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THE GAUSS–BONNET INTEGRAND FOR A CLASS OF RIEMANNIAN MANIFOLDS ADMITTING TWO ORTHOGONAL COMPLEMENTARY FOLIATIONS

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ABSTRACT. With the general assumption that the manifold admits two orthogonal complementary foliations, one of which is totally geodesic, we study the components of the curvature tensor field of the characteristic connection.

In the case where the manifold is compact, orientable of dimension 6 or 8 and the dimension of the totally geodesic foliation is 4, we relate the sign of the Euler characteristic of the manifold and that of the sectional curvature of the leaves of both foliations.

§0. **Introduction.** It is well known that if a Riemannian manifold is locally a Riemannian product of two manifolds the Euler polynomial $\chi(R)$ in the components of the curvature of the Levi-Civita connection can be written as a product of two polynomials. We are interested in the Gauss-Bonnet integrand of a Riemannian manifold admitting two orthogonal complementary foliations, one of which is totally geodesic. In order to find a good expression of this integrand, we define in §1 a connection on such a manifold, which we call characteristic connection; it is metric and makes P parallel.

We have studied the components of the curvature tensor field of this connection and the relations with the curvature of the Levi-Civita connection, finding some results that we use in §2 to obtain an expression of the Gauss-Bonnet integrand. This allows us to prove Corollary 4, where the sign of the Euler characteristic of the manifold and that of the sectional curvatures of the leaves of both foliations are related in the cases where the dimension of the manifold is 6 or 8 and the dimension of the totally geodesic foliation is 4.

§1. **The curvature tensor field of the characteristic connection.** Let $\mathcal{V}$ be a distribution on a Riemannian manifold $(\mathcal{M}, g)$ and let $\mathcal{H}$ be the orthogonal distribution to $\mathcal{V}$. At every point $m \in \mathcal{M}$ we have then $T_m\mathcal{M} = \mathcal{V}_m \oplus \mathcal{H}_m$. One

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can thus uniquely define a $(1, 1)$ tensor field P such that $P^2 = I$, $P_{|\mathscr{V}} = I$ and $P_{|\mathscr{H}} = -I$.

Then $(\mathscr{M}, g, P)$ is a Riemannian almost-product manifold where the vertical and horizontal distributions are exactly $\mathscr{V}$ and $\mathscr{H}$.

As we have noted in §0 we are interested in manifolds where $\mathscr{V}$ and $\mathscr{H}$ are integrable and one of them is totally geodesic; in the sequel we will suppose that $\mathscr{V}$ is the distribution with this property.

These kinds of almost-product manifolds are exactly those manifolds belonging to one of the thirty-six classes of Riemannian almost-product manifolds defined in [6] with regard to the algebraic properties of ∇P, where ∇ is the Levi–Civita connection. For a geometric description of these classes see also [3] and [5]; nontrivial examples of manifolds in each class can be found in [4].

Let $(\mathscr{M}, g, P)$ be a Riemannian almost-product manifold; we can consider the connection whose covariant derivative is defined by

$$\bar{\nabla}_M N = \nabla_M N + \frac{P}{2} (\nabla_M P) N.$$

We call it the characteristic connection. If we denote v and h the projections on the vertical and horizontal distributions, respectively, then $\bar{\nabla}$ can be written in a more useful form:

$$\bar{\nabla}_M A = v\, \nabla_M A; \qquad \bar{\nabla}_M X = h\, \nabla_M X \tag{1}$$

when $M \in \mathscr{X}(\mathscr{M})$, $A \in \mathscr{V}$, $X \in \mathscr{H}$.

Now, $\bar{\nabla}$ makes g and P parallel and then when restricted to both subbundles, vertical and horizontal, it induces connections which paralelize the restricted metric. We shall denote the curvature tensor of $\bar{\nabla}$ by $\bar{R}$, that is

$$\bar{R}_{MNLQ} = g(\bar{\nabla}_N \bar{\nabla}_M L, Q) - g(\bar{\nabla}_M \bar{\nabla}_N L, Q) + g(\bar{\nabla}_{[M,N]} L, Q). \tag{2}$$

It is obvious that

$$\bar{R}_{MNAX} = \bar{R}_{MNXA} = 0, \tag{3}$$

for all $A \in \mathscr{V}$, $X \in \mathscr{H}$, $M, N \in \mathscr{X}(\mathscr{M})$.

Our purpose is to compute the other components of $\bar{R}$.

Proposition 1. *Let $(\mathscr{M}, g)$, $\mathscr{V}$ and $\mathscr{H}$ be as above, then:*

a) $\bar{R}_{XYAB} = 0$.

b) $\bar{R}_{AXBC} = 0$.

c) $\bar{R}_{ABXY} = R_{ABXY}$ *and* $\bar{R}_{AXYZ} = R_{AXYZ}$.

d) *$\bar{R}$ and R coincide on vertical vectors.*

e) *On horizontal vectors, $\bar{R}$ coincides with the Riemannian curvature induced on the horizontal leaves.*

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Proof. Since $\mathcal{V}$ and $\mathcal{H}$ are foliations we can assume that the Lie brackets of the arguments are zero, thus

$$\bar{R}_{XYAB} = g(\nabla_Y v \nabla_X A, B) - g(\nabla_X v \nabla_Y A, B).$$

Since $\mathcal{V}$ is totally geodesic $h \nabla_A B = 0$ for all $A, B \in \mathcal{V}$. Thus

$$g(\nabla_A X, B) = -g(X, \nabla_A B) = 0.$$

Therefore $v \nabla_A X = v\nabla_X A = 0$ because $[A, X] = 0$. This proves a).

b) Using (1) and (2) we obtain

$$\bar{R}_{AXBC} = g(\nabla_X v \nabla_A B, C) - g(\nabla_A v \nabla_X B, C).$$

But $v \nabla_X B = 0$ as before, and $v \nabla_A B = \nabla_A B$ and so

$$\bar{R}_{AXBC} = g(\nabla_X \nabla_A B, C).$$

Now,

$$g(\nabla_X \nabla_A B, C) = Xg(\nabla_A B, C) - g(\nabla_A B, \nabla_X C) = Xg(\nabla_A B, C).$$

Since we are assuming that $\nabla_A B = \nabla_B A$ we have

$$\bar{R}_{AXBC} = \bar{R}_{BXAC}. \tag{4}$$

But $\bar{\nabla}$ is a metric connection, whence its curvature tensor field is skewsymmetric in the last two arguments

$$\bar{R}_{AXBC} = -\bar{R}_{AXCB}. \tag{5}$$

From (4) and (5) an easy computation shows that $\bar{R}_{AXBC} = 0$.

c) The expressions of $\bar{R}_{ABXY}$ and $\bar{R}_{AXYZ}$ are

$$\bar{R}_{ABXY} = g(\nabla_B \bar{\nabla}_A X, Y) - g(\nabla_A \bar{\nabla}_B X, Y),$$
$$\bar{R}_{AXYZ} = g(\nabla_X \bar{\nabla}_A Y, Z) - g(\nabla_A \bar{\nabla}_X Y, Z).$$

In a) we have seen that if $\mathcal{V}$ is totally geodesic then $\nabla_A X \in \mathcal{H}$ for all $A \in \mathcal{V}$, $X \in \mathcal{H}$, and so

$$\bar{\nabla}_A X = h \nabla_A X = \nabla_A X.$$

To finish the proof, we only need to transform the term $g(\nabla_A \bar{\nabla}_X Y, Z)$.

By (1) we have

$$g(\nabla_A \bar{\nabla}_X Y, Z) = g(\nabla_A h \nabla_X Y, Z) = g(\nabla_A \nabla_X Y, Z) - g(\nabla_A v \nabla_X Y, Z),$$

the last term vanishing because $\mathcal{V}$ is totally geodesic.

d) and e) follow directly from the fact that the characteristic connection when restricted to vertical (resp. horizontal) arguments is exactly the Levi-Civita connection of the vertical (resp. horiziontal) leaves.

§2. **The Gauss–Bonnet integrand for a Riemannian manifold with two orthogonal complementary foliations one of which is totally geodesic.** Let

$\chi(\bar{R})$ be the Euler polynomial expression in the components of the curvature of the characteristic connection

$$\chi(\bar{R})=\frac{1}{2^{3n}\pi^n n!}\sum_{\alpha,\beta\in S_{2n}}\varepsilon(\alpha)\varepsilon(\beta)\bar{R}_{\alpha_1\alpha_2\beta_1\beta_2}\cdots\bar{R}_{\alpha_{2n-1}\alpha_{2n}\beta_{2n-1}\beta_{2n}},\tag{6}$$

where $\bar{R}_{ijkl}$ are the components of $\bar{R}$ in any orthonormal basis.

PROPOSITION 2. *Let $(\mathcal{M}, g)$ be a Riemannian manifold as above. Then*

$$\chi(\bar{R})=\frac{1}{2^{3n}\pi^n p!\,q!}\left(\sum\varepsilon(\alpha')\varepsilon(\beta')\bar{R}_{\alpha'_1\alpha'_2\beta'_1\beta'_2}\cdots\bar{R}_{\alpha'_{2p-1}\alpha'_{2p}\beta'_{2p-1}\beta'_{2p}}\right)\cdot$$
$$\cdot\left(\sum\varepsilon(\alpha'')\varepsilon(\beta'')\bar{R}_{\alpha''_{2p+1}\alpha''_{2p+2}\beta''_{2p+1}\beta''_{2p+2}}\cdots\bar{R}_{\alpha''_{2n-1}\alpha''_{2n}\beta''_{2n-1}\beta''_{2n}}\right)$$

where the first sum runs over all permutations α', β' of $\{1,\ldots,2p\}$ and the second one runs over all permutations α'', β'' of $\{2p+1,\ldots,2n\}$ being $2n=$ $\dim\mathcal{M}$, $2p=\dim\mathcal{V}$, $2q=\dim\mathcal{H}$.

Proof. We choose a basis adapted to the Riemannian almost-product structure, that is

$$X_1,\ldots,X_{2p}\in\mathcal{V}\qquad X_{2p+1},\ldots,X_{2n}\in\mathcal{H}.$$

Using (3) and Proposition 1 a), b), the only terms in (6) which are different from zero are those of the form

$$\bar{R}_{\alpha'_1\alpha'_2\beta'_1\beta'_2}\cdots\bar{R}_{\alpha'_{2p-1}\alpha'_{2p}\beta'_{2p-1}\beta'_{2p}}\bar{R}_{\alpha''_{2p+1}\alpha''_{2p+2}\beta''_{2p+1}\beta''_{2p+2}}\cdots\bar{R}_{\alpha''_{2n-1}\alpha''_{2n}\beta''_{2n-1}\beta''_{2n}}$$

where α',β' are permutations of $\{1,\ldots,2p\}$ and α'',β'' are permutations of $\{2p+1,\ldots,2n\}$.

REMARK. We are only considering the case where both foliations are even dimensional because the Euler characteristic of the tangent bundle of $\mathcal{M}$ will vanish otherwise.

There is an important conjecture, still unsolved in the general case, which is as follows:

(*) Let $\mathcal{M}$ be a compact, orientable, $2n$-dimensional Riemannian manifold. If all sectional curvatures of $\mathcal{M}$ are non-negative then the Euler characteristic of $\mathcal{M}$ verifies $\chi(\mathcal{M})\geq 0$. If all sectional curvatures are non-positive then $(-1)^n\chi(\mathcal{M})\geq 0$.

The result that (*) is true for $n=2$ is due to J. W. Milnor and the proof can be found in [2] or in [1]. It is also there shown that if the manifold is of either positive or negative curvature, then $\chi(\mathcal{M})>0$.

The method used in [1], [2] consists in finding a basis for which the Euler polynomial expression in the components of the curvature of the Levi–Civita connection, $\chi(R)$, is everywhere non-negative.

In the next theorem we prove that the sign of the sectional curvature of the leaves of both foliations determines the sign of $\chi(\bar{R})$.

THEOREM 3. *Let* $(\mathcal{M}, g)$ *be a compact, oriented, Riemannian manifold, with two orthogonal complementary foliations* $\mathcal{V}$ *and* $\mathcal{H}$ *such that* $\mathcal{V}$ *is totally geodesic and* $\dim \mathcal{V} = 4$.

a) *Assume* $\dim \mathcal{M} = 6$. *If each leaf of* $\mathcal{V}$ *has non-negative or non-positive curvature and the leaves of* $\mathcal{H}$ *are of non-negative* (*resp. non-positive*) *curvature, then* $\chi(\bar{R}) \geq 0$ (*resp.* $\chi(\bar{R}) \leq 0$).

If curvatures are positive (*resp. negative*) *then* $\chi(\bar{R}) > 0$ (*resp.* $\chi(\bar{R}) < 0$).

b) *Assume* $\dim \mathcal{M} = 8$. *If each leaf, either vertical or horizontal, has sectional curvature of constant sign then* $\chi(\bar{R}) \geq 0$.

Furthermore, if the curvature of each leaf never vanishes then $\chi(\bar{R}) > 0$.

Proof. By Proposition 2 we have

$$\chi(\bar{R}) = \frac{1}{2^8 \pi^3} \bar{R}_{5656} \sum_{\alpha', \beta' \in S_4} \varepsilon(\alpha')\varepsilon(\beta') \bar{R}_{\alpha'_1\alpha'_2\beta'_1\beta'_2} \bar{R}_{\alpha'_3\alpha'_4\beta'_3\beta'_4}, \tag{7}$$

in the 6-dimensional case, and

$$\begin{aligned} \chi(\bar{R}) = \frac{1}{2^{14}\pi^4} &\left(\sum_{\alpha', \beta' \in S_4} \varepsilon(\alpha')\varepsilon(\beta') \bar{R}_{\alpha'_1\alpha'_2\beta'_1\beta'_2} \bar{R}_{\alpha'_3\alpha'_4\beta'_3\beta'_4} \right) \\ &\times \left(\sum_{\alpha'', \beta'' \in S'_4} \varepsilon(\alpha'')\varepsilon(\beta'') \bar{R}_{\alpha''_5\alpha''_6\beta''_5\beta''_6} \bar{R}_{\alpha''_7\alpha''_8\beta''_7\beta''_8} \right) \end{aligned} \tag{8}$$

in the 8-dimensional case, where S'_4 denotes the group of permutations of $\{5, 6, 7, 8\}$.

Now, using Proposition 1d), (7) and (8) can be written respectively

$$\chi(\bar{R}) = \frac{1}{2^8 \pi^3} R'_{5656} \sum_{\alpha', \beta' \in S_4} \varepsilon(\alpha')\varepsilon(\beta') R_{\alpha'_1\alpha'_2\beta'_1\beta'_2} R_{\alpha'_3\alpha'_4\beta'_3\beta'_4}, \tag{7'}$$

$$\begin{aligned} \chi(\bar{R}) = \frac{1}{2^{14}\pi^4} &\left(\sum_{\alpha', \beta' \in S_4} \varepsilon(\alpha')\varepsilon(\beta') R_{\alpha'_1\alpha'_2\beta'_1\beta'_2} R_{\alpha'_3\alpha'_4\beta'_3\beta'_4} \right) \\ &\times \left(\sum_{\alpha'', \beta'' \in S'_4} \varepsilon(\alpha'')\varepsilon(\beta'') R'_{\alpha''_5\alpha''_6\beta''_5\beta''_6} R'_{\alpha''_7\alpha''_8\beta''_7\beta''_8} \right), \end{aligned} \tag{8'}$$

where R' is the curvature tensor of the Levi–Civita connection of the horizontal leaves. If the sectional curvatures are non-negative or non-positive, it is known [1] that

$$\left(\sum_{\alpha, \beta \in S_4} \varepsilon(\alpha)\varepsilon(\beta) R_{\alpha_1\alpha_2\beta_1\beta_2} R_{\alpha_3\alpha_4\beta_3\beta_4} \right)$$

is non-negative and thus $\chi(\bar{R})$ is non-negative in the 8-dimensional case and has the same sign as R'_{5656} in the 6-dimensional case, concluding the proof of the theorem.

Using now the Gauss–Bonnet–Chern formula

$$\chi(\mathcal{M})=\int_{\mathcal{M}} \chi(\bar{R})\, dV$$

we obtain directly from Theorem 3, the following

COROLLARY 4. *Let* $(\mathcal{M}, g)$, $\mathcal{V}$ *and* $\mathcal{H}$ *be as in Theorem above, then:*

a′) *If the hypotheses of part* a) *in Theorem* 3 *hold, then* $\chi(\mathcal{M})\geq 0$ (*resp.* $\chi(\mathcal{M})\leq 0$).

b′) *If the hypotheses of part* b) *in Theorem* 3 *hold, then* $\chi(\mathcal{M})\geq 0$.

Although the conclusions in (*) and in Corollary 4 are the same, the hypotheses are different; we are going now to give two cases where the hypothesis of (*) implies that of Corollary 4.

PROPOSITION 5. *Let* $(\mathcal{M}, g)$ *be a compact, oriented, Riemannian manifold of dimension* 6 *or* 8, *of non-negative curvature with two orthogonal complementary foliations* $\mathcal{V}$ *and* $\mathcal{H}$, *such that* $\mathcal{V}$ *is* 4-*dimensional and totally geodesic and* $\mathcal{H}$ *is totally umbilical, then* $\chi(\mathcal{M})\geq 0$.

Proof. By Corollary 4, it is only necessary to show that horizontal leaves are manifolds of non-negative curvature.

For each $m\in\mathcal{M}$ choose an orthonormal basis of the normal space to the leaf of $\mathcal{H}$ through $m, \{\xi_i\}_{i=1}^4$; as $\mathcal{H}$ is totally umbilical for each ξ_i there exists $\lambda_i\in\mathbb{R}$ such that $A_{\xi_i}=\lambda_i I$.

Let X, Y be vectors tangent to that leaf, and α the second fundamental form, then

$$g(\alpha(X, X), \alpha(Y, Y))-g(\alpha(X, Y), \alpha(X, Y))=\left(\sum_{i=1}^{4}\lambda_i^2\right)(\|X\|^2\,\|Y\|^2-g(X, Y)^2)\geq 0,$$

which implies that the sectional curvature of the leaf is non-negative.

PROPOSITION 6. *Let* $(\mathcal{M}, g)$ *be a* 6-*dimensional, compact, oriented Riemannian manifold of non-positive curvature with two orthogonal complementary foliations* $\mathcal{V}$ *and* $\mathcal{H}$ *such that* $\mathcal{V}$ *is* 4-*dimensional and totally geodesic and* $\mathcal{H}$ *is minimal, then* $\chi(\mathcal{M})\leq 0$.

Proof. As before, it is only necessary to check that the curvature of the horizontal leaves has the adequate sign.

Let $\{X, Y\}$ be an orthonormal basis of the tangent space to the leaf of $\mathcal{H}$ through m, $m\in\mathcal{M}$. As the leaf is a minimal submanifold, $\alpha(X, X)=-\alpha(Y, Y)$ and then

$$g(\alpha(X, X), \alpha(Y, Y))-g(\alpha(X, Y), \alpha(X, Y))=-(\|\alpha(X, X)\|^2+\|\alpha(X, Y)\|^2)\leq 0,$$

which implies that the curvature of the leaves is non-positive.

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