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A Herman–Avila–Bochi formula for higher-dimensional pseudo-unitary and Hermitian-symplectic cocycles

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Abstract. A Herman–Avila–Bochi type formula is obtained for the average sum of the top d Lyapunov exponents over a one-parameter family of $\mathbb{G}$-cocycles, where $\mathbb{G}$ is the group that leaves a certain, non-degenerate Hermitian form of signature (c, d) invariant. The generic example of such a group is the pseudo-unitary group $\mathrm{U}(c, d)$ or, in the case $c = d$, the Hermitian-symplectic group $\mathrm{HSp}(2d)$ which naturally appears for cocycles related to Schrödinger operators. In the case $d = 1$, the formula for $\mathrm{HSp}(2d)$ cocycles reduces to the Herman–Avila–Bochi formula for $\mathrm{SL}(2, \mathbb{R})$ cocycles.

1. *Introduction*

A fundamental problem in the theory of dynamical systems is the explicit calculation of Lyapunov exponents. The Herman–Avila–Bochi formula for $\mathrm{SL}(2, \mathbb{R})$ cocycles is a remarkable formula giving an average of Lyapunov exponents over a family of $\mathrm{SL}(2, \mathbb{R})$ cocycles. It was first partly obtained as an inequality by Herman [**Her**] and later proved to be an equality by Avila and Bochi [**AB**]. More recently, a different proof was given in [**BDD**].

I will show that there is a Herman–Avila–Bochi type formula for $\mathbb{G}$-cocycles where $\mathbb{G}$ is a matrix group that leaves a non-degenerate Hermitian form $h(v, w) = v^*\mathcal{G}w$ invariant, i.e. $\mathcal{G}$ is an invertible, Hermitian matrix. The group $\mathrm{SL}(2, \mathbb{R})$ is a special case as it leaves the Hermitian form given by $G = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ invariant. If $\mathcal{G}$ has signature (c, d)† then one obtains a formula for the average sum of the first d Lyapunov exponents over a one-parameter family of cocycles. By conjugation one only has to consider the pseudo-unitary group $\mathrm{U}(c, d)$. In fact, the proof in this group is very simple and a main step is a very well-known Hilbert–Schmidt type decomposition for matrices within this group as shown in Theorem 2.1.

† This means that $\mathcal{G}$ has c positive and d negative eigenvalues.

Besides these groups an explicit form of the formula will also be stated for the Hermitian-symplectic† group HSp($2d$) leaving the Hermitian form given by $G \otimes \mathbf{1}_d$ invariant. Such cocycles appear naturally in the theory of random and quasi-periodic Schrödinger operators. In the case $d = 1$, the formula for HSp($2d$) cocycles reduces exactly to the Herman–Avila–Bochi formula for SL(2, $\mathbb{R}$) cocycles.

Recent papers show some interest in higher-dimensional cocycles and such quasi-periodic operators, e.g. [**AJS**, **DK1**, **DK2**, **HaP**, **S**]. The Herman–Avila–Bochi formula proved to be a useful tool for the theory of SL(2, $\mathbb{R}$) cocycles and the result presented may lead to some generalizations.

1.1. *Pseudo-unitary cocycles.* The Lorentz group or pseudo-unitary group of signature (c, d), denoted by $\mathrm{U}(c, d)$, is given by

$$\mathrm{U}(c, d) = \{\mathcal{T} \in \mathrm{Mat}(c + d, \mathbb{C}) : \mathcal{T}^* \mathcal{G}_{c,d} \mathcal{T} = \mathcal{G}_{c,d}\}, \quad \mathcal{G}_{c,d} = \begin{pmatrix} \mathbf{1}_c & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_d \end{pmatrix}, \tag{1.1}$$

i.e. it is the group of linear transformations that leave the standard Hermitian form of signature (c, d) invariant.

Let $(\mathbb{X}, \mathfrak{A}, \mu)$ be a probability space and let $f : \mathbb{X} \to \mathbb{X}$ be a measure-preserving map, i.e. for any $g \in L^1(d\mu)$ one has $\int_{\mathbb{X}} g(f(x))\, d\mu(x) = \int_{\mathbb{X}} g(x)\, d\mu(x)$.

Moreover, let $\mathcal{A} : \mathbb{X} \to \mathrm{U}(c, d)$ be a measurable map such that $x \mapsto \ln \|\mathcal{A}(x)\|$ is μ-integrable, i.e. $\ln \|\mathcal{A}(\cdot)\| \in L^1(d\mu)$. The set of such functions $\mathcal{A}$ will be denoted by $\mathcal{LI}(\mathbb{X}, \mathrm{U}(c, d))$ (for logarithmic integrable). This condition is sufficient for the existence of the Lyapunov exponents. Then the pair $(f, \mathcal{A})$ interpreted as map

$$(f, \mathcal{A}) : (x, v) \in \mathbb{X} \times \mathbb{C}^{c+d} \mapsto (f(x), \mathcal{A}(x)v) \tag{1.2}$$

defines a $\mathrm{U}(c, d)$-cocycle over $\mathbb{X}$. The iteration of this map gives $(f, \mathcal{A})^n = (f^n, \mathcal{A}_n)$ where

$$\mathcal{A}_n(x) = \mathcal{A}(f^{n-1}(x)) \cdots \mathcal{A}(f(x))\mathcal{A}(x). \tag{1.3}$$

The Lyapunov exponents are defined by

$$L_k(f, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{X}} \ln(\sigma_k(\mathcal{A}_n(x)))\, d\mu(x) \tag{1.4}$$

where $\sigma_k(\mathcal{A})$ denotes the kth singular value of $\mathcal{A}$. We also define

$$L^k(f, \mathcal{A}) := \sum_{j=1}^{k} L_j(f, \mathcal{A}) = L_1(f, \Lambda^k \mathcal{A}). \tag{1.5}$$

For the latter equation, $\Lambda^k \mathcal{A}$ denotes the function $x \mapsto \Lambda^k \mathcal{A}(x)$ where $\Lambda^k M$ denotes the anti-symmetric tensor power of a matrix M. More precisely, for $M \in \mathrm{Mat}(m, \mathbb{C})$, $\Lambda^k M$ is a linear operator acting on the anti-symmetric tensor product $\Lambda^k \mathbb{C}^m$ by $\Lambda^k M(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = M v_1 \wedge M v_2 \wedge \cdots \wedge M v_k$.

† This is different from the (complex-)symplectic group Sp($2d$, $\mathbb{C}$). Sp($2d$, $\mathbb{C}$) leaves the bilinear form given by $\mathcal{G} = G \otimes \mathbf{1}_d$ invariant, i.e. $\mathcal{T}^\top \mathcal{G} \mathcal{T} = \mathcal{G}$, instead of $\mathcal{T}^* \mathcal{G} \mathcal{T} = \mathcal{G}$.

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The existence of the Lyapunov exponents is guaranteed by Kingman's subadditive theorem; in fact, for μ almost every $x \in \mathbb{X}$ the function

$$L^k(f, \mathcal{A}, x) = \lim_{n\to\infty} \frac{1}{n} \ln \|\Lambda^k(\mathcal{A}_n(x))\| \tag{1.6}$$

exists and $L^k(f, \mathcal{A}) = \int_{\mathbb{X}} L^k(f, \mathcal{A}, x)\, d\mu(x)$.

I will consider averages of a one-parameter family by multiplying with certain unitary matrices. Therefore, for $\theta \in [0, 1]$ let us define

$$\mathcal{U}_\theta^{(c,d)} = \begin{pmatrix} e^{2\pi i\theta}\mathbf{1}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_d \end{pmatrix} \in \mathrm{U}(c, d). \tag{1.7}$$

Analoguously to the function $N(A)$ for $A \in \mathrm{SL}(2, \mathbb{R})$ introduced in [**AB**], for $\mathcal{T} \in \mathrm{GL}(m, \mathbb{C})$ and $\mathbb{N} \ni r \le m$ let us define the functions

$$N_r(\mathcal{T}) = \sum_{i=1}^{r} \ln\bigg(\frac{\sigma_i(\mathcal{T}) + (\sigma_i(\mathcal{T}))^{-1}}{2}\bigg). \tag{1.8}$$

Since $\sigma_i(\mathcal{T}) \ge 1$ for $i = 1, \ldots, d$ for $\mathcal{T} \in \mathrm{U}(c, d)$ (cf. Theorem 2.1) one has

$$\ln(\|\Lambda^d\mathcal{T}\|) - d\ln(2) \le N_d(\mathcal{T}) \le \ln(\|\Lambda^d\mathcal{T}\|). \tag{1.9}$$

One obtains the following analogue to the Herman–Avila–Bochi formula.

THEOREM 1.1. *Let $\mathcal{A} \in \mathcal{LI}(\mathbb{X}, \mathrm{U}(c, d))$. Then*

$$\int_0^1 L^d(f, \mathcal{U}_\theta^{(c,d)}\mathcal{A})\, d\theta = \int_0^1 N_d(\mathcal{A}(x))\, d\mu(x). \tag{1.10}$$

This will follow immediately from the definition of L^d, equation (1.9) and the following fact.

THEOREM 1.2. *Let $\mathcal{T}_1, \ldots, \mathcal{T}_n \in \mathrm{U}(c, d)$. Then*

$$\int_0^1 N_d(\mathcal{U}_\theta^{(c,d)}\mathcal{T}_1\mathcal{U}_\theta^{(c,d)}\mathcal{T}_2\cdots\mathcal{U}_\theta^{(c,d)}\mathcal{T}_n)\, d\theta = \sum_{j=1}^{n} N_d(\mathcal{T}_j) \tag{1.11}$$

and

$$\int_0^1 \ln(\rho(\Lambda^d(\mathcal{U}_\theta^{(c,d)}\mathcal{T}_1\mathcal{U}_\theta^{(c,d)}\mathcal{T}_2\cdots\mathcal{U}_\theta^{(c,d)}\mathcal{T}_n)))\, d\theta = \sum_{j=1}^{n} N_d(\mathcal{T}_j) \tag{1.12}$$

where $\rho(\cdot)$ denotes the spectral radius.

Remark. Let $\mathcal{T} \in \mathrm{U}(c, d)$, $c \ge k > d$; then $\sigma_k(\mathcal{T}) = 1$ for $c \ge k > d$. Hence, one actually obtains $N_k(\mathcal{T}) = N_d(\mathcal{T})$ and $L^k(f, \mathcal{A}) = L^d(f, \mathcal{A})$ for any k between d and c. Therefore, one could replace d by any k between d and c in (1.10) and (1.11).

For groups leaving a general non-degenerate Hermitian form invariant one immediately obtains the following corollary.

COROLLARY 1.3. *Let $\mathcal{G}$ be any invertible, $(c+d)\times(c+d)$ Hermitian matrix with signature (c, d) and $\mathbb{G}$ the group of matrices leaving the form $v^*\mathcal{G}w$ invariant, i.e. $\mathbb{G} = \{\mathcal{T} : \mathcal{T}^*\mathcal{G}\mathcal{T} = \mathcal{G}\}$. Then there exists an invertible matrix $\mathcal{B}$ such that for $\mathcal{A} \in \mathcal{LI}(\mathbb{X}, \mathbb{G})$,*

$$\int_0^1 L^d(f, \mathcal{B}^{-1}\mathcal{U}_\theta^{(c,d)}\mathcal{B}\,\mathcal{A})\,d\theta = \int_{\mathbb{X}} N_d(\mathcal{B}\,\mathcal{A}(x)\mathcal{B}^{-1})\,d\mu(x). \tag{1.13}$$

Proof. By diagonalizing $\mathcal{G}$ and contractions we find an invertible matrix $\mathcal{B}$ such that $\mathcal{G} = \mathcal{B}^*\mathcal{G}_{c,d}\mathcal{B}$ and thus $\mathcal{B}\mathbb{G}\mathcal{B}^{-1} = \mathrm{U}(c, d)$. As $\mathcal{B}\mathcal{A}\mathcal{B}^{-1} \in \mathcal{LI}(\mathbb{X}, \mathrm{U}(c, d))$, (1.13) follows. □

1.2. *Hermitian-symplectic cocycles.* The Hermitian-symplectic group $\mathrm{HSp}(2d)$ is given by

$$\mathrm{HSp}(2d) = \{\mathcal{T} \in \mathrm{Mat}(2d, \mathbb{C}) : \mathcal{T}^*\mathcal{J}\mathcal{T} = \mathcal{J}\} \quad \text{where } \mathcal{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_d \\ -\mathbf{1}_d & \mathbf{0} \end{pmatrix}, \tag{1.14}$$

hence it leaves the Hermitian form given by $\mathcal{G} = i\mathcal{J}$ invariant. As $i\mathcal{J}$ has signature (d, d), $\mathrm{HSp}(2d)$ is conjugated to $\mathrm{U}(d, d)$. The conjugation is actually unitary and given by the Cayley transform, i.e.

$$\mathcal{C}\,\mathrm{HSp}(2d)\,\mathcal{C}^* = \mathrm{U}(d, d) \quad \text{where } \mathcal{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1}_d & i\mathbf{1}_d \\ \mathbf{1}_d & -i\mathbf{1}_d \end{pmatrix} \in \mathrm{U}(2d). \tag{1.15}$$

Hermitian-symplectic cocycles appear naturally for Schrödinger operators on strips. More precisely, assume f to be invertible†, $T \in L^1(\mathbb{X}, \mathrm{GL}(d, \mathbb{C}))$, $V \in L^1(\mathbb{X}, \mathrm{Her}(d))$, where $\mathrm{Her}(d)$ denotes the set of Hermitian $d \times d$ matrices. Then we get the family of self-adjoint‡ Schrödinger operators

$$(H_x\Psi)_n = T(f^{n+1}(x))\Psi_{n+1} + V(f^n(x))\Psi_n + T^*(f^n(x))\Psi_{n-1} \tag{1.16}$$

on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^d \ni \Psi = (\Psi_n)_n$, $\Psi_n \in \mathbb{C}^d$. Solving $H_x\Psi = E\Psi$ leads to

$$\begin{pmatrix} T(f^{n+1}(x))\Psi_{n+1} \\ \Psi_n \end{pmatrix} = \mathcal{T}^E(f^n(x)) \begin{pmatrix} T(f^n(x))\Psi_n \\ \Psi_{n-1} \end{pmatrix} \tag{1.17}$$

where

$$\mathcal{T}^E(x) = \begin{pmatrix} (E\mathbf{1}_d - V(x))T^{-1}(x) & -T^*(x) \\ T^{-1}(x) & \mathbf{0} \end{pmatrix}. \tag{1.18}$$

It is not very hard to check that $\mathcal{T}^E(x) \in \mathrm{HSp}(2d)$. The behavior of the generalized eigenvectors (not necessarily in ℓ^2) is given by the cocycle $(f, \mathcal{T}^E)$ and hence these cocycles are strongly related to the spectral theory of H_x.

The role of the unitaries $\mathcal{U}_\theta^{(c,d)}$ is played by the following rotation matrices:

$$\mathcal{R}_\theta = e^{-i\pi\theta}\mathcal{C}^*\mathcal{U}_\theta^{(d,d)}\mathcal{C} = \begin{pmatrix} \cos(\pi\theta)\mathbf{1}_d & -\sin(\pi\theta)\mathbf{1}_d \\ \sin(\pi\theta)\mathbf{1}_d & \cos(\pi\theta)\mathbf{1}_d \end{pmatrix} \in \mathrm{HSp}(2d). \tag{1.19}$$

As $\mathcal{C}$ is unitary, $N_d(\mathcal{C}\mathcal{T}\mathcal{C}^*) = N_d(\mathcal{T})$ and Theorems 1.1 and 1.2 immediately imply the following result.

† If f is not invertible one can still define operators on the 'half strip' $\ell^2(\mathbb{N}) \otimes \mathbb{C}^d$.
‡ The operators are self-adjoint for μ-almost all x by the criterion in [**SB**].

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THEOREM 1.4. *Let $\mathcal{A} \in \mathcal{LI}(\mathbb{X}, \mathrm{HSp}(2d))$; then*

$$\int_0^1 L^d(f, \mathcal{R}_\theta \mathcal{A})\, d\theta = \int_{\mathbb{X}} N_d(\mathcal{A}(x))\, d\mu(x). \tag{1.20}$$

Let $\mathcal{T}_1, \ldots, \mathcal{T}_n \in \mathrm{HSp}(2d)$; then

$$\int_0^1 N_d(\mathcal{R}_\theta \mathcal{T}_1 \mathcal{R}_\theta \mathcal{T}_2 \cdots \mathcal{R}_\theta \mathcal{T}_n)\, d\theta = \sum_{j=1}^n N_d(\mathcal{T}_j) \tag{1.21}$$

and

$$\int_0^1 \ln(\rho(\Lambda^d(\mathcal{R}_\theta \mathcal{T}_1 \mathcal{R}_\theta \mathcal{T}_2 \cdots \mathcal{R}_\theta \mathcal{T}_n)))\, d\theta = \sum_{j=1}^n N_d(\mathcal{T}_j). \tag{1.22}$$

Remark. The case $d = 1$ corresponds exactly to the Herman–Avila–Bochi formula for $\mathrm{SL}(2, \mathbb{R})$ cocycles as $\mathrm{HSp}(2) = \{e^{i\varphi} A : \varphi \in \mathbb{R},\ A \in \mathrm{SL}(2, \mathbb{R})\} = S^1 \cdot \mathrm{SL}(2, \mathbb{R})$.

2. *Structure of pseudo-unitary matrices*

The following decomposition is most crucial.

THEOREM 2.1. *For $\mathcal{T} \in \mathrm{U}(c, d)$, $c \geq d$, there exist unitary $c \times c$ matrices $U_1, V_1 \in \mathrm{U}(c)$ and unitary $d \times d$ matrices $U_2, V_2 \in \mathrm{U}(d)$ and a real diagonal $d \times d$ matrix $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_d) > \mathbf{0}$ with $\gamma_i \geq \gamma_{i+1}$ such that*

$$\mathcal{T} = \begin{pmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{pmatrix} \begin{pmatrix} \cosh(\Gamma) & \mathbf{0} & \sinh(\Gamma) \\ \mathbf{0} & \mathbf{1}_{c-d} & \mathbf{0} \\ \sinh(\Gamma) & \mathbf{0} & \cosh(\Gamma) \end{pmatrix} \begin{pmatrix} V_1 & \mathbf{0} \\ \mathbf{0} & V_2 \end{pmatrix}. \tag{2.1}$$

In particular, $\sigma_i(\mathcal{T}) = e^{\gamma_i}$, $\sigma_{d+c+1-i}(\mathcal{T}) = e^{-\gamma_i} = (\sigma_i(\mathcal{T}))^{-1}$, for $i = 1, \ldots, d$, and $\sigma_i(\mathcal{T}) = 1$, for $d < i \leq c$, are the $d + c$ singular values of $\mathcal{T}$. Thus, the matrix Γ is uniquely determined. If $\mathcal{T} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A \in \mathrm{Mat}(c, \mathbb{C})$, $D \in \mathrm{Mat}(d, \mathbb{C})$, then one finds $D = U_2 \cosh(\Gamma) V_2$ implying

$$\sigma_i(D) = \cosh(\gamma_i) = \tfrac{1}{2}(\sigma_i(\mathcal{T}) + (\sigma_i(\mathcal{T}))^{-1}) \quad \text{for } i = 1, \ldots, d \tag{2.2}$$

and

$$|\det(D)| = \det(\cosh(\Gamma)) = \exp(N_d(\mathcal{T})). \tag{2.3}$$

Proof. Let $\mathcal{T} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ written in blocks of size c and d. Then $\mathcal{T} \in \mathrm{U}(c, d)$ implies that

$$B^* A = D^* C, \quad D^* D - B^* B = \mathbf{1}_d, \quad A A^* - B B^* = \mathbf{1}_c. \tag{2.4}$$

For the last equation, note that $\mathcal{T}^* \in \mathrm{U}(c, d)$ as well, hence $\mathcal{T} \mathcal{G}_{c,d} \mathcal{T}^* = \mathcal{G}_{c,d}$†. The Hilbert–Schmidt or singular value decomposition of $B \in \mathrm{Mat}(c \times d, \mathbb{C})$ gives

$$B = U_1 \begin{pmatrix} \sinh(\Gamma) \\ \mathbf{0} \end{pmatrix} V_2 \tag{2.5}$$

† Indeed, as $\mathcal{G}_{c,d} \mathcal{T}^* \mathcal{G}_{c,d} \mathcal{T} = \mathcal{G}_{c,d}^2 = \mathbf{1}$, one finds $\mathcal{T}^{-1} = \mathcal{G}_{c,d} \mathcal{T}^* \mathcal{G}_{c,d}$ and $\mathcal{T} \mathcal{G}_{c,d} \mathcal{T}^* = \mathcal{G}_{c,d}^{-1} = \mathcal{G}_{c,d}$, giving $\mathcal{T}^* \in \mathrm{U}(c, d)$.

for some unitaries $U_1 \in \mathrm{U}(c)$, $V_2 \in \mathrm{U}(d)$ and a diagonal $d \times d$ matrix Γ as described above. By the second equation in (2.4) it follows that $D^*D = V_2^*(\mathbf{1} + \sinh^2(\Gamma))V_2 = V_2^* \cosh^2(\Gamma)V_2$. Hence, defining U_2 by

$$D = U_2 \cosh(\Gamma) V_2, \tag{2.6}$$

one sees that $U_2 \in \mathrm{U}(d)$. Similarly, defining $V_1 \in \mathrm{Mat}(c, \mathbb{C})$ by

$$A = U_1 \begin{pmatrix} \cosh(\Gamma) & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{c-d} \end{pmatrix} V_1 \tag{2.7}$$

and using the third equation in (2.4), one also sees that $V_1 V_1^* = \mathbf{1}_c$, thus $V_1 \in \mathrm{U}(c)$. Finally, using the first equation in (2.4), one obtains

$$C = D^{*-1} B^* A = U_2 \begin{pmatrix} \sinh(\Gamma) & \mathbf{0} \end{pmatrix} V_1. \tag{2.8}$$

By (2.5)–(2.8), equation (2.1) follows. □

Note that, for the special case $c = d$, $\mathcal{T} \in \mathrm{U}(d, d)$, this theorem yields

$$\mathcal{T} = \begin{pmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{pmatrix} \begin{pmatrix} \cosh(\Gamma) & \sinh(\Gamma) \\ \sinh(\Gamma) & \cosh(\Gamma) \end{pmatrix} \begin{pmatrix} V_1 & \mathbf{0} \\ \mathbf{0} & V_2 \end{pmatrix}.$$

Next we want to consider the actions on d-dimensional subspaces, therefore let $G(d, c+d)$ be the Grassmannian of d-dimensional subspaces of $\mathbb{C}^{c+d}$. Such a subspace $\mathbb{V} \in G(d, c+d)$ is represented by a $(c+d) \times d$ matrix Φ of full rank d, where the d column vectors of Φ span $\mathbb{V}$. Two such matrices are equivalent, $\Phi_1 \sim \Phi_2$, in the sense that they span the same subspace, if $\Phi_1 = \Phi_2 S$ for $S \in \mathrm{GL}(d, \mathbb{C})$. We denote an equivalence class by $[\Phi]_\sim \in G(d, c+d)$. Hence, denoting the $(c+d) \times d$ matrices of full rank by $\mathrm{GL}((c+d) \times d, \mathbb{C})$, the Grassmannian $G(d, c+d)$ can be seen as the quotient $\mathrm{GL}((c+d) \times d, \mathbb{C}) \,/\, \mathrm{GL}(d, \mathbb{C})$ and therefore has complex dimension $\dim_{\mathbb{C}} G(d, c+d) = (c+d)d - d^2 = cd$. In fact, $G(d, c+d)$ can also be considered as a quotient of Lie groups and defines a holomorphic manifold (cf. [**AJS**, Appendix A]). The group $\mathrm{GL}(c+d, \mathbb{C})$ and hence in particular the group $\mathrm{U}(c, d)$ acts on $G(d, c+d)$ by $\mathcal{T}[\Phi]_\sim := [\mathcal{T}\Phi]_\sim$.

Definition 2.2. Let us define the following subset of $G(d, c+d)$:

$$\mathbb{S} = \left\{ \left[\begin{pmatrix} M \\ \mathbf{1}_d \end{pmatrix} \right]_\sim : M \in \mathrm{Mat}(c \times d, \mathbb{C}),\ M^*M < \mathbf{1}_d \right\} \subset G(d, c+d). \tag{2.9}$$

This set is the image of the classical domain

$$R_I(c, d) = \{M \in \mathrm{Mat}(c \times d, \mathbb{C}) : M^*M < \mathbf{1}_d\} \tag{2.10}$$

under the holomorphic injection

$$\varphi : \mathrm{Mat}(c \times d, \mathbb{C}) \to G(d, c+d), \quad \varphi(M) = \left[\begin{pmatrix} M \\ \mathbf{1}_d \end{pmatrix} \right]_\sim. \tag{2.11}$$

This map can be viewed as a holomorphic chart for $G(d, c+d)$.

PROPOSITION 2.3. *The action of* $\mathrm{U}(c, d)$ *leaves* $\mathbb{S}$ *invariant, i.e.* $\mathcal{T}[\Phi]_\sim \in \mathbb{S}$ *for* $\mathcal{T} \in \mathrm{U}(c, d)$, $[\Phi]_\sim \in \mathbb{S}$.

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Proof. Let $M \in R_I(c, d)$, $\mathcal{T} \in \mathrm{U}(c, d)$ and let

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathcal{T} \begin{pmatrix} M \\ \mathbf{1}_d \end{pmatrix} \qquad X \in \mathrm{Mat}(c \times d, \mathbb{C}),\ Y \in \mathrm{Mat}(d, \mathbb{C}). \tag{2.12}$$

Then

$$\begin{pmatrix} X \\ Y \end{pmatrix}^* \mathcal{G}_{c,d} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} M \\ \mathbf{1}_d \end{pmatrix}^* \mathcal{T}^* \mathcal{G}_{c,d} \mathcal{T} \begin{pmatrix} M \\ \mathbf{1}_d \end{pmatrix} = \begin{pmatrix} M \\ \mathbf{1}_d \end{pmatrix}^* \mathcal{G}_{c,d} \begin{pmatrix} M \\ \mathbf{1}_d \end{pmatrix}. \tag{2.13}$$

Hence, $X^*X - Y^*Y = M^*M - \mathbf{1}_d$ and $Y^*Y = X^*X + \mathbf{1}_d - M^*M > X^*X$ is invertible and

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} XY^{-1} \\ \mathbf{1}_d \end{pmatrix}, \qquad (XY^{-1})^*(XY^{-1}) = \mathbf{1}_d - Y^{-1*}(\mathbf{1}_d - M^*M)Y^{-1} < \mathbf{1}_d, \tag{2.14}$$

which finishes the proof. □

As a final note, let us make the following remark.

Remark. The action of $\mathrm{U}(c, d)$ on $\mathbb{S}$ corresponds to the Möbius action on the classical domain $R_I(c, d)$. Hence, for $M \in \mathrm{Mat}(c \times d, \mathbb{C})$ with $M^*M < \mathbf{1}_d$ and $\mathcal{T} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, define

$$\mathcal{T} \cdot M = (AM + B)(CM + D)^{-1}; \tag{2.15}$$

then

$$\mathcal{T}\varphi(M) = \varphi(\mathcal{T} \cdot M). \tag{2.16}$$

By the calculation in the proof of Proposition 2.3, the inverse in the Möbius action exists for $M \in R_I(c, d)$. In fact, the group $\mathrm{U}(c, d)$ represents exactly the biholomorphic maps on $R_I(c, d)$.

3. *Analytic dependence of invariant subspaces*

In this section we will finally prove the main theorems. They will follow from the mean value property of harmonic functions. We may assume, without loss of generality, that $c \geq d$ as $\mathrm{U}(c, d)$ and $\mathrm{U}(d, c)$ are groups conjugated to each other, $N_d(\mathcal{T}) = N_c(\mathcal{T})$ for $\mathcal{T} \in \mathrm{U}(c, d)$, and the conjugation maps $\mathcal{U}_\theta^{(c,d)}$ to $e^{2\pi i\theta}\mathcal{U}_{-\theta}^{(d,c)}$.

For $z \in \mathbb{C}$ let us define

$$\mathcal{B}(z) = \begin{pmatrix} z\mathbf{1}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_d \end{pmatrix}. \tag{3.1}$$

We will consider the cocycles $\mathcal{B}(z)\mathcal{A}$ for $|z| < 1$ and denote the unit disk by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. It is obvious that for $z \in \mathbb{D}$, $\mathcal{B}(z)$ maps $\overline{\mathbb{S}}$ into $\mathbb{S}$, i.e. $\mathcal{B}(z)\overline{\mathbb{S}} \subset \mathbb{S}$. Now let $\mathcal{T}_1, \ldots, \mathcal{T}_n \in \mathrm{U}(c, d)$ and let

$$\mathcal{D}(z) = \mathcal{B}(z)\mathcal{T}_1\mathcal{B}(z)\mathcal{T}_2 \cdots \mathcal{B}(z)\mathcal{T}_n. \tag{3.2}$$

LEMMA 3.1. *There is a holomorphic function $\mathbb{W} : \mathbb{D} \to G(d, c+d)$ such that $\mathbb{W}(z)$ is invariant under $\mathcal{D}(z)$ and $\mathbb{W}(z)$ is associated to the d largest absolute values of eigenvalues of $\mathcal{D}(z)$†. In particular, let $D_{\mathbb{W}}(z)$ be the restriction of $\mathcal{D}(z)$ on $\mathbb{W}(z)$; then*

$$z \mapsto \ln(\rho(\Lambda^d\mathcal{D}(z))) = \ln|\det D_{\mathbb{W}}(z)| \tag{3.3}$$

is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

† $\mathbb{W}(z)$ is basically a direct sum of generalized eigenspaces of $\mathcal{D}(z)$.

Proof. As $\mathcal{D}(z)\overline{\mathbb{S}} \subset \mathbb{S}$, using the chart φ, we see that the image of the classical domain $R_I(c, d)$ under the Möbius action of $\mathcal{D}(z)$ has compact closure in $R_I(c, d)$. By the Earle–Hamilton fixed point theorem, the map $M \mapsto \mathcal{D}(z) \cdot M$ has a unique fixed point $M(z) \in R_I(c, d)$. Moreover, the nth iteration of this map, given by $M \mapsto \mathcal{D}^n(z) \cdot M$, converges to the fixed point $M(z)$ for any $M \in R_I(c, d)$ as n tends to infinity†. Therefore, the subspace $\mathbb{W}(z) = \varphi(M(z)) = \lim_{n\to\infty} \varphi(\mathcal{D}^n(z) \cdot \mathbf{0})$ is invariant under $\mathcal{D}(z)$. Clearly, for any n, $z \mapsto \mathcal{D}^n(z) \cdot \mathbf{0}$ is holomorphic in $z \in \mathbb{D}$. As this family of functions takes only values in $R_I(c, d)$, it is a normal family by Montel's theorem. Thus, the limit is holomorphic as well. As the action of $\mathcal{D}^n(z)$ on $G(d, c + d)$ contracts a neighborhood of $\mathbb{W}(z)$, it is clear that $\mathbb{W}(z)$ is spanned by generalized eigenvectors of $\mathcal{D}(z)$ that correspond to the eigenvalues with largest absolute values. Choosing the column vectors of $\binom{M(z)}{\mathbf{1}}$ as a basis for $\mathbb{W}(z)$, $D_{\mathbb{W}}(z)$ is represented as an invertible, holomorphic $d \times d$ matrix valued function and one finds that $\ln(\rho(\Lambda^d \mathcal{D}(z))) = \ln(\rho(\Lambda^d D_{\mathbb{W}}(z))) = \ln|\det D_{\mathbb{W}}(z)|$ is harmonic in $z \in \mathbb{D}$. For the first equality, note that the spectral radius of $\Lambda^d \mathcal{D}$ is the product of the d largest absolute values of eigenvalues of $\mathcal{D}$ (counted with algebraic multiplicity). □

Remark. Assume that f is an invertible transformation, $\mathcal{A} \in \mathcal{LI}(\mathbb{X}; \mathrm{U}(c, d))$. The proof for the holomorphic dependence of $\mathbb{W}(z)$ and the related harmonic dependence of $\ln(\rho(\Lambda^d \mathcal{D}(z)))$ can be modified to a proof of harmonic dependence of $L^d(f, \mathcal{B}(z)\mathcal{A})$ on $z \in \mathbb{D}$. In fact, $\mathbb{X} \times \mathbb{S}$ is a d-conefield, $\mathcal{A}(x)\overline{\mathbb{S}} \subset \mathbb{S}$ shows that the cocycle $(f, \mathcal{B}(z)\mathcal{A})$ is d-dominated and the unstable directions $\mathbb{W}(x, z) = \lim_{n\to\infty} (\mathcal{B}(z)\mathcal{A})_n(f^{-n}(x))[\binom{\mathbf{0}}{\mathbf{1}}]_\sim$ of the corresponding dominated splitting depend holomorphically on z; cf. [**AJS**, §§3 and 6]. Choosing the basis $\binom{\varphi^{-1}(\mathbb{W}(x,z))}{\mathbf{1}}$, the restriction $\mathcal{B}(z)\mathcal{A}(x) : \mathbb{W}(x, z) \to \mathbb{W}(f(x), z)$ can be written as $d \times d$ matrix $D_{\mathbb{W}}(x, z)$, holomorphic in z, and $\int_{\mathbb{X}} \ln \|D_{\mathbb{W}}(x, z)\|\, d\mu(x) < \infty$ uniformly in z. Hence, $L^d(f, \mathcal{B}(z)\mathcal{A}) = \int_{\mathbb{X}} \ln|\det D_{\mathbb{W}}(x, z)|\, d\mu(x)$ is harmonic in z; cf. [**A**, §2].

Now Theorem 1.2 follows easily.

Proof of Theorem 1.2. As $\mathcal{B}(e^{2\pi i\theta}) = \mathcal{U}_\theta^{(c,d)}$, one has

$$\mathcal{D}(e^{2\pi i\theta}) = \mathcal{U}_\theta^{(c,d)} \mathcal{T}_1 \mathcal{U}_\theta^{(c,d)} \mathcal{T}_2 \cdots \mathcal{U}_\theta^{(c,d)} \mathcal{T}_n. \tag{3.4}$$

By harmonicity of $\ln(\rho(\Lambda^d \mathcal{D}(z)))$, we have

$$\int_0^1 \ln(\rho(\Lambda^d \mathcal{D}(e^{2\pi i\theta})))\, d\theta = \ln(\rho(\Lambda^d \mathcal{D}(0))). \tag{3.5}$$

Now, using blocks of sizes c and d, let

$$\mathcal{T}_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \quad \text{then } \mathcal{B}(0)\mathcal{T}_j = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ C_j & D_j \end{pmatrix} \tag{3.6}$$

and hence

$$\mathcal{D}(0) = \mathcal{Q}^{-1} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_1 \cdots D_n \end{pmatrix} \mathcal{Q} \quad \text{for } \mathcal{Q} = \begin{pmatrix} \mathbf{1}_c & \mathbf{0} \\ D_n^{-1} C_n & \mathbf{1}_d \end{pmatrix}. \tag{3.7}$$

† In fact, the Carathéodory metric is contracted.

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In particular, using (2.3), this gives

$$\ln(\rho(\Lambda^d \mathcal{D}(0))) = \ln|\det(D_1 \cdots D_n)| = \sum_{j=1}^{n} \ln|\det(D_j)| = \sum_{j=1}^{n} N_d(\mathcal{T}_j). \tag{3.8}$$

Now, (3.4), (3.5) and (3.8) together give (1.12). Using (3.4) and (1.12) again, one also obtains

$$\begin{aligned}\int_0^1 N_d(\mathcal{D}(e^{2\pi i\theta}))\, d\theta &= \int_0^1 \int_0^1 \ln \rho(\Lambda^d \mathcal{U}_{\theta'}^{(c,d)} \mathcal{D}(e^{2\pi i\theta}))\, d\theta\, d\theta' \\ &= \int_0^1 N_d(\mathcal{U}_{\theta'}^{(c,d)} \mathcal{T}_1) + \sum_{j=2}^{n} N_d(\mathcal{T}_j)\, d\theta' = \sum_{j=1}^{n} N_d(\mathcal{T}_j) \end{aligned}\tag{3.9}$$

which shows (1.11). □

Proof of Theorem 1.1. By subadditivity we find, for $\mathcal{A} \in \mathcal{LI}(\mathbb{X}, \mathrm{U}(c,d))$,

$$\begin{aligned}0 &\le \frac{1}{n}\int_{\mathbb{X}} N_d((\mathcal{U}_\theta^{(c,d)}\mathcal{A})_n(x))\, d\mu(x) \le \frac{1}{n}\int_{\mathbb{X}} \ln \|\Lambda^d (\mathcal{U}_\theta^{(c,d)}\mathcal{A})_n(x)\|\, d\mu(x) \\ &\le \int_{\mathbb{X}} \ln \|\Lambda^d \mathcal{U}_\theta^{(c,d)}\mathcal{A}(x)\|\, d\mu(x) \le d \int_{\mathbb{X}} \ln \|\mathcal{A}(x)\|\, d\mu(x) < \infty \end{aligned}\tag{3.10}$$

uniformly in θ and n. Hence, using (1.9) we find by dominated convergence that

$$\begin{aligned}\int_0^1 L^d(f, \mathcal{U}_\theta^{(c,d)}\mathcal{A})\, d\theta &= \int_0^1 \lim_{n\to\infty} \int_{\mathbb{X}} \frac{1}{n} N_d((\mathcal{U}_\theta^{(c,d)}\mathcal{A})_n(x))\, d\mu(x)\, d\theta \\ &= \lim_{n\to\infty} \int_{\mathbb{X}} \int_0^1 \frac{1}{n} N_d(\mathcal{U}_\theta^{(c,d)}\mathcal{A}(f^{n-1}x) \cdots \mathcal{U}_\theta^{(c,d)}\mathcal{A}(x))\, d\theta\, d\mu(x) \\ &= \lim_{n\to\infty} \int_{\mathbb{X}} \frac{1}{n} \sum_{j=0}^{n-1} N_d(\mathcal{A}(f^j(x)))\, d\mu(x) = \int_{\mathbb{X}} N_d(\mathcal{A}(x))\, d\mu(x). \end{aligned}\tag{3.11}$$

□

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