The Random Graph Intuition
for the Tournament Game

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In the *tournament game* two players, called Maker and Breaker, alternately take turns in claiming an unclaimed edge of the complete graph K_n and selecting one of the two possible orientations. Before the game starts, Breaker fixes an arbitrary tournament T_k on k vertices. Maker wins if, at the end of the game, her digraph contains a copy of T_k; otherwise Breaker wins. In our main result, we show that Maker has a winning strategy for $k = (2 - o(1)) \log_2 n$, improving the constant factor in previous results of Beck and the second author. This is asymptotically tight since it is known that for $k = (2 - o(1)) \log_2 n$ Breaker can prevent the underlying graph of Maker's digraph from containing a k-clique. Moreover, the precise value of our lower bound differs from the upper bound only by an *additive constant* of 12.

We also discuss the question of whether the random graph intuition, which suggests that the threshold for k is asymptotically the same for the game played by two 'clever' players and the game played by two 'random' players, is supported by the tournament game. It will turn out that, while a straightforward application of this intuition fails, a more subtle version of it is still valid.

Finally, we consider the *orientation game* version of the tournament game, where Maker wins the game if the final digraph – also containing the edges directed by Breaker – possesses a copy of T_k. We prove that in that game Breaker has a winning strategy for $k = (4 + o(1)) \log_2 n$.

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1. Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the classical Maker–Breaker game $(X, \mathcal{F})$, two players, called Maker and Breaker, alternately claim elements of X, with Maker going first. X is usually called the *board*, and $\mathcal{F}$ is referred to as the *family of winning sets*. Maker wins the game if she claims all elements of some winning set; otherwise Breaker wins. A well-studied class of Maker–Breaker games is that of *graph games*, where the board is the edge set of a complete graph K_n, and Maker's goal is to create a graph which possesses some fixed (usually monotone) property P. A widely investigated example of such a game is the *k-clique game* (sometimes abbreviated to *clique game*), where Maker wins if and only if, by the end of the game, her graph contains a clique of size at least k. In [15], Erdős and Selfridge considered the largest value $k_{cl} = k_{cl}(n)$ such that Maker has a winning strategy in the k_{cl}-clique game. By applying their well-known Erdős–Selfridge criterion, they obtained that $k_{cl} \leqslant (2 - o(1)) \log n$ (throughout this paper all logarithms are in base 2, unless stated otherwise). Later, Beck [6] introduced the method of *self-improving potentials* and used his technique to determine the exact value of k_{cl}.

Theorem 1.1 ([6], Theorem 6.4). $k_{cl} = \lfloor 2 \log n - 2 \log \log n + 2 \log e - 3 + o(1) \rfloor.$

Random graph intuition. There is an intriguing relation between k_{cl} and the corresponding extremal value k_{cl}^* for a game where Maker and Breaker are replaced with 'random players' which select their edge in each round uniformly at random: In this game, *RandomMaker* creates a random graph $G(n, m)$ with $m = \lceil \frac{1}{2} \binom{n}{2} \rceil$ edges. It is well known that the size of the largest clique of $G(n, \lceil \frac{1}{2} \binom{n}{2} \rceil)$ is $(2 - o(1)) \log n$ asymptotically almost surely (abbreviated as a.a.s. henceforth), so the threshold where the random k-clique game turns from a RandomMaker's win to a RandomBreaker's win is around $(2 - o(1)) \log n$ – just as in the deterministic game, as shown by Theorem 1.1.

For quite a few other games it has been found that the outcome of the random game is essentially the same as the outcome of the deterministic game (see, *e.g.*, [4, 8, 17, 18, 19]). This phenomenon, known as the *random graph intuition* or the *Erdős paradigm*, was first pointed out by Chvátal and Erdős [13], and later investigated further in many papers by Beck [2, 3, 4, 5] and Bednarska and Łuczak [8].

For a small number of games it has been established that the random graph intuition fails. An interesting example is the *diameter game* where the graph property P that Maker aims to achieve is that every pair of vertices has distance at most two (in her graph). It is known that a.a.s. $G(n, \lceil \frac{1}{2} \binom{n}{2} \rceil)$ has property P, hence RandomMaker wins the random game a.a.s. On the other hand, Balogh, Martin and Pluhár [1] proved that actually Breaker has a winning strategy in the deterministic game, which yields that the random graph intuition fails in this case. It is an interesting open problem to determine suitable criteria which guarantee for a given game that the random graph intuition (or some weaker version of it) holds.

Tournament game. In this paper we study a variant of the k-clique game and investigate whether it supports the random graph intuition. A *tournament* is a directed graph where

every pair of vertices is connected by a single directed edge. The *k-tournament game* $\mathcal{T}(k,n)$ is played on K_n. At the beginning of the game Breaker fixes an arbitrary tournament T_k on k vertices. In each round, Maker and Breaker then alternately claim one unclaimed edge (as in classical graph games), and – additionally – select one of the two possible orientations for their chosen edge. If, at the end of the game, Maker's graph contains a copy of the goal tournament T_k, she wins; otherwise, Breaker is the winner. Note that for the outcome of this particular game, the orientations of Breaker's edges are irrelevant. They become meaningful, however, for so-called *orientation games*, which we shall introduce below.

Let $k_t = k_t(n)$ denote the largest k such that Maker has a winning strategy in the game $\mathcal{T}(k,n)$. To get an indication of the value of k_t, Beck analysed the *random tournament game* in which RandomMaker and RandomBreaker each choose their edge and the corresponding orientation uniformly at random. He found that the threshold where the random game turns from a RandomMaker's win to a RandomBreaker's win is around $(1 - o(1))\log n$. Motivated by the question of whether the tournament game supports the random graph intuition, Beck [6] asked to determine k_t. Since a winning strategy for Breaker in the k-clique game only requires him to prevent Maker from achieving a single specific tournament on k vertices, we have

$$k_t \leqslant k_{cl} = (2 - o(1))\log n. \tag{1.1}$$

The second equation follows from Theorem 1.1. From the other side, Beck [6, p. 457] derived that

$$k_t \geqslant \left(\frac{1}{2} - o(1)\right)\log n.$$

In fact, he proved the stronger statement that for $k = (\frac{1}{2} - o(1))\log n$, Maker has a strategy to occupy a graph containing a copy of *every* tournament on k vertices. The lower bound on k_t was improved by the second author in [16] to $k_t \geqslant (1 - o(1))\log n$. In our main result we show that the upper bound is tight. This means that k_t is twice as large as the random graph intuition suggests.

Theorem 1.2. *For n large enough, $k_t(n) \geqslant 2\log n - 2\log\log n - 12$.*

As a direct consequence of (1.1), Theorem 1.1 and Theorem 1.2, the asymptotics of k_t are determined:

$$k_t = 2\log n - 2\log\log n + \Theta(1) = (2 - o(1))\log n.$$

Remarkably, the upper bound in the clique game and the lower bound in the tournament game differ only by an additive constant of 12 (for large n). Thus, the additional constraint that the edges have to be oriented in a particular way makes it only a little harder for Maker.

Our result seemingly refutes the random graph intuition described above. However, as a first step towards the proof of Theorem 1.2 we will define a suitable, *classical* graph game $\mathcal{G}$ (with no edge orientations involved), which has the property that every winning

strategy for Maker on $\mathcal{G}$ directly gives her a winning strategy for the tournament game. It turns out that in the random analogue of $\mathcal{G}$, the threshold where the game turns from a RandomMaker's win to a RandomBreaker's win is around $(2 - o(1)) \log n$. Thus, from a more subtle point of view, the random graph intuition can be considered valid.

Orientation games. We study a variant of the tournament game, which follows the rules of *orientation games*. Following the notation in [9], an orientation game is played on the edge set of K_n by two players, called OMaker and OBreaker, which alternately orient (or direct) a previously undirected edge. At the end of the game, OMaker wins if and only if the final digraph consisting of both OMaker's and OBreaker's edges satisfies a given property P. Various orientation games have been studied in the literature (see, *e.g.*, [9], [11], [12] and [14]).

In this paper we study an orientation-version of the tournament game, denoted by $\mathcal{T}_o(k, n)$. At the beginning of the game, OBreaker fixes a tournament T_k on k vertices. In each round, OMaker and OBreaker alternately orient previously undirected edges of the complete graph K_n. OMaker wins the game if the final tournament on n vertices (that is, the union of edges directed by both OMaker and OBreaker) contains a copy of T_k; otherwise OBreaker wins. In the spirit of the k-clique game and the k-tournament game, it is quite natural to ask for the largest integer $k_o = k_o(n)$ such that OMaker has a winning strategy for the game $\mathcal{T}_o(k, n)$.

Trivially, k_o is at least as large as the corresponding extremal number k_t for the ordinary tournament game. We will show that, asymptotically, k_o is at most twice as large as k_t.

Theorem 1.3. *Let n be large enough, and let $k \geqslant 4 \log n + 2$ be an integer. Then OBreaker has a strategy to win the game $\mathcal{T}_o(k, n)$. In fact, OBreaker has a winning strategy for every tournament on k vertices that he chooses at the beginning of the game.*

Together with Theorem 1.2, since $k_t \leqslant k_o$ as mentioned, we therefore get

$$2 \log n(1 - o(1)) \leqslant k_o \leqslant 4 \log n(1 + o(1)).$$

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.2. In Section 3 we prove Theorem 1.3. Finally, in Section 4, we discuss open problems related to the games we study.

2. Proof of Theorem 1.2

Let $n \in \mathbb{N}$ be large enough, and let k be the largest integer such that $n \geqslant k 2^{(k+9)/2}$. Note that by definition, $n < (k+1)2^{(k+10)/2}$, so $k \geqslant 2 \log n - 2 \log \log n - 12$. For clarity of presentation, we assume from now on that $n = k2^{(k+9)/2}$.

Let T_k be the tournament on k vertices that Breaker chooses at the beginning, with $V(T_k) = \{u_1, \ldots, u_k\}$. First, Maker partitions the vertex set into k equally sized parts:

$$V(K_n) = V_1 \dot{\cup} \cdots \dot{\cup} V_k.$$

Then she identifies the class V_i with the vertex u_i: Whenever Maker claims an edge between V_i and V_j, she chooses the direction according to the direction of $u_i u_j$ in T_k.

Therefore, her goal reduces to gaining a copy of a clique K_k, containing one vertex from each class V_i. Hence, she plays on the reduced board

$$X := \{\{v_i, v_j\} : v_i \in V_i, v_j \in V_j, i \neq j\}.$$

Our goal is to prove that she wins the classical Maker–Breaker game $(X, \mathcal{F})$, where $\mathcal{F}$ consists of all edge sets of k-cliques in the reduced k-partite graph:

$$\mathcal{F} := \left\{ \binom{S}{2} : S \subseteq V_1 \,\dot\cup\, \cdots \,\dot\cup\, V_k \text{ such that } |S \cap V_i| = 1, \text{ for every } 1 \leqslant i \leqslant k \right\}.$$

To this end, we will use a general criterion for Maker's win from [6]. Let us introduce the necessary notation first. For $p \in \mathbb{N}$, we define the set of *p-clusters* of $\mathcal{F}$ as

$$\mathcal{F}_2^p := \left\{ \bigcup_{1 \leqslant i \leqslant p} E_i : \{E_1, \ldots, E_p\} \in \binom{\mathcal{F}}{p}, \left| \bigcap_{1 \leqslant i \leqslant p} E_i \right| \geqslant 2 \right\}.$$

That is, $\mathcal{F}_2^p$ is the family consisting of all those subsets of X which can be represented as the union of p distinct winning sets sharing at least two elements of X. Furthermore, for any family $\mathcal{H}$ of finite sets, we consider the well-known potential function used in the Erdős–Selfridge criterion

$$T(\mathcal{H}) := \sum_{H \in \mathcal{H}} 2^{-|H|}.$$

According to Beck [6], we have the following sufficient condition for Maker's win.

Theorem 2.1 (Advanced Weak Win Criterion [6]). *Maker has a winning strategy for the Maker–Breaker game $(X, \mathcal{F})$ if there exists an integer $p \geqslant 2$ such that*

$$\frac{T(\mathcal{F})}{|X|} > p + 4p \left(T(\mathcal{F}_2^p) \right)^{1/p}. \tag{2.1}$$

In the remainder of the proof we will show that our choice of $(X, \mathcal{F})$ satisfies (2.1) for $p = 4$ and n large enough. First, we note that

$$T(\mathcal{F}) = \sum_{F \in \mathcal{F}} 2^{-|F|} = \left(\frac{n}{k} \right)^k \cdot 2^{-\binom{k}{2}} = 2^{5k}$$

and $\quad \dfrac{|X|}{T(\mathcal{F})} = \dfrac{\binom{k}{2} \left(\frac{n}{k} \right)^2}{2^{5k}} \leqslant \dfrac{k^2 \, 2^{k+9}}{2^{5k}} = o(1). \tag{2.2}$

As a first step towards the application of the Advanced Weak Win Criterion, we give an estimate on $T(\mathcal{F}_2^4)$. By definition,

$$\mathcal{F}_2^4 = \left\{ \bigcup_{1 \leqslant i \leqslant 4} E_i : \{E_1, \ldots, E_4\} \in \binom{\mathcal{F}}{4}, \left| \bigcap_{1 \leqslant i \leqslant 4} E_i \right| \geqslant 2 \right\}.$$

Note that any collection of cliques meets in two edges if and only if it meets in a triangle. Recall that the elements of $\mathcal{F}_2^4$ are referred to as *clusters*. Following the standard notation, we call a cluster a *sunflower* if there is a triangle such that any two of the four cliques meet
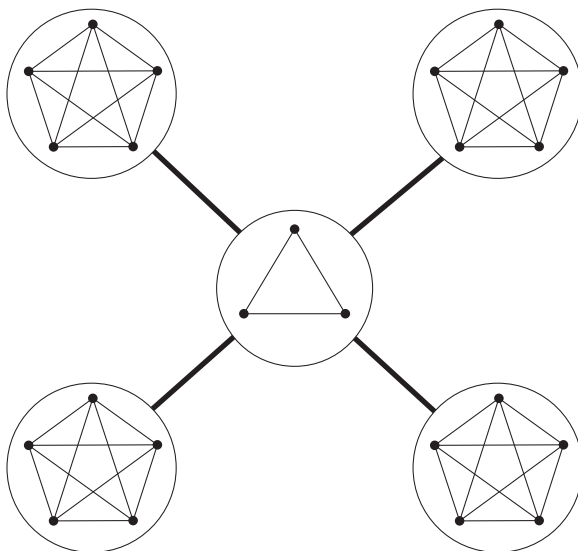
Figure 1. An example of a sunflower for $k = 8$. A thick line indicates that the vertices of the corresponding sets are pairwise connected.

in exactly this triangle. Figure 1 shows an illustration. We denote the subset of sunflowers of $\mathcal{F}_2^4$ by $\mathcal{S}_2^4$. By definition, a sunflower $F \in \mathcal{S}_2^4$ has exactly $4\binom{k}{2} - 9$ edges. In $\mathcal{F}_2^4$, there are at most

$$\binom{k}{3}\left(\frac{n}{k}\right)^3 \cdot \left(\frac{n}{k}\right)^{4(k-3)}$$

sunflowers. Therefore,

$$T(\mathcal{S}_2^4) \leqslant \binom{k}{3}\left(\frac{n}{k}\right)^{4k-9} 2^{-4\binom{k}{2}+9} =: f(n,k).$$

It will turn out that $f(n,k)$ dominates the sum $T(\mathcal{F}_2^4)$.

For every $E \in \mathcal{F}$, we let $V(E)$ denote the set of vertices corresponding to E. Note that $|V(E)| = k$ for every $E \in \mathcal{F}$. As a first step of our analysis we use the technique of Beck [6] to assign to each cluster $F = \bigcup_{1 \leqslant i \leqslant 4} E_i$ some sequence $S(F) := (m_1, m_2, m_3)$ such that

$$m_1 = |V(E_1) \cap V(E_2)|,$$
$$m_2 = |(V(E_1) \cup V(E_2)) \cap V(E_3)|,$$
$$m_3 = |(V(E_1) \cup V(E_2) \cup V(E_3)) \cap V(E_4)|.$$

Note that for a given cluster F we may have several choices for selecting $S(F)$ (depending on the considered order of the E_i). Furthermore, we let $\mathcal{F}_2^4(m_1, m_2, m_3)$ denote the subset of clusters of $\mathcal{F}_2^4$ to which we assigned the sequence (m_1, m_2, m_3). Then, obviously,

$$T(\mathcal{F}_2^4) = \sum_{m_1=3}^{k} \sum_{m_2=3}^{k} \sum_{m_3=3}^{k} T(\mathcal{F}_2^4(m_1, m_2, m_3)). \tag{2.3}$$

We now bound the cardinality of $\mathcal{F}_2^4(m_1, m_2, m_3)$.

Proposition 2.2. *For fixed $3 \leqslant m_1, m_2, m_3 \leqslant k$, we have that*

$$|\mathcal{F}_2^4(m_1, m_2, m_3)| \leqslant \binom{k}{3} \left(\frac{n}{k}\right)^{4k} \cdot \prod_{j=1}^{3} \binom{jk}{m_j - 3} \left(\frac{k}{n}\right)^{m_j}.$$

Furthermore, for any cluster $F \in \mathcal{F}_2^4(m_1, m_2, m_3)$ we have

$$|F| \geqslant 4\binom{k}{2} - \binom{m_1}{2} - \binom{m_2}{2} - \binom{m_3}{2}.$$

Proof. We fix any m_1, m_2, m_3 with $3 \leqslant m_1, m_2, m_3 \leqslant k$, and we also fix any triple v_1, v_2, v_3 of vertices from distinct classes. We now derive an upper bound on the number of those clusters in $\mathcal{F}_2^4(m_1, m_2, m_3)$ where all four cliques contain v_1, v_2, and v_3. To this end we consider the number of possibilities for selecting

$$V(E_1)\backslash\{v_1, v_2, v_3\}, \quad V(E_2)\backslash\{v_1, v_2, v_3\}, \quad V(E_3)\backslash\{v_1, v_2, v_3\}, \quad V(E_4)\backslash\{v_1, v_2, v_3\}.$$

Note that we have $(n/k)^{k-3}$ possibilities for the $k - 3$ vertices of $V(E_1)\backslash\{v_1, v_2, v_3\}$.

Suppose that for some $1 \leqslant i \leqslant 3$ we have already determined the sets

$$V(E_1)\backslash\{v_1, v_2, v_3\}, \ldots, V(E_i)\backslash\{v_1, v_2, v_3\}.$$

Then $V(E_1), \ldots, V(E_i)$ cover at most $3 + i(k - 3)$ vertices, including v_1, v_2, v_3. Therefore, we have at most

$$\binom{i(k-3)}{m_i - 3} \leqslant \binom{ik}{m_i - 3}$$

choices for those vertices of

$$(V(E_1) \cup \cdots \cup V(E_i)) \cap V(E_{i+1})$$

which are different from v_1, v_2, v_3. Finally, there are at most $(n/k)^{k-m_i}$ possibilities for

$$V(E_{i+1}) \setminus (V(E_1) \cup \cdots \cup V(E_i)).$$

Therefore, for any given m_1, m_2, m_3, every triple v_1, v_2, v_3 of vertices contributes at most

$$\left(\frac{n}{k}\right)^{k-3} \cdot \prod_{i=1}^{3} \binom{ik}{m_i - 3} \left(\frac{n}{k}\right)^{k-m_i}$$

to the number of clusters in $\mathcal{F}_2^4(m_1, m_2, m_3)$. Hence,

$$|\mathcal{F}_2^4(m_1, m_2, m_3)| \leqslant \binom{k}{3} \left(\frac{n}{k}\right)^3 \left(\frac{n}{k}\right)^{k-3} \cdot \prod_{i=1}^{3} \binom{ik}{m_i - 3} \left(\frac{n}{k}\right)^{k-m_i}$$

$$= \binom{k}{3} \left(\frac{n}{k}\right)^{4k} \cdot \prod_{i=1}^{3} \binom{ik}{m_i - 3} \left(\frac{k}{n}\right)^{m_i},$$

as claimed. For the second part of the proposition, note that $|E_1| = \binom{k}{2}$ and that every E_{i+1} contributes at least $\binom{k}{2} - \binom{m_i}{2}$ new edges to the cluster. $\qquad\square$

We now show that $f(n,k)$ dominates the sum $T(\mathcal{F}_2^4)$.

Lemma 2.3. $T(\mathcal{F}_2^4) < k^3 f(n,k)$, *provided k is large enough.*

Proof. By definition of $T(\cdot)$ and Proposition 2.2 we have that

$$T\big(\mathcal{F}_2^4(m_1, m_2, m_3)\big) \leqslant \binom{k}{3}\left(\frac{n}{k}\right)^{4k} \cdot \prod_{j=1}^{3}\left(\binom{jk}{m_j - 3}\left(\frac{k}{n}\right)^{m_j}\right) \cdot 2^{-4\binom{k}{2}+\binom{m_1}{2}+\binom{m_2}{2}+\binom{m_3}{2}}$$

$$= \binom{k}{3}\left(\frac{n}{k}\right)^{4k} 2^{-4\binom{k}{2}} \cdot \prod_{j=1}^{3}\binom{jk}{m_j - 3}\left(\frac{k}{n}\right)^{m_j} 2^{\binom{m_j}{2}}.$$

$$= f(n,k) \cdot \prod_{j=1}^{3}\binom{jk}{m_j - 3}\left(\frac{k}{n}\right)^{m_j - 3} 2^{\binom{m_j}{2}-3}. \tag{2.4}$$

We set

$$g_j(m) := \binom{jk}{m - 3}\left(\frac{k}{n}\right)^{m-3} 2^{\binom{m}{2}-3}.$$

We will show that $g_j(m) \leqslant 1$ for all $j \in \{1, 2, 3\}$ and $3 \leqslant m \leqslant k$, provided k is large enough. Indeed, for $3 \leqslant m \leqslant 15k/16$ and k large enough, we have

$$g_j(m) \leqslant (jk)^{m-3} \cdot 2^{-((k+9)/2)(m-3)} \cdot 2^{((m+2)(m-3))/2}$$
$$= (jk \cdot 2^{-(k+9)/2+(m+2)/2})^{m-3} \leqslant (jk \cdot 2^{-k/32-7/2})^{m-3} \leqslant 1. \tag{2.5}$$

For $15k/16 \leqslant m \leqslant k$ and k large enough, we obtain

$$g_j(m) \leqslant 2^{jk}(2^{-(k+9)/2+(m+2)/2})^{m-3} \leqslant 2^{3k}(2^{-7/2})^{m-3} \leqslant 2^{3k}(2^{-7/2})^{15k/16-3} \leqslant 1. \tag{2.6}$$

Now, (2.4), (2.5) and (2.6) imply that

$$T(\mathcal{F}_2^4(m_1, m_2, m_3)) \leqslant f(n,k)$$

for any sequence (m_1, m_2, m_3), provided k is large enough. Due to (2.3), we conclude that $T(\mathcal{F}_2^4) \leqslant k^3 f(n,k)$. $\qquad\square$

Finally, we show that the Advanced Weak Win Criterion (Theorem 2.1) applies with $p = 4$.

Corollary 2.4. *For n large enough,*

$$T(\mathcal{F}) > 16|X|\left((T(\mathcal{F}_2^4))^{1/4} + \frac{1}{4}\right).$$

Proof. By Lemma 2.3, the definition of $f(n,k)$ and the fact that $|X| \leqslant n^2$, we get that

$$\frac{16\,|X|\,(T(\mathcal{F}_2^4))^{1/4}}{T(\mathcal{F})} \leqslant \frac{16\,|X|\,(k^3\,f(n,k))^{1/4}}{T(\mathcal{F})} \leqslant \frac{16\,n^2\,k^{3/4}\,\left(k^3\,(n/k)^{4k-9}\,2^{-4\binom{k}{2}+9}\right)^{1/4}}{T(\mathcal{F})}$$

$$\leqslant \frac{16 \cdot 2^{9/4}\,n^2\,k^{6/4}\,(n/k)^{k-9/4}\,2^{-\binom{k}{2}}}{(n/k)^k \cdot 2^{-\binom{k}{2}}} \leqslant \frac{100n^2\,k^{6/4}}{(n/k)^{9/4}}$$

$$= \frac{100k^{15/4}}{n^{1/4}} < 100 \cdot 2^{-k/8+(14/4)\log(k)} = o(1).$$

By (2.2),

$$\frac{1/4 \cdot 16|X|}{T(\mathcal{F})} = o(1),$$

which concludes the proof.   □

3. Proof of Theorem 1.3

We first need some notation. In an $(a : b)$ Maker–Breaker game $(X, \mathcal{F})$ Maker claims a elements and Breaker claims b elements in each round. A game is called *biased* if $a \neq 1$ or $b \neq 1$.

In order to provide OBreaker with a winning strategy for the game $\mathcal{T}_o(k,n)$ we associate with $\mathcal{T}_o(k,n)$ an auxiliary *biased* Maker–Breaker game. In the first step of the proof we show that Breaker has a strategy to win the auxiliary game, and in the second step we prove that this strategy directly gives him a winning strategy for $\mathcal{T}_o(k,n)$.

We will make use of the Generalized Erdős–Selfridge Criterion proved by Beck [6].

Theorem 3.1 (Generalized Erdős–Selfridge Criterion). *Let X be a finite set and let $\mathcal{F} \subseteq 2^X$. If*

$$\sum_{F \in \mathcal{F}} (1+b)^{-|F|/a} < \frac{1}{1+b},$$

then Breaker has a winning strategy in the $(a : b)$ Maker–Breaker game $(X, \mathcal{F})$.

Let T_k be some tournament on k vertices. Consider the $(2 : 1)$ Maker–Breaker game

$$\mathcal{H}(T_k) = \mathcal{H}(T_k, n) = (X, \mathcal{F}(T_k)),$$

where

$$X := \{(u,v) : u, v \in V(K_n),\ u \neq v\}$$

is the board of the game consisting of $|X| = n(n-1)$ elements, and

$$\mathcal{F}(T_k) := \{S \subseteq X : S \text{ is a copy of } T_k\}$$

is the family of winning sets.

Claim 3.2. *For large enough n and $k \geqslant 4 \log n + 2$, Breaker has a winning strategy in $\mathcal{H}(T_k)$, for any tournament T_k on k vertices.*

Proof. We check that Theorem 3.1 applies. By definition, $|\mathscr{F}(T_k)| < n^k$, and $|F| = \binom{k}{2}$ for every $F \in \mathscr{F}(T_k)$. Therefore, and since $k \geqslant 4 \log n + 2$,

$$\sum_{F \in \mathcal{F}} 2^{-|F|/2} < n^k \cdot 2^{-k(k-1)/4} \leqslant \frac{1}{2}. \qquad \square$$

We conclude the proof of Theorem 1.3 with the following lemma.

Lemma 3.3. *Suppose T_k is a tournament on k vertices such that Breaker has a strategy to win the game $\mathcal{H}(T_k)$. Then OBreaker has a winning strategy for the game $\mathcal{T}_o(k, n)$.*

Proof. We first need some notation. By *directing an edge* (u, v) we mean that we direct the edge spanned by u and v from u to v. We note that in each round of $\mathcal{H}(T_k)$, Maker is allowed to choose either two, one, or zero elements. (Otherwise she can just claim additional, arbitrary elements, and then follow her strategy. If this strategy calls for something she occupied before, she takes an arbitrary element; no extra element is disadvantageous for her.)

Suppose, for a contradiction, that OMaker wins the game $\mathcal{T}_o(k, n)$. That is, OMaker has a winning strategy $\mathcal{S}(T)$ for every tournament T on k vertices that OBreaker may choose as the goal tournament. Let $\mathcal{S} = \mathcal{S}(T_k)$. We now describe a winning strategy $\mathcal{S}'$ for Maker in $\mathcal{H}(T_k)$. During the play, Maker simulates (in parallel) a play of the game $\mathcal{T}_o(k, n)$, where she is playing the role of both OMaker and OBreaker, with the constraint that OMaker's and OBreaker's directed edges in $\mathcal{T}_o(k, n)$ represent Maker's edges in the actual game $\mathcal{H}(T_k)$. More precisely, Maker maintains the invariant that after each of her moves in $\mathcal{H}(T_k)$, every pair $u, v \in V$ has the property that

(i) in $\mathcal{H}(T_k)$, if Breaker owns the element (u, v) then Maker owns (v, u), and

(ii) in $\mathcal{T}_o(k, n)$, there is a directed edge from u to v if and only if Maker has claimed the element (u, v) in $\mathcal{H}(T_k)$.

At the beginning of the game, Maker chooses as OBreaker the tournament T_k as goal tournament in $\mathcal{T}_o(k, n)$. Let (a, b) be the edge $\mathcal{S}$ tells OMaker to direct in her first move. Then Maker claims the element (a, b) in the actual game $\mathcal{H}(T_k)$ (at this point she does not make use of the possibility of occupying two elements), and directs (a, b) in the parallel game $\mathcal{T}_o(k, n)$ as OMaker.

Suppose that i rounds have been played, and let $(u, v) \in X$ denote the element Breaker chose in his ith move. If Maker has already claimed (v, u) in a previous round then she does not claim a single element. Otherwise, as her $(i + 1)$st move, she first occupies the element (v, u). In the parallel game $\mathcal{T}_o(k, n)$, she directs as OBreaker the edge (v, u). Then she identifies the edge (x, y) that the strategy $\mathcal{S}$ tells OMaker to direct. Finally, she directs (x, y) as OMaker in $\mathcal{T}_o(k, n)$, and claims the element (x, y) in $\mathcal{H}(T_k)$.

We note that the invariants (i) and (ii) remain satisfied after Maker's $(i + 1)$st move. So, by following $\mathcal{S}'$, Maker can guarantee that at the end of the game, these invariants still

hold. Since by assumption $\mathcal{S}$ is a winning strategy, the final digraph in $\mathcal{T}_o(k,n)$ contains a copy of T_k. Together with invariant (ii) this yields that Maker possesses all elements of some winning set in $\mathcal{H}(T_k)$. This contradicts the assumption that Breaker has a winning strategy for $\mathcal{H}(T_k)$. $\square$

4. Concluding remarks and open problems

Random graph intuition. As noted in the Introduction, our lower bound on k_t seemingly refutes the random graph intuition. However, to prove Theorem 1.2, we reduced the tournament game to a classical Maker–Breaker game on the board of the complete k-partite graph with vertex classes $V_1, \ldots, V_k$, where each V_i has size roughly n/k. Let us consider the corresponding random game on this reduced board, where in every round, each player claims a random unclaimed edge. We briefly sketch why the threshold where this game turns from a RandomMaker's win to a RandomBreaker's win is around $(2 - o(1)) \log n$. By standard techniques it can be shown that the expected number of k-cliques in RandomMaker's graph is

$$(1 + o(1)) \left(\frac{n}{k} \right)^k 2^{-\binom{k}{2}},$$

which jumps from below one to above one at $k = 2 \log n - 2 \log \log n - 1 + o(1)$. Analogously to the proof of the concentration result for the largest clique size in the random graph $G(n, m)$ [10], it can be shown that if

$$\left\{ \begin{array}{l} k \leqslant (2 - o(1)) \log n \\ k \geqslant (2 - o(1)) \log n \end{array} \right\},$$

then

$$\text{RandomMaker's graph} \left\{ \begin{array}{l} \text{contains} \\ \text{does not contain} \end{array} \right\} \text{ a } k\text{-clique a.a.s.}$$

Thus, from a more subtle point of view, the random graph intuition can be considered valid.

Tournament game. The strategy for Maker in Theorem 1.2 is independent of the actual tournament T_k Breaker chooses. On the other hand, the upper bound in (1.1) is the upper bound from the k-clique game. This means that for $k > 2 \log n - 2 \log \log n + o(1)$, Breaker has a strategy to prevent Maker from building *any* given tournament on k vertices. So both the lower and the upper bounds are universal for all tournaments on k vertices. On the other hand, there is a tournament on k_{cl} vertices which Maker can build. Consider the transitive tournament on the vertex set $\{u_1, \ldots, u_k\}$, where, say, for all indices $1 \leqslant i < j \leqslant k$ the edge between u_i and u_j is directed from u_i to u_j. Now let $\{v_1, \ldots, v_n\}$ be an arbitrary enumeration of the vertices in K_n. Then Maker can just follow the strategy provided by the k-clique game. Whenever this strategy tells her to claim the edge $\{v_i, v_j\}$ for $i < j$, she chooses the direction (v_i, v_j). It would be interesting to determine whether all tournaments are 'equally hard' for Maker. Therefore, we pose the following question.

Problem 4.1. *Does there exist a tournament T_k on $k \leqslant k_{cl}$ vertices such that Breaker has a strategy to prevent Maker from building T_k in the tournament game?*

Note that a negative answer to this question, together with Theorem 1.1, would give us the exact value of k_t. But even if there exists a tournament on at most k_{cl} vertices which Breaker can prevent, it is of particular interest to get rid of the gap in the constant term.

Problem 4.2. *Determine the exact value of k_t.*

Orientation tournament game. A similar discussion arises for the orientation version $\mathcal{T}_o(k,n)$. Since we use Breaker's strategy in the game $\mathcal{H}(T_k)$ as a black box, by Claim 3.2, OBreaker can win $\mathcal{T}_o(k,n)$ for $k \geqslant 4 \log n + 2$, for *any* choice of the tournament T_k on k vertices.

Problem 4.3. *Do there exist an integer $k_1 \in \mathbb{N}$ and two (non-isomorphic) tournaments T and T' on k_1 vertices such that in the respective orientation game, OMaker has a strategy to build T while OBreaker has a strategy to prevent OMaker from building T'?*

Furthermore, the lower and the upper bounds for $k_o(n)$ are a factor of two apart. The probabilistic analysis would suggest the breakpoint to be around $2 \log n$.

Problem 4.4. *Determine the constant $2 \leqslant c \leqslant 4$ such that $k_o(n) = (c + o(1)) \log n$.*

Universal tournament game. As noted in the Introduction, Maker has a strategy to occupy a copy of every tournament on k vertices for $k \leqslant (1/2 - o(1)) \log n$. Beck conjectured that this result is not best possible and posed the following problem.

Problem 4.5 ([6], p. 457). *Determine the largest k_u such that Maker has a strategy such that at the end of the game, her digraph contains* every *tournament on k_u vertices.*

For the upper bound, we can show that $k_u \leqslant (1 + o(1)) \log n$. We note that the number of unlabelled k-tournaments is at least

$$c(k) := 2^{\binom{k}{2}}/k! > 2^{\binom{k}{2} - k \log k}.$$

By assumption, Maker has a strategy to occupy a copy of every tournament on k_u vertices. Hence, at the end of the game, the underlying graph of Maker's graph contains $c(k_u)$ (not necessarily edge-disjoint) distinct cliques. However, a result of Bednarska and Łuczak (see Lemma 5 in [7]) asserts that there is some $k = (1 + o(1)) \log n$ such that in the ordinary graph game (where no edge orientations are involved), Breaker has a strategy to prevent Maker from claiming more than $c(k)$ distinct k-cliques. Thus, $k_u \leqslant (1 + o(1)) \log n$. To the best of our knowledge, nothing better is known for the universal tournament game.

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