Canad. Math. Bull. Vol. **18** (3), 1975

COMMENTS ON THE SPINOR STRUCTURE OF SPACE-TIME*

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ABSTRACT. A simpler proof of the theorem on the spinor structure of space-time is given. Some geometrical insights are provided.

The definition and the implications of the existence of a spinor structure on a space-time* have been elucidated by Geroch [1]. By construction, he has established that *a necessary and sufficient condition for a non-compact space-time M to admit a spinor structure is that there exists on M a global system of orthonormal tetrads* [1]. The same is true for a compact space-time M [2]. This short note provides a simpler proof of the theorem and some geometrical insights into the theorem are given.

It is well-known that the space-time M possesses a spinor structure if and only if the second Stiefel-Whitney class of M, $w_2(M)$, vanishes [3], where the q-th Stiefel-Whitney class is a characteristic cohomology class of $H^q(M, Z_2)$. The geometrical meaning for the condition $w_q(M)=0$, for $q=1, 2, 3, 4$, is equivalent to the existence of a continuous field of orthogonal $(4-(q-1))$-frames over the q-dimensional skeleton of M [5]. Consequently, saying that the space-time M admits a spinor structure is equivalent to claiming that an orthogonal triad can be placed at each point of every two-surface of M in a continuous way. To say that there exists on the space-time M a global system of orthogonal tetrads is tantamount to claiming that $w_q(M)=0$ for all q, $q=1, 2, 3, 4$. It is then obvious that we only have to prove that if the space-time M admits a spinor structure, then there exists on M a global system of orthonormal tetrads.

The q-th Stiefel-Whitney class of M is equal to the obstruction of the vector bundle ξ over M with Z_2 coefficients [4]. But the obstruction $\theta_q(\xi)$, is a characteristic cohomology class of $H^q(M, \pi_{q-1}(F))$. For the bundle of orthonormal frames on M, the structure group is the proper Lorentz group, L_0, which is homeomorphic to $P^3 \times R^3$ where P^3 is the 3-dimensional real projective space. Since the space-time M is orientable, we have $w_1(M)=0$. $w_2(M)=0$ because the space-time admits a spinor structure. $\theta_3(\xi) \in H^3(M, \pi_2(L_0))$, but $\pi_2(L_0)=0$, thus $\theta_3(\xi)=0$. Consequently $w_3(M)=0$. For non-compact space-time M, $H^4(M)=0$, thus $w_4(M)=0$.

* Partly supported by the National Science Foundation while the author was at Physics Department, Syracuse University, Syracuse, N.Y. 13210.

* A space-time M is an orientable 4-dim differentiable manifold with a metric of Lorentz signature $(-, +, +, +)$.

If M is a compact space-time, the Euler-Poincaré characteristic class of M vanishes [5, p. 203], thus $w_4(M)=0$. The proof is completed.

It is evident from the proof that the theorem is not true if the dimension of the space-time is greater than four.

References

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