal{MT}_\ell(F)$.

4.1 Classifying Spaces of Torsors in $\mathcal{MT}_\ell(F)$

In this subsection, we make remarks on classifying spaces of torsors in $\mathcal{MT}_\ell(F)$.

Lemma 4.3 *Let $(V_1, W_\bullet V_1)$ and $(V_2, W_\bullet V_2)$ be objects in $\mathcal{MT}_\ell(F)$. Let $f\colon V_1 \xrightarrow{\sim} V_2$ be an isomorphism of $\mathbf{Q}_\ell[\mathscr{G}_F]$-modules. Then f preserves the weight filtration of both-hand sides. In other words, f defines a morphism in $\mathcal{MT}_\ell(F)$.*

Proof This is easily checked by the definition of the weight filtration and by an inductive argument on the length of the weight filtration. ■

Let us fix an algebraic group $G = \mathrm{Spec}(R)$ in $\mathcal{MT}_\ell(F)$ satisfying (Pos). Then, according to Corollary A.6, the underlying algebraic group G is automatically unipotent. Recall that $G(\mathbf{Q}_\ell)$ has a natural topology on which $\mathscr{G}_F$ acts continuously (*cf.* [24, Section 3.2]). Note that this natural topology coincides with the relative topology induced by any closed immersion $G \hookrightarrow \mathrm{GL}_{n,\mathbf{Q}_\ell}$, where the topology on $\mathrm{GL}_n(\mathbf{Q}_\ell) \hookrightarrow \mathbf{Q}_\ell^{n^2}$ is induced by the product topology of $\mathbf{Q}_\ell$.

Lemma 4.4 *There exists a natural isomorphism of pointed sets*

$$H^1(F, G(\mathbf{Q}_\ell)) \cong H^1(\mathcal{MT}_\ell(F), G).$$

Here, the left-hand side is the first continuous Galois cohomology with coefficients in $G(\mathbf{Q}_\ell)$.

Proof First, we note that the underlying scheme of each G-torsor is isomorphic to the affine space over $\mathbf{Q}_\ell$, because the underlying group scheme of G is unipotent. Especially, the set of $\mathbf{Q}_\ell$-rational points of any G-torsor is non-empty.

For each continuous 1-cocycle $c\colon \mathscr{G}_F \to G(\mathbf{Q}_\ell)$, we define the new action a_c of $\mathscr{G}_F$ on $R = \mathcal{O}(G)$ by

$$a_c(\sigma)(f) := c(\sigma)^\sharp({}^\sigma f) \text{ for all } f \in R, \tag{4.1}$$

and define R_c to be the ring R equipped with the new action of $\mathscr{G}_F$ defined by a_c. Let us denote by $H^1(\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F), G)$ the set of isomorphism classes of torsors under G

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in $\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F)$ and let G_c be $\mathrm{Spec}(R_c)$ equipped with the natural right G-action. Then, since $X(\mathbf{Q}_\ell)$ is non-empty for any G-torsor X in $\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F)$, we have a natural isomorphism

$$H^1(F, G(\mathbf{Q}_\ell)) \xrightarrow{\sim} H^1(\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F), G); [c] \longmapsto [G_c]$$

(*cf.* [24, Proposition 3.15]). Therefore, to prove the lemma, it is sufficient to show that the natural map

$$H^1(\mathcal{MT}_\ell(F), G) \longrightarrow H^1(\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F), G) \tag{4.2}$$

induced by the forgetful functor

$$\mathcal{MT}_\ell(F) \longrightarrow \mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F); (V, W_\bullet V) \longmapsto V$$

is bijective. The injectivity of (4.2) follows from Lemma 4.3 directly. Hence, to show this lemma, it is sufficient to show the surjectivity of the map (4.2).

We put $W_{2n}R_c := W_{2n}R$. Then, according to Lemma A.5 and (4.1), the action of $\mathscr{G}_F$ on R_c preserves the filtration $\{W_{2n}R_c\}_{n\in\mathbf{Z}}$ and coincides with the original action of $\mathscr{G}_F$ on each graded piece $\mathrm{gr}^W_{2n} R_c$ of R_c. Therefore, the pair $(R_c, W_{2n}R_c)$ is an object in $\mathrm{Ind}(\mathcal{MT}_\ell(F))$ and this implies that (4.2) is surjective. ∎

Lemma 4.4 can be rewritten as follows. Let $\omega_0 : \mathcal{MT}_\ell(F) \to \mathrm{Vec}_{\mathbf{Q}_\ell}$ be the forgetful functor. We fix an isomorphism of fiber functors of $\mathcal{MT}_\ell(F)$

$$\Gamma : \omega_0 \xrightarrow{\sim} \omega$$

such that

$$\omega_0(\mathbf{Q}_\ell(1)) = \mathbf{Q}_\ell(1) \xrightarrow{\sim} \omega(\mathbf{Q}_\ell(1)) \xleftarrow{\sim} \mathbf{Q}_\ell; \quad \zeta_{\ell^\infty} \longmapsto 1.$$

We also denote by Γ the isomorphism

$$\pi_1(\mathcal{MT}_\ell(F), \omega_0) \xrightarrow{\sim} \pi_1(\mathcal{MT}_\ell(F), \omega); \sigma \longmapsto \Gamma \circ \sigma \circ \Gamma^{-1}$$

for simplicity. Since $\pi_1(\mathcal{MT}_\ell(F), \omega_0)$ is canonically isomorphic to the weighted completion of $\mathscr{G}_F$ with respect to the ℓ-adic cyclotomic character $\chi_\ell : \mathscr{G}_F \to \mathbf{Z}_\ell^\times \subset \mathbf{G}_m(\mathbf{Q}_\ell)$ in the sense of [12, Section 4], there exists a canonical continuous homomorphism

$$\rho_0 : \mathscr{G}_F \longrightarrow \pi_1\big(\mathcal{MT}_\ell(F), \omega_0\big)(\mathbf{Q}_\ell).$$

Therefore, by composing the isomorphism Γ, we obtain a continuous homomorphism

$$\rho : \mathscr{G}_F \longrightarrow \pi_1\big(\mathcal{MT}_\ell(F), \omega\big)(\mathbf{Q}_\ell).$$

The functor Γ also induces an isomorphism $\alpha : G \xrightarrow{\sim} \omega(G)$ of algebraic groups over $\mathbf{Q}_\ell$. Then, for each continuous 1-cocycle

$$c : \mathscr{G}_F \to G(\mathbf{Q}_\ell),$$

there exists a unique rational 1-cocycle

$$\widetilde{c} : \pi_1(\mathcal{MT}_\ell(F), \omega) \longrightarrow \omega(G),$$

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which makes the following diagram commute:

$$\begin{array}{ccc} \mathscr{G}_F & \xrightarrow{\quad\rho\quad} & \pi_1(\mathcal{MT}_\ell(F),\omega)(\mathbf{Q}_\ell) \\ \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle \widetilde{c}} \\ G(\mathbf{Q}_\ell) & \xrightarrow{\quad\alpha\quad} & \omega(G)(\mathbf{Q}_\ell). \end{array}$$

As an elementary consequence of the lemma, we have the following corollary.

Corollary 4.5 (*cf.* [11, Corollary 9.3]) *For each positive integer m, there exists a natural isomorphism*

$$\mathrm{Ext}^1_{\mathcal{MT}_\ell(F)}(\mathbf{Q}_\ell(0),\mathbf{Q}_\ell(m)) \xrightarrow{\sim} H^1(F,\mathbf{Q}_\ell(m)).$$

Proof Let $\mathbf{A}(\mathbf{Q}_\ell(m)) := \mathrm{Sp}(\mathrm{Sym}^\bullet\mathbf{Q}_\ell(-m))$ be the vector group scheme in $\mathcal{MT}_\ell(F)$ defined by $\mathbf{Q}_\ell(m)$. Then, by a standard argument, $\mathrm{Ext}^1_{\mathcal{MT}_\ell(F)}(\mathbf{Q}_\ell(0),\mathbf{Q}_\ell(m))$ is canonically isomorphic to $H^1(\mathcal{MT}_\ell(F),\mathbf{A}(\mathbf{Q}_\ell(m)))$. Since m is positive, the group scheme $\mathbf{A}(\mathbf{Q}_\ell(m))$ satisfies (Pos). Hence, we have the conclusion of the corollary by Lemma 4.4. ■

In later sections, we always identify $\mathrm{Ext}^1_{\mathcal{MT}_\ell(F)}(\mathbf{Q}_\ell(0),\mathbf{Q}_\ell(m))$ with $H^1(F,\mathbf{Q}_\ell(m))$ for each positive integer m.

4.2 Definition of ℓ-adic Étale Modified Polylogarithms

In this and the next subsections, we fix a positive integer m.

Definition 4.6 We define the ℓ-adic étale polylogarithmic quotient $\mathscr{P}_m^{\ell\text{-ét}}$, which is an algebraic group in $\mathcal{MT}_\ell(F)$, to be $\mathscr{P}_m^{\mathcal{MT}_\ell(F)} = \exp(\mathbf{Q}_\ell(1) \ltimes \prod_{n=1}^m \mathbf{Q}_\ell(n))$.

We recall another description of ℓ-adic étale polylogarithmic quotient (*cf.* [21, Section 2.1]). Let $\pi_1^\ell(\mathbf{P}^1_{01\infty,\overline{F}},\overrightarrow{01})$ be the maximal pro-ℓ quotient of the étale fundamental group of $\mathbf{P}^1_{01\infty,\overline{F}}$ with the base point $\overrightarrow{01}$. This pro-ℓ group has a standard set of free generators $\{x,y\}$ (*cf.* [27, Section 8, Picture 4], [10, Exposé XIII, Corollaire 2.12]). Then the group homomorphism

$$\mathbf{p}\colon \pi_1^\ell(\mathbf{P}^1_{01\infty,\overline{F}},\overrightarrow{01}) \longrightarrow \mathbf{Z}_\ell(1);\, x \longmapsto \zeta_{\ell^\infty},\, y \longmapsto 1$$

is $\mathscr{G}_F$-equivariant where ζ_{ℓ^∞} is the basis of $\mathbf{Z}_\ell(1)$ fixed in the beginning of this section. We put

$$\pi^{\mathrm{pol}} := \pi_1^\ell(\mathbf{P}^1_{01\infty,\overline{F}},\overrightarrow{01})/[\mathrm{Ker}(\mathbf{p}),\mathrm{Ker}(\mathbf{p})]$$

and

$$\pi^{\mathrm{pol}}(m) := \pi^{\mathrm{pol}}/\pi^{\mathrm{pol},(m)}$$

where $\{\pi^{\mathrm{pol},(n)}\}_{n=1}^\infty$ is the central descending series of π^{pol}. Then $\mathscr{P}_m^{\ell\text{-ét}}$ is canonically isomorphic to the unipotent completion $\pi^{\mathrm{pol}}(m)_{\mathbf{Q}_\ell}$ of $\pi^{\mathrm{pol}}(m)$ over $\mathbf{Q}_\ell$.

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By the above remark, the algebraic groups $\mathscr{P}_m^{\ell\text{-ét}} \cong \pi^{\mathrm{pol}}(m)_{\mathbf{Q}_\ell}$ are quotients of the unipotent completion $\pi_1^{\text{ét}}(\mathbf{P}^1_{01\infty,\overline{F}}, \overrightarrow{01})_{\mathbf{Q}_\ell}$ of the étale fundamental group of $\mathbf{P}^1_{01\infty,\overline{F}}$ with the base point $\overrightarrow{01}$ over $\mathbf{Q}_\ell$ because of the functoriality of the unipotent completion. Let

$$u_m^{\ell\text{-ét}}: \pi_1^{\text{ét}}(\mathbf{P}^1_{01\infty,\overline{F}}, \overrightarrow{01})_{\mathbf{Q}_\ell} \twoheadrightarrow \mathscr{P}_m^{\ell\text{-ét}}$$

be the canonical surjective homomorphism of affine group schemes in $\mathcal{MT}_\ell(F)$. For each $z \in \mathbf{P}^1_{01\infty}(F)$, we denote by $\mathscr{P}_m^{\ell\text{-ét}}(\overrightarrow{01}, z)$ the pushforward of the torsor $\pi_1^{\text{ét}}(\mathbf{P}^1_{01\infty,\overline{F}}; \overrightarrow{01}, z)_{\mathbf{Q}_\ell}$ by $u_m^{\ell\text{-ét}}$.

Definition 4.7 We define the series of ℓ-adic étale unipotent Albanese maps $\underline{\mathrm{Alb}}_F^{\ell\text{-ét}} = \{\mathrm{Alb}_{F,m}^{\ell\text{-ét}}\}$ by

$$\mathrm{Alb}_{F,m}^{\ell\text{-ét}}(z) := \mathscr{P}_m^{\ell\text{-ét}}(\overrightarrow{01}, z) \text{ for all } z \in \mathbf{P}^1_{01\infty}(F).$$

By the condition $(\mathrm{nd})_\ell$ (see the beginning of Section 4 for the definition of $(\mathrm{nd})_\ell$), $\underline{\mathrm{Alb}}_F^{\ell\text{-ét}}$ is a series of abstract unipotent Albanese maps (*cf.* [22, (3.1)]). Thus, we can define the ℓ-adic étale modified polylogarithms as follows.

Definition 4.8 We define the m-th ℓ-adic étale modified polylogarithm

$$\mathscr{L}_m^{\ell\text{-ét}}: \mathbf{Z}[\mathbf{P}^1_{01\infty}(F)] \longrightarrow \mathrm{coLie}(\mathcal{MT}_\ell(F))$$

to be $\mathscr{L}_m(\underline{\mathrm{Alb}}_F^{\ell\text{-ét}})$.

We define $R_m^{\ell\text{-ét}}(F)$, $A_m^{\ell\text{-ét}}(F)$, and $B_m^{\ell\text{-ét}}(F)$ by

$$R_m^{\ell\text{-ét}}(F) := R_m(F, \underline{\mathrm{Alb}}_F^{\ell\text{-ét}}), \quad A_m^{\ell\text{-ét}}(F) := A_m(F, \underline{\mathrm{Alb}}_F^{\ell\text{-ét}}),$$
$$B_m^{\ell\text{-ét}}(F) := B_m(F, \underline{\mathrm{Alb}}_F^{\ell\text{-ét}}).$$

In [22], we used the notation "ℓ-adic" instead of "ℓ-ét". By Proposition 2.11, $\mathscr{L}_m^{\ell\text{-ét}}$ induces a well-defined and injective group homomorphism

$$\mathscr{L}_m^{\ell\text{-ét}}: B_m^{\ell\text{-ét}}(F) \hookrightarrow H^1(F, \mathbf{Q}_\ell(m)) = \mathrm{Ext}^1_{\mathcal{MT}_\ell(F)}(\mathbf{Q}_\ell, \mathbf{Q}_\ell(m)).$$

Remark 4.9 By construction, the m-th ℓ-adic étale unipotent Albanese map $\mathrm{Alb}_{F,m}^{\ell\text{-ét}}$ is functorial in F. Hence, $\mathscr{L}_m^{\ell\text{-ét}}$ is also functorial in F.

4.3 Comparison with ℓ-adic Polylogarithms

In this subsection, we compare $\mathscr{L}_m^{\ell\text{-ét}}$ with the Wojtkowiak ℓ-adic polylogarithms. For the reader's convenience, we give a quick review of the ℓ-adic polylogarithm $\ell i_m(z,\gamma)$ attached to a $\mathbf{Q}_\ell$-path γ from $\overrightarrow{01}$ to z in $\mathbf{P}^1_{01\infty,\overline{F}}$. Recall that a $\mathbf{Q}_\ell$-path in $\mathbf{P}^1_{01\infty,\overline{F}}$ is defined as an element of $\pi_1(\mathbf{P}^1_{01\infty,\overline{F}}; \overrightarrow{01}, z)_{\mathbf{Q}_\ell}(\mathbf{Q}_\ell)$. Let $\iota: \pi_1^{\ell}(\mathbf{P}^1_{01\infty,\overline{F}}, \overrightarrow{01}) \hookrightarrow \mathbf{Q}_\ell\langle\langle X, Y\rangle\rangle$ be the multiplicative embedding to non-commutative formal power series defined by

$$\iota(x) := \exp(X), \quad \iota(y) := \exp(Y).$$

Let $\mathfrak{f}_\gamma : \mathscr{G}_F \to \pi_1(\mathbf{P}^1_{01\infty,\overline{F}}, \overrightarrow{01})_{\mathbf{Q}_\ell}(\mathbf{Q}_\ell)$ be the 1-cocycle defined by $\mathfrak{f}_\gamma(\sigma) := \gamma^{-1} \circ {}^\sigma\gamma$. Then *the m-th ℓ-adic polylogarithm $\ell i_m(z,\gamma)$* is defined by

$$\ell i_m(z,\gamma)(\sigma) := (-1)^{m-1} \times \text{ the coefficient of } \log(\iota(\mathfrak{f}_\gamma(\sigma))) \text{ at } \mathrm{ad}(X)^{m-1}(Y)$$

(*cf.* [28, Definition 11.0.1]).

Lemma 4.10 *Let $c_\gamma : \mathscr{G}_F \to \mathscr{P}_m^{\ell\text{-ét}}(\mathbf{Q}_\ell)$ be the composite of $\mathfrak{f}_\gamma$ and the canonical homomorphism $\mathbf{pr}_m : \pi_1(\mathbf{P}^1_{01\infty,\overline{F}}, \overrightarrow{01})_{\mathbf{Q}_\ell}(\mathbf{Q}_\ell) \twoheadrightarrow \pi^{\mathrm{pol}}(m)_{\mathbf{Q}_\ell}(\mathbf{Q}_\ell) \cong \mathscr{P}_m^{\ell\text{-ét}}(\mathbf{Q}_\ell)$. Then the image of the cohomology class $[c_\gamma] \in H^1(F, \mathscr{P}_m^{\ell\text{-ét}}(\mathbf{Q}_\ell))$ under the natural isomorphism*

$$H^1(F, \mathscr{P}_m^{\ell\text{-ét}}(\mathbf{Q}_\ell)) \cong H^1(\mathcal{MT}_\ell(F), \mathscr{P}_m^{\ell\text{-ét}})$$

*(*cf. *Lemma 4.4) coincides with $\mathrm{Alb}^{\ell\text{-ét}}_{m,F}(z)$.*

Proof The cohomology class of c_γ in $H^1(F, \mathscr{P}_m^{\ell\text{-ét}}(\mathbf{Q}_\ell))$ represents the torsor under $\mathscr{P}_m^{\ell\text{-ét}}(\mathbf{Q}_\ell)$ in $\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathscr{G}_F)$ defined as the pushforward of $\pi_1(\mathbf{P}^1_{01\infty,\overline{F}}; \overrightarrow{01}, z)_{\mathbf{Q}_\ell}(\mathbf{Q}_\ell)$ by $\mathbf{pr}_m$. Therefore, the conclusion follows from the definition of the ℓ-adic étale unipotent Albanese map. ∎

Recall that the weight filtration $W_{-2n}\mathscr{G}_F$ on $\mathscr{G}_F$ is the filtration induced by the weight filtration on $U(\mathcal{MT}_\ell(F))$ (*cf.* [12, Section 7.4]). According to [12, Proposition 7.1, Lemma 7.5], we have canonical isomorphisms

$$\mathrm{gr}^W_{-2m}\mathscr{G}_F \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \xrightarrow{\sim} \left(\mathrm{gr}^W_{-2m} U(\mathcal{MT}_\ell(F))\right)(\mathbf{Q}_\ell) \cong \mathrm{Lie}(\mathcal{MT}_\ell(F))^{(-2m)}$$

of $\mathbf{Q}_\ell$-vector spaces.

Proposition 4.11 *Let γ be a $\mathbf{Q}_\ell$-path from $\overrightarrow{01}$ to z in $\mathbf{P}^1_{01\infty,\overline{F}}$. The composite of homomorphisms*

$$W_{-2m}\mathscr{G}_F \twoheadrightarrow \mathrm{gr}^W_{-2m}\mathscr{G}_F \hookrightarrow \mathrm{Lie}(\mathcal{MT}_\ell(F))^{(-2m)} \xrightarrow{\mathscr{L}_m^{\ell\text{-ét}}(z)} \mathbf{Q}_\ell$$

coincides with the restriction of $(-1)^{m-1}\ell i_m(z,\gamma)$ to $W_{-2m}\mathscr{G}_F$. In particular, that restriction does not depend on the choice of γ.

Proof Let $c_\gamma : \mathscr{G}_F \to \mathscr{P}_m(\mathbf{Q}_\ell)$ be the continuous 1-cocycle defined by γ (*cf.* Lemma 4.10). Then there exists a unique rational 1-cocycle $\widetilde{c}_\gamma : \pi_1(\mathcal{MT}_\ell(F), \omega) \to \omega(\mathscr{P}_m^{\ell\text{-ét}})$ which make the following diagram commute:

$$\begin{array}{ccc} \mathscr{G}_F & \xrightarrow{\quad\rho\quad} & \pi_1(\mathcal{MT}_\ell(F), \omega)(\mathbf{Q}_\ell) \\ \downarrow{\scriptstyle c_\gamma} & & \downarrow{\scriptstyle \widetilde{c}_\gamma} \\ \mathscr{P}_m^{\ell\text{-ét}}(\mathbf{Q}_\ell) & \xrightarrow{\quad\alpha\quad} & \omega(\mathscr{P}_m^{\ell\text{-ét}})(\mathbf{Q}_\ell), \end{array}$$

where $\alpha : \mathscr{P}_m^{\ell\text{-ét}} \xrightarrow{\sim} \omega(\mathscr{P}_m^{\ell\text{-ét}})$ is an isomorphism of algebraic groups over $\mathbf{Q}_\ell$ induced by the fixed $\mathbf{Q}_\ell$-basis ζ_{ℓ^∞} of $\mathbf{Q}_\ell(1)$. Then, by Lemma 4.10 and equation (2.2), we have

$$\mathrm{gr}^W_{-2m}(\widetilde{c}_\gamma) = \mathscr{L}_m^{\ell\text{-ét}}(z)$$

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under the natural isomorphism

$$\big(\mathrm{gr}^W_{-2m}\, U(\mathcal{MT}_\ell(F))\big)(\mathbf{Q}_\ell) \cong \mathrm{Lie}(\mathcal{MT}_\ell(F))^{(-2m)}.$$

Therefore, we have the following commutative diagram:

$$\begin{array}{ccccc} W_{-2m}\mathscr{G}_F & \xrightarrow{\quad\rho\quad} & W_{-2m}U(\mathcal{MT}_\ell(F))(\mathbf{Q}_\ell) & \longrightarrow & \mathrm{Lie}(\mathcal{MT}_\ell(F))^{(-2m)} \\ \downarrow{\scriptstyle c_\gamma} & & \downarrow{\scriptstyle \widetilde{c}_\gamma} & & \downarrow{\scriptstyle \mathscr{L}^{\ell\text{-ét}}_m} \\ \mathbf{Q}_\ell(m) & \xrightarrow{\zeta_{\ell^\infty}^{\otimes m}\mapsto 1} & \omega(\mathbf{Q}_\ell(m)) & \xrightarrow{\quad = \quad} & \mathbf{Q}_\ell. \end{array}$$

On the other hand, by the definition of the ℓ-adic polylogarithms, we have the equality

$$c_\gamma(\sigma) = (-1)^{m-1}\ell i_m(z,\gamma)(\sigma)\zeta_{\ell^\infty}^{\otimes m}$$

for all $\sigma \in W_{-2m}\mathscr{G}_F$. Hence, we have the conclusion of the proposition. ■

Lemma 4.12 *The restriction map*

$$H^1(F, \mathbf{Z}_\ell(m)) \longrightarrow H^1\big(W_{-2m}\mathscr{G}_F, \mathbf{Z}_\ell(m)\big)$$

is injective for any positive integer m.

Proof By the Hochschild–Serre spectral sequence, it is sufficient to show the vanishing of the continuous group cohomology $H^1(\mathscr{G}_F/W_{-2m}\mathscr{G}_F, \mathbf{Z}_\ell(m))$. By using the Hochschild–Serre spectral sequence again, we have an exact sequence

$$(4.3)\quad 0 \longrightarrow H^1(\mathrm{Gal}(F(\mu_{\ell^\infty})/F), \mathbf{Z}_\ell(m)) \longrightarrow H^1(\mathscr{G}_F/W_{-2m}\mathscr{G}_F, \mathbf{Z}_\ell(m)) \longrightarrow \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Gal}(F(\mu_{\ell^\infty})/F)}\big(W_{-2}\mathscr{G}_F/W_{-2m}\mathscr{G}_F, \mathbf{Z}_\ell(m)\big).$$

Since m is non-zero, the first term of (4.3) vanishes. On the other hand, $\mathscr{G}_F$ acts on the graded quotient $\mathrm{gr}^W_{-2n}\mathscr{G}_F$ via the n-th power of the ℓ-adic cyclotomic character by the definition of the weight filtration of $\mathscr{G}_F$. Hence, there exists no non-trivial $\mathscr{G}_F$-equivariant homomorphism from $W_{-2}\mathscr{G}_F/W_{-2m}\mathscr{G}_F$ to $\mathbf{Z}_\ell(m)$. Thus, the last term of (4.3) also vanishes, and we have the conclusion of the lemma. ■

Proposition 4.13 (*cf.* [7, Theorem 2.3]) *A cohomology class $x \in H^1(F, \mathbf{Q}_\ell(m))$ is represented by a linear sum*

$$\sum_i a_i \ell i_m(z_i, \gamma_i) \colon \mathscr{G}_F \longrightarrow \mathbf{Q}_\ell(m)$$

with $a_i \in \mathbf{Q}$, $z_i \in F$, and $\gamma_i\colon \overrightarrow{01} \rightsquigarrow z_i$ if and only if x is contained in the image of $B^{\ell\text{-ét}}_m(F) \otimes_{\mathbf{Z}} \mathbf{Q}$ under $\mathscr{L}^{\ell\text{-ét}}_m$. Moreover, for each $\xi = \sum_i a_i\{z_i\} \in A^{\ell\text{-ét}}_m(F)$, we have

$$\sum_i a_i \ell i_m(z_i,\gamma_i) = (-1)^{m-1}\mathscr{L}^{\ell\text{-ét}}_m(\xi) \quad \text{in } H^1\big(F, \mathbf{Q}_\ell(m)\big).$$

Proof We first show the "if" part. Suppose that $\xi \in A^{\ell\text{-ét}}_m(F)$. The existence of good paths γ_i follows by repeating exactly the same argument of the proof of [7, Theorem 2.3] by replacing the conjectural vector space $\mathcal{L}_k$ in that paper by $(\mathbf{Z}[\mathbf{P}^1_{01\infty}(F)]/R^{\ell\text{-ét}}_k(F)) \otimes_{\mathbf{Z}} \mathbf{Q}$. Indeed, the latter vector space satisfies all the conditions that $\mathcal{L}_k$ is expected to satisfy except the realization homomorphism $B_k^{\ell\text{-ét}}(F) \to K_{2k-1}(F) \otimes_{\mathbf{Z}} \mathbf{Q}$, and it was not needed for the proof.

Next, we show the "only if" part. Let $c: \mathscr{G}_F \to \mathbf{Q}_\ell(m)$ be a 1-cocycle of the form $\sum_i a_i \ell i_m(z_i, \gamma_i)$. We put $\xi := \sum_i a_i \{z_i\}$. Then, according to Proposition 4.11, the restriction of c to $W_{-2m}\mathscr{G}_F$ coincides with the composite of $(-1)^{m-1}\mathscr{L}_m^{\ell\text{-ét}}(\xi)$ and $W_{-2m}\mathscr{G}_F \to \mathrm{Lie}(\mathcal{MT}_\ell(F))^{(-2m)}$. Since the restriction c to $\mathscr{G}_{F(\mu_{\ell^\infty})}$ factors through the abelianization of $\mathscr{G}_{F(\mu_{\ell^\infty})}$, $\mathscr{L}_m^{\ell\text{-ét}}(\xi)$ also factors through the abelianization homomorphism

$$\mathrm{Lie}(\mathcal{MT}_\ell(F))^{(-2m)} \subset \mathrm{Lie}(\mathcal{MT}_\ell(F)) \twoheadrightarrow \mathrm{Lie}(\mathcal{MT}_\ell(F))^{\mathrm{ab}}.$$

This implies that $d\mathscr{L}_m^{\ell\text{-ét}}(\xi) = 0$ and that $\xi \in A_m^{\ell\text{-ét}}(F) \otimes_{\mathbf{Z}} \mathbf{Q}$ (*cf.* Remark 2.12). Since the restriction of the cohomology class $(-1)^{m-1}\mathscr{L}_m^{\ell\text{-ét}}(\xi) \in H^1(F, \mathbf{Q}_\ell(m))$ to $W_{-2m}\mathscr{G}_F$ coincides with the restriction of the class of c, $(-1)^{m-1}\mathscr{L}_m^{\ell\text{-ét}}(\xi)$ agree with $[c]$ in $H^1(F, \mathbf{Q}_\ell(m)))$ by Lemma 4.12. ∎

4.4 Comparison with Generalized Soulé Character

In this subsection, we compare modified ℓ-adic étale polylogarithms with generalized Soulé characters. Here, we suppose that F is a subfield of $\mathbf{C}$. Let γ be an ℓ-adic path from $\overrightarrow{01}$ to $z \in \mathbf{P}^1_{01,\infty}(F)$. Recall that the *generalized Soulé character*

$$\widetilde{\chi}_m^{z,\gamma} : \mathscr{G}_F \to \mathbf{Z}_\ell(m)$$

is defined by the equations

$$\widetilde{\chi}_m^{z,\gamma}(\sigma) \mod \ell^n = \frac{\sigma\Big(\prod_{1 \le a \le \ell^n} (1 - \zeta_{\ell^n}^{\chi_\ell^{-1}(\sigma) a} z^{1/\ell^n})^{\frac{a^{m-1}}{\ell^n}}\Big)}{\Big(\prod_{1 \le a \le \ell^n} (1 - \zeta_{\ell^n}^{a} z^{1/\ell^n})^{\frac{a^{m-1}}{\ell^n}}\Big)} \otimes \zeta_{\ell^n}^{\otimes(m-1)} \tag{4.4}$$

in $\mathbf{Z}/\ell^n\mathbf{Z}(m)$ for all n, where z^{1/ℓ^n} is the ℓ^n-th root of z defined by γ. If we restrict σ to $\mathscr{G}_{F(\mu_{\ell^\infty}, z^{1/\ell^\infty})}$, then equation (4.4) coincides with equation (1.2). Similarly, we define the continuous map $\kappa_{z,\gamma} : \mathscr{G}_F \to \mathbf{Z}_\ell(1)$ by the equations

$$\kappa_{z,\gamma}(\sigma) \mod \ell^n = \sigma(z^{1/\ell^n}) z^{-1/\ell^n}$$

in $\mathbf{Z}/\ell^n\mathbf{Z}(1)$ for all n. We call $\kappa_{z,\gamma}$ the *Kummer cahracter attached to γ*. The key formula of our comparison is Nakamura–Wojtkowiak's formula of ℓ-adic polylogarithms.

Theorem 4.14 ([20, §3 Corollary]) *Let $\kappa_{z,\gamma} : \mathscr{G}_F \to \mathbf{Z}_\ell(1)$ be the Kummer character attached to γ. Then we have the following equality of functions on $\mathscr{G}_F$ valued in $\mathbf{Q}_\ell(m)$:*

$$\ell i_m(z, \gamma) = (-1)^{m-1} \Big(\sum_{j=0}^{m-1} \frac{B_j}{j!} (-\kappa_{\gamma,z})^j \frac{\widetilde{\chi}_{m-j}^{z,\gamma}}{(m-k-1)!} \Big).$$

In particular, on $\mathscr{G}_{K(\mu_{\ell^\infty}, z^{1/\ell^\infty})}$, we have

$$\ell i_m(z, \gamma) = \frac{(-1)^{m-1}}{(m-1)!} \widetilde{\chi}_m^z.$$

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Corollary 4.15 *Let* $\xi = \sum_i a_i\{z_i\}$ *be an element of* $A_m^{\ell\text{-ét}}(F)$ *and let* $L := F(\{\mu_{\ell^\infty}, z_i^{1/\ell^\infty}\}_i)$. *Then the group homomorphism*

$$\widetilde{\chi}_m^{\xi} := \sum_i a_i \widetilde{\chi}_m^{z_i} : \mathcal{G}_L \longrightarrow \mathbf{Z}_\ell(m)$$

extends to a 1*-cocycle on* $\mathcal{G}_F$. *Moreover, the cohomology class of this extension is unique and coincides with* $(m-1)!\mathcal{L}_m^{\ell\text{-ét}}(\xi)$.

Proof The first assertion is a direct consequence of Theorem 4.14 and Proposition 4.13. To show the uniqueness of the cohomology class, it is sufficient to show the restriction map

$$H^1(\mathcal{G}_F, \mathbf{Z}_\ell(m)) \longrightarrow H^1(\mathcal{G}_L, \mathbf{Z}_\ell(m)) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{G}_L, \mathbf{Z}_\ell(m))$$

is injective. This injectivity is a direct consequence of Lemma 4.12, because $\mathcal{G}_L$ contains $W_2\mathcal{G}_F$. ∎

5 Motivic Modified Polylogarithms

The final example of an abstract modified polylogarithm is the motivic modified polylogarithm. Now, we fix a number field K for the rest of this paper. We denote $K_n(K)\otimes_{\mathbf{Z}}\mathbf{Q}$ by $K_n(K)_{\mathbf{Q}}$, where $K_n(K)$ is the higher K-group of K. Let $\mathcal{MT}(K)$ be the category of mixed Tate motives over K (*cf.* [5, §1]). For all integers n and m, there exist canonical isomorphisms

(5.1)
$$\mathrm{Hom}_{\mathcal{MT}(K)}(\mathbf{Q}(n), \mathbf{Q}(m)) \cong \begin{cases} 0 & n \neq m, \\ \mathbf{Q} & n = m, \end{cases}$$

(5.2)
$$\mathrm{Ext}^1_{\mathcal{MT}(K)}(\mathbf{Q}(n), \mathbf{Q}(m)) \cong \begin{cases} 0 & n \geq m, \\ K_{2(m-n)-1}(K)_{\mathbf{Q}} & n < m, \end{cases}$$
$$\mathrm{Ext}^2_{\mathcal{MT}(K)}(\mathbf{Q}(n), \mathbf{Q}(m)) = 0.$$

In particular, $\mathcal{MT}(K)$ is a mixed Tate category over $\mathbf{Q}$. We always identify the left-hand sides of (5.1) and (5.2) with the right-hand sides by those canonical isomorphisms.

Definition 5.1 Let m be a non-negative integer. Then we define $\mathcal{P}_m^{\mathrm{mot}}$, which is an algebraic group in $\mathcal{MT}(K)$, to be $\mathcal{P}_m^{\mathcal{MT}(K)} = \exp(\mathbf{Q}(1) \ltimes \oplus_{n=1}^m \mathbf{Q}(n))$.

It is known that the motivic polylogarithmic quotient is a quotient of the motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathbf{P}^1_{01\infty,K}, \overrightarrow{01})$ which is an affine group scheme in $\mathcal{MT}(K)$ (*cf.* [5]). More precisely, the projections u_m^{H} and $u_m^{\ell\text{-ét}}$ are realizations (see the next section) of a surjective homomorphism

$$u_m^{\mathrm{mot}} : \pi_1^{\mathrm{mot}}(\mathbf{P}^1_{01\infty,K}, \overrightarrow{01}) \twoheadrightarrow \mathcal{P}_m^{\mathrm{mot}}$$

of affine schemes in $\mathcal{MT}(K)$. Deligne and Goncharov also proved the existence of the path torsor $\pi_1^{\mathrm{mot}}(\mathbf{P}^1_{01\infty,K}; \overrightarrow{01}, z)$ from $\overrightarrow{01}$ to z, which is a right torsor under

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$\pi_1^{\mathrm{mot}}(\mathbf{P}^1_{01\infty,K}, \overrightarrow{01})$ in $\mathcal{MT}(K)$ for any K-rational base point z of $\mathbf{P}^1_{01\infty,K}$ (*cf.* loc. cit.). We put

$$\mathscr{P}_m^{\mathrm{mot}}(\overrightarrow{01}, z) := u_{m,*}^{\mathrm{mot}}\big(\pi_1^{mot}(\mathbf{P}^1_{01\infty,K}; \overrightarrow{01}, z)\big).$$

By definition, $\mathscr{P}_m^{\mathrm{mot}}(\overrightarrow{01}, z)$ is a right $\mathscr{P}_m^{\mathrm{mot}}$-torsor in $\mathcal{MT}(K)$. Then we define the motivic unipotent Albanese map as follows.

Definition 5.2 Let m be a non-negative integer. *The series of motivic unipotent Albanese maps* $\underline{\mathrm{Alb}}_K^{\mathrm{mot}} := \{\mathrm{Alb}_{K,m}^{\mathrm{mot}}\}$ is defined by

$$\mathrm{Alb}_{K,m}^{\mathrm{mot}}(z) := \mathscr{P}_m^{\mathrm{mot}}(\overrightarrow{01}, z) \text{ for all } z \in \mathbf{P}_{01\infty}(K).$$

The following lemma is a well-known fact.

Lemma 5.3 (*cf.* [5, Lemma 5.12]) *The collection of motivic unipotent Albanese maps is a series of abstract unipotent Albanese maps. Furthermore, the restriction of the* 0*-th motivic unipotent Albanese map*

$$\mathrm{Alb}_{K,0}^{\mathrm{mot}} \colon \mathbf{P}^1_{01\infty}(K) = K^\times \backslash \{1\} \longrightarrow K_1(K)_{\mathbf{Q}} = K^\times \otimes_{\mathbf{Z}} \mathbf{Q}$$

is the natural map induced by the identity of $K^\times \backslash \{1\}$.

Definition 5.4 We define the *m-th motivic modified polylogarithm*

$$\mathscr{L}_m^{\mathrm{mot}} \colon \mathbf{Z}[\mathbf{P}^1_{01\infty}(K)] \longrightarrow \mathrm{coLie}(\mathcal{MT}(K))^{(2m)}$$

to be $\mathscr{L}_m(\underline{\mathrm{Alb}}_K^{\mathrm{mot}})$.

We define $R_m^{\mathrm{mot}}(K)$, $A_m^{\mathrm{mot}}(K)$, and $B_m^{\mathrm{mot}}(K)$ by

$$R_m^{\mathrm{mot}}(K) := R_m(K, \underline{\mathrm{Alb}}_K^{\mathrm{mot}}), \quad A_m^{\mathrm{mot}}(K) := A_m(K, \underline{\mathrm{Alb}}_K^{\mathrm{mot}}),$$
$$B_m^{\mathrm{mot}}(K) := B_m(K, \underline{\mathrm{Alb}}_K^{\mathrm{mot}}),$$

respectively. Then the motivic modified polylogarithm induces an injective and well-defined homomorphism

$$\mathscr{L}_m^{\mathrm{mot}} \colon B_m^{\mathrm{mot}}(K) \hookrightarrow \mathrm{Ext}^1_{\mathcal{MT}(K)}(\mathbf{Q}(0), \mathbf{Q}(m)) = K_{2m-1}(K)_{\mathbf{Q}}.$$

6 Proof of Main Theorem

6.1 Mixed Realizations of Motives

In this subsection, we recall mixed realizations of mixed Tate motives (*cf.* [4,5,13–15]). A. Huber had constructed the category $D_{\mathcal{MR}}$, which contains the derived category of systems of realizations over a field k that can be embedded into $\mathbf{C}$ (*cf.* [13, Definition 11.1.3]). The category $D_{\mathcal{M}R}$ is equipped with the canonical functors

$$\mathbf{q}_\ell \colon D_{\mathcal{MR}} \longrightarrow D(\mathrm{Spec}(k)_{\text{ét}}, \mathbf{Q}_\ell) \quad \text{and} \quad \mathbf{q}_{\mathrm{H}}^{\sigma} \colon D_{\mathcal{MR}} \longrightarrow D(MHS(\mathbf{Q}))$$

for each rational prime ℓ and $\sigma \colon k \hookrightarrow \mathbf{C}$. Here, $D(\mathrm{Spec}(k)_{\text{ét}}, \mathbf{Q}_\ell)$ and $D(MHS(k))$ are the derived category of smooth $\mathbf{Q}_\ell$-sheaves on $\mathrm{Spec}(k)_{\text{ét}}$ and mixed Hodge structures over $\mathbf{Q}$, respectively. We also define $\mathbf{q}_{\mathrm{H}_{\mathbf{R}}^{\sigma}}$ to be the composite of $\mathbf{q}_{\mathrm{H}_{\mathbf{R}}^{\sigma}}$ and the natural

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functor $D(MHS(\mathbf{Q})) \to D(MHS(\mathbf{R}))$. In [14,15], she also had constructed a functor between triangulated categories

$$\mathrm{R}_{\mathcal{MR}}\colon DM_{gm}(k) \longrightarrow D_{\mathcal{M}R}$$

called the mixed realization functor for any field k that can be embedded into $\mathbf{C}$. Let $\sigma\colon k \hookrightarrow \mathbf{C}$ be an embedding. We denote by $\mathrm{R}_{\mathrm{H}}^{\sigma}$ (resp. R_{ℓ}) the composite of $\mathrm{R}_{\mathcal{M}R}$ and

$$D_{\mathcal{MR}} \xrightarrow{\mathbf{q}_{\mathrm{H}_{\mathbf{R}}^{\sigma}}} D(MHS(\mathbf{R})) \left(\text{resp. } D_{\mathcal{MR}} \xrightarrow{\mathbf{q}_{\ell}} D(\mathrm{Spec}(k)_{\text{ét}}, \mathbf{Q}_{\ell})\right).$$

Let ℓ be a rational prime and let $\sigma\colon K \hookrightarrow \mathbf{C}$ be a field embedding. By construction, we have $\mathrm{R}_{\ell}(\mathbf{Q}(m)) = \mathbf{Q}_{\ell}(m)$ and $\mathrm{R}_{\mathrm{H}}^{\sigma}(\mathbf{Q}(m)) = \mathbf{R}(m)$ for each integer m. Therefore, R_{ℓ} and $\mathrm{R}_{\mathrm{H}}^{\sigma}$ induce canonical functors

$$\mathrm{R}_{\ell}\colon \mathcal{MT}(K) \longrightarrow \mathcal{MT}_{\ell}(K) \quad \text{and} \quad \mathrm{R}_{\mathrm{H}}^{\sigma}\colon \mathcal{MT}(K) \longrightarrow \mathcal{H}_{\mathbf{R}},$$

respectively. Then we denote by

$$\mathfrak{r}_{\ell}\colon \mathrm{Ext}^1_{\mathcal{MT}(K)}(\mathbf{Q}(0), \mathbf{Q}(m)) \longrightarrow \mathrm{Ext}^1_{\mathcal{MT}_{\ell}(K)}\left(\mathbf{Q}_{\ell}, \mathbf{Q}_{\ell}(m)\right)$$

and

$$\mathfrak{r}_{\mathrm{H}}^{\sigma}\colon \mathrm{Ext}^1_{\mathcal{MT}(K)}(\mathbf{Q}(0), \mathbf{Q}(m)) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}_{\mathbf{R}}}(\mathbf{R}, \mathbf{R}(m))$$

the canonical homomorphisms induced by R_{ℓ} and $\mathrm{R}_{\mathrm{H}}^{\sigma}$, respectively.

Theorem 6.1 ([19, Proposition 5]) *Under the canonical identifications*

$$\mathrm{Ext}^1_{\mathcal{MT}(K)}\left(\mathbf{Q}(0), \mathbf{Q}(m)\right) = K_{2m-1}(K)_{\mathbf{Q}}$$

and

$$\mathrm{Ext}^1_{\mathcal{MT}_{\ell}(K)}\left(\mathbf{Q}_{\ell}, \mathbf{Q}_{\ell}(m)\right) = H^1\left(K, \mathbf{Q}_{\ell}(m)\right),$$

$\mathfrak{r}_{\ell}$ *coincides with the ℓ-adic higher regulator defined in [25]. In particular,* $\mathfrak{r}_{\ell}$ *is injective whose image is a* $\mathbf{Q}$*-lattice of* $H^1(K, \mathbf{Q}_{\ell}(m))$.

Theorem 6.2 (*cf.* [5, 1.6]) *Under the canonical identifications*

$$\mathrm{Ext}^1_{\mathcal{MT}(K)}(\mathbf{Q}(0), \mathbf{Q}(m)) = K_{2m-1}(K)_{\mathbf{Q}}$$

and

$$\mathrm{Ext}^1_{\mathcal{H}_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(m)) = \mathbf{R}(m-1),$$

$\mathfrak{r}_{\mathrm{H}}^{\sigma}$ *coincides with the Beilinson regulator. In particular,*

$$\mathfrak{r}_{\mathrm{H},K} := \bigoplus_{\sigma: K \hookrightarrow \mathbf{Q}} \mathfrak{r}_{\mathrm{H}}^{\sigma}\colon K_{2m-1}(K)_{\mathbf{Q}} \longrightarrow \left(\bigoplus_{\sigma: K \hookrightarrow \mathbf{C}} \mathbf{R}(m-1)\right)^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})}$$

is injective whose image is a $\mathbf{Q}$*-lattice of the right-hand side.*

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6.2 Proof of Main Theorem

Let z be a K-rational base point of $\mathbf{P}^1_{01\infty,K}$. Then the tuple

$$\mathscr{P}^*_m(\overrightarrow{01},z) := \left(\mathscr{P}^{\mathrm{H}}_m(\overrightarrow{01},\sigma(z)),\mathscr{P}^{\ell\text{-ét}}_m(\overrightarrow{01},z)\right)_{\sigma:K\hookrightarrow\mathbf{C},\ \ell:\text{prime numbers}}$$

is a part of a scheme in the category of systems of realizations (*cf.* [5, 2.15]). Deligne and Goncharov proved the following theorem.

Theorem 6.3 ([5, Théorème 4.4]) *Let z be a K-rational base point of $\mathbf{P}^1_{01\infty,K}$. Then the system of realizations $\mathscr{P}^*_m(\overrightarrow{01},z)$ is motivic. More precisely, the Hodge realization attached to $\sigma\colon K\hookrightarrow\mathbf{C}$ (resp. ℓ-adic étale realization) of $\mathscr{P}^{\mathrm{mot}}_m(\overrightarrow{01},z)$ is canonically isomorphic to the torsor $\mathscr{P}^{\mathrm{H}}_m(\overrightarrow{01},\sigma(z))$ (resp. $\mathscr{P}^{\ell\text{-ét}}_m(\overrightarrow{01},z)$).*

We denote by

$$\mathrm{R}^{\sigma}_{\mathrm{H},*}\colon \mathrm{coLie}(\mathcal{MT}(K)) \longrightarrow \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})$$

and by

$$\mathrm{R}_{\ell,*}\colon \mathrm{coLie}(\mathcal{MT}(K)) \longrightarrow \mathrm{coLie}(\mathcal{MT}_{\ell}(K))$$

the canonical coLie homomorphisms induced by $\mathrm{R}^{\sigma}_{\mathrm{H}}$ and R_{ℓ}, respectively. Remark that the restrictions of $\mathrm{R}^{\sigma}_{\mathrm{H},*}$ and $\mathrm{R}_{\ell,*}$ to $K_{2m-1}(K)_{\mathbf{Q}} \subset \mathrm{coLie}(\mathcal{MT}(K))^{(2m)}$ coincide with $\mathfrak{r}^{\sigma}_{\mathrm{H}}$ and $\mathfrak{r}_{\ell}$, respectively.

Proposition 6.4 *Let K be a number field and let m be a positive integer. Let ℓ be a rational prime and let σ be an embedding $K\hookrightarrow\mathbf{C}$. Then the following diagram commutes:*

$$\begin{array}{ccc}
 & & \mathrm{coLie}(\mathcal{H}_{\mathbf{R}})^{(2m)} \\
 & \nearrow^{\mathscr{L}^{\mathcal{H}_{\mathbf{R}}}_m\circ\sigma} & \uparrow{\scriptstyle \mathrm{R}^{\sigma}_{\mathrm{H},*}} \\
\mathbf{Z}[\mathbf{P}^1_{01\infty}(K)] & \xrightarrow{\ \mathscr{L}^{\mathrm{mot}}_m\ } & \mathrm{coLie}(\mathcal{MT}(K))^{(2m)} \\
 & \searrow_{\mathscr{L}^{\ell\text{-ét}}_m} & \downarrow{\scriptstyle \mathrm{R}_{\ell,*}} \\
 & & \mathrm{coLie}(\mathcal{MT}_{\ell}(K))^{(2m)}.
\end{array}$$

Proof The commutativity of the diagram is a direct consequence of Theorem 6.3 and constructions of modified polylogarithms. ∎

Corollary 6.5 *Let K be a number field and let ℓ be a rational prime. Then, for each positive integer m greater than 1, we have equalities*

$$R^{\mathrm{mot}}_m(K) = R^{\ell\text{-ét}}_m(K) = R^{\mathrm{H}}_m(K).$$

Proof Let $\bullet$ be one of the symbols H, ℓ-ét, and mot. Since $R^{\bullet}_m(K) \subset A^{\bullet}_m(K)$ and $A^{\bullet}_m(K)$ is mapped to the first extension group by $\mathscr{L}^{\bullet}_m$, it is sufficient to show the injectivity of the restrictions of $\mathrm{R}_{\ell,*}$ and $\oplus_{\sigma}\mathrm{R}^{\sigma}_{\mathrm{H},*}$ to $K_{2m-1}(K)_{\mathbf{Q}}$. Theorems 6.1 and 6.2 guarantee those restrictions to be injective. ∎

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Proof of the Main Theorem According to Propositions 3.10, and 6.4, and Theorem 6.2, we have the equality

$$c_m(\xi) = -\mathscr{L}_m^{\mathrm{mot}}(\xi) \quad \text{in } K_{2m-1}(K) \otimes_{\mathbf{Z}} \mathbf{Q},$$

where $c_m(\xi)$ is the generalized Beilinson element defined by de Jeu (*cf.* the introduction). Therefore, according to Proposition 6.4 and Theorem 6.1, we have

$$\mathrm{reg}_m^{\ell\text{-ét}}(c_m(\xi)) = -\mathscr{L}_m^{\ell\text{-ét}}(\xi) \quad \text{in } H^1(K, \mathbf{Q}_\ell(m)). \tag{6.1}$$

Finally, by combining Corollary 4.15 and equation (6.1), we have the assertion of Main Theorem. ∎

A Group Schemes in IFVec_k

In this appendix, we show useful lemmas for describing classifying spaces of torsors under algebraic groups in a mixed Tate category. Now, we fix a field k and denote by IFVec_k the category of finite dimensional k-vector spaces equipped with increasing, saturated, and separated filtrations. Denote by $(V, W_\bullet V)$ an object of IFVec_k, and we usually denote this object by V for simplicity. Though this category is not an abelian category, we can consider group schemes in IFVec_k as we did. In this appendix, we fix an algebraic group $G = \mathrm{Sp}(R)$ in IFVec_k satisfying the condition (Pos) (*cf.* Definition 2.2); namely, R is a finitely generated Hopf algebra object in $\mathrm{Ind}(\mathrm{IFVec}_k)$ satisfying $W_0 R = k$.

Lemma A.1 *Let V_1, V_2, and V_3 be objects in* $\mathrm{Ind}(\mathrm{IFVec}_k)$ *and let*

$$f\colon V_1 \longrightarrow V_2 \otimes V_3$$

be a morphism in $\mathrm{Ind}(\mathrm{IFVec}_k)$. *For each $i = 1, 2, 3$, suppose that $W_n V_i = 0$ for all negative integers n. Then, for any k-linear homomorphism $g\colon V_3 \to k$, the composite of k-linear homomorphisms*

$$(\mathrm{id}_{V_2} \otimes g) \circ f\colon V_1 \xrightarrow{f} V_2 \otimes V_3 \xrightarrow{\mathrm{id}_{V_2} \otimes g} V_2$$

is a morphism in $\mathrm{Ind}(\mathrm{IFVec}_k)$.

Proof To prove the lemma, it is sufficient to show that $(\mathrm{id}_{V_2} \otimes g) \circ f$ preserves the filtrations of both-hand sides. Since $W_n V_i$ vanish for all negative integers n, f sends $W_n V_1$ to $\sum_{j,l\geq 0,\ j+l=n} W_j V_2 \otimes_k W_l V_3$. Therefore, $(\mathrm{id}_{V_2} \otimes g) \circ f$ sends $W_n V_1$ to $\sum_{j,l\geq 0,\ j+l=n} W_j V_2$. Since $W_\bullet V_i$ is an increasing filtration, we have $\sum_{j,l\geq 0,\ j+l=n} W_j V_2 = W_n V_2$. This completes the proof of the lemma. ∎

The following corollaries are direct consequences of Lemma A.1.

Corollary A.2 *Let g be a k-rational point of the underlying algebraic group of G. Let $g^\sharp\colon R \to R$ be the k-algebra automorphism induced by the left multiplication by g. Then the restriction of $g^\sharp$ to $W_i R$ is an automorphism of $W_i R$ for each integer i.*

Corollary A.3 *Let $X = \mathrm{Sp}(R')$ be an affine scheme in* IFVec_k *equipped with a right action of G. Suppose that $W_n R' = 0$ for all negative integer n. Then, for each $x \in X(k)$, the k-algebra homomorphism $x^\sharp\colon R' \to R$ induced by $G \to X; g \mapsto xg$ is a morphism in* $\mathrm{Ind}(\mathrm{IFVec}_k)$

Example A.4 If $W_iR \neq 0$ for some negative integer, then there exists a counter example of Corollary A.2. Let G be the additive group $\mathbf{G}_{a,k} = \mathrm{Spec}(k[t^{-1}])$. Let us define the weight of t^{-1} to be -1. Then the weight filtration $W_\bullet k[t^{-1}]$ is written as

$$W_i k[t^{-1}] = \begin{cases} t^i k[t^{-1}] & \text{if } i < 0, \\ k[t^{-1}] & \text{if } i \geq 0. \end{cases}$$

By definition, G is an algebraic group in GrVec_k.

For each $g \in k = G(k)$, we have

$$g^\sharp \colon k[t^{-1}] \longrightarrow k[t^{-1}]; t^{-1} \longmapsto g + t^{-1}.$$

Therefore, if g is not 0, then $g^\sharp(W_i k[t^{-1}])$ is not contained in $W_i k[t^{-1}]$ for each negative integer i.

Lemma A.5 *Let $G = \mathrm{Sp}(R)$ be an algebraic group in IFVec_k satisfying (Pos). Let g be a k-rational point of G. Then the automorphism on $\mathrm{gr}_i^W R$ induced by $g^\sharp$ is the identity map.*

Proof We denote by $e^*\colon R \to k$ the counit of R. Since e^* is compatible with filtrations of both-hand sides, $\mathrm{Ker}(e^*)$ is an object in $\mathrm{Ind}(\mathrm{IFVec}_k)$ such that $W_0(\mathrm{Ker}(e^*)) = 0$. Let cm_R be the comultiplication of R and let x be an element of $W_iR \setminus W_0R$. We write

$$\mathrm{cm}_R(x) - 1 \otimes x = \sum_u a_u \otimes b_u \in W_0R \otimes_k W_iR + \sum_{j+l=i,\ j \geq 1,\ l \geq 0} W_j\mathrm{Ker}(e^*) \otimes_k W_lR$$

such that $\{b_u\}_u$ are linearly independent over k. Remark that $\mathrm{cm}_R(x) \neq 1 \otimes x$ because $x \notin k = W_0R$. Since $\mathrm{cm}_R(x) - 1 \otimes x$ is contained in the kernel of $e^* \otimes 1$, all a_u are contained in the kernel of e^*. This implies that all b_u are contained in $W_{i-1}R$. Hence, we have $g^\sharp(x) - x = \sum_i g(a_i)b_i \in W_{i-1}R$ and this implies that the induced k-linear homomorphism by $g^\sharp$ on $\mathrm{gr}_{2n}^W R$ is the identity map. ■

Corollary A.6 *Let $G = \mathrm{Sp}(R)$ be an algebraic group in IFVec_k satisfying (Pos). Then the underlying algebraic group of G is unipotent.*

Proof According to Lemma A.5, W_iR is an algebraic representation of the underlying k-algebraic group G. Since $R = \cup_i W_iR$ and G is of finite type, W_NR is a faithful representation of G for sufficiently large N. Moreover, $W_0R \subset W_1R \subset \cdots \subset W_NR$ is a flag of this representation and G acts on all the graded quotients $\mathrm{gr}_i^W R$ trivially. Hence, G is unipotent. ■

The following lemma follows from the proof of Corollary A.6 easily, so we omit the proof of this lemma.

Lemma A.7 *Let $G = \mathrm{Sp}(R)$ be an algebraic group in IFVec_k satisfying (Pos). Then, for each positive integer N, there exists a natural Lie homomorphism*

$$\iota_N \colon \mathrm{Lie}(G) \longrightarrow \mathrm{End}_k(W_NR); D \longmapsto \log(\exp(D)^\sharp|_{W_NR}).$$

Furthermore, ι_N is injective for sufficiently large N.

https://doi.org/10.4153/S0008414X20000073 Published online by Cambridge University Press

Acknowledgments The author expresses his gratitude to Prof. Hiroaki Nakamura for his encouragement, valuable discussion, and comments about historical remark. He thanks Shingo Sugiyama and Sho Ogaki for careful reading of this manuscript and for valuable comments.

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