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TWISTED TRIPLE PRODUCT p-ADIC L-FUNCTIONS AND HIRZEBRUCH–ZAGIER CYCLES

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Abstract Let L/F be a quadratic extension of totally real number fields. For any prime p unramified in L, we construct a p-adic L-function interpolating the central values of the twisted triple product L-functions attached to a p-nearly ordinary family of unitary cuspidal automorphic representations of $\mathrm{Res}_{L\times F/F}(\mathrm{GL}_2)$. Furthermore, when $L/\mathbb{Q}$ is a real quadratic number field and p is a split prime, we prove a p-adic Gross–Zagier formula relating the values of the p-adic L-function outside the range of interpolation to the syntomic Abel–Jacobi image of generalized Hirzebruch–Zagier cycles.

Keywords: twisted triple product L-functions; syntomic Abel–Jacobi map; Hirzebruch–Zagier cycles

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Contents

1. Introduction

This article is part of the program pioneered by Darmon and Rotger in [7], [8] devoted to studying the p-adic variation of arithmetic invariants for automorphic representations on higher rank groups, with the aim of shedding some light on the relation between p-adic L-functions and Euler systems with applications to the equivariant BSD-conjecture.

Given a totally real number field F, the starting point of the program is to find a reductive group G having $\mathrm{GL}_{2,F}$ as a direct factor together with an automorphic L-function for which there is an explicit formula for the central L-value. The expectation is that there exists a transcendental period for which the ratio between the special value and the period becomes a meaningful algebraic number varying p-adically. More precisely, these modified central L-values should determine a rigid-analytic meromorphic function by interpolation. In the present work, we consider the group $G_{L\times F} = \mathrm{Res}_{L\times F/F}(\mathrm{GL}_{2,L\times F})$ for L/F a quadratic extension of totally real number fields. Piatetski-Shapiro and Rallis [31] studied the analytic properties of the twisted triple product L-function attached to cuspidal representations of $G_{L\times F}$ and Ichino [18] proved a formula for its central value, generalizing earlier work of Harris–Kudla [12]. The first part of the paper is devoted to the construction of a p-adic L-function, called *twisted triple product p-adic L-function*.

Several far-reaching conjectures suggest a strong link between automorphic L-functions and algebraic cycles: relevant cycles should live on a Kuga–Sato variety whose étale cohomology realizes the Galois representation (conjecturally) attached to the automorphic representation of G, out of which one constructs the L-function. Furthermore, as the central L-values should vary p-adically after a modification by an appropriate period, by tinkering with these cycles it should be possible to produce Galois cohomology classes that p-adically interpolate into a *big cohomology class*, giving rise to the p-adic L-function via Perrin-Riou's machinery. Note that such p-adic L-function and big cohomology class are defined using completely different inputs, an automorphic and a geometric one; the sole fact that in certain cases it is possible to prove these approaches produce the same object is in itself an amazing confirmation of the power of the existing conjectures.

The relation between p-adic L-functions and algebraic cycles, as we just sketched it, can be very hard to prove since it requires, among various things, a deep understanding of the cohomology of semistable models of Shimura varieties. Therefore, we decided to dedicate the second part of this work to the more humble goal of showing that the p-adic L-function, built using the automorphic input, encodes geometric information of some kind. More precisely, we compute some values of the p-adic L-function in terms of the syntomic Abel–Jacobi image of *generalized Hirzebruch–Zagier cycles*.

Our result is evidence that the twisted triple product p-adic L-function and the generalized Hirzebruch–Zagier cycles are the right objects to consider in the framework determined by $G_{L\times F}$ and the twisted triple product L-function.

In the remaining of the introduction we present our results in more detail. We fix, once and for all, a p-adic embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for every rational prime p, and a complex embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Given a number field $E/\mathbb{Q}$ we let I_E be the set of field embeddings of E into $\overline{\mathbb{Q}}$ and $t_E = \sum_{\tau\in\mathrm{I}_E} \tau \in \mathbb{Z}[\mathrm{I}_E]$. For $k, k' \in \mathbb{Z}[\mathrm{I}_E]$ we write $k \geqslant k'$ if $k_\tau \geqslant k'_\tau$ for all $\tau \in \mathrm{I}_E$, and $k > k'$ if $k \geqslant k'$ and $\exists\tau_\circ$ with $k_{\tau_\circ} > k'_{\tau_\circ}$.

1.0.1. The p-adic L-function. Let L/F be a quadratic extension of totally real number fields, $\mathfrak{Q} \lhd \mathcal{O}_L$ and $\mathfrak{N} \lhd \mathcal{O}_F$ ideals. Consider primitive eigenforms $\mathsf{g}_\circ \in S_{\ell_\circ,x_\circ}(\mathfrak{Q}; L; \overline{\mathbb{Q}})$ and $\mathsf{f}_\circ \in S_{k_\circ,w_\circ}(\mathfrak{N}; F; \overline{\mathbb{Q}})$, whose weights satisfy $n_\circ t_L = \ell_\circ - 2x_\circ$ and $m_\circ t_F = k_\circ - 2w_\circ$ for $n_\circ, m_\circ \in \mathbb{Z}$, generating irreducible cuspidal automorphic representations π, σ of $G_L(\mathbb{A}), G_F(\mathbb{A})$ respectively. We denote by π^u, σ^u their unitarizations and define a representation of $\mathrm{GL}_2(\mathbb{A}_{L\times F})$ by $\Pi = \pi^u \otimes \sigma^u$. Let $\rho : \Gamma_F \to S_3$ be the homomorphism mapping the absolute Galois group of F to the symmetric group over 3 elements associated with the étale cubic algebra $(L \times F)/F$. The L-group ${}^L(G_{L\times F})$ is given by the semi-direct product $\widehat{G} \rtimes \Gamma_F$ where Γ_F acts on $\widehat{G} = \mathrm{GL}_2(\mathbb{C})^{\times 3}$ through ρ. One can define the twisted triple product L-function $L(s, \Pi, \mathrm{r})$ of Π via the representation r of ${}^L(G_{L\times F})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, which restricts to the natural 8-dimensional representation of $\widehat{G}$ and through which Γ_F acts via ρ permuting the vectors. We assume the central character ω_Π of Π satisfies $\omega_{\Pi|\mathbb{A}_F^\times} \equiv 1$, so that the twisted triple product L-function has a functional equation and we can talk about its central value.

Definition 1.1. We say that weights $(\ell, x) \in \mathbb{Z}[\mathrm{I}_L]^2$, $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$ are F-*dominated* if there exists $r \in \mathbb{N}[\mathrm{I}_L]$ with $k = (\ell + 2r)_{|F}$ and $w = (x + r)_{|F}$. In particular, F-dominated weights satisfy $k - 2w = (\ell - 2x)_{|F}$.

Let $\eta : \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the idele character attached to the quadratic extension L/F by class field theory. Suppose that the weights of $\mathsf{g}_\circ$ and $\mathsf{f}_\circ$ are F-dominated and that the local ϵ-factors satisfy

$$\epsilon_v\left(\frac{1}{2}, \Pi_v, \mathrm{r}_v\right)\eta_v(-1) = +1 \quad \forall v \text{ finite place of } F.$$

Then Theorem 3.2 and Lemma 3.4 show that the non-vanishing of the central L-value $L(\frac{1}{2}, \Pi, \mathrm{r})$ is equivalent to the existence of *test vectors* $\breve{\mathsf{g}}_\circ, \breve{\mathsf{f}}_\circ$ in π, σ, respectively, of some level $V_{11}(\mathfrak{A})$ such that the prime factors of $\mathfrak{A}$ are among those dividing $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$. More precisely, $\breve{\mathsf{g}}_\circ$ and $\breve{\mathsf{f}}_\circ$ are cuspforms such that the Petersson inner product

$$\mathrm{I}(\phi) = \left\langle \zeta^* \left(\delta^r \breve{\mathsf{g}}_\circ\right), \breve{\mathsf{f}}_\circ^* \right\rangle, \tag{1}$$

for some $r \in \mathbb{N}[\mathrm{I}_L]$, does not vanish. In other words, we can take (1) as an avatar of the central L-value and use it to construct the p-adic L-function.

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Remark. The assumption on local ϵ-factors at the finite places of F can be satisfied by requiring the ideals $\mathrm{N}_{L/F}(\mathfrak{Q})\cdot d_{L/F}$ and $\mathfrak{N}$ to be coprime and by asking all prime ideals dividing $\mathfrak{N}$ to split in L/F.

Definition 1.2. Let $(\ell, x) \in \mathbb{Z}[\mathrm{I}_L]^2$, $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$ be weights and $\theta \in \mathbb{Z}[\mathrm{I}_L]$ be an element satisfying $\theta_{|F} = 0 \cdot t_F$. If $\theta \equiv_2 w$ holds, i.e., $\theta_\mu \equiv_2 w_{\mu_{|F}}$ for all $\mu \in \mathrm{I}_L$, we define

$$r(\theta) = \sum_{\mu \in \mathrm{I}_L} \left[\frac{w_{\mu_{|F}} + \theta_\mu}{2} - x_\mu\right] \cdot \mu \in \mathbb{Z}[\mathrm{I}_L].$$

Let p be a rational prime unramified in L, coprime to the levels $\mathfrak{Q}, \mathfrak{N}$. We write $\mathcal{P}$ (respectively $\mathcal{Q}$) for the set of prime $\mathcal{O}_L$-ideals (respectively $\mathcal{O}_F$-ideals) dividing p. We choose an element $\theta \in \mathbb{Z}[\mathrm{I}_L]$ such that $\theta_{|F} = 0 \cdot t_F$ and $\theta \equiv_2 w_\circ$, and we let $\bar{r} = \sum_{\mu \in \mathrm{I}_L} \bar{r}_\mu \cdot \mu$, with $\bar{r}_\mu \in \mathbb{Z}/(q_{\mathfrak{p}_\mu} - 1)\mathbb{Z}$, denote the reduction of $r_\circ = r_\circ(\theta)$. We suppose $\mathsf{g}_\circ$, $\mathsf{f}_\circ$ are p-nearly ordinary and we denote by $\mathscr{G} \in \overline{\mathbf{S}}_L^{\mathrm{n.o.}}(\mathfrak{Q}, \chi; \mathbf{I}_{\mathscr{G}})$ and $\mathscr{F} \in \overline{\mathbf{S}}_F^{\mathrm{n.o.}}(\mathfrak{N}, \psi; \mathbf{I}_{\mathscr{F}})$ the Hida families passing through nearly ordinary p-stabilizations $\mathsf{g}_\circ^{(p)}$ and $\mathsf{f}_\circ^{(p)}$. We have $\chi_{|Z_L(\mathfrak{Q})_{\mathrm{tor}}} = \chi_\circ \mathcal{N}_L^{n_\circ}$ and $\psi_{|Z_F(\mathfrak{N})_{\mathrm{tor}}} = \psi_\circ \mathcal{N}_F^{m_\circ}$ for characters $\chi_\circ : \mathrm{cl}_L^+(\mathfrak{Q}) \to \mathbb{C}^\times$, $\psi_\circ : \mathrm{cl}_F^+(\mathfrak{N}) \to \mathbb{C}^\times$ and we suppose that $\chi_{\circ|F} \cdot \psi_\circ \equiv 1$. We let $\mathscr{F}^* \in \overline{\mathbf{S}}_F^{\mathrm{n.o.}}(\mathfrak{A}, \psi_\circ^{-2}\psi; \mathbf{I}_{\mathscr{F}^*})$ [14, § 7F] be the twisted Hida family, where $\mathbf{I}_{\mathscr{F}^*} \cong \mathbf{I}_{\mathscr{F}}(\psi_\circ^{-2})$ as an $\mathbf{\Lambda}_{F,\psi_\circ^{-2}\psi}$-algebra.

Definition 1.3. Let $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{W}}_{\mathscr{G},\mathscr{F}^*}$ be the rigid-analytic space $\mathrm{Spf}(\mathbf{I}_{\mathscr{G}} \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_{\mathscr{F}^*})^{\mathrm{rig}}$. The subset of *$F$-dominated crystalline points* with respect to $(\theta, \bar{r})$, denoted by $\boldsymbol{\mathcal{C}}_F^{\theta,\bar{r}}$, is the subset of arithmetic points $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$ whose weights are F-dominated, $r(\theta) \in \mathbb{Z}[\mathrm{I}_L]$ is a lift of $\bar{r}$, and such that the specialization of the Hida families is old at p; that is, they are the p-stabilization of eigenforms of prime-to-p level: $\mathscr{G}_\mathrm{P} = \mathsf{g}_\mathrm{P}^{(p)}$ and $\mathscr{F}_\mathrm{Q} = \mathsf{f}_\mathrm{Q}^{(p)}$.

Set $\mathbf{K}_{\mathscr{G},\mathscr{F}^*} = (\mathbf{I}_{\mathscr{G}} \widehat{\otimes}_{\mathcal{O}} \mathbf{I}_{\mathscr{F}^*}) \otimes \mathbb{Q}$, $\mathbf{K}_{\mathscr{G}} = \mathbf{I}_{\mathscr{G}} \otimes \mathbb{Q}$ and $\mathbf{K}_{\mathscr{F}^*} = \mathbf{I}_{\mathscr{F}^*} \otimes \mathbb{Q}$. We define a $\mathbf{K}_{\mathscr{G}}$-adic cuspform $\breve{\mathscr{G}}$ (respectively $\mathbf{K}_{\mathscr{F}^*}$-adic cuspform $\breve{\mathscr{F}}^*$) passing through the nearly ordinary p-stabilization of the test vectors $\breve{\mathsf{g}}_\circ, \breve{\mathsf{f}}_\circ^*$ as in [7, § 2.6]. Then Lemma 3.7 ensures the existence of a meromorphic rigid-analytic function ${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}}, \breve{\mathscr{F}}) : \boldsymbol{\mathcal{W}} \longrightarrow \mathbb{C}_p$ whose value at crystalline points $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta) \in \mathbb{Z}[\mathrm{I}_L]$ a lift of $\bar{r}$, is

$${}_{\bar{r}}\mathscr{L}_p^\theta\big(\breve{\mathscr{G}}, \breve{\mathscr{F}}\big)(\mathrm{P}, \mathrm{Q}) = \frac{1}{\mathsf{E}(\mathsf{f}_\mathrm{Q}^*)} \frac{\big\langle e_{\mathrm{n.o.}}\zeta^*\big(d^{r(\theta)}\breve{\mathsf{g}}_\mathrm{P}^{[\mathcal{P}]}\big), \breve{\mathsf{f}}_\mathrm{Q}^*\big\rangle}{\big\langle \mathsf{f}_\mathrm{Q}^*, \mathsf{f}_\mathrm{Q}^*\big\rangle}.$$

Here the number $\mathsf{E}(\mathsf{f}_\mathrm{Q}^*)$ is defined by $\mathsf{E}(\mathsf{f}_\mathrm{Q}^*) = (1 - \beta_{\mathsf{f}_\mathrm{Q}^*}\alpha_{\mathsf{f}_\mathrm{Q}^*}^{-1})$ for $\alpha_{\mathsf{f}_\mathrm{Q}^*}, \beta_{\mathsf{f}_\mathrm{Q}^*}$ the inverses of the roots of the Hecke polynomial for $T(p)$. We are justified in calling ${}_{\bar{r}}\mathscr{L}_p^\theta\big(\breve{\mathscr{G}}, \breve{\mathscr{F}}\big)$ a p-adic L-function because it interpolates the algebraic avatar (1) of central L-values $L(\frac{1}{2}, \Pi_{\mathrm{P},\mathrm{Q}}, \mathrm{r})$ at points $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{C}}_F^{\theta,\bar{r}}$, as the next theorem shows.

Theorem 1.4. *Consider the partition $\mathcal{Q}_{\mathrm{inert}} \coprod \mathcal{Q}_{\mathrm{split}}$ of the set of $\mathcal{O}_F$-prime ideals above p determined by the splitting behavior of the primes in the quadratic extension L/F. The value of the twisted triple product p-adic L-function ${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}}, \breve{\mathscr{F}}) : \boldsymbol{\mathcal{W}} \longrightarrow \mathbb{C}_p$ at any*

$(\mathrm{P},\mathrm{Q}) \in \mathcal{C}_F^{\theta,\bar{r}}$ *satisfies*

$$ {}_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}},\breve{\mathscr{F}})(\mathrm{P},\mathrm{Q}) = \pm\frac{1}{\mathsf{E}(\mathsf{f}^*_{\mathrm{Q}})}\left(\prod_{\wp\in\mathcal{Q}_{\mathrm{inert}}}\mathcal{E}_{\wp}(\mathsf{g}_{\mathrm{P}},\mathsf{f}^*_{\mathrm{Q}})\prod_{\wp\in\mathcal{Q}_{\mathrm{split}}}\frac{\mathscr{E}_{\wp}(\mathsf{g}_{\mathrm{P}},\mathsf{f}^*_{\mathrm{Q}})}{\mathscr{E}_{0,\wp}(\mathsf{g}_{\mathrm{P}},\mathsf{f}^*_{\mathrm{Q}})}\right) \times\frac{\left\langle \zeta^*\left(\delta^{s(w-x_{|F})}\breve{\mathsf{g}}_{\mathrm{P}}\right),\breve{\mathsf{f}}^*_{\mathrm{Q}}\right\rangle}{\left\langle \mathsf{f}^*_{\mathrm{Q}},\mathsf{f}^*_{\mathrm{Q}}\right\rangle}, $$

where $s:\mathrm{I}_F\to\mathrm{I}_L$ *is any section of the restriction* $\mathrm{I}_L\to\mathrm{I}_F$, $\mu\mapsto\mu_{|F}$, *and the Euler factors appearing in the formula are defined in Lemmas 3.9 and 3.11.*

1.0.2. A p-adic Gross–Zagier formula. The second part of the paper deals with the evaluation of the p-adic L-function outside the range of interpolation. From now on, we assume $L/\mathbb{Q}$ to be a real quadratic number field.

Definition 1.5. A triple of integers $(a,b,c)\in\mathbb{Z}^3$, is said to be balanced if none among a,b,c is greater or equal than the sum of the other two. We say that the weights $(\ell,x)\in\mathbb{Z}[\mathrm{I}_L]^2$, $(k,w)\in\mathbb{Z}[\mathrm{I}_{\mathbb{Q}}]^2$ are *balanced* if there exists $r\in\mathbb{N}[\mathrm{I}_L]$, $r\neq 0$, such that $k=|\ell-2r|$, $w=|x-r|$ and the triple of integers (ℓ_1,ℓ_2,k) is balanced.

Definition 1.6. The set of *balanced crystalline points* with respect to $(\theta,\bar{r})$, denoted by $\mathcal{C}_{\mathrm{bal}}^{\theta,\bar{r}}$, is the subset of arithmetic points $(\mathrm{P},\mathrm{Q})\in\boldsymbol{\mathcal{W}}$, whose weights are balanced, $r(\theta)\in\mathbb{Z}[\mathrm{I}_L]$ is a lift of $\bar{r}$, and such that the specialization of the Hida families are old at p. This set is a disjoint union, indexed by balanced triples (ℓ,k), of subsets $\mathcal{C}_{\mathrm{bal}}^{\theta,\bar{r}}(\ell,k)$ consisting of points whose weights have the form $(\ell,x)\in\mathbb{Z}[\mathrm{I}_L]^2$, $(k,w)\in\mathbb{Z}[\mathrm{I}_{\mathbb{Q}}]^2$.

For a balanced crystalline point $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_{\mathrm{bal}}^{\theta,\bar{r}}$, the global sign of the functional equation of $L(s,\Pi_{\mathrm{P},\mathrm{Q}},\mathrm{r})$ is -1. This forces the vanishing of the central value, which one expects to be accounted for by the family of generalized Hirzebruch–Zagier cycles. Interestingly, the twisted triple product p-adic L-function is not forced to vanish on $\mathcal{C}_{\mathrm{bal}}^{\theta,\bar{r}}$ and we can try to compute its values there. Let (ℓ,k) be a balanced triple such that either ℓ is not parallel or $(\ell,k)=(2t_L,2)$. Let $\mathcal{A}\to\mathbf{Sh}_K(G_L^*)$ be the universal abelian surface over the Shimura variety for G_L^* and let $\mathcal{E}\to\mathbf{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$ be the universal elliptic curve over the modular curve, both defined over some open subset of $\mathrm{Spec}(\mathcal{O}_E)$, where $E/\mathbb{Q}$ is a large enough finite Galois extension. For all but finitely many primes p, let $\wp\lhd\mathcal{O}_E$ be the prime above p induced by the fixed p-adic embedding ι_p, and consider $\mathscr{U}_{\ell-4}\times_{\mathcal{O}_{E,\wp}}\mathscr{W}_{k-2}$ a smooth and proper compactification of $\mathcal{A}^{|\ell|-4}\times\mathcal{E}^{k-2}$. The generalized Hirzebruch–Zagier cycle of weight (ℓ,k) is a De Rham null-homologous cycle

$$ \Delta_{\ell,k}\in\mathrm{CH}_{\gamma+2}(\mathscr{U}_{\ell-4}\times_{\mathcal{O}_{E,\wp}}\mathscr{W}_{k-2})_0\otimes_{\mathbb{Z}}L $$

of dimension $\gamma+2=\frac{|\ell|+k-2}{2}$. Given a pair of eigenforms $\breve{\mathsf{g}}_{\mathrm{P}}\in S_{\ell,x}(V_1(\mathfrak{A}\mathcal{O}_L);L;E)$ and $\breve{\mathsf{f}}_{\mathrm{Q}}\in S_{k,w}(V_1(\mathfrak{A});E)$ we can produce cohomology classes ω_{P} and η_{Q}, as in Definition 5.5, such that $\pi_1^*\omega_{\mathrm{P}}\cup\pi_2^*\eta_{\mathrm{Q}}\in\mathrm{F}^{|\ell|-2-s}H_{\mathrm{dR}}^{|\ell|+k-3}\big(U_{\ell-4}\times_{E_\wp}W_{k-2}\big)$ where $s=\frac{|\ell|-k-2}{2}$; that is, the cohomology class $\pi_1^*\omega_{\mathrm{P}}\cup\pi_2^*\eta_{\mathrm{Q}}$ lives in the domain of the syntomic Abel–Jacobi

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image of $\Delta_{\ell,k}$,

$$\mathrm{AJ}_p(\Delta_{\ell,k}) : \mathrm{F}^{|\ell|-2-s} H_{\mathrm{dR}}^{|\ell|+k-3}\big(U_{\ell-4} \times_{E_\wp} W_{k-2}\big) \longrightarrow E_\wp,$$

and we can compute the number $\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega_{\mathrm{P}} \cup \pi_2^*\eta_{\mathrm{Q}})$ as follows.

Theorem 1.7. *Let $L/\mathbb{Q}$ be a real quadratic field and (ℓ, k) a balanced triple. Let p be a prime splitting in L for which the generalized Hirzebruch–Zagier cycle $\Delta_{\ell,k}$ is defined. Then for all $(\mathrm{P}, \mathrm{Q}) \in \mathcal{C}_{\mathrm{bal}}^{\theta,\bar{r}}(\ell, k)$ we have*

$$_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}}, \breve{\mathscr{F}})(\mathrm{P}, \mathrm{Q}) = \frac{\pm 1}{s!\,\mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*)} \frac{\mathscr{E}_p(\mathsf{g}_{\mathrm{P}}, \mathsf{f}_{\mathrm{Q}}^*)}{\mathscr{E}_{0,p}(\mathsf{g}_{\mathrm{P}}, \mathsf{f}_{\mathrm{Q}}^*)} \mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega_{\mathrm{P}} \cup \pi_2^*\eta_{\mathrm{Q}}).$$

Remark. The assumption on the splitting behavior of p in $L/\mathbb{Q}$ should not be necessary. It could be dispensed with by showing the overconvergence of the p-adic cuspform $d_\mu^{1-\ell_\mu}(\breve{\mathsf{g}}_{\mathrm{P}}^{[p]})$ for $\mu \in \mathrm{I}_L$. It seems reasonable to believe that by generalizing the recent work of Andreatta and Iovita [1] one could prove such a result.

Let A be an elliptic curve over L of conductor $\mathfrak{Q}$ and B a rational elliptic curve of conductor $\mathfrak{N}$, both without complex multiplication over $\overline{\mathbb{Q}}$. We denote by $(\mathrm{M}_{A,B})_p$ the Galois representation $\mathrm{AsV}_p(A)(-1) \otimes_{\mathbb{Q}_p} \mathrm{V}_p(B)$ of the absolute Galois group of $\mathbb{Q}$. We can use Theorem 1.7 to give a criterion for the Bloch–Kato Selmer group $H_f^1(\mathbb{Q}, (\mathrm{M}_{A,B})_p)$ to be of dimension one in terms of the non-vanishing of a value of one of our twisted triple product p-adic L-functions. We build on the recent work of Liu [26], where he computes the dimension of $H_f^1(\mathbb{Q}, (\mathrm{M}_{A,B})_p)$ assuming the non-vanishing of the étale Abel–Jacobi map of certain cycle closely related to our Hirzebruch–Zagier cycle of weight $(2t_L, 2)$. Let $\mathsf{g}_A \in S_{2t_L,t_L}(V_1(\mathfrak{Q}); L; \mathbb{Q})$, $\mathsf{f}_B \in S_{2,1}(V_1(\mathfrak{N}); \mathbb{Q})$ be the newforms attached to A and B by modularity and p a rational prime coprime to $\mathfrak{N} \cdot \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q}) \cdot d_{L/F}$. If $\mathsf{g}_A, \mathsf{f}_B$ are p-nearly ordinary, we denote by $\mathscr{G}, \mathscr{F}$ the Hida families passing through the p-nearly ordinary stabilizations $\mathscr{G}_{\mathrm{P}_A} = \mathsf{g}_A^{(p)}$ and $\mathscr{F}_{\mathrm{Q}_B} = \mathsf{f}_B^{(p)}$, respectively.

Corollary 1.8. *Suppose that $\mathfrak{N}$ and $\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q}) \cdot d_{L/\mathbb{Q}}$ are coprime ideals and that all the primes dividing $\mathfrak{N}$ split in L. For all but finitely many primes p that are split in L and such that $\mathsf{g}_A, \mathsf{f}_B$ are p-nearly ordinary we have*

$$_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}}, \breve{\mathscr{F}})(\mathrm{P}_A, \mathrm{Q}_B) \neq 0 \quad \Longrightarrow \quad \dim_{\mathbb{Q}_p} H_f^1(\mathbb{Q}, (\mathrm{M}_{A,B})_p) = 1,$$

where $\theta = -\mu + \mu' \in \mathbb{Z}[\mathrm{I}_L]$, $\bar{r} = -\mu$.

The arithmetic setting of this paper has recently been considered by several independent groups: [4, 10, 19]. Ignacio Sols and I.B. computed syntomic Abel–Jacobi images of some Hirzebruch–Zagier cycles in terms of p-adic modular forms, while Ishikawa constructed twisted triple product p-adic L-functions over $\mathbb{Q}$ following the refined approach of Hsieh [17]. Given the similarities between the computations of syntomic Abel–Jacobi images in the work of B.-Sols and M.F., the two groups agreed to publish together.

2. Automorphic forms

2.1. Adelic Hilbert modular forms

Let $F/\mathbb{Q}$ be a totally real number field and let I_F be the set of field embeddings of F into $\overline{\mathbb{Q}}$. We denote by G_F the algebraic group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$. We choose a square root $i \in \mathbb{C}$ of -1 which allows us to define the Poincaré half-plane $\mathfrak{H}$, we consider the complex manifold $\mathfrak{H}^{\mathrm{I}_F}$ which is endowed with a transitive action of $G_F(\mathbb{R})^+ \cong \prod_{\mathrm{I}_F} \mathrm{GL}_2(\mathbb{R})^+$ and contains the point $\mathbf{i} = (i, \ldots, i)$. For any $K \leqslant G_F(\mathbb{A}^\infty)$ compact open subgroup we denote by $S_{k,w}(K; F; \mathbb{C})$, or simply $S_{k,w}(K; \mathbb{C})$ when there is no risk of confusion, the space of holomorphic Hilbert cuspforms of weight $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$, $k - 2w = mt_F$ for some $m \in \mathbb{Z}$, and level K. It is defined as the space of functions $\mathsf{f} : G_F(\mathbb{A}) \to \mathbb{C}$ that satisfy the following list of properties:

- $\mathsf{f}(\alpha x u) = \mathsf{f}(x) j_{k,w}(u_\infty, \mathbf{i})^{-1}$ where $\alpha \in G_F(\mathbb{Q})$, $u \in K \cdot C_\infty^+$ for C_∞^+ the stabilizer of $\mathbf{i}$ in $G_F(\mathbb{R})^+$ and the automorphy factor is $j_{k,w}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) = (ad - bc)^{-w}(cz + d)^k$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_F(\mathbb{R})$, $z \in \mathfrak{H}^{\mathrm{I}_F}$;
- for every finite adelic point $x \in G_F(\mathbb{A}^\infty)$ the well-defined function $\mathsf{f}_x : \mathfrak{H}^{\mathrm{I}_F} \to \mathbb{C}$ given by $\mathsf{f}_x(z) = \mathsf{f}(x u_\infty) j_{k,w}(u_\infty, \mathbf{i})$ is holomorphic, where for each $z \in \mathfrak{H}^{\mathrm{I}_F}$ we choose $u_\infty \in G_F(\mathbb{R})^+$ such that $u_\infty \mathbf{i} = z$.
- for all adelic points $x \in G_F(\mathbb{A})$ and for all additive measures on $F\backslash\mathbb{A}_F$ we have

$$\int_{F\backslash\mathbb{A}_F} \mathsf{f}\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x \right) da = 0.$$

- If the totally real field is just the field of rational numbers, $F = \mathbb{Q}$, we need to impose the extra condition that for all finite adelic point $x \in G_{\mathbb{Q}}(\mathbb{A}^\infty)$ the function $|\mathrm{Im}(z)^{\frac{k}{2}} \mathsf{f}_x(z)|$ is uniformly bounded on $\mathfrak{H}$.

Definition 2.1. We denote by G_F^* the algebraic group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F} \times_{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}} \mathbb{G}_m$. By replacing G_F by G_F^* in the previous definition, we define $S_{k,\nu}^*(K; \mathbb{C})$ to be the space of cuspforms for G_F^* of weight $(k, \nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ and level K, for any $K \leqslant G_F^*(\mathbb{Q})$ compact open subgroup.

Note that for all pairs of weights $(k, \nu), (k, \nu') \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ there is a natural isomorphism

$$\Psi_{\nu,\nu'} : S_{k,\nu}^*(K; \mathbb{C}) \xrightarrow{\sim} S_{k,\nu'}^*(K; \mathbb{C}) \tag{2}$$

given by $\mathsf{f}(x) \mapsto \mathsf{f}(x)|\mathrm{det}(x)|_{\mathbb{A}_\mathbb{Q}}^{\nu'-\nu}$.

Each irreducible automorphic representation π spanned by some form in $S_{k,w}(K; \mathbb{C})$ has central character equal to $|-|_{\mathbb{A}_F}^{-m}$ up to finite order characters. The twist $\pi^u := \pi \otimes |-|_{\mathbb{A}_F}^{\frac{m}{2}}$ is called the unitarization of π. Note that there is an isomorphism of function spaces (not of $G_F(\mathbb{A})$-modules)

$$\begin{array}{l} \pi \xrightarrow{\sim} \pi^u \\ \mathsf{f} \mapsto \mathsf{f}^u \end{array} \quad \text{where } \mathsf{f}^u(x) = \mathsf{f}(x)|\mathrm{det}(x)|_{\mathbb{A}_F}^{\frac{m}{2}}. \tag{3}$$

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Let $\mathrm{d}x$ be the Tamagawa measure on $[G_F(\mathbb{A})] = \mathbb{A}_F^\times G_F(\mathbb{Q})\backslash G_F(\mathbb{A})$, for any two cuspforms $\mathsf{f}_1, \mathsf{f}_2 \in S_{k,w}(K;\mathbb{C})$, with $k - 2w = mt_F$, we define their Petersson inner product to be

$$\langle \mathsf{f}_1, \mathsf{f}_2\rangle = \int_{[G_F(\mathbb{A})]} \mathsf{f}_1(x)\overline{\mathsf{f}_2(x)}|\det(x)|^m_{\mathbb{A}_F}\,\mathrm{d}x = \langle \mathsf{f}_1^u, \mathsf{f}_2^u\rangle. \tag{4}$$

For an $\mathcal{O}_F$-ideal $\mathfrak{N}$ we consider the following compact open sugroups of $G_F(\widehat{\mathbb{Z}})$:

- $U_0(\mathfrak{N}) = \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in G_F(\widehat{\mathbb{Z}}) \middle|\ c \in \mathfrak{N}\widehat{\mathcal{O}_F}\right\}$,
- $V_1(\mathfrak{N}) = \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in U_0(\mathfrak{N}) \middle|\ d \equiv 1 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}}\right\}$,
- $V_{11}(\mathfrak{N}) = \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in V_1(\mathfrak{N}) \middle|\ a \equiv 1 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}}\right\}$,
- $U(\mathfrak{N}) = \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in V_{11}(\mathfrak{N}) \middle|\ b \equiv 0 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}}\right\}$

For any prime p coprime to $\mathfrak{N}$ and any compact open subgroups satisfying $V_1(\mathfrak{N}) \leqslant K \leqslant U_0(\mathfrak{N})$, we set $K(p^\alpha) = K \cap V_{11}(p^\alpha)$ and $Z_F(K) = \mathbb{A}_F^\times/\overline{F^\times \det K(p^\infty)F^\times_{\infty,+}}$. One can decompose the ideles of F as

$$\mathbb{A}_F^\times = \coprod_{i=1}^{h_F^+(\mathfrak{N})} F^\times a_i \det V_{11}(\mathfrak{N}) F^\times_{\infty,+}$$

where $a_i \in \mathbb{A}_F^{\infty,\times}$ and $h_F^+(\mathfrak{N})$ is the cardinality of $\mathrm{cl}_F^+(\mathfrak{N}) := F_+^\times\backslash\mathbb{A}_F^{\infty,\times}/\det V_{11}(\mathfrak{N})$. The ideles decomposition induces a decomposition of the adelic points of G_F

$$G_F(\mathbb{A}) = \coprod_{i=1}^{h_F^+(\mathfrak{N})} G_F(\mathbb{Q})t_i U(\mathfrak{N})G_F(\mathbb{R})^+ \quad \text{for } t_i = \begin{pmatrix} a_i^{-1} & 0\\ 0 & 1\end{pmatrix}.$$

2.1.1. Adelic q-expansion. The Shimura variety $\mathrm{Sh}_K(G_F)$, determined by G_F and a compact open subgroup K, is not compact, therefore there is a notion of q-expansion for Hilbert modular forms. Even more, Shimura found a way to package the q-expansions of each connected component of $\mathrm{Sh}_K(G_F)$ into a unique adelic q-expansion. Fix $\mathsf{d}_F \in \mathbb{A}_F^{\infty,\times}$ such that $\mathsf{d}_F\mathcal{O}_F = \mathfrak{d}_F$ is the absolute different ideal of F. Let F^{Gal} be the Galois closure of F in $\overline{\mathbb{Q}}$ and write $\mathcal{V}$ for the ring of integers or a valuation ring of a finite extension F_0 of F^{Gal} such that for every ideal $\mathfrak{a}$ of $\mathcal{O}_F$, for all $\tau \in \mathrm{I}_F$, the ideal $\mathfrak{a}^\tau\mathcal{V}$ is principal. Choose a generator $\{\mathfrak{q}^\tau\} \in \mathcal{V}$ of $\mathfrak{q}^\tau\mathcal{V}$ for each prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ and by multiplicativity define $\{\mathfrak{a}^v\} \in \mathcal{V}$ for each fractional ideal $\mathfrak{a}$ of F and each $v \in \mathbb{Z}[\mathrm{I}_F]$. Given a Hilbert cuspform $\mathsf{f} \in S_{k,w}(V_{11}(\mathfrak{N});\mathbb{C})$, one can consider for every index $i \in \{1,\ldots,h_F^+(\mathfrak{N})\}$, the holomorphic function $\mathsf{f}_i : \mathfrak{H}^{\mathrm{I}_F} \to \mathbb{C}$

$$\mathsf{f}_i(z) = y_\infty^{-w}\mathsf{f}\left(t_i\begin{pmatrix} y_\infty & x_\infty\\ 0 & 1\end{pmatrix}\right) = \sum_{\xi\in(\mathfrak{a}_i\mathfrak{d}_F^{-1})_+} a(\xi,\mathsf{f}_i)e_F(\xi z)$$

for $z = x_\infty + \mathbf{i}y_\infty$, $\mathfrak{a}_i = a_i\mathcal{O}_F$ and $e_F(\xi z) = \exp\left(2\pi i\sum_{\tau\in\mathrm{I}_F}\tau(\xi)z_\tau\right)$. Every idele y in $\mathbb{A}^\times_{F,+} := \mathbb{A}_F^{\infty,\times}F^\times_{\infty,+}$ can be written as $y = \xi a_i^{-1}\mathsf{d}u$ for $\xi \in F_+^\times$ and $u \in \det U(\mathfrak{N})F^\times_{\infty,+}$; the following functions

$$\mathsf{a}(-,\mathsf{f}) : \mathbb{A}^\times_{F,+} \longrightarrow \mathbb{C}, \quad \mathsf{a}_p(-,\mathsf{f}) : \mathbb{A}^\times_{F,+} \longrightarrow \overline{\mathbb{Q}}_p$$

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are defined by

$$\mathsf{a}(y,\mathsf{f}) := a(\xi,\mathsf{f}_i)\{y^{w-t_F}\}\xi^{t_F-w}|a_i|_{\mathbb{A}_F} \quad \text{and} \quad \mathsf{a}_p(y,\mathsf{f}) := a(\xi,\mathsf{f}_i)y_p^{w-t_F}\xi^{t_F-w}\mathcal{N}_F(a_i)^{-1}$$

if $y \in \widehat{\mathcal{O}_F}F^\times_{\infty,+}$ and zero otherwise. Here $\mathcal{N}_F : Z_F(1) \to \overline{\mathbb{Q}}_p^\times$ is the map defined by $y \mapsto y_p^{-t_F}|y^\infty|^{-1}_{\mathbb{A}_F}$. Clearly, the function $\mathsf{a}_p(-,f)$ makes sense only if the coefficients $a(\xi,f_i) \in \overline{\mathbb{Q}}$ are algebraic $\forall \xi, i$. For each $\mathcal{V}$-algebra A contained in $\mathbb{C}$ we denote by $S_{k,w}(K;A)$ the A-module $\left\{\mathsf{f} \in S_{k,w}(K;\mathbb{C}) \middle| \mathsf{a}(y,\mathsf{f}) \in A\ \forall y \in \mathbb{A}^\times_{F,+}\right\}$.

Theorem 2.2 [14, Theorem 1.1]. *Consider the map $e_F : \mathbb{C}^{\mathrm{I}_F} \longrightarrow \mathbb{C}^\times$ defined by $e_F(z) = \exp\left(2\pi i \sum_{\tau\in \mathrm{I}_F} z_\tau\right)$ and the additive character of the ideles $\chi_F : \mathbb{A}_F/F \longrightarrow \mathbb{C}^\times$ which satisfies $\chi_F(x_\infty) = e_F(x_\infty)$. Each cuspform $\mathsf{f} \in S_{k,w}(V_{11}(\mathfrak{N});\mathbb{C})$ has an adelic q-expansion of the form*

$$\mathsf{f}\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = |y|_{\mathbb{A}_F} \sum_{\xi\in F_+} \mathsf{a}(\xi y \mathsf{d}_F,\mathsf{f})\{(\xi y \mathsf{d}_F)^{t_F-w}\}(\xi y_\infty)^{w-t_F} e_F(\mathbf{i}\xi y_\infty)\chi_F(\xi x)$$

for $y \in \mathbb{A}^\times_{F,+}$, $x \in \mathbb{A}^\times_F$, where $\mathsf{a}(-,\mathsf{f}) : \mathbb{A}^\times_{F,+} \longrightarrow \mathbb{C}$ vanishes outside $\widehat{\mathcal{O}_F}F^\times_{\infty,+}$ and depends only on the coset $y^\infty \det V_{11}(\mathfrak{N})$.

2.1.2. Nearly holomorphic cuspforms. For any K compact open subgroup satisfying $V_{11}(\mathfrak{N}) \leqslant K \leqslant G_F(\mathbb{A}^\infty)$ we denote by $N_{k,w,q}(K;F;\mathbb{C})$, or $N_{k,w,q}(K;\mathbb{C})$ when F is clear, the space of nearly holomorphic cuspforms of weight $(k,w) \in \mathbb{Z}[\mathrm{I}_F]^2$ and order less than or equal to $q \in \mathbb{N}[\mathrm{I}_F]$ with respect to K. It is the space of functions $\mathsf{f} : G_F(\mathbb{A}) \to \mathbb{C}$ that satisfy the following list of properties:

- $\mathsf{f}(\alpha x u) = \mathsf{f}(x) j_{k,w}(u_\infty,\mathbf{i})^{-1}$ where $\alpha \in G_F(\mathbb{Q})$, $u \in K \cdot C^+_\infty$;
- for each $x \in G_F(\mathbb{A}^\infty)$ the well-defined function $\mathsf{f}_x(z) = \mathsf{f}(x u_\infty) j_{k,w}(u_\infty,\mathbf{i})$ can be written as

$$\mathsf{f}_x(z) = \sum_{\xi\in L(x)_+} a(\xi,\mathsf{f}_x)((4\pi y)^{-1}) e_F(\xi z)$$

  for polynomials $a(\xi,\mathsf{f}_x)(Y)$ in the variables $(Y_\tau)_{\tau\in I}$ of degree less than q_τ in Y_τ for each $\tau \in \mathrm{I}_F$ and for $L(x)$ a lattice of F.

As before f_i stands for f_{t_i} and we consider adelic Fourier coefficients

$$\mathsf{a}(y,\mathsf{f})(Y) = \{y^{w-t_F}\}\xi^{t_F-w}|a_i|_{\mathbb{A}_F} a(\xi,\mathsf{f}_i)(Y), \quad \mathsf{a}_p(y,\mathsf{f})(Y) = y_p^{w-t}\xi^{t_F-w}\mathcal{N}_F(a_i)^{-1}a(\xi,\mathsf{f}_i)(Y)$$

if $y = \xi a_i^{-1}\mathsf{d}_F u \in \widehat{\mathcal{O}_F}F^\times_{\infty,+}$ and zero otherwise. The adelic Fourier expansion of a nearly holomorphic cuspform f is given by

$$\mathsf{f}\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = |y|_{\mathbb{A}_F} \sum_{\xi\in F_+} \mathsf{a}(\xi y \mathsf{d}_F,\mathsf{f})(Y)\{(\xi y \mathsf{d}_F)^{t_F-w}\}(\xi y_\infty)^{w-t_F} e_F(\mathbf{i}\xi y_\infty)\chi_F(\xi x)$$

for $Y = (4\pi y_\infty)^{-1}$ and for A a subring of $\mathbb{C}$ one can consider the A-module $N_{k,w,q}(K;A)$ defined by $\{\mathsf{f} \in N_{k,w,q}(K;\mathbb{C}) |\ \mathsf{a}(y,\mathsf{f}) \in A[Y]\ \forall y \in \mathbb{A}^\times_{F,+}\}$.

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There are Maass–Shimura differential operators for $r \in \mathbb{N}[\mathrm{I}_F]$, $k \in \mathbb{Z}[\mathrm{I}_F]$ defined as

$$\delta_k^r = \prod_{\tau \in \mathrm{I}_F} (\delta^\tau_{k_\tau + 2r_\tau - 2} \circ \cdots \circ \delta^\tau_{k_\tau}) \quad \text{where } \delta^\tau_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy_\tau} + \frac{\partial}{\partial z_\tau}\right). \tag{5}$$

They act on a nearly holomorphic cuspform $\mathsf{f} \in N_{k,w,q}(K;\mathbb{C})$ via the expression $\mathsf{a}(y, \delta_k^r \mathsf{f})(Y) = \{y^{w-t_F+r}\}\xi^{t_F-w-r}|a_i|_{\mathbb{A}_F} a(\xi, \delta_k^r \mathsf{f}_i)(Y)$. Suppose that $\mathbb{Q} \subset A$, then Hida showed [14, Proposition 1.2] the differential operator δ_k^r maps $N_{k,w,q}(K;A)$ to $N_{k+2r,w+r,q+r}(K;A)$ and if $k_\tau > 2q_\tau$ $\forall \tau \in \mathrm{I}_F$, then there is holomorphic projector $\Pi^{\mathrm{hol}} : N_{k,w,q}(K;A) \longrightarrow S_{k,w}(K;A)$.

2.1.3. Hecke theory. Consider a compact open subgroup $K \leqslant G_F(\mathbb{A}^\infty)$ of the finite adelic points of G_F that satisfies $V_{11}(\mathfrak{N}) \leqslant K \leqslant U_0(\mathfrak{N})$. Suppose that $\mathcal{V}$ is the valuation ring corresponding to the fixed embedding $\iota_p : F^{\mathrm{Gal}} \hookrightarrow \overline{\mathbb{Q}}_p$, so that we may assume $\{y^{t_F-w}\} = 1$ whenever the ideal $y\mathcal{O}_F$ generated by y is prime to $p\mathcal{O}_F$. Let ϖ be a uniformizer of the completion $\mathcal{O}_{F,\mathfrak{q}}$ of $\mathcal{O}_F$ at a prime $\mathfrak{q}$. We are interested in Hecke operators defined by the following double cosets

$$T_0(\varpi) = \{\varpi^{w-t_F}\}\left[V_{11}(\mathfrak{N})\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}V_{11}(\mathfrak{N})\right] \quad \text{if } \mathfrak{q} \nmid \mathfrak{N},$$

$$U_0(\varpi) = \{\varpi^{w-t_F}\}\left[V_{11}(\mathfrak{N})\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}V_{11}(\mathfrak{N})\right] \quad \text{if } \mathfrak{q} \mid \mathfrak{N},$$

and for $a \in \mathcal{O}^\times_{F,\mathfrak{N}} := \prod_{\mathfrak{q}|\mathfrak{N}} \mathcal{O}^\times_{F,\mathfrak{q}}$ the double coset

$$T(a,1) = \left[V_{11}(\mathfrak{N})\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}V_{11}(\mathfrak{N})\right].$$

If the prime $\mathfrak{q}$ is coprime to the level, then the Hecke operator $T_0(\varpi)$ acting on modular forms is independent of the choice of the uniformizer ϖ and we simply denote it $T_0(\mathfrak{q})$. For any finite adelic point $z \in Z_G(\mathbb{A}^\infty)$ of the center of G_F we define the diamond operator associated to it by $\mathsf{f}_{|\langle z\rangle}(x) = \mathsf{f}(xz)$, for any modular form f. For a prime ideal $\mathfrak{q}$ such that $\mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \subset K$, we write $\langle\mathfrak{q}\rangle$ for the operator $\langle\varpi\rangle$, where ϖ is a uniformizer of $\mathcal{O}_{F,\mathfrak{q}}$. The action of the operators on adelic q-expansion is given by the following formulas. If $\mathfrak{q} \nmid \mathfrak{N}$ one can compute

$$\begin{aligned}\mathsf{a}_p(y, \mathsf{f}_{|T_0(\mathfrak{q})}) &= \mathsf{a}_p(y\varpi, \mathsf{f})\{\varpi^{w-t_F}\}\varpi_p^{t_F-w}\\ &\quad + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})\{\mathfrak{q}^{2(w-t_F)}\}\mathsf{a}_p(y\varpi^{-1}, \mathsf{f}_{|\langle\mathfrak{q}\rangle})\{\varpi^{t_F-w}\}\varpi_p^{w-t_F}\end{aligned}$$

and

$$\mathsf{a}(y, \mathsf{f}_{|T_0(\mathfrak{q})}) = \mathsf{a}(y\varpi, \mathsf{f}) + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})\{\mathfrak{q}^{2(w-t_F)}\}\mathsf{a}(y\varpi^{-1}, \mathsf{f}_{|\langle\mathfrak{q}\rangle}).$$

If $\mathfrak{q} \mid \mathfrak{N}$ one can compute

$$\mathsf{a}_p(y, \mathsf{f}_{|U_0(\varpi)}) = \mathsf{a}_p(y\varpi, \mathsf{f})\{\varpi^{w-t_F}\}\varpi_p^{t_F-w}$$

and

$$\mathsf{a}(y, \mathsf{f}_{|U_0(\varpi)}) = \mathsf{a}(y\varpi, \mathsf{f}).$$

Finally, for $a \in \mathcal{O}^\times_{F,\mathfrak{N}}$ one finds $\mathsf{a}_p(y, \mathsf{f}_{|T(a,1)}) = \mathsf{a}_p(ya, \mathsf{f})a_p^{t_F-w}$. It follows that if $\varpi \in \mathcal{O}_{F,\mathfrak{q}}$ is a uniformizer and $a \in \mathcal{O}^\times_{F,\mathfrak{q}}$ then $U_0(a\varpi) = T(a,1)U_0(\varpi)$.

The Hecke algebra $\mathsf{h}_{k,w}(K;\mathcal{V})$ is defined to be the $\mathcal{V}$-subalgebra of $\mathrm{End}_{\mathbb{C}}\big(S_{k,w}(K;\mathbb{C})\big)$ generated by the Hecke operators $T_0(\mathfrak{q})$'s for primes outside the level $\mathfrak{q}\nmid\mathfrak{N}$, $U_0(\varpi)$'s for primes dividing the level $\mathfrak{q}\mid\mathfrak{N}$, $T(a,1)$'s for $a\in\mathcal{O}_{F,\mathfrak{N}}^{\times}$ and the diamond operators. For each $\mathcal{V}$-algebra A contained in $\mathbb{C}$ one defines $\mathsf{h}_{k,w}(K;A)=\mathsf{h}_{k,w}(K;\mathcal{V})\otimes_{\mathcal{V}}A$.

Theorem 2.3 [14, Theorem 2.2]. *For any finite field extension L/F^{Gal} and any $\mathcal{V}$-subalgebra A of L, there is a natural isomorphism $S_{k,w}(K;L)\cong S_{k,w}(K;A)\otimes_A L$. Moreover, if A an integrally closed domain containing $\mathcal{V}$, finite flat over either $\mathcal{V}$ or $\mathbb{Z}_p$, then $S_{k,w}(K;A)$ is stable under $\mathsf{h}_{k,w}(K;A)$ and the pairing $(\,,\,):S_{k,w}(K;A)\times\mathsf{h}_{k,w}(K;A)\to A$ given by $(f,h)=\mathsf{a}(1,f_{|h})$ induces isomorphisms of A-modules*

$$\mathsf{h}_{k,w}(K;A)\cong S_{k,w}(K;A)^{*}\quad\textit{and}\quad S_{k,w}(K;A)\cong\mathsf{h}_{k,w}(K;A)^{*},$$

where $(-)^{*}$ denotes the A-linear dual $\mathrm{Hom}_A(-,A)$.

Every idele $y\in\widehat{\mathcal{O}_F}\cap\mathbb{A}_F^{\times}$ can be written as $y=a\prod_{\mathfrak{q}}\varpi_{\mathfrak{q}}^{e(\mathfrak{q})}u$ with $u\in\det U(\mathfrak{N})$ and $a\in\mathcal{O}_{F,\mathfrak{N}}^{\times}$. Write $\mathfrak{n}$ for the ideal $\big(\prod_{\mathfrak{q}\nmid\mathfrak{N}}\varpi_{\mathfrak{q}}^{e(\mathfrak{q})}\big)\mathcal{O}_F$, then the Hecke operator

$$T_0(y)=T(a,1)T_0(\mathfrak{n})\prod_{\mathfrak{q}\mid\mathfrak{N}}U_0(\varpi_{\mathfrak{q}}^{e(\mathfrak{q})}) \tag{6}$$

depends only on the idele y. A cuspform that is an eigenvector for all the Hecke operators is called an eigenform and it is normalized when $\mathsf{a}(1,\mathsf{f})=1$. Shimura proved [37, Proposition 2.2] that the eigenvalues for the Hecke operators are algebraic numbers, hence a normalized eigenform $\mathsf{f}\in S_{k,w}(K;\mathbb{C})$ is an element of $S_{k,w}(K;\overline{\mathbb{Q}})$ since the $T_0(y)$-eigenvalue is $\mathsf{a}(y,\mathsf{f})$ for every idele y. For an idele $y\in\widehat{\mathcal{O}_F}\cap\mathbb{A}_F^{\times}$, let $T(y)=T_0(y)\{y^{t_F-w}\}$.

Definition 2.4. Let $\mathfrak{p}\mid p$ be a prime of $\mathcal{O}_F$ coprime to the level K and $(k,w)\in\mathbb{Z}[\mathrm{I}_F]$ with $k\geqslant 2t_F$. A normalized eigenform $\mathsf{f}\in S_{k,w}(K;\overline{\mathbb{Q}})$ is nearly ordinary at $\mathfrak{p}$ if the $T_0(\mathfrak{p})$-eigenvalue is a p-adic unit with respect to the specified embedding $\iota_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. If f is nearly ordinary at $\mathfrak{p}$ for all $\mathfrak{p}\mid p$ we say that f is p-nearly ordinary.

Definition 2.5. For every idele $b\in\mathbb{A}_F^{\times}$ there is an operator $V(b)$ on cuspforms defined by

$$\mathsf{f}_{|V(b)}(x)=\mathrm{N}_{F/\mathbb{Q}}(b\mathcal{O}_F)\mathsf{f}\left(x\begin{pmatrix}b^{-1}&0\\0&1\end{pmatrix}\right)$$

that acts on p-adic q-expansions as $\mathsf{a}_p(y,\mathsf{f}_{|V(b)})=b_p^{w-t_F}\mathsf{a}_p(yb^{-1},\mathsf{f})$ (this operator is denoted by $[b]$ in [14, § 7B]). Its normalization $[b]=\{b^{t_F-w}\}V(b)$ acts on q-expansions by $\mathsf{a}(y,\mathsf{f}_{|[b]})=\mathsf{a}(yb^{-1},\mathsf{f})$.

Remark. We have $U_0(\varpi)\circ[\varpi]=U(\varpi)\circ V(\varpi)=1$.

Let $\mathsf{f}\in S_{k,w}(K,\overline{\mathbb{Q}})$ be a normalized eigenform of level prime to $\mathfrak{p}$. Set $\langle\mathfrak{p}\rangle_0:=\{\varpi_{\mathfrak{p}}^{2(w-t_F)}\}\langle\mathfrak{p}\rangle$, then the $\langle\mathfrak{p}\rangle_0$-eigenvalue of f is $\psi_{\mathsf{f},0}(\mathfrak{p})=\{\varpi_{\mathfrak{p}}^{2(w-t_F)}\}\psi_{\mathsf{f}}(\mathfrak{p})$ for $\psi_{\mathsf{f}}(\mathfrak{p})$ the $\langle\mathfrak{p}\rangle$-eigenvalue of f. The $T_0(\mathfrak{p})$-Hecke polynomial for f is given by

$$1-\mathsf{a}(\mathfrak{p},\mathsf{f})X+\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})\psi_{\mathsf{f},0}(\mathfrak{p})X^2=(1-\alpha_{0,\mathfrak{p}}X)(1-\beta_{0,\mathfrak{p}}X).$$

If f is nearly ordinary at $\mathfrak{p}$, $\mathsf{a}(\mathfrak{p},\mathsf{f})$ is a p-adic unit and we can assume that $\alpha_{0,\mathfrak{p}}$ is a p-adic unit too. The nearly ordinary $\mathfrak{p}$-stabilization of f is the cuspform $\mathsf{f}^{(\mathfrak{p})}=(1-\beta_{0,\mathfrak{p}}[\varpi_{\mathfrak{p}}])\mathsf{f}$ that has the same Hecke eigenvalues of f away from $\mathfrak{p}$ and whose $U_0(\varpi_{\mathfrak{p}})$-eigenvalue is $\alpha_{0,\mathfrak{p}}$. For $\mathcal{S}$ a finite set of prime $\mathcal{O}_F$-ideals, the $\mathcal{S}$-depletion of a cuspform f is the cuspform $\mathsf{f}^{[\mathcal{S}]}=\prod_{\mathfrak{p}\in\mathcal{S}}\big(1-V(\varpi_{\mathfrak{p}})\circ U(\varpi_{\mathfrak{p}})\big)\mathsf{f}$ whose Fourier coefficient $\mathsf{a}_p(y,\mathsf{f}^{[\mathcal{S}]})$ equals $\mathsf{a}_p(y,\mathsf{f})$ if $y_{\mathcal{S}}\in\mathcal{O}_{F,\mathcal{S}}^{\times}$ and 0 otherwise.

Lemma 2.6. *For all pairs of weights $(k,\nu),(k,\nu')\in\mathbb{Z}[\mathrm{I}_F]\times\mathbb{Z}$ we have the equality $V(p)\circ\Psi_{\nu,\nu'}=p^{\nu'-\nu}\Psi_{\nu,\nu'}\circ V(p)$ of maps from $S^*_{k,\nu}(K,\mathbb{C})$ to $S^*_{k,\nu'}(K,\mathbb{C})$.*

Proof. Follows directly from the definitions. □

2.2. Hida families

We consider compact open subgroups that satisfy $V_1(\mathfrak{N})\leqslant K\leqslant U_0(\mathfrak{N})$. The group $Z_F(K)$ has a finite torsion, so we can fix a prime p coprime to $\mathfrak{N}$ and the order of $Z_F(K)_{\mathrm{tor}}$. Let O be a valuation ring in $\overline{\mathbb{Q}}_p$ finite flat over $\mathbb{Z}_p$ containing $\iota_p(\mathcal{V})$. Consider the space of p-adic cuspforms

$$S_{k,w}(K(p^\infty);O)=\lim_{\alpha,\to}S_{k,w}(K(p^\alpha);O)$$

on which the p-adic Hecke algebra

$$\mathsf{h}_{k,w}(K(p^\infty);O)=\lim_{\leftarrow,\alpha}\mathsf{h}_{k,w}(K(p^\alpha);O)$$

naturally acts. The Hecke operators defined by $\mathbf{T}(y)=\lim_{\leftarrow}T(y)y_p^{w-t_F}$ play an important role in the theory. There is a p-adic norm on the space of p-adic cuspforms $S_{k,w}(K(p^\infty);O)$ defined by $|\mathsf{f}|_p=\sup_y\{|\mathsf{a}_p(y,\mathsf{f})|_p\}$; the resulting completed space is denoted by $\overline{S}_{k,w}(K(p^\infty);O)$ and it has a natural perfect O-pairing with the p-adic Hecke algebra [14, Theorem 3.1]. Each element $\mathsf{f}\in\overline{S}_{k,w}(K(p^\infty);O)$ induces a continuous function $\mathsf{f}:\mathfrak{J}\longrightarrow O$, defined by $y\mapsto\mathsf{a}_p(y,\mathsf{f})$, on the topological semigroup

$$\mathfrak{J}=\widehat{\mathcal{O}_F}^{\times}F^{\times}_{\infty,+}/\det V_{11}(p^\infty)F^{\times}_{\infty,+},$$

isomorphic to $\mathcal{O}_{F,p}^{\times}\times\mathscr{I}_F$ for $\mathscr{I}_F$ the free semigroup of integral ideals of F. Hence, there is a continuous embedding $\overline{S}_{k,w}(K(p^\infty);O)\hookrightarrow\mathcal{C}(\mathfrak{J};O)$ of the completed space of p-adic cuspforms into the continuous functions from $\mathfrak{J}$ to O. The image of the embedding $\overline{\mathbf{S}}_F(K;O)$ is independent of the weight (k,w) since there exists a canonical algebra isomorphism $\mathsf{h}_{k,w}(K(p^\infty);O)\cong\mathsf{h}_{2t_F,t_F}(K(p^\infty);O)$ which takes $\mathbf{T}(y)$ to $\mathbf{T}(y)$ [13, Theorem 2.3]. Hence, we write $\mathbf{h}_F(K;O)$ for $\mathsf{h}_{k,w}(K(p^\infty);O)$. From now on, $\overline{\mathbf{S}}_F(\mathfrak{N},O)$ and $\mathbf{h}_F(\mathfrak{N};O)$ stand respectively for $\overline{\mathbf{S}}_F(V_1(\mathfrak{N});O)$ and $\mathbf{h}_F(V_1(\mathfrak{N});O)$.

Remark. Nearly holomorphic cuspforms can be seen as p-adic cuspforms. For each nearly holomorphic cuspform $\mathsf{f}\in N_{k,w,q}(K(p^\alpha);F;O)$ one can define a p-adic cuspform by setting $c(\mathsf{f})=\mathcal{N}_F(y)^{-1}\sum_{\xi\in F_+^\times}\mathsf{a}_p(\xi y\mathsf{d}_F,\mathsf{f})(0)q^\xi\in\overline{\mathbf{S}}_F(K;O)$ [14, Proposition 7.3].

One can decompose the compact ring $\mathbf{h}_F(K;O)$ as a direct sum of algebras $\mathbf{h}_F(K;O)=\mathbf{h}_F^{\mathrm{n.o.}}(K;O)\oplus\mathbf{h}_F^{\mathrm{ss}}(K;O)$ in such a way that $\mathbf{T}(p)$ is a unit in $\mathbf{h}_F^{\mathrm{n.o.}}(K;O)$ and it

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is topologically nilpotent in $\mathbf{h}_F^{\mathrm{ss}}(K; O)$. Furthermore, the idempotent $e_{\mathrm{n.o.}}$ of the nearly ordinary part $\mathbf{h}_F^{\mathrm{n.o.}}(K; O)$ has the familiar expression $e_{\mathrm{n.o.}} = \lim_{n\to\infty} \mathbf{T}(p)^{n!}$. Let $\overline{\mathbf{S}}_F^{\mathrm{n.o.}}(K; O) = e_{\mathrm{n.o.}}\overline{\mathbf{S}}_F(K; O)$ be the space of nearly ordinary p-adic cuspforms.

Consider the topological group $\mathbb{G}_F(K) = Z_F(K) \times \mathcal{O}_{F,p}^{\times}$ equipped with the continuous group homomorphism $\mathbb{G}_F(K) \longrightarrow \mathbf{h}_F^{\mathrm{n.o.}}(K; O)^{\times}$ given by $\langle z, a\rangle \mapsto \langle z\rangle T(a^{-1}, 1)$. As p is prime to the order of $\mathbb{G}_F(K)_{\mathrm{tor}}$, there is a canonical decomposition $\mathbb{G}_F(K) \cong \mathbb{G}_F(K)_{\mathrm{tor}} \times \mathbf{W}_F$ for a $\mathbb{Z}_p$-torsion free subgroup $\mathbf{W}_F$. Then $\mathbf{W}_F \cong \mathbb{Z}_p^r$ for $r = [F : \mathbb{Q}] + 1 + \delta$, where δ is Leopoldt's defect for F, and we denote by $O[\![\mathbf{W}_F]\!] \cong O[\![X_1, \ldots, X_r]\!]$ the completed group ring.

Theorem 2.7 [13, Theorem 2.4]. *The universal nearly ordinary Hecke algebra* $\mathbf{h}_F^{\mathrm{n.o.}}(K; O)$ *is finite and torsion-free over* $\mathbf{\Lambda}_F = O[\![\mathbf{W}_F]\!]$.

One can write $O[\![\mathbb{G}_F(K)]\!] = \bigoplus_{\chi} \mathbf{\Lambda}_{F,\chi}$ as a direct sum ranging over all the characters of $\mathbb{G}_F(K)_{\mathrm{tor}}$ where $\mathbf{\Lambda}_{F,\chi} \cong \mathbf{\Lambda}_F$, and obtain a similar decomposition of the universal nearly ordinary Hecke algebra $\mathbf{h}_F^{\mathrm{n.o.}}(K; O) = \bigoplus_{\chi} \mathbf{h}_F^{\mathrm{n.o.}}(K; O)_{\chi}$.

Definition 2.8. Let K be a compact open subgroup satisfying $V_1(\mathfrak{N}) \leqslant K \leqslant U_0(\mathfrak{N})$ for an $\mathcal{O}_F$-ideal $\mathfrak{N}$ prime to p. Given a character $\chi : \mathbb{G}(K)_{\mathrm{tor}} \to O^{\times}$ and a $\mathbf{\Lambda}_{F,\chi}$-algebra $\mathbf{I}$, we define the space of nearly ordinary $\mathbf{I}$-adic cuspforms of tame level K and character χ to be $\overline{\mathbf{S}}_F^{\mathrm{n.o.}}(K, \chi; \mathbf{I}) = \mathrm{Hom}_{\mathbf{\Lambda}_{F,\chi}\text{-mod}}(\mathbf{h}_F^{\mathrm{n.o.}}(K; O)_{\chi}, \mathbf{I})$. We call Hida families those homomorphisms that are homomorphisms of $\mathbf{\Lambda}_{F,\chi}$-algebras.

Given $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$, with $k - 2w = mt_F$, and finite order characters $\psi : Z_F(K) \to O^{\times}$, $\psi' : \mathcal{O}_{F,p}^{\times} \to O^{\times}$ one can define a homomorphism $\mathbb{G}_F(K) \to O^{\times}$ by $(z, a) \mapsto \psi(z)\psi'(a)\mathcal{N}_F(z)^m a^{t_F - w}$, which determines an O-algebra homomorphism $\mathrm{P}_{k,w,\psi,\psi'} : O[\![\mathbb{G}_F(K)]\!] \to O$. Let us fix an algebraic closure $\overline{\mathbf{L}}$ of the fraction field $\mathbf{L}$ of $\mathbf{\Lambda}_{F,\chi}$ with an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \overline{\mathbf{L}}$. Suppose $\lambda : \mathbf{h}_F^{\mathrm{n.o.}}(K; O)_{\chi} \to \overline{\mathbf{L}}$ is an $\mathbf{\Lambda}_{F,\chi}$-linear map; since the universal nearly ordinary Hecke algebra is finite over $\mathbf{\Lambda}_{F,\chi}$, the image of λ is contained in the integral closure $\mathbf{I}_{\lambda}$ of $\mathbf{\Lambda}_{F,\chi}$ in a finite extension $\mathbf{K}_{\lambda}$ of $\mathbf{L}$.

Definition 2.9. Let $\mathbf{I}$ be a finite integrally closed extension of $\mathbf{\Lambda}_{F,\chi}$. We denote by $\mathcal{A}_{\chi}(\mathbf{I})$ the set of *arithmetic points*, i.e., the subset of $\mathrm{Hom}_{O\text{-alg}}(\mathbf{I}, \overline{\mathbb{Q}}_p)$ consisting of homomorphisms that coincide with some $\mathrm{P}_{k,w,\psi,\psi'}$ (with $k \geqslant 2t_F$, $w \leqslant t_F$) on $\mathbf{\Lambda}_{F,\chi}$.

If $\mathrm{P} \in \mathcal{A}_{\chi}(\mathbf{I}_{\lambda})$, $\mathrm{P}_{|\mathbf{\Lambda}_{F,\chi}} = \mathrm{P}_{k,w,\psi,\psi'|\mathbf{\Lambda}_{F,\chi}}$, the composite $\lambda_{\mathrm{P}} = \mathrm{P} \circ \lambda$ induces a $\overline{\mathbb{Q}}_p$-linear map $\lambda_{\mathrm{P}} : \mathsf{h}_{k,w}^{\mathrm{n.o.}}(K(p^{\alpha}); \overline{\mathbb{Q}}_p) \longrightarrow \overline{\mathbb{Q}}_p$ for some $\alpha > 0$ [13, Theorem 2.4]. Therefore, the duality between Hecke algebra and cuspforms produces a unique p-adic cuspform $\mathsf{f}_{\mathrm{P}} \in S_{k,w}^{\mathrm{n.o.}}(K(p^{\alpha}); \overline{\mathbb{Q}}_p)$ that satisfies $\mathsf{a}_p(y, \mathsf{f}_{\mathrm{P}}) = \lambda_{\mathrm{P}}(\mathbf{T}(y))$ for all integral ideles y. Furthermore, if λ is an algebra homomorphism, each specialization at an arithmetic point is an eigenform and so classical, i.e., an element of $S_{k,w}^{\mathrm{n.o.}}(K(p^{\alpha}); \overline{\mathbb{Q}})$. On the other hand, if $\mathsf{f} \in S_{k,w}(K(p^{\alpha}); \overline{\mathbb{Q}})$ is an eigenform for all Hecke operators and its $U_0(p)$-eigenvalue is a p-adic unit with respect to the fixed p-adic embedding ι_p, then there is character χ, a finite integrally closed extension $\mathbf{I}_{\mathscr{F}}$ of $\mathbf{\Lambda}_{F,\chi}$ and a nearly ordinary $\mathbf{I}_{\mathscr{F}}$-adic Hida family $\mathscr{F} : \mathbf{h}_F^{\mathrm{n.o.}}(K; O) \to \mathbf{I}_{\mathscr{F}}$ passing through f [13, Theorem 2.4].

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Definition 2.10. We define the set of *crystalline* points, $\mathcal{A}^{\circ}_{\chi}(\mathbf{I})$, to be the subset of arithmetic points $\mathrm{P} \in \mathcal{A}_{\chi}(\mathbf{I})$ such that $\mathrm{P}_{|\mathbf{\Lambda}_{F,\chi}} = \mathrm{P}_{k,w,\psi,1|\mathbf{\Lambda}_{F,\chi}}$ for ψ factoring through $\psi : \mathrm{cl}^{+}_{F}(\mathfrak{N}) \to \mathcal{O}^{\times}$ and the eigenform f_{P} is p-old.

Specializations of Hida families with trivial nebentype at p are automatically p-old when $k > 2t_F$ [13, Lemma 12.2].

2.3. Diagonal restriction

If L/F is an extension of totally real fields, there is a restriction map $\mathrm{I}_L \to \mathrm{I}_F$ which induces a group homomorphism $\mathbb{Z}[\mathrm{I}_L] \to \mathbb{Z}[\mathrm{I}_F]$ denoted by $\ell \mapsto \ell_{|F}$ and satisfies $(t_L)_{|F} = [L : F] \cdot t_F$. Let $\mathfrak{N}$ an ideal of $\mathcal{O}_F$, the natural inclusion $\zeta : \mathrm{GL}_2(\mathbb{A}_F) \hookrightarrow \mathrm{GL}_2(\mathbb{A}_L)$ defines by composition a *diagonal restriction* map $\zeta^* : S_{\ell,x}(V_{11}(\mathfrak{N}\mathcal{O}_L); L; \mathbb{C}) \longrightarrow S_{\ell_{|F},x_{|F}}(V_{11}(\mathfrak{N}); F; \mathbb{C})$.

Proposition 2.11. *Let $b \in \mathbb{A}^{\times}_F$. For any cuspform $\mathsf{g} \in S_{\ell,x}(V_{11}(\mathfrak{N}\mathcal{O}_L); L; \mathbb{C})$ we have*

$$\zeta^*(\mathsf{g}_{|V(b)}) = \mathrm{N}_{F/\mathbb{Q}}(b\mathcal{O}_F)^{1-[L:F]}(\zeta^*\mathsf{g})_{|V(b)}.$$

Proof. Follows directly from the definitions. □

Definition 2.12. Let L/F be an extension of totally real number fields and let $\mathfrak{N}$ be an $\mathcal{O}_F$-ideal. For every prime p coprime to $\mathfrak{N}$ and the orders of $Z_F(V_1(\mathfrak{N}))_{\mathrm{tor}}, Z_L(V_1(\mathfrak{N}\mathcal{O}_L))_{\mathrm{tor}}$, diagonal restriction of cuspforms induces by $\mathcal{O}$-duality a map between universal Hecke algebras $\zeta : \mathbf{h}_F(\mathfrak{N}; \mathcal{O}) \longrightarrow \mathbf{h}_L(\mathfrak{N}\mathcal{O}_L; \mathcal{O})$. The element $\zeta(\mathbf{T}(y))$ is determined by the equality

$$\mathsf{a}_p\big(1, \mathsf{g}_{|\zeta(\mathbf{T}(y))}\big) = \mathsf{a}_p\big(1, (\zeta^*\mathsf{g})_{|\mathbf{T}(y)}\big) \quad \forall \mathsf{g} \in \overline{\mathbf{S}}_L(\mathfrak{N}\mathcal{O}_L; \mathcal{O}).$$

We endow $\mathcal{O}[\![\mathbb{G}_L(V_1(\mathfrak{N}\mathcal{O}_L))]\!]$ with the $\mathcal{O}[\![\mathbb{G}_F(V_1(\mathfrak{N}))]\!]$-algebra structure given by $[(z,a)] \mapsto [(z,a)]a^{-t_F}$. The homomorphism ζ is also $\mathcal{O}[\![\mathbb{G}_F(V_1(\mathfrak{N}))]\!]$-linear because diamond operators and operators $T(a,1)$ for $a \in \mathcal{O}^{\times}_{F,p}$ commute with diagonal restriction: $(\zeta^*\mathsf{g})_{|\langle z\rangle} = \zeta^*(\mathsf{g}_{|\langle z\rangle})$, and $(\zeta^*\mathsf{g})_{|T(a,1)} = \zeta^*(\mathsf{g}_{|T(a,1)})$.

2.3.1. On differential operators.

For each $\mu \in \mathrm{I}_L$ there is an operator on p-adic cuspforms $d_\mu : \overline{\mathbf{S}}_L(\mathfrak{Q}; \mathcal{O}) \to \overline{\mathbf{S}}_L(\mathfrak{Q}; \mathcal{O})$ given on q-expansions by $\mathsf{a}_p(y, d_\mu\mathsf{g}) = y_p^{\mu}\mathsf{a}_p(y, \mathsf{g})$. The definition can be extended to all $r \in \mathbb{N}[\mathrm{I}_L]$ by setting $d^r = \prod_{\mu\in\mathrm{I}_L} d_\mu^{r_\mu}$ [14, § 6G].

Lemma 2.13. *Let $r \in \mathbb{N}[\mathrm{I}_L]$ and let $\mathsf{g} \in S_{\ell,x}(V_1(\mathfrak{Q}p^{\alpha}); L; \mathcal{O})$ be a cuspform, then*

$$e_{\mathrm{n.o.}}\Pi^{\mathrm{hol}}\zeta^*(\delta^r_\ell\mathsf{g}) = e_{\mathrm{n.o.}}\zeta^*(d^r\mathsf{g}),$$

where δ^r_ℓ is the Maass–Shimura differential operator (5).

Proof. [14, Proposition 7.3] gives $e_{\mathrm{n.o.}}\Pi^{\mathrm{hol}}\zeta^*(\delta^r_\ell\mathsf{g}) = e_{\mathrm{n.o.}}c(\zeta^*(\delta^r_\ell\mathsf{g}))$. Since $c(\zeta^*(\delta^r_\ell\mathsf{g})) = \zeta^*c(\delta^r_\ell\mathsf{g})$, we conclude by showing that $c(\delta^r_\ell\mathsf{g}) = d^r c(\mathsf{g})$. Indeed,

$$\begin{aligned}\mathsf{a}_p(y, c(\delta^r_\ell\mathsf{g})) = \mathsf{a}_p(y, \delta^r_\ell\mathsf{g})(0) &= y_p^{x-t_L+r}\mathcal{N}_L(a_i)^{-1}\xi^{t_L-x-r}a(\xi, \delta^r_\ell\mathsf{g}_i)(0)\\ &= y_p^{x-t_L+r}\mathcal{N}_L(a_i)^{-1}\xi^{t_L-x}a(\xi, \mathsf{g}_i) = \mathsf{a}_p(y, d^r c(\mathsf{g})).\end{aligned}$$ □

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3. Twisted triple product L-functions

3.1. Complex L-functions

Let L/F be a quadratic extension of totally real number fields, $\mathfrak{Q} \triangleleft \mathcal{O}_L$ and $\mathfrak{N} \triangleleft \mathcal{O}_F$ ideals. Two primitive eigenforms $\mathsf{g} \in S_{\ell,x}(V_1(\mathfrak{Q}); L; \overline{\mathbb{Q}})$ and $\mathsf{f} \in S_{k,w}(V_1(\mathfrak{N}); F; \overline{\mathbb{Q}})$ generate irreducible cuspidal automorphic representations π, σ of $G_L(\mathbb{A}), G_F(\mathbb{A})$ respectively. Let $\pi^u = \pi \otimes |\bullet|_{\mathbb{A}_L}^{n/2}$, $\sigma^u = \sigma \otimes |\bullet|_{\mathbb{A}_F}^{m/2}$ their unitarizations, where n, m are the integers satisfying $n \cdot t_L = \ell - 2x$, $m \cdot t_F = k - 2w$. One can define a unitary representation of $G_{L\times F} = \mathrm{Res}_{L\times F/F}(\mathrm{GL}_{2,L\times F})$ by $\Pi = \pi^u \otimes \sigma^u$. Let $\rho : \Gamma_F \to S_3$ be the homomorphism mapping the absolute Galois group of F to the symmetric group over 3 elements associated with the étale cubic algebra $(L \times F)/F$. The L-group ${}^L(G_{L\times F})$ is given by the semi-direct product $\widehat{G} \rtimes \Gamma_F$ where Γ_F acts on $\widehat{G} = \mathrm{GL}_2(\mathbb{C})^{\times 3}$ through ρ.

Definition 3.1. The twisted triple product L-function associated with the unitary automorphic representation Π is given by the Euler product

$$L(s, \Pi, \mathrm{r}) = \prod_v L_v(s, \Pi_v, \mathrm{r})^{-1}$$

where Π_v is the local representation at the place v of F appearing in the restricted tensor product decomposition $\Pi = \bigotimes_v' \Pi_v$ and representation r gives the action of the L-group of $G_{L\times F}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which restricts to the natural 8-dimensional representation of $\widehat{G}$ and for which Γ_F acts via ρ permuting the vectors.

Let v be a prime of F unramified in L for which Π_v is an unramified principal series, i.e., $v \nmid \mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$. We write ϖ_v for a uniformizer of F_v and q_v for the cardinality of the residue field of F_v. If $v = \mathscr{V} \cdot \overline{\mathscr{V}}$ splits in L, the $\mathrm{GL}_2(F_v)^{\times 3}$-representation Π_v can be written as $\Pi_v = \pi(\chi_{1,\mathscr{V}}, \chi_{2,\mathscr{V}}) \otimes \pi(\chi_{1,\overline{\mathscr{V}}}, \chi_{2,\overline{\mathscr{V}}}) \otimes \pi(\psi_{1,v}, \psi_{2,v})$ and the local Euler factor is given by

$$L_v(s, \Pi_v, \mathrm{r}) = \prod_{i,j,k} \left(1 - \chi_{i,\mathscr{V}}(\varpi_v)\chi_{j,\overline{\mathscr{V}}}(\varpi_v)\psi_{k,v}(\varpi_v)q_v^{-s}\right). \tag{7}$$

If v is inert in L, the $\mathrm{GL}_2(L_v) \times \mathrm{GL}_2(F_v)$-representation Π_v can be written as $\Pi_v = \pi(\chi_{1,v}, \chi_{2,v}) \otimes \pi(\psi_{1,v}, \psi_{2,v})$ and the local Euler factor is given by

$$L_v(s, \Pi_v, \mathrm{r}) = \prod_{i,j} \left(1 - \chi_{i,v}(\varpi_v)\psi_{j,v}(\varpi_v)q_v^{-s}\right) \times \prod_k \left(1 - \chi_{1,v}(\varpi_v)\chi_{2,v}(\varpi_v)\psi_{k,v}^2(\varpi_v)q_v^{-2s}\right). \tag{8}$$

Assume the central character ω_Π of Π is trivial when restricted to $\mathbb{A}_F^\times$, then the complex L-function $L(s, \Pi, \mathrm{r})$ has meromorphic continuation to $\mathbb{C}$ with possible poles at $0, \frac{1}{4}, \frac{3}{4}, 1$ and functional equation $L(s, \Pi, \mathrm{r}) = \epsilon(s, \Pi, \mathrm{r})L(1 - s, \Pi, \mathrm{r})$ [31, Theorems 5.1, 5.2, 5.3].

Remark. The relation between Satake parameters of π^u, σ^u and Hecke eigenvalues of the primitive eigenforms $\mathsf{g}^u, \mathsf{f}^u$ can be given explicitly as follows. Suppose $v \nmid \mathfrak{Q}$ and $v = \mathscr{V}\overline{\mathscr{V}}$ splits in L, then

$$\mathsf{g}^u{}_{|T(\mathscr{V})} = q_v^{1/2}\left(\chi_{1,\mathscr{V}}(\varpi_v) + \chi_{2,\mathscr{V}}(\varpi_v)\right)\mathsf{g}^u, \quad \mathsf{g}^u{}_{|T(\overline{\mathscr{V}})} = q_v^{1/2}\left(\chi_{1,\overline{\mathscr{V}}}(\varpi_v) + \chi_{2,\overline{\mathscr{V}}}(\varpi_v)\right)\mathsf{g}^u. \tag{9}$$

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Moreover, if $v \nmid \mathfrak{Q}$ and v is inert in L then

$$\mathsf{g}^{u}{}_{|T(v\mathcal{O}_L)} = q_v \left(\chi_{1,v}(\varpi_v) + \chi_{2,v}(\varpi_v)\right) \mathsf{g}^{u}. \tag{10}$$

Finally, if $v \nmid \mathfrak{N}$ a finite place of F we have

$$\mathsf{f}^{u}{}_{|T(v)} = q_v^{1/2} \left(\psi_{1,v}(\varpi_v) + \psi_{2,v}(\varpi_v)\right) \mathsf{f}^{u}. \tag{11}$$

3.2. Central L-values and period integrals

Let $D_{/F}$ be a quaternion algebra. We denote by Π^D the irreducible unitary cuspidal automorphic representation of $D^\times(\mathbb{A}_{L\times F})$ associated with Π by the Jacquet–Langlands correspondence when it exists. For a vector $\phi \in \Pi^D$ one defines its period integral as

$$\mathrm{I}^D(\phi) = \int_{[D^\times(\mathbb{A}_F)]} \phi(x)\,\mathrm{d}x$$

where $[D^\times(\mathbb{A}_F)] = \mathbb{A}_F^\times D^\times(F)\backslash D^\times(\mathbb{A}_F)$. To simplify the notation we write $\mathrm{I}(\phi)$ to denote the period integral for the quaternion algebra $\mathrm{M}_2(F)$.

Theorem 3.2. *Let $\eta : \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the quadratic character attached to L/F by class field theory. Then the following are equivalent:*

(1) *The central L-value $L(\frac{1}{2}, \Pi, \mathrm{r})$ does not vanish, and for every place v of F the local ϵ-factor satisfies $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r}) \cdot \eta_v(-1) = 1$.*

(2) *There exists a vector $\phi \in \Pi$, called a test vector, whose period integral $\mathrm{I}(\phi)$ does not vanish.*

Proof. (1) $\Longrightarrow$ (2) By Jacquet conjecture, as proved in [34, Theorem 1.1], the non-vanishing of the central value implies that there exist a quaternion algebra D/F and a vector $\phi \in \Pi^D$ such that its period integral is non-zero, i.e., $\mathrm{I}^D(\phi) \neq 0$. We want to show that the assumption on local ϵ-factors forces the quaternion algebra to be split everywhere. Ichino's formula [18, Theorem 1.1] gives an equality, up to non-zero constants,

$$\mathrm{I}^D \cdot \mathrm{I}^{\widetilde{D}} \doteq L\left(\frac{1}{2}, \Pi, \mathrm{r}\right) \cdot \prod_v \mathrm{I}_v^D$$

of linear forms in $\mathrm{Hom}_{D^\times(\mathbb{A}_F)\times D^\times(\mathbb{A}_F)}(\Pi^D \otimes \widetilde{\Pi}^D, \mathbb{C})$ where $\widetilde{\Pi}^D$ is the contragredient representation and the I_v^D's are local linear forms in $\mathrm{Hom}_{D^\times(\mathbb{A}_{F_v})\times D^\times(\mathbb{A}_{F_v})}(\Pi_v^D \otimes (\widetilde{\Pi}^D)_v, \mathbb{C})$. Suppose v is a place of F at which the quaternion algebra D ramifies, i.e., $v \mid \mathrm{disc}D$. Requiring the value of the expression $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r}) \cdot \eta_\ell(-1)$ to be equal to 1 forces the local Hom-space $\mathrm{Hom}_{D^\times(\mathbb{A}_{F_v})\times D^\times(\mathbb{A}_{F_v})}\left(\Pi_v^D \otimes (\widetilde{\Pi}^D)_v, \mathbb{C}\right)$ to be trivial [11, Theorem 1.2]; in particular it forces the local linear form I_v^D to be trivial. This produces a contradiction because the LHS of Ichino's formula is non-trivial. Indeed, choosing the complex conjugate $\bar{\phi} \in \overline{\Pi^D} \cong \widetilde{\Pi}^D$ of the test vector ϕ we compute that

$$\mathrm{I}^D \cdot \mathrm{I}^{\widetilde{D}}(\phi \otimes \bar{\phi}) = \left|\mathrm{I}^D(\phi)\right|^2 \neq 0.$$

Hence, the discriminant of D has to be trivial, i.e., $D = M_2(F)$.

(2) $\Longrightarrow$ (1) The existence of a test vector $\phi \in \Pi$ implies the non-vanishing of the central value $L(\frac{1}{2}, \Pi, \mathrm{r})$ by Jacquet conjecture. Moreover, Ichino's formula provides us with non-trivial local linear forms, the I_v's, in the local Hom-spaces $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_{F_v})\times \mathrm{GL}_2(\mathbb{A}_{F_v})}(\Pi_v \otimes (\widetilde{\Pi})_v, \mathbb{C})$ which force the equality $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r}) \cdot \eta_v(-1) = 1$ for every place v of F [11, Theorem 1.1]. $\square$

Remark. We can give sufficient conditions on the eigenforms $\mathsf{g} \in S_{\ell,x}(V_1(\mathfrak{Q}); L; \overline{\mathbb{Q}})$ and $\mathsf{f} \in S_{k,w}(V_1(\mathfrak{N}); F; \overline{\mathbb{Q}})$ such that the local ϵ-factors of the automorphic representation Π satisfy the hypothesis of Theorem 3.2. The local ϵ-factor at the archimedean places of F satisfy the hypothesis of the theorem if the weights of g and f are F-dominated (Definition 1.1). Moreover, the same is true for the ϵ-factors at the finite places if we assume that $\mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$ and $\mathfrak{N}$ are coprime and that every finite prime v dividing $\mathfrak{N}$ splits in L [33, Theorems B, D and Remark 4.1.1].

Proposition 3.3. *For all finite places v of F away from the level of Π and unramified in L/F, a newvector in Π_v is a choice of test vector for Ichino's local linear functional.*

Proof. If v is a place splitting in L, the claim follows from [32, Theorem 5.10]. We show that the proof given by Prasad can be adapted to deal with the inert case as follows. Our claim is that the image of the spherical vector under the non-trivial linear functional $\Upsilon : (\pi^u)_v \to (\widetilde{\sigma^u})_v$, unique up to scaling, is non-zero. As in [33, § 4] we can assume that $(\pi^u)_v$ is the principal series V_χ for the character of the Borel

$$\chi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \alpha(a)\beta(d)^{-1} \quad \text{for unramified characters } \alpha, \beta : L_v^\times \to \mathbb{C}^\times,$$

so that the representation V_χ can be realized in the space of functions over $\mathbb{P}^1_{L_v}$ and the spherical vector corresponds to the constant function $1_{\mathbb{P}^1_{L_v}}$. The projective line $\mathbb{P}^1_{L_v}$ can be decomposed into an open and a closed orbit for the action of $\mathrm{GL}_2(F_v)$,

$$\mathbb{P}^1_{L_v} = \left(\mathbb{P}^1_{L_v} \setminus \mathbb{P}^1_{F_v}\right) \coprod \mathbb{P}^1_{F_v},$$

which produces an exact sequence of $\mathrm{GL}_2(F_v)$-modules

$$0 \longrightarrow \mathrm{ind}_{L_v^\times}^{\mathrm{GL}_2(F_v)}(\chi') \longrightarrow V_\chi \longrightarrow \mathrm{Ind}_{\mathrm{B}(F_v)}^{\mathrm{GL}_2(F_v)}(\chi\delta_{F_v}^{1/2}) \longrightarrow 0 \tag{12}$$

for $\chi' : L_v^\times \to \mathbb{C}^\times$ the character defined by $\chi'(x) = \alpha(x)\beta(\overline{x})$. If $\mathrm{Ind}_{\mathrm{B}(F_v)}^{\mathrm{GL}_2(F_v)}(\chi\delta_{F_v}^{1/2})$ is isomorphic to the contragradient representation $(\widetilde{\sigma^u})_v$ then we are done, because $1_{\mathbb{P}^1_{L_v}} \mapsto 1_{\mathbb{P}^1_{F_v}} \neq 0$. Otherwise, suppose $\Upsilon(1_{\mathbb{P}^1_{L_v}}) = 0$. Let T_v be the Hecke operator given by the double coset

$$T_v = \left[\mathrm{GL}_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v})\right],$$

then the function

$$\frac{1}{(q_v+1)\chi\delta^{1/2}(\varpi_v)} \left(T_v(1_{\mathbb{P}^1_{L_v}}) - q_v\chi\delta^{1/2}(\varpi_v) - \chi\delta^{1/2}(1/\varpi_v)\right) \tag{13}$$

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is the constant function 1 on the $\mathrm{GL}_2(\mathcal{O}_{F_v})$-orbit of $\mathbb{P}^1_{L_v}$ consisting of those points that reduce to a point in $\mathbb{P}^1(\mathcal{O}_{L_v}/\varpi_v) \setminus \mathbb{P}^1(\mathcal{O}_{F_v}/\varpi_v)$, and the constant function zero everywhere else. Therefore, the function (13) is an element of $\mathrm{ind}_{L_v^\times}^{\mathrm{GL}_2(F_v)}(\chi')$ because of the short exact sequence (12). The function (13) is sent to zero by Υ by $\mathrm{GL}_2(F_v)$-equivariance, but at the same time that is not possible because we can explicitly describe the elements of $\mathrm{Hom}_{\mathrm{GL}_2(F_v)}(\mathrm{ind}_{L_v^\times}^{\mathrm{GL}_2(F_v)}(\chi'), (\widetilde{\sigma^u})_v)$ in terms of integration over $\mathrm{GL}_2(\mathcal{O}_{F_v})$-orbits of $\mathbb{P}^1_{L_v}$ giving a contradiction. □

3.3. p-adic L-functions

Let $\mathsf{g} \in S_{\ell,x}(V_1(\mathfrak{Q}); L; E)$, $\mathsf{f} \in S_{k,w}(V_1(\mathfrak{N}); F; E)$ be primitive eigenforms defined over a number field E whose weights are F-dominated. We assume the central character ω_Π of Π to be trivial when restricted to $\mathbb{A}_F^\times$, that the central L-value $L(\frac{1}{2}, \Pi, \mathrm{r})$ does not vanish, and that for every place v of F we have the condition $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r})\eta_v(-1) = 1$ on local ϵ-factors satisfied. Then there exists a vector $\phi \in \Pi$ such that the period integral $\mathrm{I}(\phi)$ is non-zero (Theorem 3.2). Let $\mathfrak{J}$ be the element

$$\mathfrak{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{\mathrm{I}_F} \in \mathrm{GL}_2(\mathbb{R})^{\mathrm{I}_F}.$$

For any $\mathsf{h} \in \sigma^u$ we define $\mathsf{h}^{\mathfrak{J}} \in \sigma^u$ to be the vector obtained by right translation $\mathsf{h}^{\mathfrak{J}}(g) = \mathsf{h}(g\mathfrak{J})$. If h has weight $k \in \mathbb{Z}[\mathrm{I}_F]$ then $\mathsf{h}^{\mathfrak{J}}(h)$ has weight $-k$.

Lemma 3.4. *Let $r \in \mathbb{N}[\mathrm{I}_L]$ be such that $k = (\ell + 2r)_{|F}$ and $w = (x + r)_{|F}$. Then there is an $\mathcal{O}_F$-ideal $\mathfrak{A}$ supported on a subset of the prime factors of $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$ such that a test vector ϕ can be chosen to be of the form $\phi = (\delta^r \breve{\mathsf{g}})^u \otimes (\breve{\mathsf{f}}^{\mathfrak{J}})^u$ for $\breve{\mathsf{g}} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ and $\breve{\mathsf{f}} \in S_{k,w}(V_{11}(\mathfrak{A}); F; E)$. The cuspforms $\breve{\mathsf{g}}, \breve{\mathsf{f}}$ are eigenforms for all Hecke operators outside $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$ with the same Hecke eigenvalues of g and f respectively.*

Proof. By linearity of the period integral we can assume ϕ to be a simple tensor. We can also assume $\phi = \delta^r \vartheta \otimes \nu^{\mathfrak{J}} \in \Pi$ because the archimedean linear functional appearing in Ichino's formula is non-zero if and only if the sum of the weights of the local vectors is zero. Moreover, Proposition 3.3 allows us to take ϑ_v and ν_v newvectors for all finite places that do not divide $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$. Note that spherical vectors are mapped to spherical vectors by the isomorphism $\pi \otimes \sigma \xrightarrow{\sim} \pi^u \otimes \sigma^u$ as in (3). Therefore we can write $\phi = \delta^r \vartheta \otimes \nu^{\mathfrak{J}}$ as $(\delta^r \breve{\mathsf{g}})^u \otimes (\breve{\mathsf{f}}^{\mathfrak{J}})^u$, for $\breve{\mathsf{g}} \in \pi$ and $\breve{\mathsf{f}} \in \sigma$ of levels $U(\mathfrak{B}\mathcal{O}_L)$ and $U(\mathfrak{B})$ for some $\mathcal{O}_F$-ideal $\mathfrak{B}$ supported on a subset of the places dividing $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$. We conclude by showing that we can assume that $\breve{\mathsf{g}} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ and $\breve{\mathsf{f}} \in S_{k,w}(V_{11}(\mathfrak{A}); F; E)$ for $\mathfrak{A} = \mathfrak{B}^2$. Indeed, right translation by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad b\mathcal{O}_F = \mathfrak{B},$$

induces injections

$$S_{\ell,x}(U(\mathfrak{B}\mathcal{O}_L); L; E) \hookrightarrow S_{\ell,x}(V_{11}(\mathfrak{B}^2\mathcal{O}_L); L; E),$$
$$S_{k,w}(U(\mathfrak{B}); F; E) \hookrightarrow S_{k,w}(V_{11}(\mathfrak{B}^2); F; E)$$

equivariant for the action of Hecke operators away from the level and that change the period by a non-zero constant. □

When the test vector ϕ is as in Lemma 3.4, we can rewrite the period integral $\mathrm{I}(\phi)$ as a Petersson inner product

$$\mathrm{I}(\phi) = \int_{[\mathrm{GL}_2(\mathbb{A}_F)]} (\delta^r \breve{\mathsf{g}})^u \otimes (\breve{\mathsf{f}}^{\mathfrak{I}})^u \, \mathrm{d}x = \left\langle \zeta^* \left(\delta^r \breve{\mathsf{g}}\right), \breve{\mathsf{f}}^* \right\rangle \tag{14}$$

where $\breve{\mathsf{f}}^* = \overline{(\breve{\mathsf{f}}^{\mathfrak{I}})}$ is the cuspform in $S_{k,w}(V_{11}(\mathfrak{A}); F; E)$ whose Fourier coefficients are complex conjugates of those of $\breve{\mathsf{f}}$. We conclude the section with a proposition showing that a good transcendental period for the central L-value of the twisted triple product L-function is the Petersson norm of the eigenform f^*.

Proposition 3.5. *Let E be a number field and let $\breve{\mathsf{f}} \in S_{k,w}(V_{11}(\mathfrak{A}); F; E)$ be a vector in an irreducible cuspidal automorphic representation σ spanned by a primitive cuspform $\mathsf{f} \in S_{k,w}(V_{11}(\mathfrak{N}); F; E)$. Then for any $\varphi \in S_{k,w}(V_{11}(\mathfrak{A}); F; E)$ the Petersson inner product $\langle \varphi, \breve{\mathsf{f}} \rangle$ is a E-rational multiple of $\langle \mathsf{f}, \mathsf{f} \rangle$.*

Proof. We follow the argument of [7, Lemma 2.12]. The Petersson inner product $\langle \breve{\mathsf{f}}, \varphi \rangle$ depends only on the projection $e_{\mathsf{f}}\varphi$ of φ to σ. The E-vector space $e_{\mathsf{f}} S_{k,w}(V_{11}(\mathfrak{A}); F; E)$ is spanned by the cuspforms

$$\left\{\mathsf{f}_{\mathfrak{a}} \mid \mathsf{f}_{\mathfrak{a}}(x) = \mathsf{f}(x s_{\mathfrak{a}})\right\}_{\mathfrak{a} \mid \frac{\mathfrak{A}}{\mathfrak{N}}}, \quad s_{\mathfrak{a}} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad a\mathcal{O}_F = \mathfrak{a}$$

for all ideals $\mathfrak{a}$ dividing $\mathfrak{A}/\mathfrak{N}$ ([29, Proposition 6] and [37, Proposition 2.3]). Thus, it suffices to prove the statement for $\mathsf{f}_{\mathfrak{a}_1}$ and $\mathsf{f}_{\mathfrak{a}_2}$ when $\mathfrak{a}_1, \mathfrak{a}_2 \mid \mathfrak{A}/\mathfrak{N}$. We prove the claim by induction on the prime divisors of $\mathfrak{a}_1, \mathfrak{a}_2$. If $\mathfrak{a}_1\mathfrak{a}_2 = \mathcal{O}_F$ then the claim is clear. Suppose there is a prime ideal $\mathfrak{p}$ that divides both $\mathfrak{a}_1$ and $\mathfrak{a}_2$, then $\langle \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2} \rangle = \langle \mathsf{f}_{\mathfrak{a}_1/\mathfrak{p}}, \mathsf{f}_{\mathfrak{a}_2/\mathfrak{p}} \rangle$ because the Haar measure is invariant under translation. Thus, without loss of generality, we can assume $\mathfrak{p}$ divides $\mathfrak{a}_2$ but not $\mathfrak{a}_1$. We compute the equalities

$$\mathsf{a}(\varpi_{\mathfrak{p}}, \mathsf{f}) \langle \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2} \rangle = \begin{cases} \langle T_0(\varpi_{\mathfrak{p}}) \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2} \rangle = (q_{\mathfrak{p}} + 1) \langle \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2/\mathfrak{p}} \rangle & \text{if } \mathfrak{p} \nmid \mathfrak{N} \\ \langle U_0(\varpi_{\mathfrak{p}}) \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2} \rangle = q_{\mathfrak{p}} \langle \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2/\mathfrak{p}} \rangle & \text{if } \mathfrak{p} \mid \mathfrak{N}, \end{cases}$$

that show

$$\langle \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2} \rangle = \begin{cases} \frac{\mathsf{a}(\varpi_{\mathfrak{p}}, \mathsf{f})}{q_{\mathfrak{p}}+1} \langle \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2/\mathfrak{p}} \rangle & \text{if } \mathfrak{p} \nmid \mathfrak{N} \\ \frac{\mathsf{a}(\varpi_{\mathfrak{p}}, \mathsf{f})}{q_{\mathfrak{p}}} \langle \mathsf{f}_{\mathfrak{a}_1}, \mathsf{f}_{\mathfrak{a}_2/\mathfrak{p}} \rangle & \text{if } \mathfrak{p} \mid \mathfrak{N}, \end{cases}$$

concluding the inductive step. □

3.3.1.

Construction. Suppose we are given primitive eigenforms $\mathsf{g}_\circ \in S_{\ell_\circ,x_\circ}(V_1(\mathfrak{Q}); L; \overline{\mathbb{Q}})$ and $\mathsf{f}_\circ \in S_{k_\circ,w_\circ}(V_1(\mathfrak{N}); F; \overline{\mathbb{Q}})$ with $n_\circ t_L = \ell_\circ - 2x_\circ$, $m_\circ t_F = k_\circ - 2w_\circ$ for $n_\circ, m_\circ \in \mathbb{Z}$. We choose an element $\theta \in \mathbb{Z}[\mathrm{I}_L]$ such that $\theta_{|F} = 0 \cdot t_F$, $\theta \equiv_2 w_\circ$ and set

$$r_\circ = r_\circ(\theta) = \sum_{\mu \in \mathrm{I}_L} \left[\frac{(w_\circ)_{\mu_{|F}} + \theta_\mu}{2} - (x_\circ)_\mu \right] \cdot \mu \in \mathbb{Z}[\mathrm{I}_L].$$

Let p be a rational prime unramified in L, coprime to the levels $\mathfrak{Q}, \mathfrak{N}$. We write $\mathcal{P}$ (respectively $\mathcal{Q}$) for the set of prime $\mathcal{O}_L$-ideals (respectively $\mathcal{O}_F$-ideals) dividing p. We suppose $\mathsf{g}_\circ$, $\mathsf{f}_\circ$ are p-nearly ordinary and we denote by $\mathscr{G} \in \overline{\mathbf{S}}_L^{\mathrm{n.o.}}(\mathfrak{Q}, \chi; \mathbf{I}_\mathscr{G})$ and $\mathscr{F} \in \overline{\mathbf{S}}_F^{\mathrm{n.o.}}(\mathfrak{N}, \psi; \mathbf{I}_\mathscr{F})$ the Hida families passing through nearly ordinary p-stabilizations $\mathsf{g}_\circ^{(p)}$ and $\mathsf{f}_\circ^{(p)}$. We have $\chi_{|Z_L(\mathfrak{Q})_{\mathrm{tor}}} = \chi_\circ \mathcal{N}_L^{n_\circ}$ and $\psi_{|Z_F(\mathfrak{N})_{\mathrm{tor}}} = \psi_\circ \mathcal{N}_F^{m_\circ}$ for characters $\chi_\circ : \mathrm{cl}_L^+(\mathfrak{Q}) \to \mathbb{C}^\times$, $\psi_\circ : \mathrm{cl}_F^+(\mathfrak{N}) \to \mathbb{C}^\times$ and we suppose that $\chi_{\circ|F} \cdot \psi_\circ \equiv 1$. Let $\mathscr{F}^* \in \overline{\mathbf{S}}_F^{\mathrm{n.o.}}(\mathfrak{A}, \psi_\circ^{-2}\psi; \mathbf{I}_{\mathscr{F}^*})$ [14, § 7F] be the twisted Hida family, where $\mathbf{I}_{\mathscr{F}^*} \cong \mathbf{I}_\mathscr{F}(\psi_\circ^{-2})$ as an $\mathbf{\Lambda}_{F,\psi_\circ^{-2}\psi}$-algebra.

Set $\mathbf{K}_{\mathscr{G},\mathscr{F}^*} = (\mathbf{I}_\mathscr{G} \widehat{\otimes}_O \mathbf{I}_{\mathscr{F}^*}) \otimes \mathbb{Q}$, $\mathbf{K}_\mathscr{G} = \mathbf{I}_\mathscr{G} \otimes \mathbb{Q}$ and $\mathbf{K}_{\mathscr{F}^*} = \mathbf{I}_{\mathscr{F}^*} \otimes \mathbb{Q}$. We define a $\mathbf{K}_\mathscr{G}$-adic cuspform $\breve{\mathscr{G}}$ (respectively $\mathbf{K}_{\mathscr{F}^*}$-adic cuspform $\breve{\mathscr{F}}^*$) passing through the nearly ordinary p-stabilization of the test vectors $\breve{\mathsf{g}}_\circ, \breve{\mathsf{f}}_\circ^*$ as in [7, § 2.6]. Let $\bar{r} = \sum_{\mu \in \mathrm{I}_L} \bar{r}_\mu \cdot \mu$, with $\bar{r}_\mu \in \mathbb{Z}/(q_{\mathfrak{p}_\mu} - 1)\mathbb{Z}$, denote the reduction of $r_\circ$. We define a homomorphism of $O[\![\mathbb{G}_L(V_1(\mathfrak{A}\mathcal{O}_L))]\!]$-modules ${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]} : \mathbf{h}_L(\mathfrak{A}\mathcal{O}_L; O) \longrightarrow \mathbf{K}_{\mathscr{G},\mathscr{F}^*}$ by

$${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}(\langle z \rangle \mathbf{T}(y)) = \begin{cases} \breve{\mathscr{G}}(\langle z \rangle \mathbf{T}(y))\,[\langle y_p \rangle] \otimes [\mathrm{N}_{L/F}\langle y_p \rangle^{-1/2}] \langle y_p \rangle^{\frac{\theta - t_L}{2}} \omega(y_p)^{\bar{r}} & \text{if } y_p \in \mathcal{O}_{L,p}^\times \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{K}_{\mathscr{G},\mathscr{F}^*}$ is given the $O[\![\mathbb{G}_L(V_1(\mathfrak{A}\mathcal{O}_L))]\!]$-algebra structure $[(z,a)] \mapsto \breve{\mathscr{G}}(\langle z \rangle T(a^{-1}, 1))$ $[\langle a \rangle^{-1}] \otimes [\mathrm{N}_{L/F}\langle a \rangle^{1/2}] \langle a \rangle^{\frac{-\theta + t_L}{2}} \omega(a)^{-\bar{r}}$ and $\langle\ \rangle : \mathcal{O}_{L,p}^\times \to (\mathcal{O}_{L,p}^\times)_{\text{pro-}p}$, $\omega : \mathcal{O}_{L,p}^\times \to (\mathcal{O}_{L,p}^\times)_{\mathrm{tor}}$ are the canonical projections. The composition of the natural maps $\mathbf{h}_F^{\mathrm{n.o.}}(\mathfrak{A}; O) \to \mathbf{h}_F(\mathfrak{A}; O) \to \mathbf{h}_L(\mathfrak{A}\mathcal{O}_L; O)$ with the homomorphism ${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]} : \mathbf{h}_L(\mathfrak{A}\mathcal{O}_L; O) \to \mathbf{K}_{\mathscr{G},\mathscr{F}^*}$ defines a nearly ordinary $\mathbf{K}_{\mathscr{G},\mathscr{F}^*}$-adic cuspform $e_{\mathrm{n.o.}}\zeta^*\big({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big) \in \overline{\mathbf{S}}_F^{\mathrm{n.o.}}(\mathfrak{A}, \psi_\circ^{-2}\psi; \mathbf{K}_{\mathscr{G},\mathscr{F}^*})$.

Proposition 3.6. *Let $s : \mathrm{I}_F \to \mathrm{I}_L$ be any section of the restriction $\mathrm{I}_L \to \mathrm{I}_F$, $\mu \mapsto \mu_{|F}$. For any arithmetic point $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta)$ a lift of $\bar{r}$, we have*

$$e_{\mathrm{n.o.}}\zeta^*\big({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)(\mathrm{P}, \mathrm{Q}) = e_{\mathrm{n.o.}}\zeta^*\big(d^{r(\theta)} \breve{\mathsf{g}}_\mathrm{P}^{[\mathcal{P}]}\big) = \pm e_{\mathrm{n.o.}}\zeta^*(d^{s(w - x_{|F})} \breve{\mathsf{g}}_\mathrm{P}^{[\mathcal{P}]}).$$

Proof. For an arithmetic point $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta)$ a lift of $\bar{r}$, the explicit description of ${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}$ produces the equality of modular forms $e_{\mathrm{n.o.}}\zeta^*\big({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)(\mathrm{P}, \mathrm{Q}) = e_{\mathrm{n.o.}}\zeta^*\big(d^{r(\theta)} \breve{\mathsf{g}}_\mathrm{P}^{[\mathcal{P}]}\big)$. Let now $\mu, \mu' \in \mathrm{I}_L$, $\mu \neq \mu'$, be such that $\tau = \mu_{|F} = \mu'_{|F}$. A direct computation shows that $0 = e_{\mathrm{n.o.}} d_\tau \zeta^* \mathsf{g} = e_{\mathrm{n.o.}}\zeta^*(d_\mu + d_{\mu'})\mathsf{g}$ for any g, which implies $e_{\mathrm{n.o.}}\zeta^*(d_\mu^\alpha \mathsf{g}) = (-1)^\alpha e_{\mathrm{n.o.}}\zeta^*(d_{\mu'}^\alpha \mathsf{g})$ for any $\alpha \in \mathbb{N}$. When $\mathsf{g} = \mathsf{g}^{[\mathcal{P}]}$ is $\mathcal{P}$-depleted, we also have $e_{\mathrm{n.o.}}\zeta^*(d_\mu^\alpha \mathsf{g}^{[\mathcal{P}]}) = (-1)^\alpha e_{\mathrm{n.o.}}\zeta^*(d_{\mu'}^\alpha \mathsf{g}^{[\mathcal{P}]})$ for any $\alpha \in \lim_{\leftarrow,n} \mathbb{Z}/p^n(q_{\mathfrak{p}_\mu} - 1)$ by taking p-adic limits. Thus, the second equality follows. □

Lemma 3.7. *There is an element* $J(e_{\text{n.o.}}\zeta^*({}_{\bar r}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}), \breve{\mathscr{F}}^*) \in \mathbf{I}_{\mathscr{G}}\widehat{\otimes}_O \mathrm{Frac}(\mathbf{I}_{\mathscr{F}^*})$ *such that for any crystalline point* $(\mathrm{P},\mathrm{Q}) \in \boldsymbol{\mathcal{W}}$*, with* $r(\theta) \in \mathbb{Z}[\mathrm{I}_L]$ *a lift of* $\bar r$*, we have*

$$J\big(e_{\text{n.o.}}\zeta^*\big({}_{\bar r}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big), \breve{\mathscr{F}}^*\big)(\mathrm{P},\mathrm{Q}) = \frac{\big\langle e_{\text{n.o.}}\zeta^*\big(d^{r(\theta)}\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]}\big), \breve{\mathsf{f}}_{\mathrm{Q}}^{*(p)}\big\rangle}{\big\langle \mathsf{f}_{\mathrm{Q}}^{*(p)}, \mathsf{f}_{\mathrm{Q}}^{*(p)}\big\rangle}. \tag{15}$$

Proof. We follow the argument of [7, Lemma 2.19]. Both $\breve{\mathscr{F}}^*$ and the $\mathscr{F}^*$-isotypic projection $e_{\mathscr{F}^*}\zeta^*\big({}_{\bar r}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)$ are $\mathbf{I}_{\mathscr{G}}\widehat{\otimes}_O\mathrm{Frac}(\mathbf{I}_{\mathscr{F}^*})$-linear combinations of the $\mathbf{I}_{\mathscr{F}^*}$-adic cuspforms $\mathscr{F}_{\mathfrak{a}}^*$ for $\mathfrak{a} \mid \mathfrak{A}/\mathfrak{N}$. Hence, the element J exists because we can interpolate expressions of the form

$$\big\langle \mathsf{f}_{\mathfrak{a}_1,\mathrm{Q}}^{*(p)}, \mathsf{f}_{\mathfrak{a}_2,\mathrm{Q}}^{*(p)}\big\rangle / \big\langle \mathsf{f}_{\mathrm{Q}}^{*(p)}, \mathsf{f}_{\mathrm{Q}}^{*(p)}\big\rangle$$

for $\mathrm{Q} \in \mathcal{A}^\circ(\mathbf{I}_{\mathscr{F}^*})$ using the explicit computations in the proof of Proposition 3.5 and the fact that $\mathfrak{A}$ is prime to p. □

Definition 3.8. The twisted triple product p-adic L-function attached to $\big(\breve{\mathscr{G}}, \breve{\mathscr{F}}, \theta, \bar r\big)$ is the meromorphic rigid-analytic function

$${}_{\bar r}\mathscr{L}_p^\theta(\breve{\mathscr{G}}, \breve{\mathscr{F}}) : \boldsymbol{\mathcal{W}}_{\mathscr{G},\mathscr{F}^*} \longrightarrow \mathbb{C}_p$$

determined by $J\big(e_{\text{n.o.}}\zeta^*\big({}_{\bar r}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big), \breve{\mathscr{F}}^*\big) \in \mathbf{I}_{\mathscr{G}}\widehat{\otimes}_O\mathrm{Frac}(\mathbf{I}_{\mathscr{F}^*})$.

Let $\mathsf{h}_{\mathrm{P},\mathrm{Q}} = e_{\mathsf{f}_{\mathrm{Q}}^*,\text{n.o.}}\zeta^*(d^{r(\theta)}\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]})$ with nearly ordinary p-stabilization $\mathsf{h}_{\mathrm{P},\mathrm{Q}}^{(p)} = (1 - \beta_{\mathsf{f}_{\mathrm{Q}}^*}V(p))\mathsf{h}_{\mathrm{P},\mathrm{Q}}$. By definition $e_{\text{n.o.}}\mathsf{h}_{\mathrm{P},\mathrm{Q}}^{(p)} = \mathsf{h}_{\mathrm{P},\mathrm{Q}}^{(p)}$, that results in the equality $\mathsf{h}_{\mathrm{P},\mathrm{Q}}^{(p)} = e_{\text{n.o.}}\mathsf{h}_{\mathrm{P},\mathrm{Q}}^{(p)} = (1-\beta_{\mathsf{f}_{\mathrm{Q}}^*}\alpha_{\mathsf{f}_{\mathrm{Q}}^*}^{-1})e_{\text{n.o.}}\mathsf{h}_{\mathrm{P},\mathrm{Q}}$. More explicitly, if we set $\mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*) = (1-\beta_{\mathsf{f}_{\mathrm{Q}}^*}\alpha_{\mathsf{f}_{\mathrm{Q}}^*}^{-1})$,

$$\Big(e_{\mathsf{f}_{\mathrm{Q}}^*,\text{n.o.}}\zeta^*(d^{r(\theta)}\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]})\Big)^{(p)} = \mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*)\cdot e_{\mathsf{f}_{\mathrm{Q}}^*,\text{n.o.}}\zeta^*(d^{r(\theta)}\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]})$$

that allows to rewrite the values of the p-adic L-function at every crystalline point $(\mathrm{P},\mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta) \in \mathbb{Z}[\mathrm{I}_L]$ a lift of $\bar r$, as

$${}_{\bar r}\mathscr{L}_p^\theta(\breve{\mathscr{G}}, \breve{\mathscr{F}})(\mathrm{P},\mathrm{Q}) = \frac{1}{\mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*)}\frac{\big\langle e_{\text{n.o.}}\zeta^*\big(d^{r(\theta)}\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]}\big), \breve{\mathsf{f}}_{\mathrm{Q}}^*\big\rangle}{\big\langle \mathsf{f}_{\mathrm{Q}}^*, \mathsf{f}_{\mathrm{Q}}^*\big\rangle}. \tag{16}$$

3.4. Interpolation formulas

The interpolation formulas satisfied by the twisted triple product p-adic L-function include Euler factors that depend on whether the primes in $\mathcal{P}$ are above a prime of F that is split or inert in the extension L/F. We partition the set of primes of F above p accordingly to the splitting behavior in L/F, $\mathcal{Q} = \mathcal{Q}_{\text{inert}} \coprod \mathcal{Q}_{\text{split}}$. For every prime $\mathcal{O}_F$-ideal $\wp \in \mathcal{Q}$ we denote by $q_\wp$ the cardinality of its residue field.

Inert case. For a prime ideal $\mathfrak{p} \in \mathcal{P}$ with $\wp = \mathfrak{p} \cap \mathcal{O}_F \in \mathcal{Q}_{\text{inert}}$, we write $\mathfrak{p} = \wp\mathcal{O}_L$.

Lemma 3.9. *Let* $\mathsf{g} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ *be a* $T(\mathfrak{p})$*-eigenvector,* $\mathsf{f} \in S_{k,w}(V_{11}(\mathfrak{N}); F; E)$ *a* p*-nearly ordinary eigenform. Suppose* $\mathfrak{N} \mid \mathfrak{A}$ *and that the weights of* g,f *are* F*-dominated.*

If we denote by $e_{\mathsf{f},\mathrm{n.o.}} = e_{\mathsf{f}} e_{\mathrm{n.o.}}$ *the composition of the* f*-isotypic projection with the nearly ordinary projector, we have*

$$e_{\mathsf{f},\mathrm{n.o.}}\zeta^*\big(d^r \mathsf{g}^{[\mathfrak{p}]}\big) = \mathcal{E}_\wp(\mathsf{g},\mathsf{f}) e_{\mathsf{f},\mathrm{n.o.}}\zeta^*\left(d^r\mathsf{g}\right) \quad \text{for } \mathcal{E}_\wp(\mathsf{g},\mathsf{f}) = \big(1-\alpha_{\mathsf{g}}\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}\big)\big(1-\beta_{\mathsf{g}}\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}\big),$$

where $\alpha_{\mathsf{g}}, \beta_{\mathsf{g}}$ *are the inverses of the roots of the* $T(\mathfrak{p})$*-Hecke polynomial of* g *and* α_{f} *is determined by* $(e_{\mathrm{n.o.}}\mathsf{f})_{|U(\varpi_\wp)} = \alpha_{\mathsf{f}} \cdot e_{\mathrm{n.o.}}\mathsf{f}$.

Proof. Let $\mathsf{g}_\alpha^{(\mathfrak{p})} = (1-\beta_{\mathsf{g}} V(\varpi_{\mathfrak{p}}))\mathsf{g}$, $\mathsf{g}_\beta^{(\mathfrak{p})} = (1-\alpha_{\mathsf{g}} V(\varpi_{\mathfrak{p}}))\mathsf{g}$ be the two $\mathfrak{p}$-stabilizations of g, they satisfy $U(\varpi_{\mathfrak{p}})\mathsf{g}_\bullet^{(\mathfrak{p})} = (\bullet)\mathsf{g}_\bullet^{(\mathfrak{p})}$ and $\mathsf{g} = 1/(\alpha_{\mathsf{g}}-\beta_{\mathsf{g}})\big(\alpha_{\mathsf{g}}\mathsf{g}_\alpha^{(\mathfrak{p})} - \beta_{\mathsf{g}}\mathsf{g}_\beta^{(\mathfrak{p})}\big)$. Using Proposition 2.11, we compute

$$\begin{aligned} e_{\mathsf{f},\mathrm{n.o.}}\zeta^*\left[d^r\left(1-V(\varpi_{\mathfrak{p}})\circ U(\varpi_{\mathfrak{p}})\right)\mathsf{g}_\bullet^{(\mathfrak{p})}\right] &= e_{\mathsf{f},\mathrm{n.o.}}\zeta^*\left[\left(1-(\bullet)V(\varpi_{\mathfrak{p}})\right)d^r\mathsf{g}_\bullet^{(\mathfrak{p})}\right] \\ &= e_{\mathsf{f},\mathrm{n.o.}}\left(1-(\bullet)q_\wp^{-1}V(\varpi_\wp)\right)\zeta^*(d^r\mathsf{g}_\bullet^{(\mathfrak{p})}) \\ &= \left(1-(\bullet)q_\wp^{-1}\alpha_{\mathsf{f}}^{-1}\right)e_{\mathsf{f},\mathrm{n.o.}}\zeta^*(d^r\mathsf{g}_\bullet^{(\mathfrak{p})}). \end{aligned}$$

Noting that the $\mathfrak{p}$-depletions of the $\mathfrak{p}$-stabilizations are equal, $(\mathsf{g}_\alpha^{(\mathfrak{p})})^{[\mathfrak{p}]} = (\mathsf{g}_\beta^{(\mathfrak{p})})^{[\mathfrak{p}]} = \mathsf{g}^{[\mathfrak{p}]}$, we deduce the claim:

$$\begin{aligned} e_{\mathsf{f},\mathrm{n.o.}}\zeta^*(d^r\mathsf{g}) &= \frac{1}{\alpha_{\mathsf{g}}-\beta_{\mathsf{g}}}\left(\alpha_{\mathsf{g}} e_{\mathsf{f},\mathrm{n.o.}}\zeta^*(d^r\mathsf{g}_\alpha^{(\mathfrak{p})}) - \beta_{\mathsf{g}} e_{\mathsf{f},\mathrm{n.o.}}\zeta^*(d^r\mathsf{g}_\beta^{(\mathfrak{p})})\right) \\ &= \frac{1}{\left(1-\alpha_{\mathsf{g}}\alpha_{\mathsf{f}}{}^{-1}q_\wp^{-1}\right)\left(1-\beta_{\mathsf{g}}\alpha_{\mathsf{f}}{}^{-1}q_\wp^{-1}\right)} e_{\mathsf{f},\mathrm{n.o.}}\zeta^*(d^r\mathsf{g}^{[\mathfrak{p}]}). \end{aligned}$$ □

Split case. For a prime ideal $\mathfrak{p} \in \mathcal{P}$ with $\wp = \mathfrak{p} \cap \mathcal{O}_F \in \mathcal{Q}_{\mathrm{split}}$, we write $\wp\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$.

Lemma 3.10. *Let* $\mathfrak{Q}$ *be any* $\mathcal{O}_L$*-ideal and* $\mathsf{g} \in S_{\ell,x}(V_{11}(\mathfrak{Q}); L; E)$ *a cuspform. If* $i, j \in \{1,2\}$, $i \neq j$, $U(p)\zeta^*\big((\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}\big) = 0$, *which implies* $e_{\mathrm{n.o.}}\zeta^*(\mathsf{g}_{|V(\varpi_{\mathfrak{p}_i})}) = e_{\mathrm{n.o.}}\zeta^*((U(\varpi_{\mathfrak{p}_j})\mathsf{g})_{|V(\varpi_\wp)})$. *In particular,* $e_{\mathrm{n.o.}}\zeta^*\big(\mathsf{g}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\big) = e_{\mathrm{n.o.}}\zeta^*(\mathsf{g}^{[\mathfrak{p}_1]}) = e_{\mathrm{n.o.}}\zeta^*(\mathsf{g}^{[\mathfrak{p}_2]})$.

Proof. For any $y \in \widehat{\mathcal{O}_F}F^\times_{\infty,+}$ we can compute that

$$\begin{aligned} \mathsf{a}_p\big(y, U(p)[\zeta^*\big((\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}\big)]\big) &= p_p^{t_F - x_{|F}}\mathsf{a}_p\big(py, \zeta^*\big((\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}\big)\big) \\ &= C\sum_{\mathrm{Tr}_{L/F}(\xi)=p}\mathsf{a}_p(\xi y\mathsf{d}_F^{-1}\mathsf{d}_L, (\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})})((\xi\mathsf{d}_L)_p\xi^{-1})^{t_L - x}, \end{aligned}$$

where C is a non-zero explicit constant. Suppose that $\mathsf{a}_p(\xi y\mathsf{d}_F^{-1}\mathsf{d}_L, (\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}) \neq 0$ for some $\xi \in L_+^\times$ with $\mathrm{Tr}_{L/F}(\xi) = p$, then $\xi y\mathsf{d}_F^{-1}\mathsf{d}_L \in \widehat{\mathcal{O}_L}L^\times_{\infty,+}$, $\varpi_{\mathfrak{p}_i} \mid (\xi y\mathsf{d}_F^{-1}\mathsf{d}_L)_{\mathfrak{p}_i}$ and $\varpi_{\mathfrak{p}_j} \nmid (\xi y\mathsf{d}_F^{-1}\mathsf{d}_L)_{\mathfrak{p}_j}$. Since p is unramified in L, that is equivalent to $\varpi_{\mathfrak{p}_i} \mid (\xi y)_{\mathfrak{p}_i}$ and $\varpi_{\mathfrak{p}_j} \nmid (\xi y)_{\mathfrak{p}_j}$ which implies that $\varpi_{\mathfrak{p}_i} \nmid (\mathrm{Tr}_{L/F}(\xi)y)_{\mathfrak{p}_i} = (py)_{\mathfrak{p}_i}$. This is absurd.

Regarding the second claim, for any $\mathfrak{p}_i$-stabilizations $\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}$ we have that

$$\begin{aligned} e_{\text{n.o.}}\zeta^*\big(\mathsf{g}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\big) &= e_{\text{n.o.}}\zeta^*\big((1-V(\varpi_{\mathfrak{p}_i})U(\varpi_{\mathfrak{p}_i}))\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}\big)\\ &= e_{\text{n.o.}}\zeta^*\big((1-(\bullet)V(\varpi_{\mathfrak{p}_i}))\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}\big)\\ &= e_{\text{n.o.}}\zeta^*\big(\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}\big). \end{aligned}$$

Taking the appropriate linear combination we prove the statement. □

Lemma 3.11. *Let $\mathsf{g}\in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L);L;E)$ be an eigenvector for the Hecke operators $T(\mathfrak{p}_1)$ and $T(\mathfrak{p}_2)$, $\mathsf{f}\in S_{k,w}(V_{11}(\mathfrak{N});F;E)$ a p-nearly ordinary eigenform. Suppose $\mathfrak{N}\mid\mathfrak{A}$ and that the weights of g,f are F-dominated. For α_i,β_i the inverses of the roots of the $T(\mathfrak{p}_i)$-Hecke polynomial for g, $i=1,2$, and $(e_{\text{n.o.}}\mathsf{f})_{|U(\varpi_\wp)}=\alpha_{\mathsf{f}}\cdot e_{\text{n.o.}}\mathsf{f}$ we have*

$$e_{\mathsf{f},\text{n.o.}}\zeta^*\big(d^r\mathsf{g}^{[\mathfrak{p}_i]}\big)=\frac{\mathscr{E}_\wp(\mathsf{g},\mathsf{f})}{\mathscr{E}_{0,\wp}(\mathsf{g},\mathsf{f})}e_{\mathsf{f},\text{n.o.}}\zeta^*(d^r\mathsf{g}),$$

where

$$\mathscr{E}_\wp(\mathsf{g},\mathsf{f})=\prod_{\bullet,\star\in\{\alpha,\beta\}}\left(1-\bullet_1\star_2\,\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}\right),\quad \mathscr{E}_{0,\wp}(\mathsf{g},\mathsf{f})=1-\alpha_1\beta_1\alpha_2\beta_2(\alpha_{\mathsf{f}}^{-1}q_\wp^{-1})^2.$$

Proof. Let $\mathsf{g}_{\alpha_i}^{(\mathfrak{p}_i)}=(1-\beta_iV(\varpi_{\mathfrak{p}_i}))\mathsf{g}$, $\mathsf{g}_{\beta_i}^{(\mathfrak{p}_i)}=(1-\alpha_iV(\varpi_{\mathfrak{p}_i}))\mathsf{g}$ be the two $\mathfrak{p}_i$-stabilizations of g. They satisfy $U(\varpi_{\mathfrak{p}_i})\mathsf{g}_\bullet^{(\mathfrak{p}_i)}=(\bullet)\mathsf{g}_\bullet^{(\mathfrak{p}_i)}$ and $\mathsf{g}=1/(\alpha_i-\beta_i)\big(\alpha_i\mathsf{g}_{\alpha_i}^{(\mathfrak{p}_i)}-\beta_i\mathsf{g}_{\beta_i}^{(\mathfrak{p}_i)}\big)$. Using Lemma 3.10 we compute

$$\begin{aligned} e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\left(\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right)^{[\mathfrak{p}_i]}\right] &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(1-(\bullet)V(\varpi_{\mathfrak{p}_i}))\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right]\\ &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right]-(\bullet)e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(U(\varpi_{\mathfrak{p}_j})\mathsf{g}_\bullet^{(\mathfrak{p}_i)})_{|V(\varpi_\wp)}\right]\\ &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right]-(\bullet)\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(T(\mathfrak{p}_j)-\alpha_j\beta_jV(\varpi_{\mathfrak{p}_j}))\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right]. \end{aligned}\tag{17}$$

Recall that $\mathsf{g}_\bullet^{(\mathfrak{p}_i)}$ is an eigenform for the operator $T(\mathfrak{p}_j)$ of eigenvalue $\alpha_j+\beta_j$. The chain of identities in (17) continues as:

$$\begin{aligned} e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\left(\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right)^{[\mathfrak{p}_i]}\right] &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right]-(\bullet)\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}\Big[(\alpha_j+\beta_j)e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right]+\\ &\quad -\alpha_j\beta_je_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(U(\varpi_{\mathfrak{p}_i})\mathsf{g}_\bullet^{(\mathfrak{p}_i)})_{|V(\varpi_\wp)}\right]\Big]\\ &=\left(1-(\bullet)\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}(\alpha_j+\beta_j)+\alpha_j\beta_j\left[(\bullet)\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}\right]^2\right)e_{\mathsf{f},\text{n.o.}}\zeta^*\left(d^r\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right)\\ &=\left(1-(\bullet)\alpha_j\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}\right)\left(1-(\bullet)\beta_j\alpha_{\mathsf{f}}^{-1}q_\wp^{-1}\right)e_{\mathsf{f},\text{n.o.}}\zeta^*\left(d^r\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right). \end{aligned}$$

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Finally, noting that $(\mathsf{g}_{\alpha_i}^{(\mathfrak{p}_i)})^{[\mathfrak{p}_i]} = (\mathsf{g}_{\beta_i}^{(\mathfrak{p}_i)})^{[\mathfrak{p}_i]} = \mathsf{g}^{[\mathfrak{p}_i]}$, we can put together the previous identities to prove the claim:

$$\begin{aligned} e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathsf{g}) =& \frac{1}{\alpha_i-\beta_i}\left(\alpha_i e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathsf{g}_{\alpha_i}^{(\mathfrak{p}_i)}) - \beta_i e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathsf{g}_{\beta_i}^{(\mathfrak{p}_i)})\right) \\ =& \frac{1-\alpha_i\beta_i\alpha_j\beta_j(\alpha_{\mathsf{f}}^{-1}q_{\wp}^{-1})^2}{\prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_i\star_j\,\alpha_{\mathsf{f}}^{-1}q_{\wp}^{-1})} e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathsf{g}^{[\mathfrak{p}_i]}). \end{aligned}$$ □

Theorem 3.12. *The value of the twisted triple product p-adic L-function $\mathscr{L}_p^\theta(\breve{\mathscr{G}},\breve{\mathscr{F}}) : \mathcal{W} \to \mathbb{C}_p$ at all $(\mathrm{P},\mathrm{Q}) \in \mathcal{C}_F^{\theta,\bar{r}}$ satisfies*

$$\begin{aligned} {}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}},\breve{\mathscr{F}})(\mathrm{P},\mathrm{Q}) =& \pm\frac{1}{\mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*)} \\ &\times\left(\prod_{\wp\in\mathcal{Q}_{\mathrm{inert}}}\mathcal{E}_\wp(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)\prod_{\wp\in\mathcal{Q}_{\mathrm{split}}}\frac{\mathscr{E}_\wp(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)}{\mathscr{E}_{0,\wp}(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)}\right)\frac{\left\langle\zeta^*\left(\delta^{s(w-x_{|F})}\breve{\mathsf{g}}_{\mathrm{P}}\right),\breve{\mathsf{f}}_{\mathrm{Q}}^*\right\rangle}{\langle\mathsf{f}_{\mathrm{Q}}^*,\mathsf{f}_{\mathrm{Q}}^*\rangle}, \end{aligned}$$

where $s:\mathrm{I}_F\to\mathrm{I}_L$ is any section of the restriction $\mathrm{I}_L\to\mathrm{I}_F$, $\mu\mapsto\mu_{|F}$ and the Euler factors appearing in the formula are defined in Lemmas 3.9 and 3.11.

Proof. We use (16) and Proposition 3.6 to obtain an explicit expression for the value of the p-adic L-function at a point $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$. Then Lemmas 3.9, 3.11 give us

$$\begin{aligned} {}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}},\breve{\mathscr{F}})(\mathrm{P},\mathrm{Q}) =& \pm\frac{1}{\mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*)}\frac{\left\langle e_{\mathrm{n.o.}}\zeta^*\left(d^{s(w-x_{|F})}\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]}\right),\breve{\mathsf{f}}_{\mathrm{Q}}^*\right\rangle}{\langle\mathsf{f}_{\mathrm{Q}}^*,\mathsf{f}_{\mathrm{Q}}^*\rangle} \\ =& \pm\frac{1}{\mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*)}\left(\prod_{\wp\in\mathcal{Q}_{\mathrm{inert}}}\mathcal{E}_\wp(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)\prod_{\wp\in\mathcal{Q}_{\mathrm{split}}}\frac{\mathscr{E}_\wp(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)}{\mathscr{E}_{0,\wp}(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)}\right) \\ &\times\frac{\left\langle e_{\mathrm{n.o}}\zeta^*\left(d^{s(w-x_{|F})}\breve{\mathsf{g}}_{\mathrm{P}}\right),\breve{\mathsf{f}}_{\mathrm{Q}}^*\right\rangle}{\langle\mathsf{f}_{\mathrm{Q}}^*,\mathsf{f}_{\mathrm{Q}}^*\rangle}. \end{aligned}$$

We conclude the proof applying Lemma 2.13 to compare p-adic and real analytic differential operators on cuspforms: $e_{\mathrm{n.o}}\zeta^*\left(d^{s(w-x_{|F})}\breve{\mathsf{g}}_{\mathrm{P}}\right) = e_{\mathrm{n.o}}\Pi^{\mathrm{hol}}\zeta^*\left(\delta^{s(w-x_{|F})}\breve{\mathsf{g}}_{\mathrm{P}}\right)$. □

Remark. Recall that for every $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$ there is a unitary automorphic representation $\Pi_{\mathrm{P,Q}}$ of prime-to-p level. The Euler factors in Theorem 3.12 also appear the expression for the local L-factor $L_\wp(\frac{1}{2},\Pi_{\mathrm{P,Q}},\mathrm{r})$. Indeed, if $\wp\in\mathcal{Q}_{\mathrm{inert}}$ by using (11), (10) we compute

$$\begin{aligned} \mathcal{E}_\wp(\mathsf{g},\mathsf{f}^*) =& \left(1-\alpha_{\mathsf{g}}\alpha_{\mathsf{f}^*}^{-1}q_\wp^{-1}\right)\left(1-\beta_{\mathsf{g}}\alpha_{\mathsf{f}^*}^{-1}q_\wp^{-1}\right) \\ =& \left(1-\chi_{1,\mathfrak{p}}(\varpi_\wp)\psi_{i,\wp}(\varpi_\wp)q_\wp^{-1/2}\right)\left(1-\chi_{2,\mathfrak{p}}(\varpi_\wp)\psi_{i,\wp}(\varpi_\wp)q_\wp^{-1/2}\right). \end{aligned}$$

Similarly if $\wp\in\mathcal{Q}_{\mathrm{split}}$ by using (11), (9) we obtain

$$\mathscr{E}_\wp(\mathsf{g},\mathsf{f}^*) = \prod_{\bullet,\star\in\{\alpha,\beta\}}\left(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}q_\wp^{-1}\right) = \prod_{i,j}\left(1-\chi_{i,\mathfrak{p}_1}(\varpi_\wp)\chi_{j,\mathfrak{p}_2}(\varpi_\wp)\psi_{k,\wp}(\varpi_\wp)q_\wp^{-1/2}\right).$$

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4. Geometric theory

4.1. Geometric Hilbert modular forms

Let F be a totally real number field and $G_F = \mathrm{Res}_{L/\mathbb{Q}}(\mathrm{GL}_{2,F})$. For any open compact subgroup $K \leqslant G_F(\mathbb{A}^\infty)$ we consider the Shimura variety

$$\mathrm{Sh}_K(G_F)(\mathbb{C}) = G_F(\mathbb{Q})\backslash(\mathfrak{H}^\pm)^{\mathrm{I}_F} \times G_F(\mathbb{A}^\infty)/K$$

where $\gamma \in G_F(\mathbb{Q}) = \mathrm{GL}_2(F)$ acts on $z = (z_\tau)_\tau \in (\mathfrak{H}^\pm)^{\mathrm{I}_F}$ via Moebius transformations $\gamma \cdot z = (\tau(\gamma)z_\tau)_\tau$. The complex manifold $\mathrm{Sh}_K(G_F)(\mathbb{C})$ has a canonical structure of quasi-projective variety over its reflex field $\mathbb{Q}$ [28, Chapter II, Theorem 5.5]. Let ω be the dual of the tautological quotient bundle on $\mathbb{P}^1_\mathbb{C}$ with $p : \omega \to \mathbb{P}^1_\mathbb{C}$ the natural projection. The group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{P}^1_\mathbb{C}$ via Moebius transformations and there is a natural way to define a $\mathrm{GL}_2(\mathbb{C})$-action on ω such that the projection p is equivariant. For any weight $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$ such that $k - 2w = mt_F$, one can define a line bundle

$$\underline{\omega}^{(k,w)} = \bigotimes_{\tau\in \mathrm{I}_F} \mathrm{pr}_\tau^*\left(\omega^{\otimes k_\tau} \otimes \det^{\frac{m+k_\tau}{2}}\right) \tag{18}$$

on $(\mathbb{P}^1_\mathbb{C})^{\mathrm{I}_F}$ with $G_F(\mathbb{C})$-action given as follows. For each $\tau \in \mathrm{I}_F$, the action of $G_F(\mathbb{C})$ on $\mathrm{pr}_\tau^*\big(\omega^{\otimes k_\tau} \otimes \det^{\frac{m+k_\tau}{2}}\big)$ factors through the τ-copy of $\mathrm{GL}_2(\mathbb{C})$, which in turn acts as the product of $\det^{\frac{m+k_\tau}{2}}$ and the k_τ-th power of the natural action on ω. One has to twist the action by such a power of the determinant because it allows the line bundle to descend to the Galois closure F^{Gal} of F over $\mathbb{Q}$. Indeed, consider the subgroup $Z_s = \mathrm{Ker}\big(\mathrm{N}_{F/\mathbb{Q}} : \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m) \to \mathbb{G}_m\big)$ of the center $Z = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ of G_F and denote by G_F^c the quotient of G_F by Z_s. The action of $G_F(\mathbb{C})$ on $\underline{\omega}^{(k,w)}$ factors through $G_F^c(\mathbb{C})$, thus $\underline{\omega}^{(k,w)}$ descends to an algebraic invertible sheaf on $\mathrm{Sh}_K(G_F)_\mathbb{C}$ if K is sufficiently small by [28, Chapter III, Proposition 2.1], and it has a canonical model over F^{Gal} by [28, Chapter III, Theorem 5.1].

Suppose $F \neq \mathbb{Q}$, then for every field E, $F^{\mathrm{Gal}} \subset E \subset \mathbb{C}$, and sufficiently small compact open subgroup $K \leqslant G_F(\mathbb{A}^\infty)$, one can give a geometric interpretation of Hilbert modular forms of weight (k, w), level K, defined over E as $M_{k,w}(K; E) = H^0(\mathrm{Sh}_K(G_F)_E, \underline{\omega}^{(k,w)})$. To deal with cuspforms and treat the case $F = \mathbb{Q}$, one has to consider compactifications of the Shimura variety $\mathrm{Sh}_K(G_F)_\mathbb{Q}$, which we discuss in § 4.2.

4.1.1. Integral models. Fix p a rational prime unramified in F and consider a level structure of type $K = K^pK_p$, where K^p is an open compact subgroup of $G_F\big(\widehat{\mathcal{O}_F}^p\big)$ and $K_p = \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$. The determinant map $\det : G_F \to \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ induces a bijection between the set of geometric connected components of $\mathrm{Sh}_K(G_F)$ and $\mathrm{cl}^+_F(K)$, the strict class group of K, $\mathrm{cl}^+_F(K) = F^\times_+\backslash\mathbb{A}^{\infty,\times}_F/\det(K)$. Since $\det(K) \subseteq \widehat{\mathcal{O}_F}^\times$, there is a surjection $\mathrm{cl}^+_F(K) \twoheadrightarrow \mathrm{cl}^+_F$ to the strict ideal class group of F, which one uses to label the geometric components of the Shimura variety $\mathrm{Sh}_K(G_F)$. Fix fractional ideals $\mathfrak{c}_1, \ldots, \mathfrak{c}_{h^+_F}$, coprime to p, forming a set of representatives of cl^+_F. Then by strong approximation there is a decomposition

$$\mathrm{Sh}_K(G_F)(\mathbb{C}) = G(\mathbb{Q})^+\backslash\mathfrak{H}^{\mathrm{I}_F} \times G_F(\mathbb{A}^\infty)/K = \coprod_{[\mathfrak{c}]\in \mathrm{cl}^+_F} \mathrm{Sh}^{\mathfrak{c}}_K(G_F)(\mathbb{C}),$$

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where each $\mathrm{Sh}_K^{\mathfrak{c}}(G_F)(\mathbb{C})$ is the disjoint union of quotients of $\mathfrak{H}^{\mathrm{I}_F}$ by groups of the form $\Gamma(g,K)=gKg^{-1}\cap G(\mathbb{Q})^+$. A different choice $\mathfrak{c}'$ of fractional ideal representing $[\mathfrak{c}]\in \mathrm{cl}_F^+$ produces a canonically isomorphic manifold $\mathrm{Sh}_K^{\mathfrak{c}'}(G_F)(\mathbb{C})\cong \mathrm{Sh}_K^{\mathfrak{c}}(G_F)(\mathbb{C})$ [38, Remark 2.8]. Suppose K^p is sufficiently small so there exists a smooth, quasi-projective $\mathbb{Z}_{(p)}$-scheme $\mathcal{M}_K^{\mathfrak{c}}$ representing the moduli problem of isomorphism classes of quadruples $(A,\iota,\lambda,\alpha_{K^p})_{/S}$ where (A,ι) is a Hilbert–Blumenthal abelian variety over S of dimension $g=[F:\mathbb{Q}]$, λ a $\mathfrak{c}$-polarization and α_{K^p} a level-K^p structure, [38, § 2.3].

The group of totally positive units $\mathcal{O}_{F,+}^\times$ acts on $\mathcal{M}_K^{\mathfrak{c}}$ by modifying the $\mathfrak{c}$-polarization. The subgroup $(K\cap\mathcal{O}_F^\times)^2$ of $\mathcal{O}_{F,+}^\times$ acts trivially, where by $K\cap\mathcal{O}_F^\times$ we mean the intersection of K and $\mathcal{O}_F^\times\hookrightarrow Z(\mathbb{A}^\infty)$ in $G_F(\mathbb{A}^\infty)$. Therefore, the finite group $\mathcal{O}_{F,+}^\times/(K\cap\mathcal{O}_F^\times)^2$ acts on the moduli scheme $\mathcal{M}_K^{\mathfrak{c}}$ and the stabilizer of each geometric connected component is $\big(\det(K)\cap\mathcal{O}_{F,+}^\times\big)/(K\cap\mathcal{O}_F^\times)^2$.

Proposition 4.1. *There is an isomorphism between the quotient of* $\mathcal{M}_K^{\mathfrak{c}\mathfrak{d}^{-1}}(\mathbb{C})$ *by the finite group* $\mathcal{O}_{F,+}^\times/(K\cap\mathcal{O}_F^\times)^2$ *and* $\mathrm{Sh}_K^{\mathfrak{c}}(G_F)(\mathbb{C})$. *Moreover, if* $\det(K)\cap\mathcal{O}_{F,+}^\times=(K\cap\mathcal{O}_F^\times)^2$, *then the quotient map* $\mathcal{M}_K^{\mathfrak{c}\mathfrak{d}^{-1}}(\mathbb{C})\to\mathrm{Sh}_K^{\mathfrak{c}}(G_F)(\mathbb{C})$ *induces an isomorphism between any geometric connected component of* $\mathcal{M}_K^{\mathfrak{c}\mathfrak{d}^{-1}}(\mathbb{C})$ *and its image.*

Proof. This is [38, Proposition 2.4] with a shift in the indices by the absolute different. It is necessary for the conventions for the complex uniformization used in [15, § 4.1.3]. □

Definition 4.2. Let p be a rational prime unramified in F and $K=K^pK_p$ a compact open subgroup of $G_F(\mathbb{A}^\infty)$ such that K^p is sufficiently small, $K_p=\mathrm{GL}_2(\mathcal{O}_F\otimes\mathbb{Z}_p)$ and $\det(K)\cap\mathcal{O}_{F,+}^\times=(K\cap\mathcal{O}_F^\times)^2$. The integral model of the Shimura variety $\mathrm{Sh}_K(G_F)$ over $\mathbb{Z}_{(p)}$ is the quotient of $\mathcal{M}_{K,F}=\coprod_{[\mathfrak{c}]\in\mathrm{cl}_F^+}\mathcal{M}_K^{\mathfrak{c}}$ by $\mathcal{O}_{F,+}^\times/(K\cap\mathcal{O}_F)^2$, which we denote $\mathbf{Sh}_K(G_F)$.

Note that the assumptions on the level K in the definition are always satisfied up to replacing K^p by an open compact subgroup [38, Lemma 2.5]. Moreover, by Proposition 4.1, the scheme $\mathbf{Sh}_K(G_F)$ is smooth quasi-projective over $\mathbb{Z}_{(p)}$ and has an abelian scheme with real multiplication over it.

Remark. The scheme $\mathcal{M}_K^{\mathfrak{d}^{-1}}$ is an integral model of the Shimura variety for the algebraic group G_F^* of level $K\cap G_F^*(\mathbb{A}^\infty)$ [35]. We denote it by $\mathbf{Sh}_K(G_F^*)$ and we let $\xi:\mathbf{Sh}_K(G_F^*)\to\mathbf{Sh}_K(G_F)$ be the natural morphism.

4.1.2. Diagonal morphism. Let L/F be an extension of totally real fields with $[F:\mathbb{Q}]=g$. Consider the map of algebraic groups $\zeta:G_F\longrightarrow G_L$ defined by the natural inclusion $\zeta(B):\mathrm{GL}_2(B\otimes_{\mathbb{Q}}F)\to\mathrm{GL}_2(B\otimes_{\mathbb{Q}}L)$ of groups for any $\mathbb{Q}$-algebra B. For compact open subgroups $K\leqslant G_L(\mathbb{A}^\infty)$ and $K'\leqslant K\cap G_F(\mathbb{A}^\infty)$ we have a commutative diagram

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$$\begin{array}{ccc} \mathrm{Sh}_{K'}(G_F)(\mathbb{C}) & \xrightarrow{\zeta} & \mathrm{Sh}_K(G_L)(\mathbb{C}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ \mathrm{cl}_F^+(K') & \xrightarrow{\zeta} & \mathrm{cl}_L^+(K) \\ \downarrow & & \downarrow \\ \mathrm{cl}_F^+ & \xrightarrow{\zeta} & \mathrm{cl}_L^+ \end{array} \tag{19}$$

hence for every fractional ideal $\mathfrak{c}$ of F there is an induced map $\zeta : \mathrm{Sh}^{\mathfrak{c}}_{K'}(G_F)(\mathbb{C}) \to \mathrm{Sh}^{\mathfrak{c}}_{K}(G_L)(\mathbb{C})$. Suppose that $K \leqslant G_L(\mathbb{A}^\infty)$ and $K' \leqslant K \cap G_F(\mathbb{A}^\infty)$ satisfy the assumptions in Definition 4.2. There is a morphism of $\mathbb{Z}_{(p)}$-schemes $\zeta : \mathbf{Sh}_{K'}(G_F) \longrightarrow \mathbf{Sh}_K(G_L)$ induced by morphisms $\tilde{\zeta} : \mathcal{M}_{K',F} \to \mathcal{M}_{K,L}$ that maps any quadruple $[A, \iota, \lambda, \alpha_{(K')^p}]_{/S}$ over a $\mathbb{Z}_{(p)}$-scheme S to the quadruple $\tilde{\zeta}\left([A, \iota, \lambda, \alpha_{(K')^p}]\right) = [A', \iota', \lambda', \alpha'_{K^p}]_{/S}$ over S defined as follows. First, the abelian scheme A' is $A \otimes_{\mathcal{O}_F} \mathcal{O}_L$, then we can describe the $\mathcal{O}_L$-action on $\mathcal{O}_L$ via a ring homomorphism $\bar{\iota} : \mathcal{O}_L \to \mathrm{M}_g(\mathcal{O}_F)$ by choosing an $\mathcal{O}_F$-basis of $\mathcal{O}_L$; the choice of basis induces an identification between $A \otimes_{\mathcal{O}_F} \mathcal{O}_L$ and A^g. Thus, the ring homomorphism $\iota' : \mathcal{O}_L \to \mathrm{End}_S(A')$ is defined as the arrow that makes the following diagram commute

$$\begin{array}{ccc} \mathcal{O}_L & \xrightarrow{\bar{\iota}} & \mathrm{M}_g(\mathcal{O}_F) \\ & \searrow^{\iota'} & \downarrow \\ & & \mathrm{M}_g\big(\mathrm{End}_S(A)\big) \cong \mathrm{End}_S(A') \end{array}$$

Following [5, Lemma 5.11], one can compute the dual abelian scheme $\left(A'\right)^\vee \cong A^\vee \otimes_{\mathcal{O}_F} \mathfrak{d}^{-1}_{L/F}$ and realize that if $\lambda : (\mathfrak{c}, \mathfrak{c}^+) \xrightarrow{\sim} (\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_F}(A, A^\vee), \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_F}(A, A^\vee)^+)$ is a $\mathfrak{c}$-polarization of A then $\lambda' = \lambda \otimes \mathrm{id}$ is a $\mathfrak{c} \otimes_{\mathcal{O}_F} \mathfrak{d}^{-1}_{L/F}$-polarization of $A' = A \otimes_{\mathcal{O}_F} \mathcal{O}_L$. Finally, it is enough to define $\tilde{\zeta}$ for principal $\mathfrak{N}$-level structures, for $\mathfrak{N}$ an $\mathcal{O}_F$-ideal. A principal $\mathfrak{N}$-level structure is an $\mathcal{O}_F$-linear isomorphism of group schemes $(\mathcal{O}_F/\mathfrak{N})^2 \xrightarrow{\sim} A[\mathfrak{N}]$ which induces an isomorphism $\mathcal{O}_F/\mathfrak{N} \xrightarrow{\sim} \mu_{\mathfrak{N}} \otimes_{\mathbb{Z}} \mathfrak{c}^{-1}\mathfrak{d}_F^{-1}$, using Weil pairing and polarization. By tensoring such an isomorphism with $\mathcal{O}_L$ over $\mathcal{O}_F$ we obtain a principal $\mathfrak{N}$-level structure on A'.

Remark. For any fractional ideal $\mathfrak{c}$ of F there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathfrak{c}\mathfrak{d}_F^{-1}}_{K',F} & \xrightarrow{\tilde{\zeta}} & \mathcal{M}^{\mathfrak{c}\mathfrak{d}_L^{-1}}_{K,L} \\ \downarrow & & \downarrow \\ \mathbf{Sh}^{\mathfrak{c}}_{K'}(G_F) & \xrightarrow{\zeta} & \mathbf{Sh}^{\mathfrak{c}}_{K}(G_L) \end{array}, \quad \text{implying} \quad \begin{array}{ccc} \mathbf{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}}) & \xrightarrow{\tilde{\zeta}} & \mathcal{M}^{\mathfrak{d}_L^{-1}}_{K,L} \\ & \searrow^{\zeta} & \downarrow{\scriptstyle \xi} \\ & & \mathbf{Sh}_K(G_L) \end{array} \quad \text{when } F = \mathbb{Q}.$$

4.2. Compactifications and p-adic theory

Sometimes we drop part of the decorations from the symbols denoting Shimura varieties when we believe it does not cause confusion, both to simplify the notation and to state

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facts that hold for both groups G and G^*. We denote by $\mathbf{Sh}_K^*$ the minimal compactification of $\mathbf{Sh}_K$ which is normal and projective. By choosing some auxiliary data Σ, one can construct an arithmetic toroidal compactification $\mathbf{Sh}_{K,\Sigma}^{\mathrm{tor}}$ smooth and projective over $\mathbb{Z}_{(p)}$. It comes equipped with a natural map $\oint : \mathbf{Sh}_K^{\mathrm{tor}} \longrightarrow \mathbf{Sh}_K^*$ and an open immersion $\mathbf{Sh}_K \hookrightarrow \mathbf{Sh}_K^{\mathrm{tor}}$ such that the boundary $\mathrm{D} = \mathbf{Sh}_K^{\mathrm{tor}} \setminus \mathbf{Sh}_K$ is a relative simple normal crossing Cartier divisor. The Hilbert–Blumenthal abelian scheme $\mathcal{A}$ over $\mathbf{Sh}_K$ extends to a semi-abelian scheme $\mathcal{A}^{\mathrm{sa}} \to \mathbf{Sh}_K^{\mathrm{tor}}$ with an $\mathcal{O}_F$-action, a K-level structure and a zero section $e : \mathbf{Sh}_K^{\mathrm{tor}} \to \mathcal{A}^{\mathrm{sa}}$ ([35]; [23], Chapter VI). There is a canonical way to extend the rank 2 vector bundle of relative de Rham cohomology $\mathbb{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathbf{Sh}_{K,R})$ to an $(\mathcal{O}_{\mathbf{Sh}_K^{\mathrm{tor}}} \otimes_{\mathbb{Z}} \mathcal{O}_F)$-module $\mathbb{H}^1$ locally free of rank 2 over $\mathbf{Sh}_{K,R}^{\mathrm{tor}}$ together with a logarithmic Gauss–Manin connection and Kodaira–Spencer isomorphism. If $\underline{\omega} = e^*\big(\Omega^1_{\mathcal{A}^{\mathrm{sa}}/\mathbf{Sh}_K^{\mathrm{tor}}}\big)$ is the cotangent space at the origin of the universal semi-abelian scheme, the vector bundle $\mathbb{H}^1$ has an $\mathcal{O}_F$-equivariant Hodge filtration

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathbb{H}^1 \longrightarrow \mathrm{Lie}((\mathcal{A}^{\mathrm{sa}})^\vee) \longrightarrow 0.$$

Let R be an $\mathcal{O}_{F^{\mathrm{Gal}},(p)}$-algebra in which the discriminant $d_{F/\mathbb{Q}}$ is invertible. For a coherent $(\mathcal{O}_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}} \otimes_{\mathbb{Z}} \mathcal{O}_F)$-module M, we denote by $M = \bigoplus_{\tau\in \mathrm{I}_F} M_\tau$ its canonical decomposition for the $\mathcal{O}_F$-action [21, Lemma 2.0.8]: M_τ is the direct summand of M on which $\mathcal{O}_F$ acts via $\tau : \mathcal{O}_F \to R \to \mathcal{O}_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}}$. Then the τ-component of the Hodge filtration is

$$0 \longrightarrow \underline{\omega}_\tau \longrightarrow \mathbb{H}^1_\tau \longrightarrow \textstyle\bigwedge^2(\mathbb{H}^1_\tau) \otimes \underline{\omega}_\tau^{-1} \longrightarrow 0\ .$$

For a weight $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$ with $k - 2w = mt_F$, we define the integral model of the line bundle (18) by

$$\underline{\omega}_G^{(k,w)} := \bigotimes_{\tau\in \mathrm{I}_F} \Big((\wedge^2\mathbb{H}^1_\tau)^{-\frac{m+k_\tau}{2}} \otimes \underline{\omega}_\tau^{k_\tau} \Big)$$

as a sheaf over $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)$. The geometric definition of cuspforms is given by $S_{k,w}(K; R) = H^0\big(\mathbf{Sh}_K^{\mathrm{tor}}(G)_R, \underline{\omega}_G^{(k,w)}(-\mathrm{D})\big)$.

Remark. A general compact open subgroup $K \leqslant G(\mathbb{A}^\infty)$ of prime-to-p level does not satisfy the assumptions in Definition 4.2. Anyway, one can work with modular forms of level K by considering a subgroup K' that does satisfy them and then take K/K'-invariants [38, § 6.4].

Definition 4.3. Let R be an $\mathcal{O}_{F^{\mathrm{Gal}},(p)}$-algebra and let $(k, \nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ be any weight. We fix one $\tau_\circ \in \mathrm{I}_F$ and set $\wedge^2\mathbb{H}^1_\circ := \wedge^2\mathbb{H}^1_{\tau_\circ}$. We define a line bundle over $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G^*)$ by

$$\underline{\omega}_{G^*}^{(k,\nu)} := (\wedge^2\mathbb{H}^1_\circ)^{\nu-|k|} \otimes \bigotimes_{\tau\in \mathrm{I}_F} \underline{\omega}_\tau^{k_\tau}.$$

It provides a geometric incarnation of cuspforms on G^* of weight $(k, \nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ by setting $S^*_{k,\nu}(K; R) = H^0\big(\mathbf{Sh}_K^{\mathrm{tor}}(G^*)_R, \underline{\omega}_{G^*}^{(k,\nu)}(-\mathrm{D})\big)$.

According to [38], a weight $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$, $k - 2w = mt_F$, is cohomological if $2 - m \geqslant k_\tau \geqslant 2$ for all $\tau \in \mathrm{I}_F$. For any cohomological weight we define the vector bundle $\mathcal{F}_G^{(k,w)}$ on $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)$ by $\mathcal{F}_G^{(k,w)} := \bigotimes_{\tau \in \mathrm{I}_F} \mathcal{F}_\tau^{(k,w)}$ for $\mathcal{F}_\tau^{(k,w)} := (\wedge^2 \mathbb{H}_\tau^1)^{\frac{2-m-k_\tau}{2}} \otimes \mathrm{Sym}^{k_\tau - 2} \mathbb{H}_\tau^1$. Similarly, a weight $(k, \nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ is cohomological if $k \geqslant 2t_F$ and $\nu \geqslant |k - t_F|$. For any cohomological weight we define the vector bundle $\mathcal{F}_{G^*}^{(k,\nu)}$ on $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G^*)$ by $\mathcal{F}_{G^*}^{(k,\nu)} := (\wedge^2 \mathbb{H}_\circ^1)^{\nu + |t_F - k|} \otimes \bigotimes_{\tau \in \mathrm{I}_F} \mathrm{Sym}^{k_\tau - 2} \mathbb{H}_\tau^1$. The extended Gauss–Manin connection on $\mathbb{H}^1$ induces by functoriality logarithmic integrable connections $\nabla : \mathcal{F}_G^{(k,w)} \to \mathcal{F}_G^{(k,w)} \otimes \Omega^1_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)}(\log \mathrm{D})$ and $\nabla : \mathcal{F}_{G^*}^{(k,\nu)} \to \mathcal{F}_{G^*}^{(k,\nu)} \otimes \Omega^1_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G^*)}(\log \mathrm{D})$ out of which one can form the complexes

$$\mathrm{DR}^\bullet\left(\mathcal{F}_G^{(k,w)}\right) = \left[0 \to \mathcal{F}_G^{(k,w)} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{F}_G^{(k,w)} \otimes \Omega^g_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)}(\log \mathrm{D}) \to 0\right], \tag{20}$$

$$\mathrm{DR}^\bullet\left(\mathcal{F}_{G^*}^{(k,\nu)}\right) = \left[0 \to \mathcal{F}_{G^*}^{(k,\nu)} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{F}_{G^*}^{(k,\nu)} \otimes \Omega^g_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G^*)}(\log \mathrm{D}) \to 0\right] \tag{21}$$

equipped with their natural Hodge filtration. We denote by $\mathrm{DR}_c^\bullet\left(\mathcal{F}_G^{(k,w)}\right)$ (respectively $\mathrm{DR}_c^\bullet\left(\mathcal{F}_{G^*}^{(k,\nu)}\right)$) the complex obtained from (20) (respectively (21)) by tensoring with $\mathcal{O}_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)}(-\mathrm{D})$ (respectively $\mathcal{O}_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G^*)}(-\mathrm{D})$). One can associate to $\mathrm{DR}^\bullet\left(\mathcal{F}_G^{(k,w)}\right)$, $\mathrm{DR}^\bullet\left(\mathcal{F}_{G^*}^{(k,\nu)}\right)$ and their compactly supported versions, dual BGG complexes. We recall the definition of $\mathrm{BGG}(\mathcal{F}_{G^*}^{(k,\nu)})$ and we refer to [38, § 2.15] for the definition of $\mathrm{BGG}(\mathcal{F}_G^{(k,w)})$. The compactly supported version is obtained by tensoring with the sheaf of functions vanishing at the boundary divisor. For any subset $J \subset \mathrm{I}_F$, let $s_J \in \{\pm 1\}^{\mathrm{I}_F}$ be the element whose τ-component is -1 if $\tau \notin J$ and 1 if $\tau \in J$. For $0 \leqslant j \leqslant g$ we put

$$\mathrm{BGG}^j(\mathcal{F}_{G^*}^{(k,\nu)}) = \bigoplus_{J \subset \mathrm{I}_F, \# J = j} \underline{\omega}_{G^*}^{s_J \cdot (k,\nu)} e_J$$

for e_J the Cech symbol and $\underline{\omega}_{G^*}^{s_J \cdot (k,\nu)} = (\wedge^2 \mathbb{H}_\circ^1)^{\nu - |\mathrm{I}_F \setminus J| - \sum_{\tau \in J} k_\tau} \otimes \bigotimes_{\tau \notin J} \underline{\omega}_\tau^{2-k_\tau} \otimes \bigotimes_{\tau \in J} \underline{\omega}_\tau^{k_\tau}$. There are differential operators $d : \mathrm{BGG}^j(\mathcal{F}_{G^*}^{(k,\nu)}) \to \mathrm{BGG}^{j+1}(\mathcal{F}_{G^*}^{(k,\nu)})$ given on local sections by $d : \mathsf{f} e_J \mapsto \sum_{\tau \notin J} \Theta_{\tau, k_{\tau - 1}}(\mathsf{f}) e_\tau \wedge e_J$ where $\Theta_{\tau, k_\tau - 1}(\mathsf{f}) = \frac{(-1)^{k_\tau - 2}}{(k_\tau - 2)!} \sum_\xi \tau_0(\xi)^{k_\tau - 1}(\xi)$ $a_\xi q^\xi$ if the local section is written as $\mathsf{f} = \sum_\xi a_\xi q^\xi$.

Theorem 4.4 ([38, Theorem 2.16]; [25, Remark 5.24]). *Let R be an F^{Gal}-algebra, then for $\mathcal{S} = \mathcal{F}_G^{(k,w)}$ (respectively $\mathcal{F}_{G^*}^{(k,\nu)}$) there are canonical quasi-isomorphic embeddings $\mathrm{BGG}^\bullet(\mathcal{S}) \hookrightarrow \mathrm{DR}^\bullet(\mathcal{S})$, $\mathrm{BGG}_c^\bullet(\mathcal{S}) \hookrightarrow \mathrm{DR}_c^\bullet(\mathcal{S})$ of complexes of abelian sheaves on $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)$ (respectively $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G^*)$). Moreover, the Hodge spectral sequences for both complexes degenerate at the first page.*

4.2.1. p-Adic theory. Katz's idea for a geometric theory of p-adic modular forms [20] consists in removing from the relevant Shimura variety the preimages, under the specialization map, of those points in the special fiber that correspond to non-ordinary abelian varieties.

Let $E \subset \mathbb{C}$ be a number field containing F^{Gal}. The fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ determines a prime ideal $\wp \mid p$ of E. We denote by $E_\wp$ the completion, $\mathcal{O}_\wp$ the ring of integers and κ the residue field. Let $\mathcal{A}^{\mathrm{sa}}_{\kappa}$ be the semi-abelian scheme over the special fiber $\mathbf{Sh}^{\mathrm{tor}}_{K,\kappa}$ of the Shimura variety. The determinant of the map, induced by Verschiebung $\mathrm{V} : (\mathcal{A}^{\mathrm{sa}}_{\kappa})^{(p)} \to \mathcal{A}^{\mathrm{sa}}_{\kappa}$ between cotangent spaces at the origin, corresponds to a characteristic p Hilbert modular form $\mathrm{Ha} \in H^0(\mathbf{Sh}^{\mathrm{tor}}_{K,\kappa}, \det(\underline{\omega})^{\otimes(p-1)})$, called the Hasse invariant. The ordinary locus $\mathbf{Sh}^{\mathrm{tor,ord}}_{K,\kappa}$ is the complement of the zero locus of the Hasse invariant. Let $\mathscr{S}^{\mathrm{tor}}_{K}$ denote the formal completion of $\mathbf{Sh}^{\mathrm{tor}}_{K,\mathcal{O}_{\wp}}$ along its special fiber and $j :]\mathbf{Sh}^{\mathrm{tor,ord}}_{K,\kappa}[\hookrightarrow \mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$ the inverse image of the ordinary locus under the specialization map $\mathrm{sp} : \mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}} \to \mathbf{Sh}^{\mathrm{tor}}_{K,\kappa}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$; one defines $j^{\dagger}\mathcal{F}$ to be the sheaf whose sections on an admissible open $U \subset \mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$ are the direct limit of $\Gamma(V \cap U, \mathcal{F})$ computed over strict neighborhoods V of $]\mathbf{Sh}^{\mathrm{tor,ord}}_{K,\kappa}[$ in $\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$.

For the minimal compactification $\mathbf{Sh}^{*}_{K,\kappa}$ one can similarly define the ordinary locus $\mathbf{Sh}^{*,\mathrm{ord}}_{K,\kappa}$ of the special fiber, which is an affine scheme, since $\det(\underline{\omega})$ is an ample line bundle on $\mathbf{Sh}^{*}_{K,\kappa}$. This is a very convenient feature because it implies the existence of a fundamental system of strict affinoid neighborhoods of $]\mathbf{Sh}^{*,\mathrm{ord}}_{K,\kappa}[$.

Theorem 4.5. *We recall that overconvergent cuspforms of weight* $(k,w) \in \mathbb{Z}[\mathrm{I}_F]^2$ *are defined as* $S^{\dagger}_{k,w}(K; E_{\wp}) = H^0\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\big(\underline{\omega}^{(k,w)}_{G}(-\mathrm{D})\big)\big)$. *For any cohomological weight* $(k,w) \in \mathbb{Z}[\mathrm{I}_F]^2$, $k - 2w = mt_F$, *the hypercohomology group* $\mathbb{H}^{g}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\mathrm{DR}^{\bullet}_{c}\big(\mathcal{F}^{(k,w)}_{G}\big)\big)$ *can be computed either as*

$$\frac{S^{\dagger}_{k,w}(K; E_{\wp})}{\sum_{\tau\in \mathrm{I}_F} \Theta_{\tau,k_\tau-1}\big(S^{\dagger}_{s_\tau\cdot(k,w)}(K; E_{\wp})\big)} \quad \textit{or} \quad \frac{H^0_{\mathrm{rig}}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\big(\mathcal{F}^{(k,w)}_{G} \otimes \Omega^{g}_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G)}\big)\big)}{\nabla H^0_{\mathrm{rig}}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\big(\mathcal{F}^{(k,w)}_{G} \otimes \Omega^{g-1}_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G)}\big)\big)}.$$

Proof. This is essentially [38, Theorem 3.5]. For completeness we write down the argument for the second computation. Theorem 4.4 states that we have a quasi-isomorphism of complexes $\mathrm{DR}^{\bullet}_{c}(\mathcal{F}^{(k,w)}_{G}) \cong \mathrm{BGG}^{\bullet}_{c}(\mathcal{F}^{(k,w)}_{G})$, thus the isomorphisms $\mathbb{H}^{g}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\mathrm{DR}^{\bullet}_{c}\big(\mathcal{F}^{(k,w)}_{G}\big)\big) \cong \mathbb{H}^{g}\big(\mathscr{S}^{*}_{K,\mathrm{rig}}, j^{\dagger}\oint_{*}\mathrm{BGG}_{c}\big(\mathcal{F}^{(k,w)}_{G}\big)\big) \cong \mathbb{H}^{g}\big(\mathscr{S}^{*}_{K,\mathrm{rig}}, j^{\dagger}\oint_{*}\mathrm{DR}^{\bullet}_{c}\big(\mathcal{F}^{(k,w)}_{G}\big)\big)$ follow by applying the Leray spectral sequence for the composition $\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}} \to \mathscr{S}^{*}_{K,\mathrm{rig}} \to \mathrm{Spa}\mathbb{Q}_p$ and the vanishing of the higher derived images of subcanonical automorphic bundles [24, Theorem 8.2.1.2]. We conclude that

$$\mathbb{H}^{g}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\mathrm{DR}^{\bullet}_{c}\big(\mathcal{F}^{(k,w)}_{G}\big)\big) \cong \frac{H^0_{\mathrm{rig}}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\big(\mathcal{F}^{(k,w)}_{G} \otimes \Omega^{g}_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G)}\big)\big)}{\nabla H^0_{\mathrm{rig}}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\big(\mathcal{F}^{(k,w)}_{G} \otimes \Omega^{g-1}_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G)}\big)\big)}$$

because there is a fundamental system of affinoid neighborhoods of the ordinary locus on the minimal compactification. □

Remark. Replacing the group G by G^* in Theorem 4.5, the conclusion still holds for any cohomological weight $(k,\nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ and the group $\mathbb{H}^{g}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\mathrm{DR}^{\bullet}_{c}\big(\mathcal{F}^{(k,\nu)}_{G^*}\big)\big)$, if we define overconvergent cuspforms for G^* as $S^{*,\dagger}_{k,\nu}(K; E_{\wp}) = H^0\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\big(\underline{\omega}^{(k,\nu)}_{G^*}(-\mathrm{D})\big)\big)$.

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Lemma 4.6. *Let $\mathfrak{p} \mid p$ be a prime $\mathcal{O}_F$-ideal. The partial Frobenius* $\mathrm{Fr}_{\mathfrak{p}}$ *[38, § 3.12] acts on the image of $S^{\dagger}_{k,w}(K, E_{\wp})$ in the hypercohomology group $\mathbb{H}^g\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\mathrm{DR}^{\bullet}_c(\mathcal{F}^{(k,w)}_G)\big)$ as $\mathrm{Fr}_{\mathfrak{p}} = \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})V(\mathfrak{p})$.*

Proof. Taking into account the action of the partial Frobenius on $j^{\dagger}\Omega^g_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G)}$, the same computation as in [6, Remark p. 339] shows that $\mathrm{Fr}_{\mathfrak{p}}$ acts on the image of $S^{\dagger}_{k,w}(K, E_{\wp})$ in $\mathbb{H}^g\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}\mathrm{DR}^{\bullet}_c(\mathcal{F}^{(k,w)}_G)\big)$ as $\varpi_{\mathfrak{p}}^{k-t_F+\frac{(2-m)t_F-k}{2}}[\mathfrak{p}]$, since $[\mathfrak{p}]$ is the operator that acts on q-expansion by $\mathsf{a}(y, \mathsf{f}_{|[\mathfrak{p}]}) = \mathsf{a}(y\varpi_{\mathfrak{p}}^{-1}, \mathsf{f})$. We conclude noting that $[\mathfrak{p}] = \varpi_{\mathfrak{p}}^{t_F-w}V(\mathfrak{p})$ as operators on $S^{\dagger}_{k,w}(K, E_{\wp})$. □

If we denote by $U_{\mathfrak{p}}$ the operator defined in [38, § 3.18], the equality $U_{\mathfrak{p}}\mathrm{Fr}_{\mathfrak{p}} = \langle\mathfrak{p}^{-1}\rangle\varpi_{\mathfrak{p}}^{t_F}$ of [38, Lemma 3.20] implies that $U(\mathfrak{p}) = U_{\mathfrak{p}}\langle\mathfrak{p}\rangle$ as operators on $S^{\dagger}_{k,w}(K; E_{\wp})$. In particular, we can restate [38, Corollary 3.24] by saying that if $\mathsf{f} \in S^{\dagger}_{s_J\cdot(k,w)}(K; E_{\wp})$ is a generalized eigenform for $U_0(\mathfrak{p})$ with non-zero eigenvalue $\lambda_{\mathfrak{p}}$, then

$$\mathrm{val}_p(\lambda_{\mathfrak{p}}) \geqslant \sum_{\tau\in \mathrm{I}_{F,\mathfrak{p}}\setminus J} (k_\tau - 1) \tag{22}$$

where $\mathrm{I}_{F,\mathfrak{p}}$ is the subset of those embeddings $F \hookrightarrow \overline{\mathbb{Q}}$ that induce the prime $\mathfrak{p}$ when composed with the fixed p-adic embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Corollary 4.7. *Let $F/\mathbb{Q}$ be a real quadratic field in which $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$ splits. Let $\mathsf{f} \in S_{k,w}(K; \overline{\mathbb{Q}})$ an eigenform of prime to p level. Then the p-adic cuspforms $d_1^{1-k_1}(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]})$, $d_2^{1-k_2}(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]})$ are overconvergent.*

Proof. We prove the corollary building on an idea of Loeffler *et al.* [27, Proposition 4.5.3]. Let $1 - \mathsf{a}(\varpi_{\mathfrak{p}_2}, \mathsf{f})X + \epsilon_{\mathsf{f}}(\mathfrak{p}_2)\varpi_{\mathfrak{p}_2}^{k-t_F}X^2 = (1 - \alpha_{0,2}X)(1 - \beta_{0,2}X)$ be the Hecke polynomial of f for $T_0(\mathfrak{p}_2)$. We denote by $\mathsf{f}_{\alpha_2}, \mathsf{f}_{\beta_2}$ the two $\mathfrak{p}_2$-stabilizations of f and without loss of generality suppose $\mathrm{val}_p(\alpha_{0,2}) \leqslant \mathrm{val}_p(\beta_{0,2})$. If we write $\Theta_i = \Theta_{\tau_i,k_{\tau_i}-1}$ for $i = 1, 2$, then the classes of $\mathsf{f}^{[\mathfrak{p}_1]}_{\alpha_2}, \mathsf{f}^{[\mathfrak{p}_1]}_{\beta_2}$ are trivial in the quotient $\frac{S^{\dagger}_{k,w}(K;E_{\wp})}{\mathrm{Im}(\Theta_1)+\mathrm{Im}(\Theta_2)}$ because they are annihilated by the invertible operator $U_0(\mathfrak{p}_1)$. Consider the Hecke-equivariant projections $\mathrm{pr}_i : \mathrm{Im}(\Theta_1) + \mathrm{Im}(\Theta_2) \to \frac{\mathrm{Im}(\Theta_i)}{\mathrm{Im}(\Theta_1)\cap\mathrm{Im}(\Theta_2)}$ for $i = 1, 2$. We immediately see that $\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}_{\alpha_2}) = 0$ because of the lower bound (22) on the slopes of $U_0(\mathfrak{p}_2)$, therefore $\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) = \frac{\beta_{2,0}}{\beta_{2,0}-\alpha_{2,0}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}_{\beta_2})$ which implies $U_0(\mathfrak{p}_2)\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) = \beta_{0,2}\cdot\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]})$. We claim that $[\mathfrak{p}_2]\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) = \frac{1}{\beta_{0,2}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]})$. Indeed, the equality of Hecke operators $T_0(\mathfrak{p}_2) = U_0(\mathfrak{p}_2) + \alpha_{0,2}\beta_{0,2}[\mathfrak{p}_2]$ allows us to compute that

$$\begin{aligned}
[\mathfrak{p}_2]\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) &= \frac{1}{\alpha_{0,2}\beta_{0,2}}\Big[T_0(\mathfrak{p}_2)\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) - U_0(\mathfrak{p}_2)\frac{\beta_{2,0}}{\beta_{2,0}-\alpha_{2,0}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}_{\beta_2})\Big]\\
&= \frac{1}{\alpha_{0,2}\beta_{0,2}}\Big[\mathsf{a}(\mathfrak{p}_2, \mathsf{f})\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) - \beta_{0,2}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]})\Big] = \frac{1}{\beta_{0,2}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}).
\end{aligned}$$

Thus, $\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]}) = 0$. By exchanging the roles of the two primes $\mathfrak{p}_1, \mathfrak{p}_2$ we also have that $\mathrm{pr}_1(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]}) = 0$, which proves $\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]} \in \mathrm{Im}(\Theta_1) \cap \mathrm{Im}(\Theta_2)$. □

5. A p-adic Gross–Zagier Formula

5.1. De Rham realization of modular forms

Let E be a number field, following Voevodsky [39] we consider two categories of motives over E: the category of effective Chow motives denoted by $\mathrm{CHM}^{\mathrm{eff}}$ with a natural functor $\mathrm{h} : \mathrm{SmProj}_{/E} \to \mathrm{CHM}^{\mathrm{eff}}$ from the category $\mathrm{SmProj}_{/E}$ of smooth and projective schemes over E, and the triangulated category $\mathrm{DM}^{\mathrm{eff}}$ of effective geometric motives with the natural functor $\mathrm{M}_{\mathrm{gm}} : \mathrm{Sm}_{/E} \to \mathrm{DM}^{\mathrm{eff}}$ from the category $\mathrm{Sm}_{/E}$ of smooth schemes over E. Since number fields have characteristic zero, these two categories are related by a full embedding $\mathrm{CHM}^{\mathrm{eff}} \to \mathrm{DM}^{\mathrm{eff}}$ that makes the diagram

$$\begin{array}{ccc} \mathrm{SmProj}_{/\mathbb{Q}} & \longrightarrow & \mathrm{Sm}_{/\mathbb{Q}} \\ {\scriptstyle \mathrm{h}}\downarrow & & \downarrow{\scriptstyle \mathrm{M}_{\mathrm{gm}}} \\ \mathrm{CHM}^{\mathrm{eff}} & \longrightarrow & \mathrm{DM}^{\mathrm{eff}} \end{array}$$

commute [39, Proposition 2.1.4 and Remark].

Let F be a totally real number field of degree g over $\mathbb{Q}$ and let E be any field containing F^{Gal}. The Shimura variety $\mathrm{Sh}_K(G^*)_{\mathbb{Q}}$ has a universal Hilbert–Blumenthal abelian scheme $\mathcal{A} \to \mathrm{Sh}_K(G^*)$, the $\mathcal{O}_F$-action induces a ring homomorphism $F \hookrightarrow \mathrm{End}_{\mathrm{Sh}_K(G^*)}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote by $\mathrm{CMH}(\mathrm{Sh}_K(G^*))$ the category of Chow motives over $\mathrm{Sh}_K(G^*)$ [9]. Since the decomposition of the Chow motive $\mathrm{h}(\mathcal{A}/\mathrm{Sh}_K(G^*)) = \bigoplus_i \mathrm{h}_i(\mathcal{A}/\mathrm{Sh}_K(G^*))$ of $\mathcal{A}$ over $\mathrm{Sh}_K(G^*)$ is functorial [9, Theorem 3.1], there is an isomorphism of $\mathbb{Q}$-vector spaces [22, Proposition 2.2.1]

$$\mathrm{End}_{\mathrm{Sh}_K(G^*)}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{End}_{\mathrm{CHM}(\mathrm{Sh}_K(G^*))}\big(\mathrm{h}_1(\mathcal{A}/\mathrm{Sh}_K(G^*))\big) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

One denotes by $e_\tau \in \mathrm{End}_{\mathrm{CHM}(\mathrm{Sh}_K(G^*))}(\mathrm{h}_1(\mathcal{A}/\mathrm{Sh}_K(G^*))) \otimes_{\mathbb{Z}} E$, $\tau \in \mathrm{I}_F$, the idempotents coming from $\prod_\tau F = F \otimes E \hookrightarrow \mathrm{End}_{\mathrm{CHM}(\mathrm{Sh}_K(G^*))}(\mathrm{h}_1(\mathcal{A}/\mathrm{Sh}_K(G^*))) \otimes_{\mathbb{Z}} E$.

Definition 5.1. Let $k \in \mathbb{N}[\mathrm{I}_F]$, $k \geqslant 2t_F$. The relative motive $\mathcal{V}^k \in \mathrm{CHM}(\mathrm{Sh}_K(G^*))_E$ is defined as

$$\mathcal{V}^k = \bigotimes_{\tau \in \mathrm{I}_F} \mathrm{Sym}^{k_\tau - 2}\mathrm{h}_1(\mathcal{A}/\mathrm{Sh}_K(G^*))^{e_\tau}$$

following the conventions of [22, p. 72] for the symmetric products. The motive $\mathcal{V}^k$ is a direct factor of $\mathrm{h}(\mathcal{A}^{|k-2t_F|}/\mathrm{Sh}_K(G^*))$, where $\mathcal{A}^{|k-2t_F|}$ denotes the $(|k|-2g)$-fold fiber product of $\mathcal{A}$ over $\mathrm{Sh}_K(G^*)$, thus it corresponds to an idempotent $e_k \in \mathrm{CH}^{g(|k|-2g)}\left(\mathcal{A}^{|k-2t_F|} \times_{\mathrm{Sh}_K(G^*)} \mathcal{A}^{|k-2t_F|}\right) \otimes_{\mathbb{Z}} E$ such that $\mathrm{M}_{\mathrm{gm}}(\mathcal{A}^{|k-2t_F|})^{e_k} = \mathcal{V}^k$.

Proposition 5.2 [41, Corollary 3.9]. *Suppose $k > 2t_F$ and let U_{k-2g} be any smooth compactification of $\mathcal{A}^{|k-2t_F|}$, then the graded part of weight zero with respect to the motivic weight structure on $\mathrm{CHM}_E^{\mathrm{eff}}$, $\mathrm{Gr}_0\mathrm{M}_{\mathrm{gm}}\left(\mathcal{A}^{|k-2t_F|}\right)^{e_k}$, is canonically a direct factor of the Chow motive $\mathrm{M}_{\mathrm{gm}}(U_{k-2g})$. Hence, it corresponds to an idempotent $\theta_k \in \mathrm{CH}^{g(|k|-2g+1)}(U_{k-2g} \times_{\mathbb{Q}} U_{k-2g}) \otimes_{\mathbb{Z}} E$.*

Proposition 5.3. *Suppose $F=\mathbb{Q}$ and let $k>2$ be an integer. For any smooth compactification W_{k-2} of the $(k-2)$th-fold product of the universal elliptic curve $\mathcal{E}$ over the modular curve $\mathrm{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$, there exists an idempotent $\theta_k\in\mathrm{CH}^{k-1}(W_{k-2}\times_{\mathbb{Q}} W_{k-2})\otimes_{\mathbb{Z}}\mathbb{Q}$ such that $\theta_k^*H^*_{\mathrm{dR}}(W_{k-2}/\mathbb{Q})=\theta_k^*H^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q})$ is functorially isomorphic to parabolic cohomology $H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K'},(\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}},\nabla)\big)$ with its Hodge filtration [2, § 2.1].*

Proof. Proposition 5.2 provides an idempotent θ_k such that $\theta_k^*\mathrm{M}_{\mathrm{gm}}(W_{k-2})=\mathrm{Gr}_0\mathrm{M}_{\mathrm{gm}}(\mathcal{E}^{k-2})^{e_k}$. We claim that the proof of [2, Lemma 2.2] applies to our situation. Indeed, the main ingredient of that proof is a result of Scholl [36, Theorem 3.1.0], which can be applied to any smooth compactification W_{k-2} since the motive considered by Scholl is isomorphic to $\mathrm{Gr}_0\mathrm{M}_{\mathrm{gm}}(\mathcal{E}^{k-2})^{e_k}$ by [40, Corollary 3.4(b)]. Note that the idempotent e in [40, Definition 3.1] acts as the idempotent e_k on $\mathrm{M}_{\mathrm{gm}}(\mathcal{E}^{k-2})$ because the action of the torsion appearing in e is trivial since $\mathcal{E}^{k-2}\to\mathrm{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$ is an abelian scheme. □

Proposition 5.4. *Let $L/\mathbb{Q}$ be a real quadratic extension and $\ell\in\mathbb{N}[\mathrm{I}_L],\ell>2t_F$ a non-parallel weight. For any smooth compactification $U_{\ell-4}$ of the $(|\ell|-4)$th-fold product of the universal abelian surface over $\mathrm{Sh}_K(G_L^*)$, there exists an idempotent $\theta_\ell\in\mathrm{CH}^{2(|\ell|-3)}(U_{\ell-4}\times_{\mathbb{Q}}U_{\ell-4})\otimes_{\mathbb{Z}}L$ such that $\theta_\ell^*H^{i+|\ell|-4}_{\mathrm{dR}}(U_{\ell-4}/\mathbb{Q})$ is functorially isomorphic to $\mathbb{H}^i(\mathrm{Sh}_K,\mathrm{DR}^\bullet(\mathcal{F}^{(\ell,|\ell-t_L|)}_{G_L^*}))$ with its Hodge filtration.*

Proof. Since the weight ℓ is not parallel, Proposition 5.2 and [41, Theorem 3.6] provide an idempotent θ_ℓ such that $\theta_\ell^*\mathrm{M}_{\mathrm{gm}}(U_{\ell-4})=\mathcal{V}^\ell$. Then Kings proved in [22, Corollary 2.3.4] that the $(i+|\ell|-4)$-th cohomology of the de Rham realization of $\mathcal{V}^\ell$ is isomorphic to $\mathbb{H}^i(\mathrm{Sh}_K,\mathrm{DR}^\bullet(\mathcal{F}^{(\ell,|\ell-t_L|)}_{G_L^*}))$. □

5.2. Generalized Hirzebruch–Zagier cycles

Let $L/\mathbb{Q}$ be a real quadratic extension, $K\subset V_{11}(\mathfrak{A}\mathcal{O}_L)$ a small enough (Definition 4.2) congruence subgroups, $K'=K\cap\mathrm{GL}_2(\mathbb{A}^\infty)$, and let $\xi:\mathrm{Sh}_K(G_L^*)\to\mathrm{Sh}_K(G_L)$ be the map of Shimura varieties derived from the inclusion $G_L^*\hookrightarrow G_L$. Let $\breve{\mathsf{g}}\in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L);L;\overline{\mathbb{Q}})$ be a eigenform of either parallel weight $\ell=2t_L$ or non-parallel weight $\ell>2t_L$ such that $\ell-2x=nt_L$. Let $\breve{\mathsf{f}}\in S_{k,w}(V_{11}(\mathfrak{A});\overline{\mathbb{Q}})$ be an elliptic eigenform for the good Hecke operators, such that $k-2w=m$, and we denote by f the newform corresponding to the system of eigenvalues. We suppose that the weights of g and f are balanced. We consider $E/\mathbb{Q}$ a finite Galois extension containing the Fourier coefficients of g and f. We want to realize these modular forms in the de Rham cohomology of some proper and smooth variety. The pullback $\xi^*\breve{\mathsf{g}}$ lives in $S^*_{\ell,x}(K;L;E)$, which by (2) is isomorphic to $S^*_{\ell,\ell-t_L}(K;L;E)$. Thanks to Theorem 4.4 we can realize the latter space as a subgroup of the hypercohomology group $\mathbb{H}^2(\mathrm{Sh}^{\mathrm{tor}}_{K,E},\mathrm{DR}^\bullet(\mathcal{F}^{\ell,|\ell-t_L|}_{G_L^*}))$, which is simply the de Rham cohomology group $H^2_{\mathrm{dR}}(\mathrm{Sh}^{\mathrm{tor}}_K(G_L^*)/E)$ when $\ell=2t_L$. Instead, when $\ell>2t_L$ is not parallel, let $U_{\ell-4}$ be any smooth compactification of $\mathcal{A}^{\ell-4}$; then, we can invoke Proposition 5.4 to establish that the differential attached to $\Psi_{x,|\ell-t_L|}(\xi^*\breve{\mathsf{g}})$, where $\Psi_{x,|\ell-t_L|}$ is defined in (2), lives in $\mathrm{F}^{|\ell|-2}H^{|\ell|-2}_{\mathrm{dR}}(U_{\ell-4}/E)$. Similarly, if $k=2$, $\Psi_{w,1}(\breve{\mathsf{f}})\in S_{2,1}(K';E)\subset\mathrm{F}^1H^1_{\mathrm{dR}}(\mathrm{Sh}^{\mathrm{tor}}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})/E)$, while when $k>2$ we can consider any smooth compactification

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W_{k-2} of $\mathcal{E}^{k-2}$ to see that the class of the differential $\omega_{\Psi_{w,k-1}(\breve{\mathsf{f}})}$ lives in $H^{k-1}_{\mathrm{dR}}(W_{k-2}/E)$, by Proposition 5.3.

Definition 5.5. Choose a prime p coprime to M. Let $E_\wp$ be the closure of $\iota_p(E)$ in $\overline{\mathbb{Q}}_p$ and suppose that $\breve{\mathsf{g}}, \breve{\mathsf{f}}$ are p-nearly ordinary. We write ω for the differential $\omega_{\Psi_{x,|\ell-t_L|}(\xi^*\breve{\mathsf{g}})}$ and we take η to be the class in the $\Psi_{w,k-1}(\mathsf{f})$-isotypic part of $H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K'}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big)^{\mathrm{u.r.}}$ whose image in the 0-th graded piece, $H^1(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp}, \underline{\omega}^{2-k}_{\mathrm{GL}_{2,\mathbb{Q}}})$, is equal to the image of $\frac{\mathrm{vol}(K')}{\langle \mathsf{f}^*, \mathsf{f}^*\rangle} \cdot \overline{\omega_{\Psi_{w,k-1}(\breve{\mathsf{f}}^*)}}$.

The class $\eta \in H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K'}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big)^{\mathrm{u.r.}}$ satisfies

$$\mathrm{Fr}_p(\eta) = \alpha_{\mathsf{f}^*} p^{w-1}\eta, \tag{23}$$

where the eigenvalue is a p-adic unit since f^* is p-nearly ordinary. Indeed, by definition $\eta = [c \cdot \Psi_{w,k-1}(\breve{\mathsf{f}}_\beta)]$ for some non-zero constant c, and applying Lemmas 2.6 and 4.6 we can compute

$$\begin{aligned}\mathrm{Fr}_p(\eta) &= pV(p)[c \cdot \Psi_{w,k-1}(\breve{\mathsf{f}}_\beta)] = p \cdot p^{k-1-w}[c \cdot \Psi_{w,k-1}(V(p)\breve{\mathsf{f}}_\beta)]\\ &= p^{k-w}[c \cdot \Psi_{w,k-1}(U(p)^{-1}\breve{\mathsf{f}}_\beta)] = p^{k-w}\beta_{\mathsf{f}}^{-1}\eta = \alpha_{\mathsf{f}^*} p^{w-1}\eta,\end{aligned}$$

since $\beta_{\mathsf{f}}^{-1} = \alpha_{\mathsf{f}}\psi_{\mathsf{f}}(p)^{-1}p^{-1} = \alpha_{\mathsf{f}^*} p^{-m-1}$.

For all $s \geqslant 0$ we want to consider the cohomology class

$$\pi_1^*\omega \cup \pi_2^*\eta \in \mathrm{F}^{|\ell|-2-s} H^{|\ell|+k-3}_{\mathrm{dR}}\big(U_{\ell-4} \times_{E_\wp} W_{k-2}\big).$$

Our goal is to define a null-homologous cycle on $U_{\ell-4} \times_E W_{k-2}$ whose syntomic Abel–Jacobi map can be evaluated at $\pi_1^*\omega \cup \pi_2^*\eta$. Let $\mathscr{Z}_{\ell,k}$ be a proper smooth model of $U_{\ell-4} \times_{E_\wp} W_{k-2}$ over $\mathcal{O}_{E_\wp}$ of relative dimension d, and denote by $Z_{\ell,k}$ its generic fiber. For all integers $i \geqslant 0$, the syntomic cohomology groups of $\mathscr{Z}_{\ell,k}$ sit in a short exact sequence of the form

$$0 \longrightarrow H^{2i-1}_{\mathrm{dR}}(Z_{\ell,k})/F^i \xrightarrow{\ \iota\ } H^{2i}_{\mathrm{syn}}(\mathscr{Z}_{\ell,k}, i) \xrightarrow{\ \pi\ } \mathrm{F}^i H^{2i}_{\mathrm{dR}}(Z_{\ell,k}).$$

The syntomic cycle class map [3, Proposition 5.4] is compatible with the de Rham cycle class map producing a commuting diagram

$$\begin{array}{ccc}\mathrm{CH}^i(\mathscr{Z}_{\ell,k}) & \xrightarrow{\mathrm{cl}_{\mathrm{syn}}} & H^{2i}_{\mathrm{syn}}(\mathscr{Z}_{\ell,k}, i)\\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \pi}\\ \mathrm{CH}^i(Z_{\ell,k}) & \xrightarrow{\mathrm{cl}_{\mathrm{dR}}} & F^i H^{2i}_{\mathrm{dR}}(Z_{\ell,k})\end{array}$$

where on the left hand side are the Chow groups of algebraic cycles modulo rational equivalence. The restriction of the syntomic cycle class map $\mathrm{cl}_{\mathrm{syn}}$ to the subgroup of de

Rham null-homologous cycles $\mathrm{CH}^i(\mathscr{Z}_{\ell,k})_0$, i.e., the kernel of the composition $\mathrm{cl}_{\mathrm{dR}} \circ \mathrm{Res}$, has image landing in $H^{2i-1}_{\mathrm{dR}}(Z_{\ell,k})/F^i$. The syntomic Abel–Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^i(\mathscr{Z}_{\ell,k})_0 \longrightarrow \left(F^{d-i+1} H^{2(d-i)+1}_{dR}(Z_{\ell,k})\right)^{\vee} \tag{24}$$

is obtained by identifying the target using Poincaré duality.

We determine the positive integer s and make sure the numerology works. The dimension of the variety $U_{\ell-4} \times_E W_{k-2}$ is $d = 2|\ell| + k - 7$, therefore the cycle we want has to be of dimension $d - i$ such that $2(d-i)+1 = |\ell| + k - 3$, and $s \geqslant 0$ has to satisfy $|\ell| - 2 - s = (d-i)+1$. Hence

$$(d-i) = \frac{|\ell| + k - 4}{2}, \quad s = \frac{|\ell| - k - 2}{2} \tag{25}$$

with $s \geqslant 0$ since the weights are balanced.

5.2.1. Definition of the cycles. We treat separately the case $(\ell, k) = (2t_L, 2)$ and the general case $(\ell, k) > (2t_L, 2)$ with ℓ not parallel. Set $\gamma + 1 = \frac{|\ell|+k-4}{2}$ and consider the finite map

$$\varphi : \mathcal{E}^{\gamma} \longrightarrow \mathcal{A}^{|\ell|-4} \times_E \mathcal{E}^{k-2},$$
$$(x, P_1, \ldots, P_{\gamma}) \mapsto (\zeta(x), P'_1 \otimes 1, \ldots, P'_{|\ell|-4} \otimes 1; x, P'_{|\ell|-3}, \ldots, P'_{2\gamma})$$

where $(P'_1, \ldots, P'_{2\gamma}) = (P_1, \ldots, P_{\gamma}, P_1, \ldots, P_{\gamma})$ and $P'_i \otimes 1$ is the point $P'_i \otimes 1 \to \mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_F \to \mathcal{A}$. The definition makes sense because $2\gamma = |\ell| - 4 + k - 2$. The variety $\mathcal{E}^{\gamma}$ has dimension equal to $\gamma + 1$ and we will define the null-homologous cycle by first compactifying and then by applying an appropriate correspondence. Let W_0 be the smooth and projective compactification of the modular curve $\mathrm{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$. We consider $W_{\gamma}, U_{\ell-4}, W_{k-2}$ smooth and projective compactifications of $\mathcal{E}^{\gamma}, \mathcal{A}^{|\ell|-4}, \mathcal{E}^{k-2}$ respectively, such that W_{γ} has a map $W_{\gamma} \to W_0$ extending $\mathcal{E}^{\gamma} \to \mathrm{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$; then the map φ defines a rational morphism $\varphi : W_{\gamma} \dashrightarrow U_{\ell-4} \times_E W_{k-2}$. Using Hironaka's work on resolution of singularities [16, Chapter 0.5, Question (E)], we can assume the rational map φ has a representative $\varphi : W_{\gamma} \longrightarrow U_{\ell-4} \times_E W_{k-2}$ defined everywhere, up to replacing the smooth and projective compactification of $\mathcal{E}^{\gamma}$. Furthermore, by desingularizing the fibers over the cusps, we can assume that $W_{\gamma} \to W_0$ is smooth. By spreading out, there is an open of $\mathrm{Spec}(\mathcal{O}_E)$ over which all our geometric objects can be defined simultaneously and retain their relevant features: we have smooth and projective models $\mathscr{W}_{\gamma}, \mathscr{U}_{\ell-4}, \mathscr{W}_{k-2}$ of $W_{\gamma}, U_{\ell-4}, W_{k-2}$ respectively, the map φ extends to a map $\tilde{\varphi} : \mathscr{W}_{\gamma} \longrightarrow \mathscr{U}_{\ell-4} \times \mathscr{W}_{k-2}$ and $\mathscr{W}_{\gamma} \to \mathscr{W}_0$ is smooth.

When $\ell = 2t_L$ and $k = 2$ we define correspondences on $\mathscr{U}_0 \times \mathscr{W}_0$ as follows. We assume the number field E is large enough such that $U_{0/E}$ (respectively $W_{0/E}$) is the disjoint union of its geometrically connected components $U_{0/E} = \coprod_i U_{0,i}$ (respectively $W_{0/E} = \coprod_j W_{0,j}$) and we pick an E-rational point $a_i \in U_{0,i}$ (respectively $b_j \in W_{0,j}$) for every such component. Consider the following morphisms: for every pair (i, j) indexing a geometrically irreducible component of $Z = U_0 \times_E W_0$, we define $q_{i,j} : Z \to U_{0,i} \times_E W_{0,j} \hookrightarrow Z$ as the map that restricts to the natural inclusion of $U_{0,i} \times_E W_{0,j}$ into Z and

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maps any other geometrically irreducible component to the point (a_i, b_j). Similarly, we define $q_{a_i,j} : Z \to \{a_i\} \times W_{0,j} \hookrightarrow Z$, $q_{i,b_j} : Z \to U_{0,i} \times \{b_j\} \hookrightarrow Z$ and $q_{a_i,b_j} : Z \to \{a_i\} \times \{b_j\} \hookrightarrow Z$. Consider $P_{i,j} = \mathrm{graph}(q_{i,j})$, $P_{a_i,j} = \mathrm{graph}(q_{a_i,j})$, $P_{i,b_j} = \mathrm{graph}(q_{i,b_j})$, $P_{a_i,b_j} = \mathrm{graph}(q_{a_i,b_j})$, all correspondences in $\mathrm{CH}^6(Z \times_E Z)$. We set

$$P = \sum_{i,j} \left(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j}\right),$$

that acts on $\mathrm{CH}^\bullet(Z)$ by $P_* = \mathrm{pr}_{2,*}(P \cdot \mathrm{pr}_1^*)$; in particular, for any cycle $S \in \mathrm{CH}^\bullet(Z)$, we have

$$P_*(S) = \sum_{i,j} \left[(q_{i,j})_* - (q_{a_i,j})_* - (q_{i,b_j})_* + (q_{a_i,b_j})_*\right](S).$$

For i, j running in the set of indices of the geometrically connected components of U_0 and W_0 the correspondences $(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j})$ are idempotents and orthogonal to each other, hence $P \circ P = P$ in $\mathrm{CH}^6(Z \times_E Z)$, i.e., P is a projector. We denote by $\tilde{P}$ the correspondence on $\mathscr{U}_0 \times \mathscr{W}_0$ defined over some open of $\mathrm{Spec}(\mathcal{O}_E)$ obtained by spreading out P.

When $(\ell, k) > (2t_L, 2)$ with ℓ non-parallel, we obtain a correspondence on $\mathscr{U}_{\ell-4} \times \mathscr{W}_{k-2}$ by spreading out those correspondences considered in § 5.1. Indeed, the idempotents $\theta_\ell \in \mathrm{CH}^{2(|\ell|-3)}(U_{\ell-4} \times_E U_{\ell-4}) \otimes_{\mathbb{Z}} L$ and $\theta_k \in \mathrm{CH}^{|k|-1}(W_{k-2} \times_E W_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Q}$ extend to elements $\tilde{\theta}_\ell \in \mathrm{CH}^{2(|\ell|-3)}(\mathscr{U}_{\ell-4} \times \mathscr{U}_{\ell-4}) \otimes_{\mathbb{Z}} L$ and $\tilde{\theta}_k \in \mathrm{CH}^{|k|-1}(\mathscr{W}_{k-2} \times \mathscr{W}_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Q}$ respectively.

Definition 5.6. For all but finitely many primes p, we define the Hirzebruch–Zagier cycle of weight $(2t_L, 2)$ to be

$$\Delta_{2t_L,2} = \tilde{P}_* \tilde{\varphi}_*[\mathscr{W}_0] \in \mathrm{CH}^2(\mathscr{U}_0 \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_0).$$

Proposition 5.7. *The Hirzebruch–Zagier cycle* $\Delta_{2t_L,2} \in \mathrm{CH}^2(\mathscr{U}_0 \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_0)$ *is de Rham null-homologous.*

Proof. To verify that $\mathrm{cl}_{\mathrm{dR}}(\Delta_{2t_L,2})$ is zero in $H^4_{\mathrm{dR}}(Z/E_\wp)$, it suffices to show that $P_* H^4_{\mathrm{dR}}(Z/E) = 0$ since our cycle starts his life over E. After base-change to $\mathbb{C}$, via the fixed complex embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, Poincaré duality tells us that it is enough to prove the projector annihilates the second singular homology, i.e., $P_* H_2(Z(\mathbb{C})) = 0$. By Kunneth formula and the fact that each connected component of $U_0(\mathbb{C})$ is simply connected, we compute that $P_* H_2(Z(\mathbb{C})) = P_*(H_0(U_0(\mathbb{C})) \otimes H_2(W_0(\mathbb{C})) \oplus H_2(U_0(\mathbb{C})) \otimes H_0(W_0(\mathbb{C})))$, which we can show to be zero by the explicit definition of the projector P. Indeed, let $[x] \otimes [C] \in H_0(U_0(\mathbb{C})) \otimes H_2(W_0(\mathbb{C}))$ be a simple tensor for $x \in U_0(\mathbb{C})$ a point, then for all i, j we find

$$\begin{aligned} &\left(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j}\right)([x] \otimes [C]) \\ &= \left((q_{i,j})_* - (q_{a_i,j})_* - (q_{i,b_j})_* + (q_{a_i,b_j})_*\right)([x] \otimes [C]) \\ &= [a_i] \otimes [C_j] - [a_i] \otimes [C_j] = 0, \end{aligned}$$

where $(q_{i,b_j})_*([x] \otimes [C]) = 0 = (q_{a_i,b_j})_*([x] \otimes [C])$ because the dimension of the pushforward drops. Similarly, if $[D] \otimes [y] \in H_2(U_0(\mathbb{C})) \otimes H_0(W_0(\mathbb{C}))$ is a simple tensor for $y \in W_0(\mathbb{C})$ a point, then $\left(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j}\right)([D] \otimes [y]) = 0$ for all i, j. □

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Definition 5.8. Let $\ell \in \mathbb{Z}[I_L]$, $\ell > 2t_L$, be a non-parallel weight and $k > 2$ an integer such that (ℓ, k) is a balanced triple. For all but finitely many primes p, the generalized Hirzebruch–Zagier cycle of weight (ℓ, k) is

$$\Delta_{\ell,k} = (\tilde{\theta}_\ell, \tilde{\theta}_k)_* \tilde{\varphi}_*[\mathscr{W}_\gamma] \in \mathrm{CH}^i(\mathscr{U}_{\ell-4} \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_{k-2}) \otimes_{\mathbb{Z}} L.$$

Proposition 5.9. *Let $\ell \in \mathbb{Z}[I_L]$, $\ell > 2t_L$, be a non-parallel weight and $k > 2$ an integer such that (ℓ, k) is a balanced triple. The generalized Hirzebruch–Zagier cycle $\Delta_{\ell,k} \in \mathrm{CH}^i(\mathscr{U}_{\ell-4} \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_{k-2}) \otimes_{\mathbb{Z}} L$ is de Rham null-homologous.*

Proof. The class $\mathrm{cl}_{\mathrm{dR}}(\Delta_{\ell,k})$ belongs to $(\theta_\ell, \theta_k)_* H^{2i}_{\mathrm{dR}}(U_{\ell-4} \times_{E_\wp} W_{k-2})$ and by Poincaré duality, it is trivial if and only if

$$(\theta_\ell, \theta_k)^* H^{2(d-i)}_{\mathrm{dR}}(U_{\ell-4} \times_{E_\wp} W_{k-2}) = \bigoplus_{\mu+\nu=2(d-i)} (\theta_\ell)^* H^{\mu}_{\mathrm{dR}}(U_{\ell-4}) \otimes (\theta_k)^* H^{\nu}_{\mathrm{dR}}(W_{k-2}) \quad (26)$$

is trivial. By Propositions 5.4 and 5.3, we have $\theta_\ell^* H^\mu_{\mathrm{dR}}(U_{\ell-4}) = \mathbb{H}^{\mu-|\ell|+4}\big(\mathrm{Sh}_K, \mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t)}_{G_L^*})\big)$ and $\theta_k^* H^*_{\mathrm{dR}}(W_{k-2}) = \theta_k^* H^{k-1}_{\mathrm{dR}}(W_{k-2}) = H^1_{\mathrm{par}}\big(\mathrm{Sh}_K^{\mathrm{tor}}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big)$. Hence, $\nu = k-1$ forces μ to be $\mu = |\ell| - 3$ and the group

$$\theta_\ell^* H^{|\ell|-3}_{\mathrm{dR}}(U_{\ell-4}) = \mathbb{H}^1(\mathrm{Sh}_K, \mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t)}_{G_L^*})) \quad (27)$$

is trivial. Indeed, by [30, A6.20], the cohomology group $\mathbb{H}^1(\mathrm{Sh}_K, \mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t)}_{G_L^*}))$ is identified with the intersection cohomology of the Baily–Borel compactification of $\mathrm{Sh}_K(G_L^*)$, that in turn is trivial in degree 1 by computations using Lie algebra cohomology [30, §§5.11, 6.5, 6.6]. □

5.2.2. Evaluation of syntomic Abel–Jacobi. We are interested in computing $\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta)$ and to relate it to some value of the twisted triple product p-adic L-function outside the range of interpolation. Let $\tilde{\omega}$ (respectively $\tilde{\eta}$) be a lift of ω (respectively η) to fp-cohomology; since the Hirzebruch–Zagier cycle is null-homologous the computation is independent of the choice of lifts. We start by treating the case $(\ell, k) = (2t_L, 2)$:

$$\begin{aligned}
\mathrm{AJ}_p(\Delta_{2t_L,2})(\pi_1^*\omega \cup \pi_2^*\eta) &= \langle \mathrm{cl}_{\mathrm{syn}}(\Delta_{2t_L,2}), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} = \langle \tilde{P}_*\mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_0]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_0]), \sum_{i,j}(\tilde{P}_{i,j} - \tilde{P}_{a_i,j} - \tilde{P}_{i,b_j} + \tilde{P}_{a_i,b_j})^*(\pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta})\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_0]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
\text{[3, Equation (20)]} \quad &= \mathrm{tr}_{\mathscr{W}_0}(\tilde{\varphi}^*(\pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta})) = \mathrm{tr}_{\mathscr{W}_0}(\tilde{\zeta}^*\tilde{\omega} \cup \tilde{\eta}).
\end{aligned}$$

The fourth equality is justified by the vanishing $H^1_{\mathrm{fp}}(\mathrm{Spec}(\mathcal{O}_{E,\wp}), 0) = 0 = H^2_{\mathrm{fp}}(\mathrm{Spec}(\mathcal{O}_{E,\wp}), 2)$, which imply that $\sum_{i,j} \tilde{P}^*_{i,j} = (\mathrm{id}_{\mathscr{U}_0 \times \mathscr{W}_0})^*$ and that all the other pullbacks are zero.

To deal with the general case, we first need to analyze the action of the correspondences $\tilde{\theta}_k, \tilde{\theta}_\ell$ on fp-cohomology. The exact sequence in [3, (8)] induces a functorial isomorphism $H^{k-1}_{\mathrm{fp}}(\mathscr{W}_{k-2}, 0) \cong H^{k-1}_{\mathrm{dR}}(W_{k-2})$, we denote by $\tilde{\eta}$ the preimage of $\eta \in \theta_k^* H^{k-1}_{\mathrm{dR}}(W_{k-2})$ that satisfies $\tilde{\theta}_k^*\tilde{\eta} = \tilde{\eta}$ since $\theta_k^*\eta = \eta$. By functoriality of the short exact sequence [3, (8)], there is a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{\mathrm{dR}}^{|\ell|-3}(U_{\ell-4})/F^{|\ell|-2-s} & \xrightarrow{\iota} & H_{\mathrm{fp}}^{|\ell|-2}(\mathscr{U}_{\ell-4}, |\ell|-2-s) & \xrightarrow{\pi} & \mathrm{F}^{|\ell|-2-s}H_{\mathrm{dR}}^{|\ell|-2}(U_{\ell-4}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_\ell^*=0} & & \downarrow{\scriptstyle \tilde{\theta}_\ell^*} & \swarrow & \downarrow{\scriptstyle \theta_\ell^*} & & \\
0 & \longrightarrow & H_{\mathrm{dR}}^{|\ell|-3}(U_{\ell-4})/F^{|\ell|-2-s} & \xrightarrow{\iota} & H_{\mathrm{fp}}^{|\ell|-2}(\mathscr{U}_{\ell-4}, |\ell|-2-s) & \xrightarrow{\pi} & \mathrm{F}^{|\ell|-2-s}H_{\mathrm{dR}}^{|\ell|-2}(U_{\ell-4}) & \longrightarrow & 0
\end{array}$$

where the leftmost vertical arrow is zero because of the vanishing (27). Therefore, there is a canonical lift $\tilde{\omega} = \tilde{\theta}_\ell^*\omega$ to $H_{\mathrm{fp}}^{|\ell|-2}(\mathscr{U}_{\ell-4}, |\ell|-2-s)$ of any class $\omega \in \theta_\ell^*\mathrm{F}^{|\ell|-2-s}H_{\mathrm{dR}}^{|\ell|-2}(U_{\ell-4})$, with the property $\tilde{\theta}_\ell^*\tilde{\omega} = \tilde{\omega}$. At this point we can compute

$$\begin{aligned}
\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta) &= \langle \mathrm{cl}_{\mathrm{syn}}(\Delta_{\ell,k}), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle (\tilde{\theta}_\ell, \tilde{\theta}_k)_*\mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_\gamma]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_\gamma]), \pi_1^*\tilde{\theta}_\ell^*\tilde{\omega} \cup \pi_2^*\tilde{\theta}_k^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_\gamma]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \mathrm{tr}_{\mathscr{W}_\gamma}(\tilde{\varphi}^*(\pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta})) = \mathrm{tr}_{\mathscr{W}_\gamma}(\tilde{\varphi}_1^*\tilde{\omega} \cup \tilde{\varphi}_2^*\tilde{\eta}),
\end{aligned}$$

where $\tilde{\varphi}_i = (\pi_i \circ \tilde{\varphi})$. The fundamental exact sequence of fp-cohomology induces an isomorphism $\iota : H_{\mathrm{dR}}^{|\ell|-3}(W_\gamma) \xrightarrow{\sim} H_{\mathrm{fp}}^{|\ell|-2}(\mathscr{W}_\gamma, |\ell|-2-s)$, since the filtered piece $\mathrm{F}^n H_{\mathrm{dR}}^j(W_\gamma)$ is trivial for $n > \dim_{E_\wp} W_\gamma$ and indeed $|\ell|-2-s$ is greater than $\dim_{E_\wp} W_\gamma = \gamma+1$. Therefore, if we write $\tilde{\varphi}_1^*\tilde{\omega} = \iota\Upsilon(\omega)$, we can rewrite the quantity we want to evaluate as

$$\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta) = \mathrm{tr}_{W_\gamma}(\Upsilon(\omega) \cup_{\mathrm{dR}} \varphi_2^*\eta) = \langle \Upsilon(\omega), \varphi_2^*\eta\rangle_{\mathrm{dR}}, \tag{28}$$

for the Poincaré pairing $\langle\ ,\ \rangle_{\mathrm{dR}} : H_{\mathrm{dR}}^{|\ell|-3}(W_\gamma) \times H_{\mathrm{dR}}^{k-1}(W_\gamma) \xrightarrow{\cup} H_{\mathrm{dR}}^{|\ell|+k-4}(W_\gamma) \xrightarrow{\mathrm{tr}_{\mathrm{dR}}} E_\wp$.

5.3. Description of $\mathbf{AJ}_p(\Delta_{\ell,k})$ in terms of p-adic modular forms

Let $\mathscr{Y}_{K'} \hookrightarrow \mathbf{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})_{\mathcal{O}_{E_\wp}}$ be the $\mathcal{O}_{E_\wp}$-scheme defined as the complement of the supersingular points and let $\mathscr{E} \to \mathscr{Y}_{K'}$ be the universal elliptic curve over it.

Proposition 5.10. *There are natural inclusions of parabolic cohomology in the de Rham cohomology of proper and smooth compactifications of Kuga–Sato varieties*

$$\begin{gathered}
H_{\mathrm{par}}^1\big(\mathrm{Sh}_{K',E_\wp}^{\mathrm{tor}}, (\mathcal{F}_{\mathrm{GL}_{2,\mathbb{Q}}}^{(k,k-1+s)}, \nabla)\big) \hookrightarrow H_{\mathrm{dR}}^{|\ell|-3}(W_\gamma), \\
H_{\mathrm{par}}^1\big(\mathrm{Sh}_{K',E_\wp}^{\mathrm{tor}}, (\mathcal{F}_{\mathrm{GL}_{2,\mathbb{Q}}}^{(k,k-1)}, \nabla)\big) \hookrightarrow H_{\mathrm{dR}}^{k-1}(W_\gamma),
\end{gathered}$$

compatible with Poincaré duality.

Proof. Let $\mathscr{D}_{\gamma,\kappa}$ be the inverse image of cusps and supersingular points under $\mathscr{W}_{\gamma,\kappa} \to \mathscr{W}_{0,\kappa}$; then $\mathscr{D}_{\gamma,\kappa} = \mathscr{W}_{\gamma,\kappa} \setminus \mathscr{E}_\kappa^\gamma$ and it is a smooth and projective subscheme of codimension

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1 in $\mathscr{W}_{\gamma,\kappa}$. Consider the diagram

$$\begin{array}{ccccc} H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{W}_{\gamma,\kappa}) & \longrightarrow & H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{E}^{\gamma}_{\kappa}) & \longrightarrow & H^{|\ell|-4}_{\mathrm{rig}}(\mathscr{D}_{\gamma,\kappa})(-1) \\ \uparrow & & \uparrow & & \\ H^1_{\mathrm{par}}(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp},(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}},\nabla)) & \hookrightarrow & \mathbb{H}^1(\mathscr{S}^{\mathrm{tor}}_{K',\mathrm{rig}}, j^{\dagger}\mathrm{DR}^{\bullet}(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}})) & & \end{array}$$

where the top horizontal arrow is exact and comes from excision. The composition

$$H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp},(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}},\nabla)\big) \longrightarrow H^{|\ell|-4}_{\mathrm{rig}}(\mathscr{D}_{r,k})(-1)$$

is identically zero because the two cohomology groups are pure of different weights. Thus, $H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp},(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}},\nabla)\big) \hookrightarrow H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{W}_{\gamma,\kappa}) \cong H^{|\ell|-3}_{\mathrm{dR}}(W_\gamma)$. A similar argument provides the other inclusion $H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp},(\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}},\nabla)\big) \hookrightarrow H^{k-1}_{\mathrm{rig}}(\mathscr{W}_{\gamma,\kappa}) \cong H^{k-1}_{\mathrm{dR}}(W_\gamma)$. □

It is clear that $\varphi_2^*\eta \in H^{k-1}_{\mathrm{dR}}(W_\gamma)$ is equal to $\eta \in H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp},(\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}},\nabla)\big) \hookrightarrow H^{k-1}_{\mathrm{dR}}(W_\gamma)$, so our task is to describe $\Upsilon(\omega) \in H^{|\ell|-3}_{\mathrm{dR}}(W_\gamma)$ using p-adic modular forms.

Let $\mathscr{X}_K \hookrightarrow \mathbf{Sh}_K(G^*_L)_{\mathcal{O}_{E_\wp}}$ be the $\mathcal{O}_{E_\wp}$-scheme defined as the complement of the supersingular locus and $\zeta : \mathscr{Y}_{K'} \longrightarrow \mathscr{X}_K$ the diagonal morphism. Let $\mathscr{A} \to \mathscr{X}_K$ be the universal abelian surface, we have a commuting diagram

$$\begin{array}{ccc} \mathscr{E}^{\gamma} & \xrightarrow{\tilde\varphi_1} & \mathscr{A}^{|\ell|-4} \\ \downarrow{\scriptstyle \upsilon} & & \downarrow{\scriptstyle \upsilon} \\ \mathscr{W}_\gamma & \xrightarrow{\tilde\varphi_1} & \mathscr{U}_{\ell-4} \end{array} \quad \text{that induces} \quad \begin{array}{ccc} \tilde\theta^*_\ell \tilde H^{|\ell|-2}_{\mathrm{f},Q}(\mathscr{U}_{\ell-4},|\ell|-2-s) & \xrightarrow{\upsilon^*} & \tilde\theta^*_\ell \tilde H^{|\ell|-2}_{\mathrm{f},Q}(\mathscr{A}^{|\ell|-4},|\ell|-2-s) \\ \downarrow{\scriptstyle \tilde\varphi_1^*} & & \downarrow{\scriptstyle \tilde\varphi_1^*} \\ \tilde H^{|\ell|-2}_{\mathrm{f},Q}(\mathscr{W}_\gamma,|\ell|-2-s) & \xrightarrow{\upsilon^*} & \tilde H^{|\ell|-2}_{\mathrm{f},Q}(\mathscr{E}^{\gamma},|\ell|-2-s) \end{array}$$

where we consider the Gros-style version of fp-cohomology [3, § 9] for a suitable choice of polynomial Q. We choose to work with the Gros-style version because for schemes that can be embedded in a smooth and proper scheme it is defined using rigid complexes in place of de Rham ones; in particular, the two versions coincide for proper and smooth schemes.

The pullback $\upsilon^*\tilde\omega \in \tilde\theta^*_\ell \tilde H^{|\ell|-2}_{\mathrm{f},Q}(\mathscr{A}^{|\ell|-4},|\ell|-2-s)$ can be directly described in terms of p-adic modular forms. Indeed, we can write $\upsilon^*\tilde\omega = [\omega, f]$ for $\omega \in H^0\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}(\mathcal{F}^{(\ell,\ell-t_L)}_{G^*_L} \otimes \Omega^2(\log \mathrm{D}))\big)$ and $f \in H^0\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^{\dagger}(\mathcal{F}^{(\ell,\ell-t_L)}_{G^*_L} \otimes \Omega^1(\log \mathrm{D}))\big)$ satisfying $Q(\mathrm{Fr}_p)\omega = \nabla f$ as the group $\tilde\theta^*_\ell H^{\ell-i}_{\mathrm{rig}}(\mathscr{A}^{|\ell|-4}_k/E_\wp)$ is the same as the cohomology of the rigid realization of the motive $\mathcal{V}^\ell$ over $]\mathbf{Sh}_K(G^*_L)^{\mathrm{ord}}_k[$, that is, the rigid cohomology $\mathbb{H}^i\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}; j^{\dagger}\mathrm{DR}^{\bullet}(\mathcal{F}^{(\ell,\ell-t_L)}_{G^*_L})\big)$, for $i = 1, 2$.

To express the class $\upsilon^*\tilde\omega$ explicitly we need to make a judicious choice of a polynomial. From now on we assume that p splits in $L/\mathbb{Q}$, $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$. By observing the form of the Euler factors appearing in Theorem 3.12 and the formulas in Corollary 4.7 we are led to consider the polynomial $P(T) = \prod_{\bullet,\star\in\{\alpha,\beta\}}(1 - \bullet_1 \star_2 T)$. Following [27, Proposition 4.5.5], if we set $T = T_1T_2$, we can write $P(T_1,T_2) = a_2(T_1,T_2)P_1(T_1) + b_1(T_1,T_2)P_2(T_2)$ for

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$P_i(T_i) = (1-\alpha_i T_i)(1-\beta_i T_i)$ and

$$a_2(T_1,T_2) = \alpha_1\beta_1\alpha_2\beta_2(\alpha_2+\beta_2)T_1^2T_2^3 - \alpha_1\beta_1\alpha_2\beta_2T_1^2T_2^2 - \alpha_2\beta_2(\alpha_1+\beta_1)T_1T_2^2 + 1,$$
$$b_1(T_1,T_2) = \alpha_1^2\beta_1^2\alpha_2\beta_2T_1^4T_2^2 - \alpha_1\beta_1(\alpha_2+\beta_2)T_1^2T_2 - \alpha_1\beta_1T_1^2 + (\alpha_1+\beta_1)T_1.$$

The index 2 in a_2 (respectively the index 1 in b_1) is there to remind us that the monomials composing the polynomial are of the form $T_1^{a_1}T_2^{a_2}$ with $a_1 \leqslant a_2$ (respectively $T_1^{b_1}T_2^{b_2}$ with $b_1 > b_2$). The polynomial $P(T_1,T_2)$ is symmetric in the indices $1, 2$, hence we can also write $P(T_1,T_2) = a_1(T_1,T_2)P_2(T_1) + b_2(T_1,T_2)P_1(T_2)$ where $a_1(T_1,T_2)$ (respectively $b_2(T_1,T_2)$) is obtained from $a_2(T_1,T_2)$ (respectively $b_1(T_1,T_2)$) by swapping all the indices. Therefore,

$$\begin{aligned}P(T_1,T_2)^2 &= a_1a_2P_1P_2 + a_2P_1b_2P_1 + a_1P_2b_1P_2 + b_1b_2P_1P_2\\ &= a_1a_2P_1P_2 + (P-b_1P_2)b_2P_1 + (P-b_2P_1)b_1P_2 + b_1b_2P_1P_2\\ &= (a_1a_2 - b_1b_2)P_1P_2 + P(b_2P_1 + b_1P_2)\\ &= P(1-\alpha_1\beta_1\alpha_2\beta_2T^2)P_1P_2 + P(b_2P_1+b_1P_2).\end{aligned}$$

We are going to use the handy identity

$$P(T_1,T_2) = (1-\alpha_1\beta_1\alpha_2\beta_2T^2)P_1(T_1)P_2(T_2) + (b_2(T_1,T_2)P_1(T_1) + b_1(T_1,T_2)P_2(T_2)).$$

The class of $\omega_{\breve{\mathsf{g}}^{[\mathfrak{p}_i]}}$ is zero in $\mathbb{H}^2(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger\mathrm{DR}^\bullet_c(\mathcal{F}^{(\ell,x)}))$, hence there are overconvergent cuspforms $\mathsf{g}_j^{(i)} \in S^\dagger_{s_{\tau_j}\cdot(\ell,x)}(K;E_\wp)$ such that $\breve{\mathsf{g}}^{[\mathfrak{p}_i]} = d_1^{\ell_1-1}(\mathsf{g}_1^{(i)}) + d_2^{\ell_2-1}(\mathsf{g}_2^{(i)})$. Furthermore, $d_1^{1-\ell_1}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}$ is overconvergent by Corollary 4.7. It follows we can write $P(V(p))\breve{\mathsf{g}}$ as

$$\begin{aligned}P(V(p))\breve{\mathsf{g}} &= (1-\alpha_1\beta_1\alpha_2\beta_2V(p)^2)\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]} + b_2(V(\mathfrak{p}_1),V(\mathfrak{p}_2))\breve{\mathsf{g}}^{[\mathfrak{p}_1]} + b_1(V(\mathfrak{p}_1),V(\mathfrak{p}_2))\breve{\mathsf{g}}^{[\mathfrak{p}_2]}\\ &= d_1^{\ell_1-1}(\mathsf{h}) + d_1^{\ell_1-1}(\mathsf{h}_1) + d_2^{\ell_2-1}(\mathsf{h}_2),\end{aligned}$$

where $\mathsf{h} = (1-\alpha_1\beta_1\alpha_2\beta_2V(p)^2)d_1^{1-\ell_1}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}$, $\mathsf{h}_1 = b_2\mathsf{g}_1^{(1)} + b_1\mathsf{g}_1^{(2)}$ and $\mathsf{h}_2 = b_2\mathsf{g}_2^{(1)} + b_1\mathsf{g}_2^{(2)}$.

Proposition 5.11. *Let $L/\mathbb{Q}$ be a real quadratic extension and $g \in S^\dagger_{\ell,x}(K,L;E_\wp)$ an overconvergent cuspform whose class ω_g in $\mathbb{H}^2\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G_L), j^\dagger\mathrm{DR}^\bullet_c(\mathcal{F}^{(\ell,x)})\big)$ is trivial. By Theorem 4.5 there are p-adic modular forms $g_j \in S^\dagger_{s_j\cdot(\ell,x)}(K;E_\wp)$ for $j=1,2$, such that $g = d_1^{\ell_1-1}(g_1) + d_2^{\ell_2-1}(g_2)$, which we can use to explicitly construct sections $G_j \in H^0\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G_L), j^\dagger\big(\mathcal{F}^{(\ell,x)}\otimes\Omega^1_{\tau_j}\big)\big)$, $j=1,2$, that satisfy the equation $\omega_g = \nabla(G_1+G_2)$ in $H^0\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G_L), j^\dagger\big(\mathcal{F}^{(\ell,x)}\otimes\Omega^2\big)\big)$.*

Proof. For $j=1,2$, let ω_j,η_j be a local basis of the τ_j-part of the first de Rham cohomology of the universal abelian surface. Set $v_j^{(a,b)} = \omega_j^a\eta_j^b$, $w_j = \omega_j\wedge\eta_j$ and consider the sections

$$\begin{aligned}G_1 \;=\; &\sum_{i=0}^{\ell_1-2}(-1)^i\frac{(\ell_1-2)!}{(\ell_1-2-i)!}d_1^{\ell_1-2-i}(g_1)\\ &\times\big(w_2^{\frac{2-n-\ell_2}{2}}\otimes v_2^{(\ell_2-2,0)}\otimes w_1^{\frac{2-n-\ell_1}{2}}\otimes v_1^{(\ell_1-2-i,i)}\big)\otimes\frac{dq_2}{q_2}.\end{aligned}$$

$$G_2 = -\sum_{i=0}^{\ell_2-2}(-1)^i \frac{(\ell_2-2)!}{(\ell_2-2-i)!} d_1^{\ell_2-2-i}(g_2) \times \left(w_1^{\frac{2-n-\ell_1}{2}} \otimes v_1^{(\ell_1-2,0)} \otimes w_2^{\frac{2-n-\ell_2}{2}} \otimes v_2^{(\ell_2-2-i,i)}\right) \otimes \frac{dq_1}{q_1},$$

of $H^0\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G_L), j^\dagger\big(\mathcal{F}^{(\ell,x)} \otimes \Omega^1\big)\big)$. Differentiating them we obtain telescopic sums which collapse to

$$\nabla(G_j) = d_j^{\ell_j-1}(g_j) \bigotimes_{c=1}^{2} \left(w_c^{\frac{2-n-\ell_c}{2}} \otimes v_c^{(\ell_c-2,0)}\right) \otimes \left(\frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2}\right).$$

Therefore, $\omega_g = \nabla(G_1) + \nabla(G_2)$ as claimed. □

It follows that there are sections $G_{\mathsf{h}}, G_{\mathsf{h}_1}, G_{\mathsf{h}_2}$ associated with h, h_1, h_2 respectively, that satisfy $P(p^{-t_L}\mathrm{Fr}_p)\omega_{\mathsf{g}} = \nabla(G_{\mathsf{h}} + G_{\mathsf{h}_1} + G_{\mathsf{h}_2})$ since $\mathrm{Fr}_p = p^{t_L}V(p)$ in cohomology (Lemma 4.6). The pullback by the morphism $\xi : \mathrm{Sh}_K(G_L^*) \to \mathrm{Sh}_K(G_L)$ gives $P(p^{-t_L}\mathrm{Fr}_p)\omega_{\xi^*\mathsf{g}} = \nabla(G_{\xi^*\mathsf{h}} + G_{\xi^*\mathsf{h}_1} + G_{\xi^*\mathsf{h}_2})$ and to land in the right cohomology group we need to change the central character using the isomorphism $\Psi = \Psi_{x,|\ell-t_L|}$. Lemma 2.6 implies

$$P(p^{x-\ell}\mathrm{Fr}_p)\omega_{\Psi\xi^*\mathsf{g}} = \nabla(G_{\Psi\xi^*\mathsf{h}} + G_{\Psi\xi^*\mathsf{h}_1} + G_{\Psi\xi^*\mathsf{h}_2}).$$

We set $G = G_{\Psi\xi^*\mathsf{h}} + G_{\Psi\xi^*\mathsf{h}_1} + G_{\Psi\xi^*\mathsf{h}_2}$ and we let $\varepsilon_\ell : \bigotimes_\tau (\mathbb{H}^1_\tau)^{\ell_\tau-2} \to \bigotimes_\tau \mathrm{Sym}^{\ell_\tau-2}\mathbb{H}^1_\tau$ be the symmetrization projector which identifies the target sheaf with a subsheaf of the first. Finally, if we set $Q(T) = P(p^{x-\ell}T)$, then the cohomology class $\upsilon^*\tilde{\omega}$ is represented by $[\omega, \varepsilon_\ell G]$ in $\tilde{H}^{|\ell|-2}_{\mathrm{f},Q}\big(\mathscr{A}^{|\ell|-4}, |\ell|-2-s\big)$.

Proposition 5.12. *The class $\nu^*(\tilde{\varphi}_1^*\tilde{\omega})$ is represented by $[0, \tilde{\varphi}_1^*\varepsilon_\ell G]$ in $\tilde{H}^{|\ell|-2}_{\mathrm{f},Q}\big(\mathscr{E}^\gamma, |\ell|-2-s\big)$ and the image of $\tilde{\varphi}_1^*\varepsilon_\ell G$ under the unit-root splitting is equal to the p-adic modular form*

$$\mathrm{Spl}_{\mathrm{ur}}(\tilde{\varphi}_1^*\varepsilon_\ell G) = (-1)^s s!\Psi_{w,k-1+s}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big)$$

in $S^{\text{p-adic}}_{k,k-1+s}(K', E_\wp)$.

Proof. The class $\nu^*(\tilde{\varphi}_1^*\tilde{\omega}) = \tilde{\varphi}_1^*\upsilon^*(\tilde{\omega}) = [0, \tilde{\varphi}_1^*\varepsilon_\ell G]$ because $\varphi_1^*\omega = 0$ as a section of $\varphi_1^*\big(\mathcal{F}^{(\ell,\ell-t_L)}_{G_L^*} \otimes \Omega^2\big) = 0$. The diagonal morphism $\tilde{\varphi}_1 : \mathscr{E}^\gamma \to \mathscr{A}^{|\ell|-4}$ is a map of $\mathscr{X}_K$-schemes, so the pullback $\tilde{\varphi}_1^* : H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{A}_\kappa^{|\ell|-4}) \to H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{E}_\kappa^\gamma)$ is compatible with the pullbacks between the terms of the Leray spectral sequences for $\mathscr{A}^{|\ell|-4} \to \mathscr{X}_K \to \mathrm{Spec}\mathcal{O}_{E_\wp}$ and $\mathscr{E}^\gamma \to \mathscr{X}_K \to \mathrm{Spec}\mathcal{O}_{E_\wp}$. Since $\zeta : \mathscr{Y}_{K'} \to \mathscr{X}_K$ is a finite morphism, we have an induced map

$$\tilde{\varphi}_1^* : \mathbb{H}^1\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}; j^\dagger\mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t_L)}_{G_L^*})\big) \longrightarrow \mathbb{H}^1\big(\mathscr{S}^{\mathrm{tor}}_{K',\mathrm{rig}}; j^\dagger\mathrm{DR}^\bullet(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}})\big).$$

It is possible to describe explicitly the pullback $\tilde{\varphi}_1^*\varepsilon_\ell G$ as in [7, Proposition 2.9] and a direct calculation reveals that

$$\begin{aligned}\mathrm{Spl}_{\mathrm{ur}}\zeta^*(\varepsilon_{\underline{\ell}}G_j) &= (-1)^s s!\zeta^*\Psi_{x,|\ell-t_L|}\big(d_1^{\ell_1-2-s}(\xi^*\mathsf{h}) + d_1^{\ell_1-2-s}(\xi^*\mathsf{h}_1) + d_2^{\ell_2-2-s}(\xi^*\mathsf{h}_2)\big)\\ &= (-1)^s s!\Psi_{x-s-1,k-1+s}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big)\\ &= (-1)^s s!\Psi_{w,k-1+s}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big),\end{aligned}$$

as p-adic modular forms. □

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Remark. We proved that the image of $\Upsilon(\omega)$ under $H^{|\ell|-3}_{\rm rig}(\mathscr{W}_{\gamma,\kappa}) \to H^{|\ell|-3}_{\rm rig}(\mathscr{E}^{\gamma}_{\kappa})$ is given by $[\tilde{\varphi}^*_1\varepsilon_\ell G] \in H^1_{\rm par}\big(\mathrm{Sh}^{\rm tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \subset \mathbb{H}^1\big(\mathscr{S}^{\rm tor}_{K',\rm rig}, j^\dagger \mathrm{DR}^\bullet_c(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}})\big)$.

Lemma 5.13. *Let* $(\omega,\eta) \in H^1_{\rm par}\big(\mathrm{Sh}^{\rm tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \times H^1_{\rm par}\big(\mathrm{Sh}^{\rm tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big)$ *be a pair such that* $\mathrm{Fr}_p\eta = \alpha\eta$ *for* α *a* p*-adic unit, then* $\langle \omega,\eta\rangle = \langle e_{\rm n.o.}\omega,\eta\rangle$.

Proof. We have the equalities of operators $\mathrm{Fr}_p = pV(p)$ and $U_0(p) = p^r U(p)$, therefore the computation

$$\begin{aligned}\langle \omega,\eta\rangle &= \alpha^{-1}\langle \omega, \mathrm{Fr}_p\eta\rangle = \alpha^{-1}\mathrm{Fr}_p\langle \mathrm{Fr}_p^{-1}\omega,\eta\rangle\\ &= \alpha^{-1}p^{r+1}\langle \mathrm{Fr}_p^{-1}\omega,\eta\rangle = \alpha^{-1}p^{r+1}\langle p^{-1}U(p)\omega,\eta\rangle = \alpha^{-1}\langle U_0(p)\omega,\eta\rangle,\end{aligned}$$

implies that $\langle\omega,\eta\rangle = \lim_{n\to\infty}\alpha^{-n!}\langle U_0(p)^{n!}\omega,\eta\rangle = \langle e_{\rm n.o.}\omega,\eta\rangle$. □

Theorem 5.14. *Let* $L/\mathbb{Q}$ *be a real quadratic extension. Consider* $\breve{\mathsf{g}} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; \overline{\mathbb{Q}})$ *a cuspform of either parallel weight* $\ell = 2t_L$ *or non-parallel weight* $\ell > 2t_L$ *over* L *and* $\breve{\mathsf{f}} \in S_{k,w}(V_{11}(\mathfrak{A}); \overline{\mathbb{Q}})$ *an elliptic eigenform for the good Hecke operators. Suppose their weights are balanced and choose a prime* p *splitting in* F*,* $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$*, coprime to* $\mathfrak{A}$*, such that both cuspforms are* p*-nearly ordinary and the cycle* $\Delta_{\ell,k}$ *is defined. Then*

$$\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega\cup\pi_2^*\eta) = s!(-1)^s\frac{1-\alpha_1\beta_1\alpha_2\beta_2(\alpha_{\mathsf{f}^*}^{-1}p^{-1})^2}{\prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\alpha_{\mathsf{f}^*}^{-1}p^{-1})}\frac{\langle e_{\rm n.o.}\zeta^*(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}), \breve{\mathsf{f}}^*\rangle}{\langle \mathsf{f}^*,\mathsf{f}^*\rangle},$$

where ω *and* η *are the classes in Definition 5.5 and* $s = \frac{|\ell|-k-2}{2}$.

Proof. Recall that (28) states that $\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega\cup\pi_2^*\eta) = \langle \Upsilon(\omega),\eta\rangle_{\rm dR}$, where the Poincaré pairing takes values in $E_\wp(-(\gamma+1))$, a one dimensional space on which Fr_p acts as multiplication by $p^{\gamma+1}$. The isomorphism $\iota: H^{|\ell|-3}_{\rm dR}(W_\gamma) \xrightarrow{\sim} H^{|\ell|-2}_{{\rm f},Q}(\mathscr{W}_\gamma, |\ell|-2-s)$ is given by $\iota(-) = [0, Q(\mathrm{Fr}_p)(-)]$, therefore $Q(\mathrm{Fr}_p)\Upsilon(\omega) = \tilde{\varphi}_1^*(\epsilon_\ell G)$. On the one hand, $\langle Q(\mathrm{Fr}_p)\Upsilon(\omega),\eta\rangle_{\rm dR} = Q(p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w})\langle\Upsilon(\omega),\eta\rangle_{\rm dR}$, because we computed in (23) that $\mathrm{Fr}_p\big(\nu^*\varphi_2^*\eta\big) = \alpha_{\mathsf{f}^*}p^{w-1}\big(\nu^*\varphi_2^*\eta\big)$. On the other hand,

$$\begin{aligned}\langle Q(\mathrm{Fr}_p)\Upsilon(\omega),\eta\rangle_{\rm dR} &= \langle\tilde{\varphi}_1^*(\epsilon_\ell G),\eta\rangle_{\rm dR}\\ &= s!(-1)^s\frac{\langle \Psi_{w,k-1}e_{\rm n.o.}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big), \Psi_{w,k-1}(\breve{\mathsf{f}}^*)\rangle}{\langle\mathsf{f}^*,\mathsf{f}^*\rangle}\\ &= s!(-1)^s\frac{\langle e_{\rm n.o.}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big), \breve{\mathsf{f}}^*\rangle}{\langle\mathsf{f}^*,\mathsf{f}^*\rangle}.\end{aligned}$$

Indeed, the class of $\tilde{\varphi}_1^*(\epsilon_\ell G)$ in $\mathbb{H}^1\big(\mathscr{S}^{\rm tor}_{K',\rm rig}, j^\dagger\mathrm{DR}^\bullet_c(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}})\big)$ is represented by an overconvergent cuspform whose nearly ordinary projection is equal to $e_{\rm n.o.}\mathrm{Spl}_{\rm ur}\tilde{\varphi}_1^*(\epsilon_\ell G)$ (see [7, Lemma 2.7]), then Lemma 5.13 justifies the computation.

For $j = 1,2$ the nearly ordinary projection $e_{\rm n.o.}\zeta^* d_j^{\ell_j-2-s}(\mathsf{h}_j) = e_{\rm n.o.}\zeta^*[d_j^{\ell_j-2-s}(b_2\mathsf{g}_j^{(1)} + b_1\mathsf{g}_j^{(2)})]$ is zero thanks to Lemma 3.10 because the cuspform $\mathsf{g}_j^{(i)}$ is $\mathfrak{p}_i$-depleted,

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$i = 1, 2$ and b_ι, $\iota = 1, 2$, can be written as a polynomial only in the variables $V(p)$, $V(\mathfrak{p}_\iota)$ divisible by $V(\mathfrak{p}_\iota)$. Moreover,

$$\begin{aligned} e_{\mathsf{f}^*,\mathrm{n.o.}}\zeta^*(d_1^{\ell_1-2-s}(\mathsf{h})) &= e_{\mathsf{f}^*,\mathrm{n.o.}}\zeta^*(1-\alpha_1\beta_2\alpha_2\beta_2 V(p)^2)(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}) \\ &= (1-\alpha_1\beta_1\alpha_2\beta_2(\alpha_{\mathsf{f}^*}^{-1}p^{-1})^2)e_{\mathsf{f}^*,\mathrm{n.o.}}\zeta^*(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}). \end{aligned}$$

Finally, the last bit we need to unravel is the polynomial $Q(p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w})$; we compute

$$\begin{aligned} Q(p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w}) &= \prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\, p^{x-\ell}p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w}) \\ &= \prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}p^{-n+\frac{m}{2}-1}) = \prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}p^{-1}) \end{aligned}$$

since under our assumptions $2n = m$. Hence, putting all together

$$\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega\cup\pi_2^*\eta) = s!(-1)^s\frac{1-\alpha_1\beta_1\alpha_2\beta_2(\alpha_{\mathsf{f}^*}^{-1}p^{-1})^2}{\prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}p^{-1})}\frac{\langle e_{\mathrm{n.o.}}\zeta^*(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}),\breve{\mathsf{f}}^*\rangle}{\langle\mathsf{f}^*,\mathsf{f}^*\rangle}.$$

□

Remark. The right-hand side of the equality in Theorem 5.14 is independent of the particular choice of small enough levels K, K' because of the normalization of the cohomology class η (Definition 5.5).

Corollary 5.15. *Let $L/\mathbb{Q}$ be a real quadratic field and (ℓ, k) a balanced triple. Let p be a prime splitting in L for which the generalized Hirzebruch–Zagier cycle $\Delta_{\ell,k}$ is defined. Then for all* $(\mathrm{P},\mathrm{Q}) \in \mathcal{C}_{\mathrm{bal}}^{\theta,\bar{r}}(\ell,k)$ *we have*

$$_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}},\breve{\mathscr{F}})(\mathrm{P},\mathrm{Q}) = \frac{\pm 1}{s!\mathsf{E}(\mathsf{f}_{\mathrm{Q}}^*)}\frac{\mathscr{E}_p(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)}{\mathscr{E}_{0,p}(\mathsf{g}_{\mathrm{P}},\mathsf{f}_{\mathrm{Q}}^*)}\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega_{\mathrm{P}}\cup\pi_2^*\eta_{\mathrm{Q}}).$$

Proof. It follows from the formula (16), Proposition 3.6 and Theorem 5.14. □

6. An application to Bloch–Kato Selmer groups

Let A be an elliptic curve over L of conductor $\mathfrak{Q}$ and B a rational elliptic curve of conductor $\mathfrak{N}$, both without complex multiplication over $\overline{\mathbb{Q}}$. We denote by $(\mathrm{M}_{A,B})_p$ the Galois representation $\mathrm{AsV}_p(A)(-1)\otimes_{\mathbb{Q}_p}\mathrm{V}_p(B)$ of the absolute Galois group of $\mathbb{Q}$. We can use Corollary 5.15 to give a criterion for the Bloch–Kato Selmer group $H_f^1(\mathbb{Q},(\mathrm{M}_{A,B})_p)$ to be of dimension one in terms of the non-vanishing of a value of one of our twisted triple product p-adic L-functions. This builds on the recent work of Liu [26], where he computes the dimension of $H_f^1(\mathbb{Q},(\mathrm{M}_{A,B})_p)$ assuming the non-vanishing of the étale Abel–Jacobi map of certain cycle $\Delta_{A,B}$.

Let $\mathsf{g}_A \in S_{2t_L,t_L}(V_1(\mathfrak{Q}); L; \mathbb{Q})$, $\mathsf{f}_B \in S_{2,1}(V_1(\mathfrak{N}); \mathbb{Q})$ be the newforms attached to A and B by modularity, π_A, σ_B the automorphic representations they respectively generate. Let p be a rational prime coprime to $\mathfrak{N}\cdot\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})\cdot d_{L/F}$, if $\mathsf{g}_A, \mathsf{f}_B$ are p-nearly ordinary we denote by $\mathscr{G}, \mathscr{F}$ the Hida families passing through the p-nearly ordinary stabilizations

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$\mathscr{G}_{\mathrm{P}_A} = \mathsf{g}_A^{(p)}$ and $\mathscr{F}_{\mathrm{Q}_B} = \mathsf{f}_B^{(p)}$. We recall some of the definitions in [26]. Let X be the minimal resolution of the Baily–Borel compactification of the Hilbert modular surface over L of Γ_0-level $\mathfrak{N}\cdot \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})$, Y the compactified modular curve of Γ_0-level $\mathfrak{N}\cdot \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})$ and $\zeta : Y \to X$ the diagonal morphism. According to Liu, there are idempotents $\mathscr{P}_A \in \mathrm{Corr}(X, X)$, $\mathscr{P}_B \in \mathrm{Corr}(Y, Y)$ acting as projectors

$$\mathscr{P}_{A,*} : H^*_{\mathrm{dR}}(X) \to H^2_{\mathrm{dR}}(X)[\pi_A], \quad \mathscr{P}_{B,*} : H^*_{\mathrm{dR}}(Y) \to H^1_{\mathrm{dR}}(Y)[\sigma_B].$$

The null-homologous cycle $\Delta_{A,B} \in \mathrm{CH}^2(X \times Y)\otimes \mathbb{Q}$ is defined as $\Delta_{A,B} = (\mathscr{P}_A \times \mathscr{P}_B)_*\Delta$ for $\Delta = \mathrm{graph}(\zeta)$. By spreading out we can consider smooth models $\mathscr{X}, \mathscr{Y}$ over $\mathbb{Z}_p$ for almost all p, and $\tilde{\mathscr{P}}_A \times \tilde{\mathscr{P}}_B \in \mathrm{Corr}(\mathscr{X}\times\mathscr{Y}, \mathscr{X}\times\mathscr{Y})$.

Corollary 6.1. *Suppose that $\mathfrak{N}$ and $\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})\cdot d_{L/\mathbb{Q}}$ are coprime ideals and that all the primes dividing $\mathfrak{N}$ split in L. For all but finitely many primes p that are split in L and such that $\mathsf{g}_A, \mathsf{f}_B$ are p-nearly ordinary we have*

$$_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}}, \breve{\mathscr{F}})(\mathrm{P}_A, \mathrm{Q}_B) \neq 0 \quad \Longrightarrow \quad \dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}, (\mathrm{M}_{A,B})_p) = 1,$$

where $\theta = -\mu + \mu' \in \mathbb{Z}[\mathrm{I}_L]$, $\bar{r} = -\mu$.

Proof. Let $\tilde{\varphi} : \mathscr{Y} \to \mathscr{X}\times\mathscr{Y}$ be the map $(\tilde{\zeta}, \mathrm{id}_{\mathscr{Y}})$, and set $\tilde{\Delta}_{A,B} = (\tilde{\mathscr{P}}_A \times \tilde{\mathscr{P}}_B)_*\tilde{\varphi}_*[\mathscr{Y}]$. For any $\omega \in H^2_{\mathrm{dR}}(X)[\pi_A]$, $\eta \in H^1_{\mathrm{dR}}(Y)[\sigma_B]$ and lifts $\tilde{\omega}, \tilde{\eta}$ to fp-cohomology we can compute

$$\begin{aligned}\mathrm{AJ}_p(\Delta_{A,B})(\pi_1^*\omega\cup\pi_2^*\eta) &= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\Delta}_{A,B}), \pi_1^*\tilde{\omega}\cup\pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}}\\ &= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{Y}]), (\mathscr{P}_A\times\mathscr{P}_B)^*(\pi_1^*\tilde{\omega}\cup\pi_2^*\tilde{\eta})\rangle_{\mathrm{fp}}\\ &= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{Y}]), \pi_1^*\tilde{\omega}\cup\pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} = \mathrm{tr}_{\mathscr{Y}}(\tilde{\zeta}^*\tilde{\omega}\cup\tilde{\eta})\end{aligned}$$

as in § 5.2.2. If $\alpha_1 : \mathscr{U}_0 \to \mathscr{X}$, $\alpha_2 : \mathscr{W}_0 \to \mathscr{Y}$ are the natural finite degeneracy maps, we know that $\mathrm{AJ}_p(\Delta_{2t_L,2})(\pi_1^*(\alpha_1^*\omega)\cup\pi_2^*(\alpha_2^*\eta)) = \mathrm{tr}_{\mathscr{W}_0}(\tilde{\zeta}^*(\alpha_1^*\tilde{\omega})\cup(\alpha_2^*\tilde{\eta}))$. Therefore,

$$\mathrm{AJ}_p(\Delta_{2t_L,2})(\pi_1^*(\alpha_1^*\omega)\cup\pi_2^*(\alpha_2^*\eta)) = \deg(\alpha_1)\deg(\alpha_2)\cdot \mathrm{AJ}_p(\Delta_{A,B})(\pi_1^*\omega\cup\pi_2^*\eta)$$

and the LHS vanishes if and only if the RHS vanishes. It follows that the non-vanishing of the p-adic L-function implies the non-vanishing of the syntomic Abel–Jacobi image of both $\Delta_{2t_L,2}$ and $\Delta_{A,E}$ by Corollary 5.15, which in turn forces the non-vanishing of the p-adic étale Abel–Jacobi image of the cycle $\Delta_{A,E}$ [2, § 3.4]. Then Liu's theorem [26, Theorem 1.5] gives the dimension of the Bloch–Kato Selmer group. □

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