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ON A BANACH SPACE PROPERTY OF TRUBNIKOV

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Trubnikov's property $(U, \lambda, \alpha, \beta)$ is investigated. In particular, it is shown that property $(U, \lambda, \alpha, \alpha - 1)$ with $\alpha > 1$ is equivalent to α-uniform smoothness. It is also shown that property $(U, 1, \alpha, 1)$ with $\alpha > 1$ is equivalent to the space being a Hilbert space. The dual property $(U^*, \lambda, \alpha, \alpha - 1)$ is also introduced and it is shown that a Banach space X has $(U^*, \lambda, \alpha, \alpha - 1)$ if and only if X is α-uniformly convex.

1. INTRODUCTION

Let X be a real Banach space and let α, λ, β be real numbers with $\alpha \geqslant 1$. Trubnikov [6] introduced the concept of property $(U, \lambda, \alpha, \beta)$ as follows.

DEFINITION 1.1: A Banach space X has property $(U, \lambda, \alpha, \beta)$ if

$$\|x + y\|^\alpha + \lambda\|x - y\|^\alpha \geqslant 2^\beta\big(\|x\|^\alpha + \|y\|^\alpha\big), \quad x, y \in X. \tag{1.1}$$

Setting $u := x + y$ and $v := x - y$ in (1.1), we see that property $(U, \lambda, \alpha, \beta)$ is equivalent to the following inequality:

$$\|x + y\|^\alpha + \|x - y\|^\alpha \leqslant 2^{\alpha-\beta}\big(\|x\|^\alpha + \lambda\|y\|^\alpha\big), \quad x, y \in X. \tag{1.2}$$

Hence each Banach space has property $(U, 1, \alpha, 0)$; this is because

$$\|u + v\|^\alpha \leqslant 2^{\alpha-1}\big(\|u\|^\alpha + \|v\|^\alpha\big), \quad u, v \in X, \tag{1.3}$$

which implies that

$$\|u + v\|^\alpha + \|u - v\|^2 \leqslant 2^\alpha\big(\|u\|^\alpha + \|v\|^\alpha\big), \quad u, v \in X.$$

Note that we can always assume that $\lambda \geqslant 1$. We can also assume that $\beta \geqslant 0$. Indeed, the best possible value of β such that (1.1) holds is given by

$$2^\beta = \inf\Big\{\frac{\|x + y\|^\alpha + \lambda\|x - y\|^\alpha}{\|x\|^\alpha + \|y\|^\alpha} : \|x\|^\alpha + \|y\|^\alpha \neq 0\Big\}. \tag{1.4}$$

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Since $\lambda \geqslant 1$, we have from (1.1) that, for all $x, y \in X$,

$$\|x+y\|^\alpha + \lambda\|x-y\|^\alpha \geqslant \|x+y\|^\alpha + \|x-y\|^\alpha \geqslant \|x\|^\alpha + \|y\|^\alpha.$$

This implies $2^\beta \geqslant 1$; hence $\beta \geqslant 0$. Note that, given λ and α, a Banach space X has property $(U, \lambda, \alpha, \beta)$ if and only if the infimum on the right-hand side of (1.4) is positive and β is given by (1.4).

We now recall the following identity in a Hilbert space H:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad t \in [0,1],\ x, y \in H. \tag{1.5}$$

We also recall some inequalities in l^p and L^p spaces (see [**7**, **3**] for more details).

1. If $2 \leqslant p < \infty$, then for all $x, y \in l^p$ (or L^p) and $t \in [0,1]$, there holds the inequality:

$$\|tx + (1-t)y\|^2 \geqslant t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)(p-1)\|x-y\|^2. \tag{1.6}$$

2. If $1 < p < 2$, then for all $x, y \in l^p$ (or L^p) and $t \in [0,1]$, there holds the inequality:

$$\|tx + (1-t)y\|^p \geqslant t\|x\|^p + (1-t)\|y\|^p - c_p W_p(t)\|x-y\|^p, \tag{1.7}$$

where

$$W_p(t) = t(1-t)^p + t^p(1-t) \quad \text{and} \quad c_p = \frac{1+s_p^{p-1}}{(1+s_p)^{p-1}} > 1,$$

with s_p being the unique solution of the equation

$$(p-2)s^{p-1} + (p-1)s^{p-2} - 1 = 0, \quad 0 < s < 1.$$

From (1.5)–(1.7) we can draw the following conclusion.

Proposition 1.2.

(i) *A Hilbert space H has property $(U, 1, 2, 1)$.*

(ii) *If $2 \leqslant p < \infty$, then both l^p and L^p have property $(U, p-1, 2, 1)$.*

(iii) *If $1 < p < 2$, then both l^p and L^p have property $(U, c_p, p, p-1)$.*

Remark 1.3. For the spaces L^p or l^p, Clarkson's inequalities [**2**, Theorem 2] also relate to property $(U, \lambda, \alpha, \beta)$. Indeed, Clarkson proves the following inequalities for L^p or l^p with $2 \leqslant p < \infty$:

$$2\left(\|x\|^p + \|y\|^p\right) \leqslant \|x+y\|^p + \|x-y\|^p \leqslant 2^{p-1}\left(\|x\|^p + \|y\|^p\right). \tag{1.8}$$

(For $1 < p < 2$ these inequalities hold in the reverse sense.) Hence L^p or l^p has $(U, 1, p, 1)$ for $2 \leqslant p < \infty$ and $(U, 1, p, p-1)$ for $1 < p < 2$. Note that $(U, 1, p, p-1)$ implies $(U, c_p, p, p-1)$ for $c_p > 1$.

Trubnikov [**6**] introduced property $(U, \lambda, \alpha, \beta)$ to obtain the convergence rate of an iterative approximation method for nonlinear equations in Banach spaces. This direction has recently been pursued by several authors (see, for example, Schu [**5**] and references therein).

The purpose of this paper is to further study property $(U, \lambda, \alpha, \beta)$. In particular, we show that a Banach space X has property $(U, \lambda, \alpha, \alpha - 1)$ with $1 < \alpha \leqslant 2$ if and only if X is α-uniformly smooth and that property $(U, 1, \alpha, 1)$ with $\alpha > 1$ implies that X is a Hilbert space. (For some related information on uniform convexity and uniform smoothness, the reader is referred to [**1, 2, 4**].)

2. $(U, \lambda, \alpha, \alpha - 1)$ and α-Uniform Smoothness

Recall that the *modulus of smoothness* of a Banach space X is defined by

$$\rho_X(\tau) = \sup\Big\{\frac{1}{2}\big(\|x + \tau y\| + \|x - \tau y\|\big) - 1 : \|x\| = \|y\| = 1\Big\}, \quad \tau > 0.$$

A Banach space X is said to be *uniformly smooth* if

$$\lim_{\tau \to 0^+} \frac{\rho_X(\tau)}{\tau} = 0.$$

For a given number $q > 1$, recall that X is q-uniformly smooth if, for some constant $c > 0$,

$$\rho_X(\tau) \leqslant c\tau^q, \quad \tau > 0.$$

It is known that $1 < q \leqslant 2$. It is also known that a Hilbert space and l^p (or L^p) for $2 \leqslant p < \infty$ are 2-uniformly smooth; while if $1 < p < 2$, l^p (or L^p) is p-uniformly smooth (see [**7**]).

The following is an inequality characterisation of q-uniform smoothness (see [**7, 3**]).

Proposition 2.1. *Let X be a Banach space and let $q \in (1, 2]$ be a real number. Then X is q-uniformly smooth if and only if there exists a constant $c > 0$ with the property:*

$$\big\|tx + (1-t)y\big\|^q \geqslant t\|x\|^q + (1-t)\|y\|^q - cW_q(t)\|x - y\|^q \tag{2.1}$$

for all $t \in [0, 1]$ and $x, y \in X$, where $W_q(t) = t(1-t)^q + t^q(1-t)$.

Now we state and prove the main result of this paper.

Theorem 2.2. *Let X be a Banach space and let $\alpha > 1$ be a real number. Then X has property $(U, \lambda, \alpha, \alpha - 1)$ for some $\lambda > 0$ if and only of X is α-uniformly smooth.*

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PROOF: Assume that X is α-uniformly smooth. Substituting $t = 1/2$ in (2.1) with q replaced with α, we obtain

$$\|x+y\|^\alpha \geqslant 2^{\alpha-1}\big(\|x\|^\alpha + \|y\|^\alpha\big) - c\|x-y\|^\alpha, \quad x, y \in X.$$

It follows that X has property $(U, \lambda, \alpha, \alpha-1)$ with $\lambda = c$.

Conversely, assume that X has property $(U, \lambda, \alpha, \alpha-1)$ for some $\lambda > 0$ and $\alpha > 1$. By (1.2), we have (note that $\alpha - \beta = 1$)

$$\|x+\tau y\|^\alpha + \|x-\tau y\|^\alpha \leqslant 2\big(\|x\|^\alpha + \lambda\tau^\alpha\|y\|^\alpha\big), \quad x, y \in X, \ \tau > 0.$$

In particular,

$$\text{(2.2)} \qquad \frac{1}{2}\big(\|x+\tau y\|^\alpha + \|x-\tau y\|^\alpha\big) - 1 \leqslant \lambda\tau^\alpha, \quad x, y \in X, \ \|x\| = \|y\| = 1, \ \tau > 0.$$

Let

$$D = \big\{(t,s) \in \mathbf{R}^2 : 0 \leqslant t, s \leqslant 1+\tau, t+s \geqslant 2\big\},$$

where we assume $\tau \in [0,1]$.

CLAIM. $\alpha\big[(t+s)/2 - 1\big] \leqslant (t^\alpha + s^\alpha)/2 - 1$ on D.

The proof of the Claim is elementary. We include it for completeness. Let

$$h(t,s) = \alpha\Big[\frac{1}{2}(t+s) - 1\Big] - \frac{1}{2}(t^\alpha + s^\alpha) + 1, \quad (t,s) \in D.$$

We shall show that $\max\big\{h(t,s) : (t,s) \in D\big\} \leqslant 0$. Since it is easy to see that h does not have critical points in the interior of D, it suffices to show that $\max\big\{h(t,s) : (t,s) \in \partial D\big\} \leqslant 0$, where ∂D is the boundary of D given by $\partial D = D_1 \cup D_2 \cup D_3$, where

$$\begin{aligned} D_1 &= \big\{(t, 1+\tau) : 1-\tau \leqslant t \leqslant 1+\tau\big\}, \\ D_2 &= \big\{(1+\tau, s) : 1-\tau \leqslant s \leqslant 1+\tau\big\}, \\ D_3 &= \big\{(t,s) : t+s = 2, \ 1-\tau \leqslant t, s \leqslant 1+\tau\big\}. \end{aligned}$$

On D_1 we have

$$\overline{h}(t) \equiv h(t, 1+\tau) = \alpha\Big[\frac{1}{2}(t+1+\tau) - 1\Big] - \frac{1}{2}\big[t^\alpha + (1+\tau)^\alpha\big] + 1, \quad 1-\tau \leqslant t \leqslant 1+\tau.$$

Since $\overline{h}'(t) = (\alpha/2)(1 - t^{\alpha-1})$, $\overline{h}$ is decreasing for $t \geqslant 1$ and increasing for $t \leqslant 1$. Hence

$$\overline{h}(t) \leqslant \overline{h}(1) = \frac{1}{2}\big[(1+\alpha\tau) - (1+\tau)^\alpha\big] \leqslant 0.$$

Similarly, we can prove that $h(t,s) \leqslant 0$ on D_2 and D_3. Hence $\max\big\{h(t,s) : (t,s) \in \partial D\big\} \leqslant 0$ and the Claim has thus been proved.

Now for $\|x\| = \|y\| = 1$, $\|x \pm \tau y\| \leqslant 1 + \tau$ and $\|x + \tau y\| + \|x - \tau y\| \geqslant 2$, it follows from the Claim and (2.2) that

$$\frac{1}{2}\big(\|x + \tau y\| + \|x - \tau y\|\big) - 1 \leqslant \frac{\lambda}{\alpha}\tau^{\alpha}, \quad x, y \in X,\ \|x\| = \|y\| = 1.$$

This implies that

$$\rho_X(\tau) \leqslant \frac{\lambda}{\alpha}\tau^{\alpha}, \quad 0 \leqslant \tau \leqslant 1$$

and X is α-uniformly smooth. ∎

We conclude this section by showing that in property $(U, \lambda, 1, \beta)$ one can assume $\beta = \alpha - 1 = 0$.

Theorem 2.3. *Assume that a Banach space X has $(U, \lambda, 1, \beta)$. Then X also has $(U, \lambda, 1, 0)$.*

Proof: By the definition of property $(U, \lambda, 1, \beta)$, we see that the largest $\widetilde{\beta}$ such that property $(U, \lambda, 1, \widetilde{\beta})$ holds for X is determined by

$$2^{\widetilde{\beta}} = \inf\Big\{\frac{\|x + y\| + \lambda\|x - y\|}{\|x\| + \|y\|} : x, y \in X,\ \|x\| + \|y\| \neq 0\Big\}. \tag{2.3}$$

All we need to show is that the infimum in (2.3) equals 1 (and hence $\widetilde{\beta} = 0 = \alpha - 1$). Indeed, if we rewrite the function on the right-hand side of (2.3) as

$$1 + \frac{\|x + y\| + \lambda\|x - y\| - \big(\|x\| + \|y\|\big)}{\|x\| + \|y\|}$$

and observe (note that $\lambda \geqslant 1$) that

$$\|x + y\| + \lambda\|x - y\| \geqslant \|x + y\| + \|x - y\| \geqslant \|x\| + \|y\|,$$

we conclude that the infimum in (2.3) is attained at any point (x, x) with $x \neq 0$ and that it does indeed equal 1. ∎

3. Property $(U, 1, \alpha, 1)$

A Hilbert space has property $(U, 1, 2, 1)$. The following result shows that property $(U, 1, \alpha, 1)$ with $1 < \alpha \leqslant 2$ characterises Hilbert spaces. Note that Clarkson's inequalities (1.8) show that if L^p (or l^p) has property $(U, 1, p, 1)$ with $1 < p \leqslant 2$, then $p = 2$ and L^p reduces to a Hilbert space. Below we extend this to the general case.

Theorem 3.1. *If a Banach space X has property $(U, 1, \alpha, 1)$ for some $1 < \alpha \leqslant 2$, then X is a Hilbert space.*

Proof: By property $(U, 1, \alpha, 1)$ we have

$$\|x + y\|^{\alpha} + \|x - y\|^{\alpha} \geqslant 2\big(\|x\|^{\alpha} + \|y\|^{\alpha}\big), \quad x, y \in X. \tag{3.1}$$

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Taking $x = y$, we see that $2^\alpha \geqslant 4$ which implies $\alpha \geqslant 2$. Hence $\alpha = 2$. So we can rewrite (3.1) as

(3.2) $$\|x+y\|^2 + \|x-y\|^2 \geqslant 2(\|x\|^2 + \|y\|^2), \quad x, y \in X.$$

Setting $x = (u+v)/2$ and $y = (u-v)/2$ in (3.2), we obtain

(3.3) $$\|u+v\|^2 + \|u-v\|^2 \leqslant 2(\|u\|^2 + \|v\|^2), \quad u, v \in X.$$

Taken together, the inequalities (3.2) and (3.3) are equivalent to the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad x, y \in X.$$

Therefore X is a Hilbert space. ∎

4. Dual Property

In this section we introduce the dual property of $(U, \lambda, \alpha, \beta)$.

Definition 4.1: A Banach space X is said to have property dual $(U, \lambda, \alpha, \beta)$, denoted by $(U^*, \lambda, \alpha, \beta)$, if, for some $\lambda > 0$, $\alpha > 1$ and $\beta > 0$, there holds

(4.1) $$\|x+y\|^\alpha + \lambda\|x-y\|^\alpha \leqslant 2^\beta(\|x\|^\alpha + \|y\|^\alpha), \quad x, y \in X.$$

In analogy with Theorem 2.2, we shall show that property $(U^*, \lambda, \alpha, \beta)$ is equivalent to α-uniform convexity. But first recall that the *modulus of convexity* of X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\right\}.$$

Recall also that X is *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$ and for $1 < p < \infty$, X is *p-uniformly convex* if, for some constant $c > 0$,

$$\delta_X(\varepsilon) \geqslant c\varepsilon^p, \quad \varepsilon > 0.$$

It is known that a Hilbert space and l^p (or L^p) for $1 < p \leqslant 2$ are 2-uniformly convex; while l^p (or L^p) for $2 \leqslant p < \infty$ is p-uniformly convex (see [**7**]).

Clarkson's inequalities (1.8) show that as a uniformly convex Banach space, L^p (or l^p) also has the dual property $(U^*, 1, p, p-1)$ for $2 \leqslant p < \infty$ and $(U^*, 1, p, 1)$ for $1 < p < 2$. Note that L^p (or l^p) is both uniformly convex and uniformly smooth. Our purpose in this section is to extend these properties for L^p (or l^p) to the more general class of α-uniformly convex Banach spaces. To this end, we need an inequality characterisation of p-uniform convexity (see [**7, 3**]).

Proposition 4.2. *Let X be a Banach space and let $1 < p < \infty$ be a real number. Then X is p-uniformly convex if and only if there exists a constant $c > 0$ with the property:*

(4.2) $$\|tx + (1-t)y\|^p \leqslant t\|x\|^p + (1-t)\|y\|^p - cW_p(t)\|x-y\|^p$$

for all $t \in [0,1]$ *and* $x, y \in X$, *where* $W_p(t) = t^p(1-t) + t(1-t)^p$.

THEOREM 4.3. *Let* X *be a Banach space and let* $1 < \alpha < \infty$ *be a real number. Then* X *has property* $(U^*, \lambda, \alpha, \alpha - 1)$ *if and only of* X *is* α*-uniformly convex.*

PROOF: Assume first that X is α-uniformly convex. From (4.2) it follows that

$$\left\|\frac{x+y}{2}\right\|^\alpha \leqslant \frac{1}{2}\|x\|^\alpha + \frac{1}{2}\|y\|^\alpha - \frac{c}{2^\alpha}\|x-y\|^\alpha, \quad x, y \in X.$$

This implies that X has $(U^*, \lambda, \alpha, \alpha - 1)$ with $\lambda = c$.

Conversely, assume that X has $(U^*, \lambda, \alpha, \alpha - 1)$. If $\|x\| = \|y\| = 1$ and $\|x - y\| = \varepsilon$, we have by (4.1),

$$\left\|\frac{x+y}{2}\right\|^\alpha \leqslant 1 - \lambda\left(\frac{\varepsilon}{2}\right)^\alpha.$$

This implies that

$$\delta_X(\varepsilon) \geqslant 1 - \left[1 - \lambda\left(\frac{\varepsilon}{2}\right)^\alpha\right]^{1/\alpha} \geqslant \frac{\lambda}{\alpha}\left(\frac{\varepsilon}{2}\right)^\alpha.$$

Hence X is α-uniformly convex. ∎

COROLLARY 4.4. *Let* X *be a Banach space and let* $\alpha > 1$ *be a real number. Then* X *has property* $(U, \lambda, \alpha, \alpha - 1)$ *for some* $\lambda > 0$ *if and only if* X^* *has property* $(U^*, \lambda', \alpha', \alpha' - 1)$ *for some* $\lambda' > 0$, *where* $1/\alpha + 1/a' = 1$.

PROOF: By Theorem 2.2, we see that X has $(U, \lambda, \alpha, \alpha - 1)$ if and only if X is α-uniformly smooth, which is equivalent to X^* being α'-uniformly convex, which is in turn, by Theorem 4.3, equivalent to X^* having property $(U^*, \lambda', \alpha', \alpha' - 1)$ for some $\lambda' > 0$. ∎

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