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# SPHERICALIZATION AND FLATTENING PRESERVE UNIFORM DOMAINS IN NONLOCALLY COMPACT METRIC SPACES

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**Abstract**

The main aim of this paper is to investigate the invariant properties of uniform domains under flattening and sphericalization in nonlocally compact complete metric spaces. Moreover, we show that quasi-Möbius maps preserve uniform domains in nonlocally compact spaces as well.

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## 1. Introduction

The present investigation is motivated by the work of John [17] and Martio and Sarvas in [21] on uniform domains in Euclidean spaces and the later investigation on this topic. After the appearance and prominent roles played by these articles, many other characterizations of uniform domains were established by a number of researchers (see [11, 13, 20, 25–27]). The importance of this class of domains in function theory is well documented (see, for example, [11, 12, 23]). It is worth recalling that their main motivation in [17, 21] for studying these domains was to establish global injectivity properties for locally injective mappings. Moreover, uniform subdomains of the Heisenberg groups, as well as more general Carnot groups, have become a focus of study (see, for example, [6, 7, 10, 14]). Bonk *et al.* [2] introduced the notion of uniformity in the locally compact metric space setting and obtained that there is a two way correspondence between uniform spaces and Gromov hyperbolic proper geodesic spaces.

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In [21], Martio and Sarvas proved that quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ preserve uniform domains. Then it follows from the equivalent condition of uniform domains introduced by Martio [20] that uniformity of domains in $\mathbb{R}^n$ is preserved by quasi-Möbius maps. Further, this result has been generalized to Banach spaces by Väisälä [25]. Recently, Xie [30] investigated the invariance of uniform domains under quasisymmetric and quasi-Möbius maps between locally compact metric spaces. The main tool in his proof relies on the concept of metric space *inversion* (or flattening) which was first introduced by Buckley *et al.* [5]. This class of conformal deformation is dual to *sphericalization* in a certain sense. Simultaneously, they also obtained the preservations of uniformity under the flattening and sphericalization transformations in the set-up of locally compact metric spaces.

It turns out that the notion of metric space inversion is very useful in order to extend the study of bounded metric spaces to the study of unbounded spaces, and also to reduce the setting of quasisymmetric mappings to quasi-Möbius maps. For instance, Bonk *et al.* [2] characterized bounded uniform proper domains of $\mathbb{R}^n$ in terms of Gromov hyperbolicity of quasihyperbolic metrics and the quasisymmetric correspondence between Euclidean boundaries and Gromov boundaries. With the aid of the inversion transformations, Väisälä subsequently generalized this result, in [29], to Banach spaces for arbitrary uniform domains by means of quasi-Möbius equivalence of the norm boundary and Gromov boundary. Moreover, Herron *et al.* [15] demonstrated that, in an annular convex metric space, uniform domains are precisely those Gromov hyperbolic domains whose quasiconformal structure on the Gromov boundary agrees with that on the metric boundary.

On the other hand, let us remark here that flattening and sphericalization in metric spaces has recently attained considerable interest in analysis of quasimetric spaces, for instance, in questions related to Poincaré inequality, Loewner condition, quasiconvexity, doubling and Ahlfors regularity of measures (see, for example, [8, 9, 18, 31]). The main purpose of this paper is to generalize the results of Väisälä and Xie and to consider the behavior of uniform domains in nonlocally compact metric spaces under quasi-Möbius maps. It is worthwhile to mention that Buckley and Herron [4] obtained several characterizations for uniformity in nonlocally compact metric spaces. More recently, the third author and Rasila studied the connection between uniform domains and quasi-Möbius maps in Banach spaces and answered two related questions raised by Väisälä in [27].

This article is an attempt to prove further results in this direction. Our first main result is to investigate the invariance of uniform domains under flattening of metric spaces. Note that we **do not** assume local compactness in our paper. We refer to Section 2 for basic information including notation, definitions, terminology and some auxiliary lemmas.

Theorem 1.1. *Suppose that $X$ is a complete metric space and $\Omega \subset X$ is a domain (open and connected set) with* $\operatorname{card}(\partial\Omega) \geq 2$ *and $p \in \partial\Omega$.*

(1) *If $\Omega$ is $c$-uniform, then $(\Omega, d_p)$ is $c_1$-uniform with $c_1 = c_1(c)$.*

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(2) *If* $\Omega$ *is bounded and* $(\Omega, d_p)$ *is c-uniform, then* $\Omega$ *is* $c_2$*-uniform with* $c_2 = C_0(\mathrm{diam}(\Omega)/b(p))$*, where* $C_0 = C_0(c)$ *and* $b(p) = \sup\{d(p,q) | q \in \partial\Omega\}$.

(3) *If* $\Omega$ *is unbounded and* $(\Omega, d_p)$ *is c-uniform, then* $\Omega$ *is* $c_3$*-uniform with* $c_3 = c_3(c)$.

REMARK 1.2. We remark that Theorem 1.1 coincides with [5, Theorem 5.1(a)] in the setting of locally compact metric spaces. As is well known, the local compactness assumption assures the existence of quasihyperbolic geodesics. But quasihyperbolic geodesics need not exist in a general metric space (see [24, 2.9]). In order to overcome this disadvantage, in this paper we substitute quasihyperbolic geodesics by *short arcs*. The class of short arcs was introduced by Väisälä [28], and we see that the existence of such a class of arcs is obvious in metric spaces. This idea is very useful in related research (see [4, 15, 19, 27]).

From Theorem 1.1, we also conclude that the sphericalization transformations preserve uniform domains because sphericalization can be realized as a special case of flattening (see more information in Section 2).

COROLLARY 1.3. *Suppose that X is a complete metric space and* $\Omega \subset X$ *is a domain with* $\mathrm{card}\,(\partial\Omega) \geq 2$ *and* $p \in \partial\Omega$*. If* $\Omega$ *is unbounded, then* $(\Omega, d)$ *is c-uniform if and only if* $(\Omega, \widehat{d_p})$ *is* $c'$*-uniform, where the constants c and* $c'$ *depend only on each other.*

As an application to Corollary 1.3, we deduce the invariance of uniform domains for quasi-Möbius maps (see Section 2.2) in quasiconvex metric spaces.

THEOREM 1.4. *Suppose that* $X_1$ *and* $X_2$ *are c-quasiconvex complete metric spaces, that* $\Omega_i \subsetneq X_i$ *are proper domains with* $\partial\Omega_i$ *containing at least two points and that* $f : \Omega_1 \to \Omega_2$ *is a* $\theta$*-quasi-Möbius homeomorphism. If* $\Omega_1$ *is* $c_1$*-uniform, then* $\Omega_2$ *is* $c_2$*-uniform for some constant* $c_2$*. When* $\Omega_1$ *is unbounded, then the constant* $c_2$ *depends only on c,* $c_1$ *and* $\theta$.

REMARK 1.5. In the locally compact metric spaces setting, this result was considered by Xie [30].

The remaining part of this paper is organized as follows. Section 2 contains basic definitions and auxiliary lemmas that are used later in the discussion. In Section 3, we present the proof of Theorem 1.1(1), which we formulate as Theorem 3.2. The proofs of Theorems 1.1(2) and 1.1(3) are presented in Section 4, which we state as Theorems 4.4 and 4.5, respectively. Finally, the proof of Theorem 1.4 is presented in Section 5.

## 2. Preliminaries

**2.1. Notation.** In what follows, $(X, d)$ always denotes a complete metric space with the metric $d$. We often write the distance between $x$ and $y$ as $d(x, y)$ and the distance from a point $x$ to a set $A$ as $d(x, A)$. For $A \subset X$, $\partial A = X \setminus A$ denotes the metric boundary of $A$. We always assume that $\infty$ denotes an element not in $X$. The one-point extension of $X$ is the set $\dot{X} = X \cup \{\infty\}$. The topology of $\dot{X}$ consists of all open sets in $X$ and of all sets $U$ containing $\infty$ such that $\dot{X} \setminus U = X \setminus U$ is closed and bounded in $X$. The diameter of a set $A \subset X$ is denoted by diam $A$.

A *curve* in $X$ means a continuous map $\gamma : I \to X$ from an interval $I \subset \mathbb{R}$ to $X$. If $\gamma$ is an embedding of $I$, it is also called an *arc*. We write $\gamma : x \curvearrowright y$ if $\gamma$ is an arc joining $x$ and $y$. If needed, this notation also gives an orientation for $\gamma$ from $x$ to $y$. By convention, we also denote the image set $\gamma(I)$ of $\gamma$ by $\gamma$ itself. The *length* $\ell_d(\gamma)$ of $\gamma$ with respect to the metric $d$ is defined in an obvious way. Here the parameter interval $I$ is allowed to be open, closed or half-open. We denote the subarc of $\gamma$ between $x$ and $y$ by $\gamma[x, y]$. A metric space $(X, d)$ is called *rectifiably connected* if every pair of points in $X$ can be joined with a curve $\gamma$ in $X$ with $\ell(\gamma) < \infty$.

Next, we recall the definition of quasiconvex metric spaces from [16].

Definition 2.1. An arc $\alpha \subset X$ with endpoints $x$ and $y$ is said to be *c-quasiconvex* if there is $c \geq 1$ such that

$$\ell(\alpha) \leq cd(x, y).$$

A metric space $(X, d)$ is called *c-quasiconvex* if each pair of points can be joined by a $c$-quasiconvex arc.

Let the letters $A, B, C, \ldots$ denote positive numerical constants. Similarly, let $C(a, b, c, \ldots)$ denote universal positive functions of the parameters $a, b, c, \ldots$. Sometimes we write $C = C(a, b, c, \ldots)$ to emphasize the parameters on which $C$ depends and abbreviate $C(a, b, c, \ldots)$ to $C$. The notation $(x, y, z, w) \mapsto (x', y', z', w')$ means that $x, y, z, w$ are substituted by $x', y', z', w'$, respectively.

**2.2. Quasi-Möbius, quasisymmetric and bilipschitz.** Given a metric space $(X, d)$, the *cross ratio* $r(x, y, z, w)$ of each of the four distinct points $x, y, z, w \in X$ is defined as

$$r(x, y, z, w) = \frac{d(x, z)d(y, w)}{d(x, y)d(z, w)}.$$

It is often convenient to consider cross ratios also in the extended space $\dot{X}$. If $x, y, z, w$ are points in $\dot{X}$ and if one of the points $x, y, z, w$ is $\infty$, the cross ratio is defined by ignoring the factor which concerns the distance from $\infty$: for example,

$$r(x, y, z, \infty) = \frac{d(x, z)}{d(x, y)}.$$

Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces, let $X_0 \subset \dot{X}_1$ and let $f : (X_0, d_1) \to (\dot{X}_2, d_2)$ be a homeomorphism. Given a homeomorphism $\eta : [0, \infty) \to [0, \infty)$, we say that $f$ is *$\eta$-quasi-Möbius* if, for $x, y, z, w \in X_0$,

$$r(f(x), f(y), f(z), f(w)) \leq \eta(r(x, y, z, w)).$$

If $f$ preserves all cross ratios, it is called a *Möbius map*.

We say that $f$ is *$\eta$-quasisymmetric* if

$$\frac{d_2(f(x), f(z))}{d_2(f(x), f(y))} \leq \eta\Big(\frac{d_1(x, z)}{d_1(x, y)}\Big) \quad \text{whenever } x, y, z \in X_1.$$

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The mapping $f$ is said to be *L-bilipschitz* if there exists an $L \geq 1$ such that, for all $x, y \in X_1$,

$$\frac{1}{L} d_1(x, y) \leq d_2(f(x), f(y)) \leq L d_2(x, y).$$

Several mapping properties of these mappings are presented, for example, in [22, Theorem 2.2] and [27, Theorem 6.23].

**2.3. Quasihyperbolic distance, *h*-short arcs and uniform domains.** Let $\Omega \subset X$ be a proper domain with nonempty boundary. The *quasihyperbolic length* of a rectifiable curve $\gamma$ in $\Omega$ is the number $\ell_{k_\Omega}(\gamma)$ defined by

$$\ell_{k_\Omega}(\gamma) = \int_\gamma \frac{ds(z)}{\delta_\Omega(z)},$$

where $\delta_\Omega(z)$ denotes the distance from $z$ to the boundary $\partial\Omega$ of $\Omega$. For any $z_1$, $z_2$ in $\Omega$, the *quasihyperbolic distance* $k_\Omega(z_1, z_2)$ between $z_1$ and $z_2$ is defined by

$$k_\Omega(z_1, z_2) = \inf\{\ell_{k_\Omega}(\gamma) : \ \gamma : z_1 \curvearrowright z_2, \gamma \subset \Omega\}.$$

If $\gamma : x \curvearrowright y$ is a rectifiable curve in $\Omega \subsetneq X$, then (see [27])

$$\ell_{k_\Omega}(\gamma) \geq \log\Big(1 + \frac{\ell(\gamma)}{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}\Big).$$

Moreover,

$$k_\Omega(x, y) \geq j_\Omega(x, y) =: \log\Big(1 + \frac{d(x, y)}{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}\Big) \geq \Big|\log \frac{\delta_\Omega(x)}{\delta_\Omega(y)}\Big|, \tag{2-1}$$

for all $x, y \in \Omega$.

We also record the following useful result.

Lemma A [16, Lemma 3.8]. *Suppose that $\Omega$ is a domain in a $c$-quasiconvex metric space $(X, d)$ with $\partial\Omega \neq \emptyset$ and that $x$, $y \in \Omega$. If $d(x, y) \leq (1/3c)\delta_\Omega(x)$ or $k_\Omega(x, y) \leq 1$, then*

$$\frac{1}{2} \frac{d(x, y)}{\delta_\Omega(x)} < k_\Omega(x, y) \leq 3c \frac{d(x, y)}{\delta_\Omega(x)}.$$

Next, we recall the definition of short arcs presented in [29] or [4].

Definition 2.2. An arc $\alpha : x \curvearrowright y$ in $\Omega$ is *h-short* for $h \geq 0$ if

$$\ell_{k_\Omega}(\alpha) \leq k_\Omega(x, y) + h.$$

We see from [28, Lemma 2.4] that every subarc of an $h$-short arc is also $h$-short. The existence of such arcs follows from the fact that $(\Omega, k_\Omega)$ is a length metric space.

Definition 2.3. Let $\Omega$ be a domain in $X$ with $\partial\Omega \neq \emptyset$, and let $c \geq 1$.

(1) $\gamma : [0, 1] \to \Omega$ is a *c-uniform arc* if $\gamma$ satisfies the following two conditions.

(a) Turning condition: $\ell(\gamma) \leq c d(\gamma(0), \gamma(1))$.

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(b) Cigar condition: for $t \in [0,1]$, $\min\{\ell(\gamma[0,t]), \ell(\gamma[t,1])\} \le c\delta_\Omega(\gamma(t))$.

(2) $\Omega$ is a *$c$-uniform domain* if every pair of points in $\Omega$ can be joined by a $c$-uniform arc and, moreover, $\Omega$ is *uniform* if it is $c$-uniform for some $c \ge 1$.

The following result concerning the uniformity of short arcs in uniform domains will be frequently used in what follows.

Lemma 2.4. *Assume that $(X,d)$ is a complete metric space and that $\Omega \subset X$ is a $c$-uniform domain with $\partial\Omega \ne \emptyset$.*

(1) *Let $x, y \in \Omega$ with $h_0 = \min\{k_\Omega(x,y), 1\}$ and $0 < h \le h_0$. If $\alpha : x \curvearrowright y$ in $\Omega$ is an $h$-short arc, then $\alpha$ is $\mu_1$-uniform with $\mu_1 = \mu_1(c)$.*

(2) *Suppose that $\gamma$ is an $h$-short $\mu_2$-uniform arc in $\Omega$ with $h \le 1$. For $u, v$ in $\gamma$, if $k_\Omega(u,v) \ge h$, then the subarc $\gamma[u,v]$ of $\gamma$ is also $h$-short $\mu_2$-uniform.*

Proof. We observe that $\delta_\Omega(x) > 0$ for each $x \in \Omega$ and thus $\Omega$ is locally complete which shows that statement (1) follows from [4, Corollary 3.3]. Moreover, it is not difficult to see that (2) is a direct consequence of (1) because every subarc of an $h$-short arc is also $h$-short. □

We remark that the constant $\mu_1$ in Lemma 2.4, as indicated in [4, Corollary 3.3], can be taken as $3\exp(200c^6)$. Finally, we conclude this part with the following lemma.

Lemma B [2, Lemma 2.13]. *If $\Omega$ is a $c$-uniform domain in $(X,d)$ with $\partial\Omega \ne \emptyset$, then, for all $x, y \in \Omega$,*

$$k_\Omega(x,y) \le 4c^2 \log\left(1 + \frac{d(x,y)}{\min\{\delta_\Omega(x), \delta_\Omega(y)\}}\right).$$

**2.4. Flattening and sphericalization.** The original idea of *sphericalization* and *flattening* (or *inversion*) in metric spaces comes from the work of Bonk and Kleiner [3] in defining a metric on the one-point compactification of an unbounded locally compact metric space. The first class of deformation, sphericalization, is a generalization of the deformation from the Euclidean distance on $\mathbb{R}^n$ to the chordal distance on $\mathbb{S}^n$. The second class of flattening deformation is a generalization of inversion on punctured $\mathbb{S}^n$.

*2.4.1. Metric space inversions.* Let $(X,d)$ be a metric space. For a fixed point $p \in X$, define

$$I_p(X) = \begin{cases} X \setminus \{p\} & \text{if } X \text{ is bounded,} \\ (X \setminus \{p\}) \cup \{\infty\} & \text{if } X \text{ is unbounded} \end{cases}$$

and

$$f_p(x,y) = f_p(y,x) = \begin{cases} \dfrac{d(x,y)}{d(x,p)d(y,p)} & \text{if } x, y \in X \setminus \{p\}, \\ \dfrac{1}{d(x,p)} & \text{if } y = \infty \text{ and } x \in X \setminus \{p\}, \\ 0 & \text{if } x = \infty = y. \end{cases}$$

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Let us recall the following useful result from [5, Lemma 3.2].

LEMMA C [5, Lemma 3.2]. *There is a distance function $d_p$ on $I_p(X)$ such that:*

(1) *for all $x, y \in I_p(X)$,*

$$\tfrac{1}{4} f_p(x, y) \le d_p(x, y) \le f_p(x, y);$$

(2) *the identity map $(I_p(X), d) \longrightarrow (I_p(X), d_p)$ is a $16t$-quasi-Möbius homeomorphism.*

*2.4.2. Metric space sphericalizations.* We need the following metric space sphericalizations due to Bonk and Kleiner [3], which have been discussed also by Buckley *et al.* [5]. Assume that $(X, d)$ is unbounded and $p$ is a fixed point in $X$. Set

$$S_p(X) = X \cup \{\infty\}.$$

We define a function $s_p$ as follows.

$$s_p(x, y) = s_p(y, x) = \begin{cases} \dfrac{d(x, y)}{[1 + d(x, p)][1 + d(y, p)]} & \text{if } x, y \in X, \\ \dfrac{1}{1 + d(x, p)} & \text{if } y = \infty \text{ and } x \in X, \\ 0 & \text{if } x = \infty = y. \end{cases}$$

Similarly, we have the following lemma.

LEMMA D. *There exists a metric $\widehat{d_p}$ on $S_p(X)$ such that:*

(1) [3, (2.3)], *see also* [5] *for all $x, y \in S_p(X)$,*

$$\tfrac{1}{4} s_p(x, y) \le \widehat{d_p}(x, y) \le s_p(x, y);$$

(2) [3, *Lemma 2.2*] *the identity map $(X, d) \longrightarrow (X, \widehat{d_p})$ is $16t$-quasi-Möbius.*

Moreover, it is mentioned in [5] that the notion of inversion is dual to the sphericalization in the sense of the following two ideas: first, sphericalization can be realized as a special case of inversion [5, 3.8]; second, repeated inversions using appropriate points produces a space which is bilipschitz equivalent to the original space (see [5, Properties 3.7, 3.8, 3.9]). Sphericalization and flattening have a lot of applications in the area of geometric function theory and analysis on metric spaces (see [1, 5, 15, 18, 30, 31]).

## 3. Flattening and uniform domains I

The goal of this section is to prove that the image of a uniform domain under flattening is still uniform in nonlocally compact spaces. The proof of this result relies on the following lemma.

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Lemma 3.1. *Suppose that $X$ is a complete metric space and that $\Omega \subset X$ is a domain with* $\operatorname{card}(\partial\Omega) \geq 2$. *Let $u, v \in \Omega$, $p \in \partial\Omega$ be points such that $d(u,p) \leq d(v,p) \leq 81 d(u,p)$, and let $\beta$ be a $\mu$-uniform arc ($\mu \geq 1$) in $(\Omega, d)$ connecting $u$ and $v$. Then $\beta$ is $c_0$-uniform in $(\Omega, d_p)$ with $c_0 = c_0(\mu)$.*

Proof. For convenience, we write $\lambda = d(u,p)$ and $\tau = d(v,p)$. Then

$$\lambda \leq \tau \leq 81\lambda.$$

By [5, Lemma 5.8(c)], we only need to prove that there exist positive constants $r$ and $R$ such that

$$\beta \subset A(p; r, R) := \{x : r \leq d(x,p) \leq R\}. \tag{3-1}$$

We note from [5, Lemma 5.8(c)] and (3-1) that $\beta$ is $c_0$-uniform with $c_0 = 8\mu(R/r)^2$, where the ratio $R/r$ is independent of $\lambda$. In the following, we prove (3-1).

Since $\beta$ is $\mu$-uniform and thus $\mu$-quasiconvex, it follows that, for $w \in \beta$,

$$d(w,p) \leq \tfrac{1}{2}\ell(\beta) + \tau \leq \tfrac{1}{2}\mu d(u,v) + \tau \leq (41\mu + 81)\lambda =: R.$$

Now we are going to get a lower bound for $d(w,p)$. To this end, we consider two possibilities. If $\min\{\ell(\beta[u,w]), \ell(\beta[w,v])\} \leq \frac{1}{2}\lambda$, then

$$d(w,p) \geq \max\{\lambda - d(u,w), \tau - d(v,w)\} \geq \tfrac{1}{2}\lambda.$$

If $\min\{\ell(\beta[u,w]), \ell(\beta[w,v])\} > \frac{1}{2}\lambda$, then we find that

$$d(w,p) \geq \delta_\Omega(w) \geq \frac{1}{2\mu}\lambda.$$

Hence we obtain (3-1) by taking $r = (1/2\mu)\lambda$ and $R = (41\mu + 81)\lambda$, and this completes the proof of Lemma 3.1. □

Using Lemma 3.1, we can carry out the proof of the following result.

Theorem 3.2. *Suppose that $X$ is a complete metric space and that $\Omega \subset X$ is a domain with* $\operatorname{card}(\partial\Omega) \geq 2$ *and $p \in \partial\Omega$. If* $\operatorname{card}(\partial\Omega) \geq 2$ *and $(\Omega, d)$ is $c$-uniform, then $(\Omega, d_p)$ is $c'$-uniform with $c' = c'(c)$.*

Proof. Let $x, y \in \Omega$ and write $t = d(x,p)$ and $s = d(y,p)$. Without loss of generality, we may assume that $t \leq s$. By Lemma 2.4(1), we know that there is a 1-short $\mu_1$-uniform arc $\gamma$ in $(\Omega, d)$ connecting $x$ and $y$ with $\mu_1 = \mu_1(c)$. To prove the uniformity of $(\Omega, d_p)$, we show that there exists a constant $c' = c'(c)$ such that $\gamma$ is a $c'$-uniform arc in $(\Omega, d_p)$.

If $s \leq 81t$, then the uniformity of $\gamma$ follows immediately from Lemma 3.1.

In the following, we may therefore assume that $s > 81t$. Then Lemma C yields that

$$d_p(x,y) \geq \frac{d(x,y)}{4d(x,p)d(y,p)} \geq \frac{s-t}{4st} > \frac{1}{5t}. \tag{3-2}$$

Let $n$ be an integer with $3^n t < s \leq 3^{n+1} t$. Thus we have $n \geq 4$. For each $i \in \{1, 2, \ldots, n-1\}$, let $x_i$ be the first point from $x$ to $y$ in $\gamma$ with

$$d(x_i, p) = 3^i t.$$

For convenience, we let $x_0 = x$, $x_n = y$ and $\gamma_i = \gamma[x_{i-1}, x_i]$. Then, for each $i \in \{1, 2, \ldots, n\}$, we get from (2-1) and Lemma C that:

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(i)

$$k_\Omega(x_{i-1}, x_i) \geq \log\Big(1 + \frac{d(x_{i-1}, x_i)}{\delta_\Omega(x_{i-1})}\Big) \geq \log\Big(1 + \frac{3^i t - 3^{i-1} t}{3^{i-1} t}\Big) = \log 3 > 1;$$

(ii)

$$\frac{1}{2 \cdot 3^i t} = \frac{1}{4}\Big\{\frac{1}{d(x_{i-1}, p)} - \frac{1}{d(x_i, p)}\Big\} \leq d_p(x_{i-1}, x_i) \leq \frac{1}{d(x_{i-1}, p)} + \frac{1}{d(x_i, p)} \leq \frac{4}{3^i t}.$$

Moreover, it follows from (i) and Lemma 2.4(2) that every subarc $\gamma_i$ is $\mu_1$-uniform in $(\Omega, d)$. Since $d(x_i, p) \leq 9d(x_{i-1}, p)$, again we obtain from Lemma 3.1 that each $\gamma_i$ is $c'_1$-uniform in $(\Omega, d_p)$. Thus (3-2) and (ii) imply that, for any $i \in \{0, 1, \ldots, n-1\}$,

$$\ell_{d_p}(\gamma[x_i, y]) = \sum_{r=i+1}^{n} \ell_{d_p}(\gamma_r) \leq \frac{2}{3^i t} c'_1 \leq \frac{10}{3^i} c'_1 d_p(x, y). \tag{3-3}$$

In particular, for $i = 0$, (3-3) deduces the inequality

$$\ell_{d_p}(\gamma) \leq 10 c'_1 d_p(x, y).$$

Therefore, it remains to show the cigar condition of $\gamma$ in $(\Omega, d_p)$. That is, there is some constant $c' = c'(c)$ such that, for every $z \in \gamma$,

$$\min\{\ell_{d_p}(\gamma[x, z]), \ell_{d_p}(\gamma[z, y])\} \leq c' \delta_{\Omega, d_p}(z).$$

To this end, we divide the discussions into three cases.

*Case 1.* $z \in \gamma_1 \cup \gamma_2$.

In this case, we see from (ii) that

$$\ell_{d_p}(\gamma[z, x_3]) \geq \ell_{d_p}(\gamma_3) \geq d_p(x_2, x_3) \geq \frac{1}{54t},$$

and so by (3-3),

$$\ell_{d_p}(\gamma[z, y]) \leq \ell_{d_p}(\gamma) \leq \frac{2}{t} c'_1 \leq 108 c'_1 \ell_{d_p}(\gamma[z, x_3]).$$

This implies that

$$\min\{\ell_{d_p}(\gamma[x, z]), \ell_{d_p}(\gamma[z, y])\} \leq 108 c'_1 \min\{\ell_{d_p}(\gamma[x, z]), \ell_{d_p}(\gamma[z, x_3])\}.$$

Since $d(x_3, p) = 27d(x, p)$, Lemma 3.1 guarantees that $\gamma[x, x_3]$ is $\mu_1$-uniform in $(\Omega, d_p)$. Hence we obtain

$$\min\{\ell_{d_p}(\gamma[x, z]), \ell_{d_p}(\gamma[z, y])\} \leq 108 c'_1 \mu_1 \delta_{\Omega, d_p}(z).$$

*Case 2.* $z \in \gamma_{n-1} \cup \gamma_n$.

Again by (ii), we get

$$\ell_{d_p}(\gamma[x_{n-3}, z]) \geq \ell_{d_p}(\gamma_{n-2}) \geq d_p(x_{n-2}, x_{n-3}) \geq \frac{1}{2 \cdot 3^{n-2} t},$$

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and so (3-3) implies that

$$\ell_{d_p}(\gamma[z,y]) \le \ell_{d_p}(\gamma[x_{n-2},y]) \le \frac{2}{3^{n-2}t}c_1' \le 4c_1'\ell_{d_p}(\gamma[x_{n-3},z])$$

from which it follows that

$$\min\{\ell_{d_p}(\gamma[x,z]), \ell_{d_p}(\gamma[z,y])\} \le 4c_1' \min\{\ell_{d_p}(\gamma[x_{n-3},z]), \ell_{d_p}(\gamma[z,y])\}.$$

Moreover, we find from Lemma 3.1 and $d(y,p) \le 81d(x_{n-3},p)$ that $\gamma[x_{n-3},y]$ is $\mu_1$-uniform in $(\Omega, d_p)$. Furthermore,

$$\min\{\ell_{d_p}(\gamma[x,z]), \ell_{d_p}(\gamma[z,y])\} \le 36c_1'\mu_1\delta_{\Omega,d_p}(z),$$

as desired.

*Case 3.* $z \in \gamma_3 \cup \cdots \cup \gamma_{n-2}$.

If $n = 4$, then the proof is complete due to the former arguments. So, in the following, we assume that $n \ge 5$. Let $m \in \{3, \ldots, n-2\}$ be the integer such that $z \in \gamma_m$. Since $d(x_{m+1},p) \le 81d(x_{m-2},p)$, it follows from Lemma 3.1 that $\gamma[x_{m-2},x_{m+1}]$ is $\mu_1$-uniform in $(\Omega, d_p)$.

If $\ell_{d_p}(\gamma[x_{m-2},z]) \le \ell_{d_p}(\gamma[z,x_{m+1}])$, then we know from (ii) that

$$\mu_1\delta_{\Omega,d_p}(z) \ge \ell_{d_p}(\gamma[x_{m-2},z]) \ge \ell_{d_p}(\gamma[x_{m-2},x_{m-1}]) \ge d_p(x_{m-2},x_{m-1}) \ge \frac{1}{2\cdot 3^{m-1}t},$$

and so (3-3) implies that

$$\ell_{d_p}(\gamma[z,y]) \le \ell_{d_p}(\gamma[x_{m-1},y]) \le \frac{2}{3^{m-1}t}c_1' \le 4c_1'\mu_1\delta_{\Omega,d_p}(z).$$

If $\ell_{d_p}(\gamma[x_{m-2},z]) > \ell_{d_p}(\gamma[z,x_{m+1}])$, then a similar argument yields that

$$\mu_1\delta_{\Omega,d_p}(z) \ge \ell_{d_p}(\gamma[z,x_{m+1}]) \ge \ell_{d_p}(\gamma[x_m,x_{m+1}]) \ge \frac{1}{2\cdot 3^{m+1}t},$$

which implies that

$$\ell_{d_p}(\gamma[z,y]) \le \ell_{d_p}(\gamma[x_{m-1},y]) \le 36c_1'\mu_1\delta_{\Omega,d_p}(z),$$

as required. The proof of Theorem 3.2 is complete. □

## 4. Flattening and uniform domains II

In this section, we consider the converse of Theorem 3.2 and prove that if $(\Omega, d_p)$ is uniform, then $(\Omega, d)$ is uniform.

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**4.1. $\Omega$ is bounded.** In this subsection, we always assume that $\Omega$ is bounded with card $(\partial\Omega) \geq 2$ and that $(\Omega, d_p)$ is $c$-uniform. Let $u, v \in \Omega$. Then, by Lemma 2.4(1), there exists an $h_{u,v}$-short $\mu_1$-uniform arc $\gamma$ in $(\Omega, d_p)$ connecting $u$ and $v$ with $\mu_1 = \mu_1(c)$ and

$$h_{u,v} \leq \min\Big\{k_{\Omega,d_p}(u,v), \log\Big(1 + \frac{b(p)}{4\mathrm{diam}(\Omega)}\Big)\Big\},$$

where $k_{\Omega,d_p}$ denotes the quasihyperbolic metric of $\Omega$ in the metric $d_p$ and $b(p) = \sup\{d(p,q) : q \in \partial\Omega\}$. We note that $\partial\Omega$ contains at least two points so that $0 < b(p) \leq \mathrm{diam}(\Omega)$. Hence we have $h_{u,v} \leq \log(1 + 1/4)$. In the following, we give some useful results concerning the properties of the above $h_{u,v}$-short $\mu_1$-uniform arc $\gamma$.

Lemma E [5, Lemma 5.13(b)]. *Let $u, v \in \Omega$, and let $\gamma$ be an $h_{u,v}$-short $\mu_1$-uniform arc $\gamma$ in $(\Omega, d_p)$ connecting $u$ and $v$. Suppose that:*

(1) *$d(u,p) \leq d(v,p)$; and*
(2) *there exists a number $K \geq 1$ such that, for all $z \in \gamma$, $d(z,p) \leq Kd(u,p)$.*

*Then $\gamma$ is $\mu_1 K^2$-quasiconvex and $c_2$-uniform in $(\Omega, d)$, where $c_2 = 2\mu_1 K^2(8\mu_1 + 1)^2$.*

Next, we state and prove a couple of technical lemmas.

Lemma 4.1. *Let $u, v \in \Omega$, and let $\gamma$ be an $h_{u,v}$-short $\mu_1$-uniform arc $\gamma$ in $(\Omega, d_p)$ connecting $u$ and $v$. Suppose that $d(u,p) \leq \frac{1}{8}d(v,p)$ and that $d(z,p) \leq 2d(v,p)$ for all $z \in \gamma$. Then $\gamma$ is $c_3$-quasiconvex and $c_4$-uniform in $(\Omega, d)$, where $c_3 = 2^8\mu_1$ and $c_4 = 2^{24}\mu_1^2(8\mu_1 + 1)^2$.*

Proof. Let $t = d(u,p)$ and let $n$ be the integer such that

$$2^n t \leq d(v,p) < 2^{n+1} t.$$

Then $n \geq 3$. For $1 \leq i \leq n-1$, we use $u_i$ to denote the first point in $\gamma$ from $u$ to $v$ such that $d(u_i, p) = 2^i t$.

Set $u_0 = u$, $u_n = v$ and $\gamma_i = \gamma[u_{i-1}, u_i]$ for each $1 \leq i \leq n$. Then we know that, for all $i \in \{1, \ldots, n\}$,

$$\tfrac{1}{2}d(u_i,p) \leq d(u_{i-1},u_i) \leq \tfrac{3}{2}d(u_i,p) \quad \text{and} \quad 2^{i-1}t \leq d(u_{i-1},u_i) \leq 3 \cdot 2^{i-1}t. \tag{4-1}$$

Moreover, for each $i < j \in \{0, \ldots, n\}$, it follows from Lemma C that

$$d_p(u_i,u_j) \geq \frac{d(u_i,u_j)}{4d(u_i,p)d(u_j,p)} \geq \frac{d(u_j,p) - d(u_i,p)}{4d(u_i,p)d(u_j,p)} \geq \frac{1}{4d(u_j,p)},$$

and for all $q \in \partial\Omega \setminus \{p\}$,

$$\delta_{\Omega,d_p}(u_j) \leq d_p(u_j,q) \leq \frac{d(u_j,q)}{d(u_j,p)d(p,q)}.$$

Then we deduce from these inequalities that

$$k_{\Omega,d_p}(u_i,u_j) \geq \log\Big(1 + \frac{d_p(u_i,u_j)}{\delta_{\Omega,d_p}(u_j)}\Big) \geq \log\Big(1 + \frac{d(p,q)}{4d(u_j,q)}\Big),$$

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and so

$$k_{\Omega,d_p}(u_i,u_j) \geq \log\Big(1+\frac{b(p)}{4\mathrm{diam}(\Omega)}\Big) \geq h_{u,v}.$$

Therefore, we see from Lemma 2.4 that, for each $i < j \in \{2,\ldots,n\}$, $\gamma[u_j,u_j]$ is $h_{u,v}$-short $\mu_1$-uniform.

Next, we check the quasiconvexity of $\gamma$ in $(\Omega, d)$. By the choice of $u_i$, we have, for all $z \in \gamma[u_{i-1},u_i]$,

$$d(z,p) \leq 2d(u_{i-1},p).$$

Then with the substitution

$$(\gamma,u,v,K) \mapsto (\gamma_i,u_{i-1},u_i,2)$$

(we note that this means that the arc $\gamma$, the points $u$, $v$ and the constant $K$ are replaced by $\gamma_i$, $u_{i-1}$, $u_i$ and 8, respectively), Lemma E gives

$$\ell(\gamma_i) \leq 4\mu_1 d(u_{i-1},u_i) \leq 2^{i+4}\mu_1 t,$$

and thus

$$\ell\Big(\bigcup_{r=1}^{i}\gamma_r\Big) = \sum_{r=1}^{i}\ell(\gamma_r) \leq \mu_1 t\sum_{r=1}^{i}2^{r+4} \leq 2^{i+5}\mu_1 t \leq 2^{6+i-n}\mu_1 d(u,v), \tag{4-2}$$

where the last inequality holds since

$$d(u,v) \geq d(v,p) - d(u,p) \geq 2^n t - t \geq 2^{n-1}t.$$

By taking $i = n$, we see from (4-2) that $\gamma$ is $c_3$-quasiconvex with $c_3 = 2^6\mu_1$.

Now, to prove the uniformity of $\gamma$, we only need to deal with the cigar condition of $\gamma$. Indeed we shall show that, for any $z \in \gamma$,

$$\min\{\ell(\gamma[u,z]),\ell(\gamma[v,z])\} \leq c_4\delta_\Omega(z), \tag{4-3}$$

where $c_4 = 2^{24}\mu_1^2(8\mu_1+1)^2$.

Let $z \in \gamma$. Note that there is a $\varrho \in \{1,\ldots,n\}$ such that $z \in \gamma_\varrho$. We separate the discussions into three cases by considering the position of $z$ in $\gamma$.

*Case 4.* $z \in \gamma_1$.

In this case, we see from (4-2) that

$$\ell(\gamma[u,z]) \leq \ell(\gamma_1) \leq 2^{10}\mu_1 t,$$

and at the same time (4-1) leads to

$$\ell(\gamma[z,u_2]) \geq \ell(\gamma[u_1,u_2]) \geq d(u_1,u_2) \geq t.$$

It follows that

$$\ell(\gamma[z,u_2]) \geq \frac{1}{2^9\mu_1}\ell(\gamma[u,z]).$$

By the choice of $u_2$, for $w \in \gamma[u, u_2]$,

$$d(w, p) \leq 4d(u, p).$$

Then we infer from Lemma E with the substitution

$$(\gamma, u, v, K) \mapsto (\gamma[u_0, u_2], u_0, u_2, 4)$$

that

$$\delta_\Omega(z) \geq \frac{1}{2^5\mu_1(8\mu_1 + 1)^2} \min\{\ell(\gamma[u, z]), \ell(\gamma[z, u_2])\} \geq \frac{1}{2^{14}\mu_1^2(8\mu_1 + 1)^2}\ell(\gamma[u, z]),$$

which implies that

$$\ell(\gamma[u, z]) \leq 2^{14}\mu_1^2(8\mu_1 + 1)^2\delta_\Omega(z), \tag{4-4}$$

as needed.

*Case 5.* $z \in \gamma_{n-1} \cup \gamma_n$.

For $z \in \gamma$, we have $d(z, p) \leq 2d(v, p)$ and thus

$$d(z, p) \leq 16d(u_{n-2}, p).$$

By the substitution

$$(\gamma, u, v, K) \mapsto (\gamma[u_{n-2}, y], u_{n-2}, y, 16),$$

we can get from Lemma E that

$$\ell(\gamma[z, y]) \leq \ell(\gamma[u_{n-2}, y]) \leq 2^8\mu_1 d(u_{n-2}, y) < 2^{n+10}\mu_1 t.$$

Moreover, by (4-1) we obtain

$$\ell(\gamma[u_{n-3}, z]) \geq \ell(\gamma[u_{n-3}, u_{n-2}]) \geq d(u_{n-3}, u_{n-2}) \geq 2^{n-3}t,$$

which yields

$$\ell(\gamma[z, v]) \leq 2^{13}\mu_1\ell(\gamma[u_{n-3}, z]).$$

Again by Lemma E with the substitution

$$(\gamma, u, v, K) \mapsto (\gamma[u_{n-3}, y], u_{n-3}, y, 32),$$

we obtain that

$$\delta_\Omega(z) \geq \frac{1}{2^{11}\mu_1(8\mu_1 + 1)^2} \min\{\ell(\gamma[u_{n-3}, z]), \ell(\gamma[z, v])\} \geq \frac{1}{2^{24}\mu_1^2(8\mu_1 + 1)^2}\ell(\gamma[z, v]),$$

so that

$$\ell(\gamma[z, v]) \leq 2^{24}\mu_1^2(8\mu_1 + 1)^2\delta_\Omega(z).$$

*Case 6.* $z \in \gamma_2 \cup \cdots \cup \gamma_{n-2}$ when $n \geq 4$.

We see from the choice of $u_\varrho$ that, for $z \in \gamma[u_{\varrho-2}, u_{\varrho+1}])$,

$$d(z, p) \leq 8d(u_{\varrho-2}, p).$$

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Then it follows from Lemma E with the substitution

$$(\gamma, u, v, K) \mapsto (\gamma[u_{\varrho-2}, u_{\varrho+1}], u_{\varrho-2}, u_{\varrho+1}, 8)$$

that

$$2^7\mu_1(8\mu_1+1)^2\delta_\Omega(z) \geq \min\{\ell(\gamma[u_{\varrho-2}, z]), \ell(\gamma[z, u_{\varrho+1}])\}. \tag{4-5}$$

Since

$$\ell(\gamma[u_{\varrho-2}, z]) \geq \ell(\gamma[u_{\varrho-2}, u_{\varrho-1}]) \geq d(u_{\varrho-2}, u_{\varrho-1}) \geq 2^{\varrho-2}t \tag{4-6}$$

and

$$\ell(\gamma[z, u_{\varrho+1}]) \geq \ell(\gamma[u_{\varrho}, u_{\varrho+1}]) \geq 2^s t, \tag{4-7}$$

we get that

$$\ell(\gamma[u,z]) \leq \ell(\gamma[u,u_\varrho]) = \sum_{j=1}^{s}\ell(\gamma_j) \leq \mu_1 t\sum_{j=1}^{s}2^{j+6} \leq 2^{s+7}\mu_1 t. \tag{4-8}$$

The combination of (4-5)–(4-8) implies that

$$\ell(\gamma[u,z]) \leq 2^{16}\mu_1^2(8\mu_1+1)^2\delta_\Omega(z), \tag{4-9}$$

which gives (4-3). Hence we have established the validity of Lemma 4.1. □

For the convenience of the reader, we may combine the derivation of (4-4) and (4-9) into the following form.

Corollary 4.2. *Under the assumptions of Lemma 4.1, let $u_i$ ($i \in \{0, 1, \ldots, n\}$) denote the consecutive points in $\gamma$ determined at the beginning of Lemma 4.1. Then, for all $z \in \gamma[u_0, u_{n-2}]$,*

$$\ell(\gamma[u,z]) \leq c_5\delta_\Omega(z),$$

*where $c_5 = 2^{16}\mu_1^2(8\mu_1+1)^2$.*

Lemma 4.3. *Let $u, v \in \Omega$, and let $\gamma$ be an $h_{u,v}$-short $\mu_1$-uniform arc $\gamma$ in $(\Omega, d_p)$ connecting $u$ and $v$. Let $z_0 \in \gamma$ such that $d(z_0, p) = \sup_{z\in\gamma} d(z, p)$. We have the following.*

(1) *If $d(u, v) \leq t/8\mu_1 \min\{d(u, p), d(v, p)\}$, then*

$$d(z, p) \leq 2\min\{d(u, p), d(v, p)\} \quad \text{for all } z \in \gamma.$$

(2) *If $k_{\Omega,d_p}(z_0, u) \leq h_{u,v}$, then $d(z, p) \leq 2d(u, p)$ for all $z \in \gamma$.*

(3) *If $k_{\Omega,d_p}(z_0, v) \leq h_{u,v}$, then $d(z, p) \leq 2d(v, p)$ for all $z \in \gamma$.*

Proof. We first prove (4.3). Without loss of generality, we may assume that $d(u, p) \leq d(v, p)$. Then we know from Lemma C that

$$d_p(u,v) \leq \frac{d(u,v)}{d(u,p)d(p,v)} \leq \frac{1}{8\mu_1 d(u,p)}.$$

For $z \in \gamma$, we infer from the uniformity of $\gamma$ in $(\Omega, d_p)$ and Lemma C that

$$\frac{d(u,z)}{4d(u,p)d(p,z)} \leq d_p(u,z) \leq \ell_{d_p}(\gamma) \leq \mu_1 d_p(u,v) \leq \frac{1}{8d(u,p)},$$

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which implies that

$$d(u,z) \le \tfrac{1}{2} d(p,z).$$

Thus we have

$$d(u,p) \ge d(p,z) - d(u,z) \ge \tfrac{1}{2} d(p,z),$$

as desired.

Next, we shall check (2). Since $k_{\Omega,d_p}(z_0,u) \le h_{u,v}$, we have

$$\log\Big(1 + \frac{d_p(z_0,u)}{\delta_{\Omega,d_p}(z_0)}\Big) \le k_{\Omega,d_p}(z_0,u) \le h_{u,v} \le \log\Big(1 + \frac{b(p)}{4\mathrm{diam}(\Omega)}\Big).$$

Then, for any $q \in \partial\Omega \setminus \{p\}$, Lemma C guarantees that

$$\frac{d(z_0,u)}{4t d(z_0,p)} \le d_p(z_0,u) \le \frac{b(p)}{4\mathrm{diam}(\Omega)} \delta_{\Omega,d_p}(z_0) \le \frac{b(p)}{4\mathrm{diam}(\Omega)} \cdot \frac{d(z_0,q)}{d(z_0,p)d(p,q)}.$$

By taking the supremum with respect to $q$ in $\partial\Omega \setminus \{p\}$, we obtain that

$$d(z_0,u) \le d(u,p),$$

and thus, for $z \in \gamma$,

$$d(p,z) \le d(z_0,p) \le d(u,p) + d(z_0,u) \le 2d(u,p).$$

Therefore, we obtain (2).

By symmetry, we know that the proof of (3) is similar to the proof of (2). Hence, the proof of the lemma is complete. □

Theorem 4.4. *Suppose that $X$ is a complete metric space and that $\Omega \subset X$ is a domain with* card $(\partial\Omega) \ge 2$ *and $p \in \partial\Omega$. If $\Omega$ is bounded and $(\Omega, d_p)$ is $c$-uniform, then $(\Omega, d)$ is $c''$-uniform with $c'' = C_0(\mathrm{diam}(\Omega)/b(p))$, where $C_0 = C_0(c)$ and $b(p) = \sup\{d(p,q) | q \in \partial\Omega\}$.*

Proof. Let $x, y \in \Omega$. Then by Lemma 2.4(1) we know that there exists an $h_{x,y}$-short $\mu_1$-uniform arc $\gamma$ in $(\Omega, d_p)$ connecting $x$ and $y$ with $\mu_1 = \mu_1(c)$ and that

$$h_{x,y} \le \min\Big\{k_{\Omega,d_p}(x,y), \log\Big(1 + \frac{b(p)}{4\mathrm{diam}(\Omega)}\Big)\Big\}.$$

We shall show that this $\gamma$ is $c''$-uniform in $(\Omega, d)$ with $c''$ depending only on $c$.

To this end, we let $z_0$ in $\gamma$ be such that, for all $z \in \gamma$, $d(z_0,p) \ge d(z,p)$. Set $s = d(y,p)$, $t = d(x,p)$, $\sigma = d(z_0,p)$ and $\tau = (\mathrm{diam}(\Omega)/b(p))$. Without loss of generality, we may assume that $t \le s$. Then we divide the proof into the following three cases.

*Case a.* Either $d(x,y) \le t/8\mu_1$ or $d(x,y) > t/8\mu_1$ and $k_{\Omega,d_p}(z_0,x) \le h_{x,y}$.

In this case, it follows from Lemmas 4.3 and E that $\gamma$ is $8\mu_1(8\mu_1+1)^2$-uniform in $(\Omega, d)$.

*Case b.* $d(x,y) > t/8\mu_1$ and $k_{\Omega,d_p}(z_0,y) \le h_{x,y}$.

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If $d(y,p) \le 8t$, then we obtain that, for all $z \in \gamma$,

$$d(z,p) \le 16t.$$

By taking $K = 16$, it follows from Lemma E that $\gamma$ is $2^9\mu_1(8\mu_1+1)^2$-uniform in $(\Omega, d)$.

Now, we assume that $d(y,p) > 8t$. Then, by Lemmas 4.1 and 4.3, we see that $\gamma$ is $c_4$-uniform in $(\Omega, d)$, where $c_4 = 2^{24}C_1(8C_1+1)^2$.

*Case c.* $d(x,y) > t/8\mu_1$, $k_{\Omega,d_p}(z_0,x) > h_{x,y}$ and $k_{\Omega,d_p}(z_0,y) > h_{x,y}$.

First, note that, by Lemma 2.4(2), both $\gamma[z_0,x]$ and $\gamma[y,z_0]$ are $h_{x,y}$-short $\mu_1$-uniform arcs. Then we claim that

$$\sigma \le 8\mu_1 s\tau. \tag{4-10}$$

We may assume that $\sigma \ge 8s$ since otherwise the claim is clear. Since $t \le s$, we know that

$$\tfrac{7}{8}\sigma \le \sigma - d(x,p) \le d(z_0,x) \le \sigma + d(x,p) \le \tfrac{9}{8}\sigma. \tag{4-11}$$

Similarly,

$$\tfrac{7}{8}\sigma \le d(z_0,y) \le \tfrac{9}{8}\sigma.$$

Thus we infer from Lemma C that

$$\ell_{d_p}(\gamma[z_0,x]) \ge d_p(z_0,x) \ge \frac{d(z_0,x)}{4\sigma t} > \frac{1}{8t}$$

and

$$\ell_{d_p}(\gamma[z_0,y]) \ge d_p(z_0,y) > \frac{1}{8s}.$$

Then the uniformity of $\gamma$ in $(\Omega, d_p)$ gives

$$\mu_1\delta_{\Omega,d_p}(z_0) \ge \min\{\ell_{d_p}(\gamma[z_0,x]), \ell_{d_p}(\gamma[z_0,y])\} > \frac{1}{8s},$$

and thus

$$\delta_{\Omega,d_p}(z_0) \ge \frac{1}{8\mu_1 s}. \tag{4-12}$$

On the other hand, for any $q \in \partial\Omega \setminus \{p\}$, Lemma C shows that

$$\delta_{\Omega,d_p}(z_0) \le d_p(z_0,q) \le \frac{d(z_0,q)}{\sigma d(p,q)},$$

and so

$$\delta_{\Omega,d_p}(z_0) \le \frac{\tau}{\sigma}. \tag{4-13}$$

Therefore, (4-10) follows from (4-12) and (4-13).

Moreover, we need to check the following estimates for the arc length of $\gamma[x,z_0]$ and $\gamma[y,z_0]$: that is,

$$\ell(\gamma[z_0,x]) \le 2^9\mu_1\sigma \quad \text{and} \quad \ell(\gamma[y,z_0]) \le 2^9\mu_1\sigma. \tag{4-14}$$

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This can be seen as follows. Since $\sigma \geq 8s \geq 8t$, it follows from Lemma 4.1 and (4-11) that

$$\ell(\gamma[z_0, x]) \leq 2^8 \mu_1 d(z_0, x) \leq 2^9 \mu_1 \sigma$$

and

$$\ell(\gamma[z_0, y]) \leq 2^8 \mu_1 d(z_0, y) \leq 2^9 \mu_1 \sigma.$$

Thus the proof of (4-14) is complete.

Now, we are ready to show the uniformity of $\gamma$ in $(\Omega, d)$. First, we see from (4-10) and (4-14) that

$$\ell(\gamma) = \ell(\gamma[z_0, x]) + \ell(\gamma[y, z_0]) \leq 2^{10} \mu_1 \sigma \leq 2^{13} \mu_1^2 s\tau. \tag{4-15}$$

By considering the cases $s \geq 2t$ and $s < 2t$, we can deduce from the triangle inequality $d(x, y) \geq s - t$ and $d(x, y) \geq t/8\mu_1$ that

$$d(x, y) \geq \frac{s}{16\mu_1}.$$

This, together with (4-15), implies that

$$\ell(\gamma) \leq 2^{17} \mu_1^3 \tau d(x, y),$$

which shows that $\gamma$ is $c_6$-quasiconvex with $c_6 = 2^{17} \mu_1^3 \tau$.

It remains to prove the cigar condition of $\gamma$. Let $m$, $n$ be two integers such that

$$2^m t \leq \sigma < 2^{m+1} t \quad \text{and} \quad 2^n s \leq \sigma < 2^{n+1} s.$$

Thus we have $m \geq 3$ and $n \geq 3$. Then, for each $i \in \{0, \ldots, m\}$, let $u_i$ be the first point from $x$ to $z_0$ in $\gamma[z_0, x]$ with

$$d(u_i, p) = 2^i t.$$

Similarly, for each $j \in \{0, \ldots, n\}$, we let $v_j$ be the first point from $y$ to $z_0$ in $\gamma[y, z_0]$ with

$$d(v_j, p) = 2^j s.$$

By applying Corollary 4.2 to $\gamma[z_0, x]$ and $\gamma[y, z_0]$, respectively, we get that

$$\ell(\gamma[x, z]) \leq c_5 \delta_\Omega(z) \quad \text{for } z \in \gamma[x, u_{m-2}]$$

and

$$\ell(\gamma[y, z]) \leq c_5 \delta_\Omega(z) \quad \text{for } z \in \gamma[y, v_{n-2}].$$

So we only need to consider the case when $z \in \gamma[u_{m-2}, v_{n-2}]$. By the choice of the points $u_{m-3}$, $v_{n-3}$ and $z_0$, we have

$$\max\{d(u_{m-3}, p), d(v_{n-3}, p)\} \leq d(z_0, p) \leq 16 \min\{d(u_{m-3}, p), d(v_{n-3}, p)\}.$$

Then Lemma E is available to $\gamma[u', v']$ with the substitution $K = 16$, which yields

$$\min\{\ell(\gamma[u_{m-3}, z]), \ell(\gamma[z, v_{n-3}])\} \leq 2^9 \mu_1 (8\mu_1 + 1)^2 \delta_\Omega(z).$$

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Without loss of generality, we may assume that $\ell(\gamma[u_{m-3}, z]) \leq \ell(\gamma[z, v_{n-3}])$. Thus

$$\begin{aligned}2^9\mu_1(8\mu_1 + 1)^2\delta_\Omega(z) &\geq \ell(\gamma[u_{m-3}, z]) \geq \ell(\gamma[u_{m-3}, u_{m-2}])\\ &\geq d(u_{m-3}, u_{m-2}) \geq 2^{m-3}t.\end{aligned}$$

This, together with (4-15), shows that

$$\begin{aligned}\min\{\ell(\gamma[x, z]), \ell(\gamma[z, y])\} &\leq \tfrac{1}{2}\ell(\gamma) \leq 2^9\mu_1\sigma\\ &\leq 2^{m+10}\mu_1 t\\ &\leq 2^{22}\mu_1^2(8\mu_1 + 1)^2\delta_\Omega(z).\end{aligned}$$

It follows that $\gamma$ is $c''$-uniform, where $c'' = \max\{2^{17}\mu_1^3\tau, 2^{22}\mu_1^2(8\mu_1 + 1)^2\}$. The proof of Theorem 4.4 is complete. □

**4.2. Ω is unbounded.** In this subsection, we consider the case when Ω is unbounded.

THEOREM 4.5. *Suppose that $X$ is a complete metric space and that $\Omega \subset X$ is a domain with* card $(\partial\Omega) \geq 2$ *and $p \in \partial\Omega$. If $\Omega$ is unbounded and $(\Omega, d_p)$ is $c$-uniform, then $(\Omega, d)$ is $c''$-uniform with $c''$ depending only on $c$.*

PROOF. Let $p' \in \mathrm{Inv}_{\,p}(X)$ correspond to $\infty \in \widehat{X}$. Since $(\Omega, d_p)$ is $c$-uniform, we get from Theorem 4.4 that $(\Omega, (d_p)_{p'})$ is $c'$-uniform with $c' = c'(c)$. Then it follows from [5, Proposition 3.7] that $(\Omega, d)$ is $c''$-uniform with $c''$ depending only on $c$. □

## 5. Quasi-Möbius preserves uniform domains

In this section, we prove that uniform domains are preserved under quasi-Möbius mappings in quasiconvex metric spaces. To this end, some useful lemmas are needed.

LEMMA F [4, Theorem 3.1]. *Suppose that $(X, d)$ is a $c$-quasiconvex complete metric spaces and that $\Omega \subset X$ is a domain. Then the following conditions are equivalent:*

(1) *$\Omega$ is $a$-uniform;*
(2) *$k_\Omega(x, y) \leq a_1 j_\Omega(x, y) + b$; and*
(3) *$k_\Omega(x, y) \leq 4a^2 j_\Omega(x, y)$,*

*where the constants $a$ and $a_1$, $b$ depend on each other and $c$.*

LEMMA 5.1. *Suppose that $(X_i, d_i)$ are $c$-quasiconvex complete metric spaces, that $\Omega_i \subset X_i$ are domains $(i = 1, 2)$ and that $f : \Omega_1 \to \Omega_2$ is an $\eta$-quasisymmetric homeomorphism. Then there exist constants $M > 0$ and $C \geq 0$ depending only $c$ and $\eta$ such that*

$$\frac{k_{\Omega_1}(x, y) - C}{M} \leq k_{\Omega_2}(f(x), f(y)) \leq M k_{\Omega_1}(x, y) + C$$

*for all $x, y \in \Omega_1$.*

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Proof. By symmetry, we only need to prove the second inequality. By [16, Lemma 3.9], we know that $(\Omega, k_{\Omega_1})$ is $\lambda$-quasiconvex for all $\lambda \geq 1$. Then, in view of [27, Lemma 2.3], it suffices to find a constant $s$ depending only on $c$ and $\eta$ such that $k_{\Omega_2}(f(x), f(y)) \leq 1$ whenever $k_{\Omega_1}(x, y) \leq s$.

To this end, let $s$ with $0 < s < \frac{1}{2}$ be a constant such that $\eta(2s) \leq 1/6c$, and let $x, y \in \Omega_1$ with $k_{\Omega_1}(x, y) \leq s$. Then Lemma A implies that

$$\frac{d_1(x, y)}{\delta_{\Omega_1}(x)} \leq 2k_{\Omega_1}(x, y) \leq 2s.$$

By [27, Lemma 6.12], $f$ extends to an $\eta$-quasisymmetric homeomorphism $f : \overline{\Omega}_1 \to \overline{\Omega}_2$. Let $w \in \partial\Omega_1$ with $d_2(f(x), f(w)) \leq 2d_{\Omega_2}(f(x))$. Thus we obtain

$$\begin{aligned}\frac{d_2(f(x), f(y))}{\delta_{\Omega_2}(f(x))} &\leq 2\frac{d_2(f(x), f(y))}{d_2(f(x), f(w))} \leq 2\eta\Big(\frac{d_1(x, y)}{d_1(x, w)}\Big)\\ &\leq 2\eta\Big(\frac{d_1(x, y)}{\delta_{\Omega_1}(x)}\Big) \leq 2\eta(2s) < \frac{1}{3c}.\end{aligned}$$

Again by Lemma A, we have $k_{\Omega_2}(f(x), f(y)) \leq 1$, and thus the proof of Lemma 5.1 is complete. □

Lemma 5.2. *Suppose that $(X_i, d_i)$ are $c$-quasiconvex complete metric spaces, that $\Omega_i \subset X_i$ are domains $(i = 1, 2)$ and that $f : \Omega_1 \to \Omega_2$ is an $\eta$-quasisymmetric homeomorphism. Then there exist constants $M_1 \geq 1$ and $C_1 \geq 0$ depending only $c$ and $\eta$ such that*

$$\frac{j_{\Omega_1}(x, y) - C_1}{M_1} \leq j_{\Omega_2}(f(x), f(y)) \leq M_1 j_{\Omega_1}(x, y) + C_1$$

*for all $x, y \in \Omega_1$.*

Proof. By [27, Lemma 6.14], we may assume that $\eta(t) = C_0 \max\{t^\alpha, t^{1/\alpha}\}$, where $C_0 \geq 1$, $0 < \alpha \leq 1$. Let $x, y \in \Omega_1$. We may assume that $\delta_{\Omega_2}(f(x)) \leq \delta_{\Omega_2}(f(y))$. By [27, Lemma 6.12], $f$ extends to an $\eta$-quasisymmetric homeomorphism $f : \overline{\Omega}_1 \to \overline{\Omega}_2$. Let $w \in \partial\Omega_1$ with $d_2(f(x), f(w)) \leq 2\delta_{\Omega_2}(f(x))$. Denote $r = (d_1(x, y)/\delta_{\Omega_1}(x))$. Thus we have $d(x, y) \leq rd(x, w)$. Therefore, we obtain

$$\begin{aligned}\frac{d_2(f(x), f(y))}{\delta_{\Omega_2}(f(x))} &\leq 2\frac{d_2(f(x), f(y))}{d_2(f(x), f(w))} \leq 2\eta\Big(\frac{d_1(x, y)}{\delta_1(x, w)}\Big)\\ &\leq 2C_0 \max\{r^\alpha, r^{1/\alpha}\}.\end{aligned}$$

This yields

$$\begin{aligned}j_{\Omega_2}(f(x), f(y)) &= \log\Big(1 + \frac{d_2(f(x), f(y))}{\delta_{\Omega_2}(f(x))}\Big)\\ &\leq \frac{1}{\alpha}\log\Big(1 + \frac{d_1(x, y)}{\delta_{\Omega_1}(x)}\Big) + \log(1 + 2C_0).\end{aligned}$$

Finally, by symmetry, Lemma 5.2 holds by letting $M_1 = 1/\alpha$ and $C_1 = \log(1 + 2C_0)$. □

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Lemma 5.3. *Suppose that* $(X_i, d_i)$ *are c-quasiconvex complete metric spaces and that* $\Omega_i \subset X_i$ *are domains* $(i = 1, 2)$*, and suppose that* $f : \Omega_1 \to \Omega_2$ *is an* $\eta$*-quasisymmetric homeomorphism. If* $\Omega_1$ *is* $c_1$*-uniform, then* $\Omega_2$ *is* $c_2$*-uniform with* $c_2$ *depending only on* $c$*,* $c_1$ *and* $\eta$*.*

Proof. The proof of this lemma follows from Lemmas F, 5.1 and 5.2. □

Now we are ready to prove Theorem 1.4.

**5.1. Proof of Theorem 1.4.** We divide the proof into three cases.

*Case 7.* Both $\Omega_1$ and $\Omega_2$ are bounded.

Because both $\Omega_1$ and $\Omega_2$ are bounded, the quasi-Möbius mapping $f$ is, in fact, quasisymmetric and thus the desired conclusion follows from Lemma 5.3, in this case.

*Case 8.* Among $\Omega_1$ and $\Omega_2$, one of them is bounded while the other is unbounded.

By symmetry, we only need to consider the case where $\Omega_1$ is bounded and $\Omega_2$ is unbounded. Choose a point $p \in \partial\Omega_2$ and set $d'_2 = \hat{d}_{2,p}$. We know from Lemma D that the identity map $\mathrm{id} : (\Omega_2, d_2) \to (\Omega_2, d'_2)$ is $16t$-quasi-Möbius. Hence by composition, we get a map $g$ from $(\Omega_1, d_1) \to (\Omega_2, d'_2)$ which is also quasi-Möbius. Moreover, $(\Omega_2, d'_2)$ is bounded, so $g$ is quasisymmetric. Since $(\Omega_1, d_1)$ is uniform, it follows from Lemma 5.3 that $(\Omega_2, d'_2)$ is also uniform. Then, by Corollary 1.3, we get that $(\Omega_2, d_2)$ is uniform.

*Case 9.* Both $\Omega_1$ and $\Omega_2$ are unbounded.

Choose points $p_i \in \partial\Omega_i$ and set $d'_i = \hat{d}_{i,p_i}$, where $i = 1, 2$. We know from Lemma D that the identity map $\mathrm{id} : (\Omega_i, d_i) \to (\Omega_i, d'_i)$ is $16t$-quasi-Möbius. Hence by composition, we get a map $g$ from $(\Omega_1, d'_1) \to (\Omega_2, d'_2)$ which is also quasi-Möbius. Moreover, $(\Omega_i, d'_i)$ are bounded, so $g$ is quasisymmetric. Since $(\Omega_1, d_1)$ is uniform, it follows from Corollary 1.3 and Lemma 5.3 that $(\Omega_2, d'_2)$ is also uniform. Then, again by Corollary 1.3, we get that $(\Omega_2, d_2)$ is uniform.

The proof of Theorem 1.4 is complete. □

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