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SLOPE FILTRATIONS IN FAMILIES

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Abstract This paper concerns arithmetic families of φ-modules over reduced affinoid spaces. For such a family, we first prove that the slope polygons are lower semicontinuous around any rigid point. We further prove that if the slope polygons are locally constant around a rigid point, then around this point, the family has a global slope filtration after base change to some extended Robba ring.

Keywords: slope filtration; ϕ-modules

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0. Introduction

The slope filtrations for Frobenius modules over the Robba ring were originally introduced in the context of rigid cohomology by Kedlaya as the key ingredient of his proof of Crew's conjecture [13]. Roughly speaking, the slope filtrations give a partial analogue, for Frobenius-semilinear actions on finite free modules over the Robba ring, of the eigenspace decompositions of linear transformations. It was discovered by Berger, through his construction of (φ, Γ)-modules associated with p-adic Galois representations, that the slope filtration theorem is also a fundamental ingredient for p-adic Hodge theory. For instance, it allowed Berger to prove Fontaine's conjecture that being de Rham implies being potentially semistable and to give a new proof of the Colmez–Fontaine theorem that being weakly admissible implies being admissible. Recently, the work of Fontaine and Fargues [10] has revealed more p-adic Hodge theoretic aspects of the slope filtration theorem. Namely, they reformulate it in terms of the Harder–Narasimhan filtrations for vector bundles over the *fundamental curve of p-adic Hodge theory*.

This paper grew out of an attempt to generalize the slope filtration theorem to families of Frobenius modules with an eye towards applications to families of p-adic representations (i.e. relative p-adic Hodge theory). There are actually two distinct forms of 'families' of p-adic representations. One is the continuous representations of absolute Galois groups of finite extensions of $\mathbb{Q}_p$ on finite locally free modules over affinoid algebras over $\mathbb{Q}_p$ such as the families of p-adic representations associated with p-adic families of automorphic forms; these are called *arithmetic families*. Another one is the continuous representations of étale fundamental groups of affinoid spaces over finite extensions of $\mathbb{Q}_p$ on finite-dimensional $\mathbb{Q}_p$-vector spaces; these are called *geometric families*. In [2], Berger and Colmez constructed a functor from arithmetic families of p-adic representations to families of (φ, Γ)-modules. For geometric families, the (φ, Γ)-module functor is constructed in [19]. It turns out that the (φ, Γ)-modules associated with these two kinds of families of p-adic representations have quite different features. Loosely speaking, the 'coefficients' for arithmetic families of (φ, Γ)-modules are of characteristic 0, and φ acts trivially on them, whereas the 'coefficients' for geometric families of (φ, Γ)-modules are of characteristic p, and φ acts on them as the pth power Frobenius.

In this paper, inspired by the construction of Berger–Colmez, we consider slope filtrations for arithmetic families of φ-modules. First of all, it is straightforward to see that for such families, a necessary condition for having global slope filtrations, at least locally around rigid points, is the local constancy of slope polygons. However, it is not difficult to see that this is not true in general (see § 2.4 for an example). Due to this fact, our first main result then concerns variations of slope polygons. To state the result, we first introduce some notation (see the body of the paper for more details). We fix a complete discretely valued field K of mixed characteristic $(0, p)$ to be the base field of the Robba ring, and fix a relative Frobenius lift φ on $\mathcal{R}_K$. Fix a reduced affinoid space $M(A)$ over $\mathbb{Q}_p$ to be the base for the families. Let $\mathcal{R}_{A_K}$ be the Robba ring over A_k, and set the φ-action on $\mathcal{R}_{A_K}$ as the continuous extension of $\mathrm{id} \otimes \varphi$ on $A \otimes_{\mathbb{Q}_p} \mathcal{R}_K$. By a *family of* φ*-modules* over $\mathcal{R}_{A_K}$ we mean a vector bundle M_A over $\mathcal{R}_{A_K}$ equipped with a semilinear φ-action such that the natural map $\varphi^* M_A \to M_A$ is an isomorphism. For any $x \in M(A)$, we set M_x, the fibre of M_A at x, as the base change of M_A to $k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}_K$.

Theorem 0.0.1 (Theorem 2.3.10). *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$. Then for any $x \in M(A)$, there is a Weierstrass subdomain $M(B)$ containing x such that the HN-polygon of M_y lies above the HN-polygon of M_x with the same endpoint for any $y \in M(B)$.*

If M_x is pure, the above theorem then implies that the fibres of M_A are also pure of the same slope around x. In fact, a stronger result holds if $k(x) \subset A$. Namely, M_A is *globally pure* around x.

Theorem 0.0.2 (Theorem 2.2.12). *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$. Suppose that M_x is pure of slope s for some $x \in M(A)$ with $k(x) \subset A$, then there exists a Weierstrass subdomain $M(B)$ containing x such that the base change of M_A to $M(B)$ admits a finite free (c, d)-pure model N_B where $d > 0$, $(c, d) = 1$ and $c/d = s$. In particular, M_B is globally pure of slope s.*

Although the slope polygons are not locally constant in general, we prove that one can shrink the Weierstrass subdomain $M(B)$ in Theorem 0.0.1 such that the set of $y \in M(B)$ where the slope polygon of M_y coincides with the slope polygon of M_x is a Zariski closed subset of $M(B)$. Furthermore, we have a global slope filtration on this Zariski closed subset after base change to some *extended Robba ring*. This forms our second main result. To state the result, fix an *admissible extension* L of K such that its residue field is *strongly difference-closed*, and let $\widetilde{\mathcal{R}}_L$ be the extended Robba ring over L.

Theorem 0.0.3 (Theorem 2.3.15 for the RF case). *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$, and let $x \in M(A)$. Then there exists a Weierstrass subdomain $M(B)$ containing x such that the set of $y \in M(B)$ where the HN-polygon of M_y coincides with the HN-polygon of M_x forms a Zariski closed subset $M(C)$ of $M(B)$, and*

$$\widetilde{M}_C = M_A \otimes_{\mathcal{R}_{A_K}} (C \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{R}}_L)$$

admits a unique slope filtration which lifts the HN filtration of the φ-module

$$\widetilde{M}_x = M_x \otimes_{k(x)\otimes_{\mathbb{Q}_p}\mathcal{R}_K} (k(x) \otimes_{\mathbb{Q}_p} \widetilde{\mathcal{R}}_L).$$

In the case where K is a finite unramified extension of $\mathbb{Q}_p$, and the φ-action on $\mathcal{R}_K$ is an absolute Frobenius lift (this is the case for the φ-modules arising from p-adic Hodge theory), we can use a canonical and smaller period ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ instead of $\widetilde{\mathcal{R}}_L$ in the statement of Theorem 0.0.3. More precisely, we have the following theorem which we expect to be useful for p-adic Hodge theory.

Theorem 0.0.4 (Theorem 2.3.15 for the AF case). *Suppose that K is a finite unramified extension of $\mathbb{Q}_p$, and the φ-action on $\mathcal{R}_K$ is an absolute Frobenius lift. Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$, and let $x \in M(A)$. Then there exists a Weierstrass subdomain $M(B)$ containing x such that the set of $y \in M(B)$ where the HN-polygon of M_y coincides with the HN-polygon of M_x forms a Zariski closed subset $M(C)$ of $M(B)$, and*

$$\widetilde{M}_C = M_A \otimes_{\mathcal{R}_{A_K}} (C \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})$$

admits a unique slope filtration which lifts the HN filtration of the φ-module

$$\widetilde{M}_x = M_x \otimes_{k(x)\otimes_{\mathbb{Q}_p}\mathcal{R}_K} (k(x) \otimes_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}).$$

One can ask similar questions for Berkovich points rather than rigid points. However, since the residue field of a general Berkovich point is not necessarily discretely valued, this requires a slope theory for Frobenius modules over the Robba ring $\mathcal{R}_K$ for non-discretely valued K. By passing to the spherical completion of K, we may reduce to the case where K is spherically complete. In this case, it is not difficult to show the existence of Harder–Narasimhan filtrations for Frobenius modules over $\mathcal{R}_K$ (Theorem 1.2.15). However, we cannot prove the equivalence of semistability and purity which is the key of Kedlaya's original slope theory. Another issue is that the pure locus is not necessarily open (see [17, Remark 7.5] for more details) which prevents the semicontinuity of variation of slope polygons in the topology of Berkovich spaces. A possible solution for this issue is to use Huber's adic spaces instead of Berkovich spaces as shown in the work of Hellmann [11].

We now sketch the structure of the paper. In [14], the slope theory for absolute Frobenius was fully developed. A large part of this theory, especially the slope filtration theorem, was then generalized to relative Frobenius in [15]. In § 1, we further generalize some of the results of [14], especially comparisons of special and generic HN-polygons,

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to the relative Frobenius case. In § 1.1, we give the definitions of various base rings. In § 1.2, we prove the existence of HN filtrations for φ-modules over $\mathcal{R}_K$ for spherically complete K, and review the slope filtration theorem for discretely valued K. In § 1.3, we generalize the classical Dieudonné–Manin decomposition theorem to spherically complete difference fields that have strongly difference-closed residue fields. This result is irrelevant to the main results of this paper, but may be of independent interest. We define various extended base rings in § 1.4. In § 1.5, we review the slope theory for φ-modules over the extended Robba ring $\widetilde{\mathcal{R}}_K$, and we prove that the HN filtrations are split when K has strongly difference-closed residue field. In § 1.6, for when K has strongly difference-closed residue field, using the slope decomposition for φ-modules over $\widetilde{\mathcal{E}}_K$, we prove the existence of reverse filtrations for φ-modules over the extended bounded Robba ring $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$. In § 1.7, we prove that the special HN-polygon lies above the generic HN-polygon with the same endpoint.

We prove our main results in § 2. In § 2.1, we first establish some basic results for various base rings with coefficients in certain Banach algebras. Then we introduce the definition of families of φ-modules. We prove Theorem 0.0.2 in § 2.2. In § 2.3, we first prove Theorem 0.0.1. We then prove Theorem 0.0.3 and Theorem 0.0.4 in a uniform way. To do this, we introduce the notation $\widetilde{\mathcal{R}}$ which represents $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ in the case where K is a finite unramified extension of $\mathbb{Q}_p$ and φ is an absolute Frobenius lift (the AF case), and represents $\widetilde{\mathcal{R}}_L$ for general K, φ (the RF case). In § 2.4, we construct a family of φ-modules where the HN-polygons are not locally constant over the base.

Convention 0.0.5. Throughout this paper, let K be a complete non-Archimedean valued field of mixed characteristic $(0, p)$. Let $\mathcal{O}_K$ be its valuation ring. Let $\mathfrak{m}_K$ be the maximal ideal of $\mathcal{O}_K$, and let $k = \mathcal{O}_K/\mathfrak{m}_K$ be the residue field. Let v denote the valuation on $\widehat{\overline{K}}$ extending the one on K. Let $\pi \in \mathfrak{m}_K$ satisfying $v(\pi) = 1$. From § 1.4 on, we further assume that K is discretely valued and π is a uniformizer. In § 1, we set the norm on $\widehat{\overline{K}}$ as $|\cdot| = p^{-v(\cdot)}$. In § 2, we further assume that K is a p-adic field, and we renormalize the norm on K such that $|p| = p^{-1}$ to fit the standard normalization on $\mathbb{Q}_p$.

1. Slope theory of φ-modules

In this section we develop the slope theory for the relative Frobenius lift. We caution that in §§ 1.1 and 1.2, we do not assume that K is discretely valued, which makes things a bit subtler.

1.1. The base rings

Definition 1.1.1. For any interval $I \subseteq (0, \infty]$, let $\mathcal{R}_K^I$ be the ring of the Laurent series $f = \sum_{i \in \mathbb{Z}} a_i T^i$ for which $a_i \in K$ and $v(a_i) + si \to \infty$ as $i \to \pm\infty$ for all $s \in I$. Geometrically, $\mathcal{R}_K^I$ is the ring of K-holomorphic functions on the annulus $\{T \in \overline{K} | v(T) \in I\}$. For any $s \in I$, the valuation w_s on $\mathcal{R}_K^I$ is defined as

$$w_s(f) = \min_{i \in \mathbb{Z}}\{v(a_i) + si\}.$$

The corresponding multiplicative norm is $|f|_s = \max_{i\in\mathbb{Z}}\{|a_i|p^{-is}\}$. For $I=(0,r]$, we denote $\mathcal{R}_K^{(0,r]}$ by $\mathcal{R}_K^r$ for simplicity. We call the union $\mathcal{R}_K=\bigcup_{r>0}\mathcal{R}_K^r$ the *Robba ring* over K.

Definition 1.1.2. For any $r>0$, let $\mathcal{R}_K^{\mathrm{bd},r}$ be the subring of $\mathcal{R}_K^r$ consisting of the Laurent series $f=\sum_{i\in\mathbb{Z}} a_iT^i$ with $\{v(a_i)\}_{i\in\mathbb{Z}}$ bounded below. Set

$$w(f)=\inf_{i\in\mathbb{Z}}\{v(a_i)\},$$

and set $|f|=\sup_{i\in\mathbb{Z}}\{|a_i|\}$. Let $\mathcal{R}_K^{\mathrm{int},r}$ be the subring of $\mathcal{R}_K^r$ consisting of all f with $w(f)\geqslant 0$. Let $\mathcal{R}_K^{\mathrm{bd}}=\bigcup_{r>0}\mathcal{R}_K^{\mathrm{bd},r}$ and $\mathcal{R}_K^{\mathrm{int}}=\bigcup_{r>0}\mathcal{R}_K^{\mathrm{int},r}$. We call $\mathcal{R}_K^{\mathrm{bd}}$ the *bounded Robba ring* over K.

Proposition 1.1.3. *For any $f\in\mathcal{R}_K^{\mathrm{bd}}$, we have $\lim_{r\to 0^+} w_r(f)=w(f)$. As a consequence, w is additive and $|\cdot|$ is multiplicative on $\mathcal{R}_K^{\mathrm{bd}}$.*

Proof. For any $\epsilon>0$, pick i_0 such that $v(a_{i_0})<w(f)+\frac{\epsilon}{2}$. Let $r_0=\frac{\epsilon}{|2i_0|+1}$, and we may suppose that $f\in\mathcal{R}_K^{\mathrm{bd},r_0}$ by shrinking ϵ. It thus follows that for any $r\in(0,r_0]$, $w_r(f)\leqslant ri_0+v(a_{i_0})<w(f)+\epsilon$. On the other hand, choose $N\in\mathbb{N}$ sufficiently large that $r_0i+v(a_i)\geqslant w(f)$ for any $i\leqslant -N$. Therefore for any $r\leqslant r_1=\min\{r_0,\frac{\epsilon}{N}\}$, if $i\leqslant -N$, then $ri+v(a_i)\geqslant r_0i+v(a_i)\geqslant w(f)$; if $i\geqslant -N$, then $ri+v(a_i)\geqslant w(f)-rN\geqslant w(f)-\epsilon$. We thus deduce that $|w_r(f)-w(f)|\leqslant\epsilon$ for any $r\in(0,r_1]$, yielding the desired result. □

Definition 1.1.4. Let $\mathcal{E}_K$ be a the ring of Laurent series $f=\sum_{i\in\mathbb{Z}}a_iT^i$ for which $\{v(a_i)\}_{i\in\mathbb{Z}}$ is bounded below and $v(a_i)\to\infty$ as $i\to-\infty$. Set $w(f)=\inf_{i\in\mathbb{Z}}\{v(a_i)\}$, and set $|f|=\sup_{i\in\mathbb{Z}}\{|a_i|\}$. Let $\mathcal{O}_{\mathcal{E}_K}=\{f\in\mathcal{E}_K|w(f)\geqslant 0\}$.

Remark 1.1.5. It is clear that $\mathcal{R}_K^{\mathrm{bd},r}$ is a subring of $\mathcal{E}_K$ consisting of the series such that $v(a_i)+ri\to\infty$ as $i\to-\infty$. In addition, the natural inclusion $\mathcal{R}_K^{\mathrm{bd}}\to\mathcal{E}_K$ is an isometry with respect to w and identifies $\mathcal{E}_K$ with the w-completion of $\mathcal{R}_K^{\mathrm{bd}}$. In particular, w is a valuation on $\mathcal{E}_K$, and its corresponding multiplicative norm is $|\cdot|$.

Remark 1.1.6. If K is discretely valued, both $\mathcal{R}_K^{\mathrm{bd}}$ and $\mathcal{E}_K$ are discretely valued fields.

Definition 1.1.7. For any interval $I\subseteq(0,\infty]$, we equip $\mathcal{R}_K^I$ with the Fréchet topology defined by $|\cdot|_s$ for all $s\in I$, and $\mathcal{R}_K^I$ is complete for this topology. If $I=[r_1,r_2]$ is a closed interval, $\mathcal{R}_K^I$ becomes a K-Banach algebra with the norm $\max\{|\cdot|_{r_1},|\cdot|_{r_2}\}$. We equip $\mathcal{R}_K=\bigcup_{r>0}\mathcal{R}_K^r$ with the locally convex inductive limit topology (in the sense of [5, §II.4]). In particular, a sequence converges in $\mathcal{R}_K$ if and only if it is a convergent sequence in $\mathcal{R}_K^r$ for some $r>0$. For any $r>0$, we equip $\mathcal{R}_K^{\mathrm{bd},r}$ with the norm $\max\{|\cdot|,|\cdot|_r\}$ under which it is a K-Banach algebra. The topology defined by this norm is the weakest topology such that the natural maps $\mathcal{R}_K^{\mathrm{bd},r}\to\mathcal{R}_K^r$ and $\mathcal{R}_K^{\mathrm{bd},r}\to\mathcal{E}_K$ are continuous. We equip $\mathcal{R}_K^{\mathrm{bd}}=\bigcup_{r>0}\mathcal{R}_K^{\mathrm{bd},r}$ with the locally convex inductive limit topology.

Proposition 1.1.8. $\mathcal{R}_K^{\times}=(\mathcal{R}_K^{\mathrm{bd}})^{\times}$. *In particular, if K is discretely valued, the units of $\mathcal{R}_K$ are precisely the non-zero elements of $\mathcal{R}_K^{\mathrm{bd}}$.*

Proof. Note that for any $f = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K^r$, $s \mapsto w_s(f)$ is a concave function on $(0, r]$. Suppose that f is a unit in $\mathcal{R}_K^r$ with inverse g. It follows that the sum of two concave functions $w_s(f), w_s(g)$ is the constant function 0. Thus both $w_s(f), w_s(g)$ are affine in s. Hence $v(a_i) = \lim_{s\to 0^+}(v(a_i) + si) \geqslant \lim_{s\to 0^+} w_s(f)$ for any a_i, yielding that $f \in \mathcal{R}_K^{\mathrm{bd},r}$. □

1.2. φ-modules over the Robba ring

Definition 1.2.1. Fix an integer $q > 1$. A *relative (q-power) Frobenius lift* on the Robba ring $\mathcal{R}_K$ is a homomorphism $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ of the form

$$\sum_{i\in\mathbb{Z}} a_i T^i \mapsto \sum_{i\in\mathbb{Z}} \varphi_K(a_i) S^i,$$

where φ_K is an isometric endomorphism on K and S lies in $\mathcal{R}_K^{\mathrm{int}}$ satisfying $w(S - T^q) > 0$. If q is a power of p, we define an *absolute (q-power) Frobenius lift* as a relative Frobenius lift for which φ_K is a q-power Frobenius lift.

Remark 1.2.2. Note that $w(T^{-q}(S - T^q)) > 0$. Thus by Proposition 1.1.3, we have $w_r(T^{-q}(S - T^q)) > 0$ for r sufficiently small. This yields $w_r(S) = w_r(T^q) = qr$; hence φ maps $\mathcal{R}_K^r$ to $\mathcal{R}_K^{\mathrm{qr}}$ for r sufficiently small.

Henceforth we fix a relative Frobenius lift φ on $\mathcal{R}_K$ such that φ_K is an automorphism. From Definition 1.2.1, it is clear that φ restricts to an isometry on $\mathcal{R}_K^{\mathrm{bd}}$ with respect to $|\cdot|$. Hence φ extends to an automorphism on $\mathcal{E}_K$ by continuity which we again denote by φ.

Definition 1.2.3. A *difference algebra/field* is an algebra/field R equipped with an endomorphism φ. We say that R is *inversive* if φ is an automorphism. A *difference module* over R is a finite free R-module M equipped with an R-linear map $\varphi^*M \to M$, which we also think of as a semilinear action φ on M; the semilinearity means that for $r \in R$ and $m \in M$, $\varphi(rm) = \varphi(r)\varphi(m)$. We say that M is *dualizable* if $\varphi^*M \to M$ is an isomorphism. By a *φ-module* over R we mean a dualizable difference module over R.

Definition 1.2.4. For any φ-module M over a difference algebra R, we define $\mathrm{H}^0(M)$ and $\mathrm{H}^1(M)$ by setting $\mathrm{H}^0(M) = M^{\varphi=1}$ and $\mathrm{H}^1(M) = \frac{M}{(\varphi-1)M}$ respectively. It is clear that $\mathrm{H}^1(M)$ classifies the extensions of the trivial φ-module R by M in the category of φ-modules over R.

Definition 1.2.5. For any $R \in \{\mathcal{E}_K, \mathcal{R}_K^{\mathrm{bd}}, \mathcal{R}_K\}$, if M is a φ-module over R of rank $n > 0$, let v be a generator of $\wedge^n M$, and suppose that $\varphi(v) = \lambda v$ for some $\lambda \in R^\times$. It follows from Proposition 1.1.8 that $R^\times \subseteq \mathcal{E}_K^\times$. We then define the *degree* of M by setting $\deg(M) = -w(\lambda)$ which is independent of the choice of v because φ is an isometry on $\mathcal{E}_K$, and we define the *slope* of M by setting $\mu(M) = \deg(M)/\mathrm{rank}\,(M)$.

Remark 1.2.6. The sign convention used here for degrees of φ-modules is opposite to that used in the previous work of Kedlaya [13–15]. We change it here to match the sign

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convention used in the coming work [18] which matches the sign convention used in geometric invariant theory, in which the ample line bundle $\mathcal{O}(1)$ on any projective space has degree 1.

Definition 1.2.7. For any difference algebra R over K and any $n \in \mathbb{Z}$, define the rank 1 φ-module $R(n)$ by setting the φ-action as

$$\varphi(rv) = \pi^{-n}\varphi(r)v, \quad r \in R \tag{1.2.7.1}$$

for some generator v. For any φ-module M over R, set the φ-module as $M(n) = M \otimes_R R(n)$.

Lemma 1.2.8. *There exists an $r_\varphi > 0$ such that for any $a \in \widehat{\overline{K}}$ with $0 < v(a) < r_\varphi$, the equation $\varphi(T) = a$ has q roots (with multiplicity) in $\widehat{\overline{K}}$. Furthermore, each of the roots has valuation $v(a)/q$.*

Proof. Note that the conditions $\varphi(T) \in \mathcal{R}_K^{\mathrm{int}}$ and $w(\varphi(T) - T^q) > 0$ imply that the Newton polygon for $\varphi(T)$ has a minimal positive slope r_0. It therefore follows that if $0 < v(a) < qr_0$, the Newton polygon for $\varphi(T) - a$ has $v(a)/q$ as the minimal positive slope with multiplicity q. Hence by the theory of Newton polygons, the equation $\varphi(T) = a$ has q roots (with multiplicity), and each of the roots has valuation $v(a)/q$. Therefore we can choose r_φ to be qr_0. □

Lemma 1.2.9. *Let f be a non-zero element of $\mathcal{R}_K$. If $\varphi(f) = \lambda f$ for some $\lambda \in \mathcal{R}_K^\times$ with $w(\lambda) \leqslant 0$, then $f \in \mathcal{R}_K^\times$.*

Proof. We first get that $f \in \mathcal{R}_K^{\mathrm{bd}}$ by [15, Proposition 1.2.6] (although it is proved under the hypothesis that K is discretely valued, the proof works in our situation). For any $g \in \mathcal{R}_K^{\mathrm{bd},r}$, it follows from [16, Lemma 8.2.6(c)] that $g \in (\mathcal{R}_K^{\mathrm{bd},r})^\times$ if and only if its Newton polygon has no slopes in $[0, r]$. Now suppose that the contrary of the lemma is true. We may choose some $0 < r_0 < r_\varphi$ such that f has a root of valuation r_0 and $\lambda \in (\mathcal{R}_K^{\mathrm{bd},r_0})^\times$. We then deduce from the equality $\varphi(f) = \lambda f$ and Lemma 1.2.8 that f has at least q roots with valuation r_0/q. Iterating this argument, for any $n \in \mathbb{N}$, we get that f has at least q^n roots (with multiplicity) with valuation r_0/q^n. Since $\sum_{n\in\mathbb{N}} q^n \times (r_0/q^n) = \infty$, we get that the sum of the slopes of f in $[0, r_0]$ is not finite. However, since f is a non-zero element of $\mathcal{R}_K^{\mathrm{bd}}$, the sum of its slopes in $[0, r_0]$ is finite. This yields a contradiction. □

Definition 1.2.10. Let $R \in \{\mathcal{E}_K, \mathcal{R}_K\}$, and let M be a non-zero φ-module over R. We say that M is *semistable* if $\mu(N) \leqslant \mu(M)$ for any non-zero φ-submodule N. We say that M is *stable* if $\mu(N) < \mu(M)$ for any proper non-zero φ-submodule N.

Proposition 1.2.11. *Any rank 1 φ-module M over $\mathcal{R}_K$ is stable.*

Proof. On tensoring M with $M^\vee = \mathrm{Hom}_{\mathcal{R}_K}(M, \mathcal{R}_K)$, this reduces to proving the proposition for $M \cong \mathcal{R}_K$. Now suppose that $N \subseteq \mathcal{R}_K$ is a non-zero φ-submodule such that $\mu(N) \geqslant \mu(\mathcal{R}_K) = 0$. Choose a generator f of N, and write $\varphi(f) = \lambda f$ for some $\lambda \in \mathcal{R}_K^\times$;

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then $w(\lambda) \leqslant 0$ since $\mu(N) \geqslant 0$. It thus follows from Lemma 1.2.9 that f is invertible in $\mathcal{R}_K$, yielding $N = \mathcal{R}_K$. In other words, $\mu(N) < \mu(\mathcal{R}_K)$ unless $N = \mathcal{R}_K$, as desired. □

Corollary 1.2.12. *Suppose that $N \subseteq M$ are two φ-modules over $\mathcal{R}_K$ of the same rank; then $\mu(N) \leqslant \mu(M)$, with equality if and only if $N = M$.*

Proof. Suppose that rank $M = n$. Then apply the above proposition to $\wedge^n N \subseteq \wedge^n M$. □

Definition 1.2.13. Let $R \in \{\mathcal{E}_K, \mathcal{R}_K\}$. For any non-zero φ-module M over R, a *semistable filtration* of M is a filtration $0 = M_0 \subset M_1 \cdots \subset M_l = M$ of M by saturated φ-submodules, such that each successive quotient M_i/M_{i-1} is a semistable φ-module of some slope s_i. The *slope multiset* of a semistable filtration of M is the multiset in which each slope of a successive quotient occurs with multiplicity equal to the rank of that quotient, and we call the associated Newton polygon of the slope multiset (see [14, Definition 3.5.1]) the *slope polygon* of this filtration.

Proposition 1.2.14. *If K is spherically complete, then every non-zero φ-module M over $\mathcal{R}_K$ has a unique maximal φ-submodule which has the maximal slope. Furthermore, it is semistable and saturated.*

Proof. The spherical completeness of K implies that $\mathcal{R}_K$ is a Bézout domain by [21, Théorème 2]. Hence if N is a finite $\mathcal{R}_K$-submodule of M, both N and its saturation are finite free $\mathcal{R}_K$-modules. It follows that the saturation of any φ-submodule of M and the sum of any two φ-submodules of M are still φ-submodules of M.

We proceed by induction on the rank of M. The initial case follows from Proposition 1.2.11. Now suppose that rank $M = d$ for some $d \geqslant 2$ and the proposition is true for φ-modules having rank $\leqslant d-1$. Let $\mu(M) = s$. If M is semistable, then we are done. Otherwise, let P be a φ-submodule of slope $>s$ and of maximal rank. By Corollary 1.2.12, the saturation $\widetilde{P}$ of P satisfies $\mu(\widetilde{P}) \geqslant \mu(P)$. Replacing P with $\widetilde{P}$, we may suppose that P is saturated. Hence rank $P \leqslant d-1$; otherwise we must have $P = M$, yielding $\mu(M) = \mu(P) > s$ which is a contradiction. By the inductive assumption, P has a unique maximal φ-submodule P_1 which has the maximal slope. We claim that P_1 is also the unique maximal φ-submodule of M which has the maximal slope. Suppose that the contrary of the claim is true. Let Q be a φ-submodule of M such that either $\mu(Q) > \mu(P_1)$ or $\mu(Q) = \mu(P_1)$ and $Q \nsubseteq P_1$; then $\mu(Q) \geqslant \mu(P_1)$ and $Q \nsubseteq P$. Consider the following exact sequence of φ-submodules:

$$0 \longrightarrow P \cap Q \longrightarrow P \oplus Q \longrightarrow P + Q \longrightarrow 0.$$

Since $\mu(P \cap Q) \leqslant \mu(P_1) \leqslant \mu(Q)$, we get

$$\begin{aligned}
\deg(P+Q) &= \mathrm{rank}\,(P)\mu(P) + \mathrm{rank}\,(Q)\mu(Q) - \mathrm{rank}\,(P \cap Q)\mu(P \cap Q) \\
&\geqslant \mathrm{rank}\,(P)\mu(P) + (\mathrm{rank}\,(Q) - \mathrm{rank}\,(P \cap Q))\mu(Q) \\
&\geqslant (\mathrm{rank}\,(P) + \mathrm{rank}\,(Q) - \mathrm{rank}\,(P \cap Q))\mu(P) \\
&= \mathrm{rank}\,(P+Q)\mu(P);
\end{aligned}$$

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hence $\mu(P+Q) \geqslant \mu(P) > s$. However, since P is saturated and $Q \not\subseteq P$, it follows that $\operatorname{rank}(Q+P) > \operatorname{rank} P$ which contradicts the maximality of $\operatorname{rank} P$. This yields the claim which finishes the inductive step.

Now let $P \subseteq M$ be the unique maximal φ-submodule which has the maximal slope. The saturation $\widetilde{P}$ of P satisfies $\mu(\widetilde{P}) \geqslant P$. This forces $\widetilde{P} = P$ by the maximality of P. Hence P is saturated. The semistability of P follows directly. □

Theorem 1.2.15. *If K is spherically complete, then every φ-module M over $\mathcal{R}_K$ admits a unique HN filtration.*

Proof. The uniqueness follows from formal properties of slopes and Corollary 1.2.12. In fact, by the definition of HN filtration, for any $i \geqslant 1$, M_i can be characterized as the preimage of the unique maximal φ-submodule of M/M_{i-1} which has the maximal slope. This also suggests a way of showing the existence. We take M_1 to be the maximal φ-submodule of M which has the maximal slope. Since M_1 is saturated by Proposition 1.2.14, M/M_1 is a φ-module over $\mathcal{R}_K$. Then we take M_2 to be the preimage of the maximal φ-submodule of M/M_1 which has the maximal slope. Iterating this process, we get the HN filtration of M. □

Definition 1.2.16. Suppose that K is spherically complete. For any φ-module M over $\mathcal{R}_K$, the slopes s_i of the HN filtration are called the *slopes* of M and the slope polygon of the HN filtration is called the *HN-polygon* of M.

Proposition 1.2.17. *Suppose that K is spherically complete. The for any φ-module M over $\mathcal{R}_K$, the HN-polygon of M lies above the slope polygon of any semistable filtration of M, with the same endpoint.*

Proof. This is a formal consequence of the definition of HN filtration. We refer to [14, Proposition 3.5.4] for a proof. We caution that both our sign convention of slopes and the definition of slope polygons are 'opposite' to those used in [14]. □

Definition 1.2.18. Let M be a φ-module over $\mathcal{E}_K$ (resp. $\mathcal{R}_K^{\mathrm{bd}}$). For $c, d \in \mathbb{Z}$ with $d > 0$, a (c, d)-*pure model* of M is a finite free $\mathcal{O}_{\mathcal{E}_K}$-submodule (resp. $\mathcal{R}_K^{\mathrm{int}}$-submodule) M_0 of M with $M_0 \otimes_{\mathcal{O}_{\mathcal{E}_K}} \mathcal{E}_K = M$ (resp. $M_0 \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K^{\mathrm{bd}} = M$), so the φ-action on M induces an isomorphism $\pi^c (\varphi^d)^* M_0 \cong M_0$. For a φ-module M over $\mathcal{R}_K$, a (c, d)-*pure model* of M is a $\mathcal{R}_K^{\mathrm{int}}$-submodule M_0 with $M_0 \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K = M$, so $M_0 \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K^{\mathrm{bd}}$ is stable under φ and the φ-action induces an isomorphism $\pi^c (\varphi^d)^* M_0 \cong M_0$. For $s \in \mathbb{Q}$, we say that M is *pure of slope s* if M admits a (c, d)-pure model for some (and hence any) $c, d \in \mathbb{Z}$ with $d > 0$ and $s = c/d$. If $s = 0$, we also say that M is *étale*, and a $(0, 1)$-pure model is also called an *étale model*.

Proposition 1.2.19. *If M is a pure φ-module over $\mathcal{R}_K$, then M is semistable.*

Proof. We follow the proof of [15, Theorem 1.6.10(a)]. Suppose that M admits a φ-submodule N such that $\mu(N) > \mu(M)$. By replacing M with $\wedge^{\operatorname{rank} N} M$, we may assume that $\operatorname{rank} N = 1$. By twisting, we may further assume that N is trivial. Hence $\mathrm{H}^0(N) \neq 0$.

Choose a non-zero φ-invariant vector $v \in N$. By replacing φ with φ^a for a suitable positive integer a, we may assume that $\mu(M) = n \in \mathbb{Z}_{<0}$. We choose a $(n, 1)$-pure model M_0 of M. Let $\mathrm{e} = \{e_1, \ldots, e_m\}$ be a basis of M_0, and write $\varphi(e_i) = \sum_{j=1}^m e_j F_{ji}$ for $F_{ji} \in \mathcal{R}_K^{\mathrm{bd}}$. By the definition of pure models, we see that $w(F_{ji}) \geqslant -n$ for all j, i. By [15, Proposition 1.5.4] (this proposition ultimately relies on [15, Proposition 1.2.6] whose proof works for general K), we get that $v \in M_0 \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K^{\mathrm{bd}}$. Write $v = \sum_{i=1}^m c_i e_i$; then $\varphi(v) = v$ implies $c_i = \sum_{i=1}^m F_{ij}\varphi(c_j)$. This yields $\min_i\{w(c_i)\} \geqslant -n + \min_j\{w(c_j)\}$ which is a contradiction. □

The converse of Proposition 1.2.19 is more difficult. It is only known for discretely valued K thanks to the following slope filtration theorem of Kedlaya [15, Theorem 1.7.1].

Theorem 1.2.20. *If K is discretely valued, then every semistable φ-module over $\mathcal{R}_K$ is pure. In particular, every φ-module M over $\mathcal{R}_K$ admits a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ by saturated φ-submodules whose successive quotients are pure with $\mu(M_1/M_0) > \cdots > \mu(M_l/M_{l-1})$.*

The following propositions will be used later.

Proposition 1.2.21. *Suppose that K is discretely valued, and that M is a pure φ-module over $\mathcal{R}_K$. If M_1 and M_2 are two pure models of M, then $M_1 \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K^{\mathrm{bd}} = M_2 \otimes_{\mathcal{R}_K^{\mathrm{int}}} \mathcal{R}_K^{\mathrm{bd}}$.*

Proof. This follows from [15, Proposition 1.5.5]. □

Proposition 1.2.22. *Suppose that K is discretely valued, and let M be a φ-module over $\mathcal{R}_K$. The following are true.*

(1) *Let a be a positive integer. Then M is semistable of slope s if and only if it is semistable of slope as as a φ^a-module.*
(2) *Suppose that M has slopes $s_1 \geqslant \cdots \geqslant s_n$ counted with multiplicity. Then for any $1 \leqslant d \leqslant n$, the slope multiset of $\wedge^d M$ is $\{s_{i_1} + \cdots + s_{i_d} | 1 \leqslant i_1 < \cdots < i_d \leqslant n\}$.*

Proof. For (1), it suffices to show that M is pure of slope s if and only if it is pure of slope as as a φ^a-module; this is [15, Lemma 1.6.3]. For (2), see [15, Remark 1.7.2]. □

It is clear that the purity of φ-modules is preserved by tensor products. Hence for discretely valued K, it follows from Theorem 1.2.20 that the semistability of φ-modules over $\mathcal{R}_K$ is also preserved by tensor products.

Question 1.2.23. *Suppose that K is merely spherically complete. Do we still have the equivalence of purity and semistability for φ-modules over $\mathcal{R}_K$? If this fails to be true, is the semistability of φ-modules over $\mathcal{R}_K$ still preserved by tensor products?*

1.3. Dieudonné–Manin decomposition

Definition 1.3.1. Let R be a difference algebra. A difference module over R is *trivial* if it admits a φ-invariant basis. We say that R is *weakly difference-closed* if every dualizable difference module over R is trivial. We say that R is *strongly difference-closed* if R is inversive and weakly difference-closed.

We fix a difference field F which is complete for a φ-invariant non-Archimedean absolute value $|\cdot|_F$. Then φ induces an endomorphism on the residue field k_F of F; we view k_F as a difference field with this endomorphism.

Lemma 1.3.2. *Suppose that F is spherically complete. Then the following are true.*

(1) *If k_F is weakly difference-closed, then for any $a \in F$, there exists $x \in F$ with $|x|_F = |a|_F$ such that $\varphi(x) - x = a$.*
(2) *If k_F is inversive, so is F.*

Proof. We first prove (1). We equip F with a partial order: for any $x, y \in F$, we say that $x > y$ if

$$|\varphi(x) - x - a|_F < |x - y|_F \leqslant |\varphi(y) - y - a|_F.$$

We first show that the set $\{x||x|_F \leqslant |a|_F\}$ has a maximal element. Suppose that $x_1 < x_2 < \ldots$ is an infinite chain in $\{x||x|_F \leqslant |a|_F\}$. Let $r_i = |\varphi(x_i) - x_i - a|_F$. It follows that $\mathrm{B}(x_1, r_1) \supset \mathrm{B}(x_2, r_2) \supset \cdots$. Since F is spherically complete, $\bigcap_{i=1}^{\infty} \mathrm{B}(x_i, r_i)$ is non-empty. Pick an $x_0 \in \bigcap_{i=1}^{\infty} \mathrm{B}(x_i, r_i)$. Then $|x_0 - x_i|_F \leqslant r_i$ for any $i \geqslant 1$. Hence

$$|\varphi(x_0) - x_0 - a|_F = |(\varphi(x_i) - x_i - a) + \varphi(x_0 - x_i) - (x_0 - x_i)|_F \leqslant r_i$$

for any $i \geqslant 1$. On the other hand, since $|x_i - x_{i+1}|_F > r_{i+1}$, we get $|x_0 - x_i|_F = |(x_0 - x_{i+1}) - (x_i - x_{i+1})|_F = |x_i - x_{i+1}|_F$. Hence

$$|\varphi(x_0) - x_0 - a|_F \leqslant r_{i+1} < |x_i - x_{i+1}|_F = |x_0 - x_i|_F,$$

yielding $x_0 > x_i$ for any $i \geqslant 1$. We therefore prove the claim by Zorn's lemma. Let x' be a maximal element of the set $\{x||x|_F \leqslant |a|_F\}$. We claim that $\varphi(x') - x' = a$. If this is not the case, let $b = \varphi(x') - x' - a$. Since k_F is weakly difference-closed, by [16, Lemma 14.3.3(b),(c)], we may choose some y with $|y|_F = 1$ such that $|\frac{\varphi(b)}{b}\varphi(y) - y + 1|_F < 1$. Let $y' = x' + by$. Then

$$|\varphi(y') - y' - a|_F = \left| b\left(\frac{\varphi(b)}{b}\varphi(y) - y + 1\right)\right|_F < |b|_F = |x' - y'|_F = |\varphi(x') - x' - a|_F.$$

This implies that $y' > x'$ which contradicts the maximality of x. Hence $\varphi(x') - x' = a$; it is clear that $|x'|_F = |a|_F$.

The proof of (2) is similar. Let $a \in F$. We equip F with a partial order: for any $x, y \in F$, we say that $x > y$ if

$$|\varphi(x) - a|_F < |x - y|_F \leqslant |\varphi(y) - a|_F.$$

Suppose that $x_1 < x_2 < \cdots$ is an infinite chain in $\{x||x|_F \leqslant |a|_F\}$. Let $r_i = |\varphi(x_i) - a|_F$. It follows that $\mathrm{B}(x_1, r_1) \supset \mathrm{B}(x_2, r_2) \supset \cdots$. Pick an $x_0 \in \bigcap_{i=1}^{\infty} \mathrm{B}(x_i, r_i)$. Then $|x_0 - x_i|_F \leqslant r_i$ for any $i \geqslant 1$. A similar argument shows that $x_0 > x_i$ for any $i \geqslant 1$. By Zorn's lemma we choose a maximal element $x' \in \{x||x|_F \leqslant |a|_F\}$. Now suppose that $b' = \varphi(x') - a$ is non-zero. Since k_F is inversive, we may choose some $y \in F$ with $|y|_F = 1$ such that $|\frac{\varphi(b)\varphi(y)}{b} - 1|_F < 1$. It follows that $x' - by > x'$ which is a contradiction. Hence $\varphi(x') = a$. □

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Convention 1.3.3. For any valuation v (resp. norm $|\cdot|$) and a matrix $A=(A_{ij})$, we use $v(A)$ (resp. $|A|$) to denote the minimal valuation (resp. maximal norm) among the entries.

Lemma 1.3.4. *If F is spherically complete, and if k_F is weakly difference-closed, then for any $A\in \mathrm{GL}_d(\mathcal{O}_F)$, there exists $U\in \mathrm{GL}_d(\mathcal{O}_F)$ such that $U^{-1}A\varphi(U)=I_d$.*

Proof. The reduction of A in $\mathrm{GL}_d(k_F)$ defines a dualizable difference module over k_F. Since k_F is weakly difference-closed, this module is trivial. This implies that we may choose some $U_1\in \mathrm{GL}_d(\mathcal{O}_F)$ such that $|U_1^{-1}A\varphi(U_1)-I_d|_F=c<1$. We will inductively construct a convergent sequence $U_1,U_2,\ldots\in \mathrm{GL}_d(\mathcal{O}_F)$ such that

$$|U_i-U_{i+1}|_F\leqslant c^i,\quad |U_i^{-1}A\varphi(U_i)-I_d|_F\leqslant c^i$$

for every $i\geqslant 1$. Choose $u\in F$ such that $|u|_F=c$. Given U_i, by Lemma 1.3.2(1), there exists some $X_i\in \mathrm{M}_d(u^i\mathcal{O}_F)$ such that

$$\varphi(X_i)-X_i+(U_i^{-1}A\varphi(U_i)-I_d)=0.$$

Put $U_{i+1}=U_i(I_d+X_i)$. It follows that $U_{i+1}\in \mathrm{GL}_d(\mathcal{O}_F)$ and $|U_{i+1}-U_i|_F\leqslant c^i$, $|U_{i+1}^{-1}A\varphi(U_{i+1})-I_d|_F\leqslant c^{i+1}$. Let $U=\lim_{i\to\infty}U_i$. Then $U^{-1}A\varphi(U)=I_d$. □

Definition 1.3.5. Let R be a difference algebra. For $\lambda\in R$ and a positive integer d, define $V_{\lambda,d}$ to be the difference module over R with a basis $e_1,\ldots,e_d$ so that

$$\varphi(e_1)=e_2,\ldots,\varphi(e_{d-1})=e_d,\quad \varphi(e_d)=\lambda e_1;$$

and any such a basis is called a *standard basis* of $V_{\lambda,d}$. For a difference module V over R, a *Dieudonné–Manin decomposition* of V is a direct sum decomposition $V\cong\oplus_{i=1}^n V_{\lambda_i,d_i}$ for some λ_i,d_i, $1\leqslant i\leqslant n$.

The following theorem generalizes the usual Dieudonné–Manin classification theorem for difference modules over complete discretely valued difference fields (e.g. [16, Theorem 14.6.3]) to difference modules over spherically complete difference fields.

Theorem 1.3.6. *If F is spherically complete, and if k_F is strongly difference-closed, then every dualizable difference module V over F admits a Dieudonné–Manin decomposition.*

Proof. First note that if V is pure of spectral norm 1 (see [16, Definition 14.4.6] for the definition), then V admits a basis on which φ acts via an element of $\mathrm{GL}(\mathcal{O}_F)$ ([16, Proposition 14.4.16]); hence V is trivial by Lemma 1.3.4. Furthermore, Lemma 1.3.2(1) implies that $\mathrm{H}^1(V)=V/(\varphi-1)$ is trivial in this case.

Now we follow the lines of the proof of [16, Theorem 14.6.3]. By Lemma 1.3.2(2), F is inversive. Hence by [16, Theorem 14.4.13], the task reduces to showing the theorem for those V which are pure of spectral norm $s>0$. Let m be the smallest positive integer such that $s^m\in|F|_F$, and choose $u\in F$ such that $|u|_F=s^m$. Then the first paragraph implies that $u^{-1}\varphi^m$ fixes some non-zero vector v of V. This induces a non-zero map from $V_{u,m}$ to V. It follows from [16, Lemma 14.6.2] that $V_{u,m}$ is irreducible. Hence this map is injective. Repeating this argument we get that V is a successive extension of copies of $V_{u,m}$. Note that $\mathrm{Ext}^1(V_{u,m}, V_{u,m}) = \mathrm{H}^1(V_{u,m}^\vee \otimes V_{u,m}) = 0$ since $V_{u,m}^\vee \otimes V_{u,m}$ is pure of spectral norm 1. We thus deduce that V is a direct sum of copies of $V_{u,m}$. □

1.4. Extended base rings

Henceforth we assume that K is discretely valued and π is a uniformizer of K.

Definition 1.4.1. For any interval $I \subset (0, \infty]$, let $\widetilde{\mathcal{R}}_K^I$ be the set of formal sums

$$f = \sum_{i\in\mathbb{Q}} a_i u^i$$

with $a_i \in K$ satisfying the following conditions.

(1) For any $c > 0$, the set of $i \in \mathbb{Q}$ such that $|a_i| \geqslant c$ is well-ordered (has no infinite decreasing subsequence).

(2) For any $s \in I$, $v(a_i) + si \to \infty$ as $i \to \pm\infty$, and $\inf_{i\in\mathbb{Q}}\{v(a_i) + si\} > -\infty$.

Then $\inf_{i\in\mathbb{Q}}\{v(a_i) + si\}$ is attained at some i because K is discretely valued. These series form a ring under formal series addition and multiplication. For any $s \in I$, set the valuation $w_s(f) = \min_{i\in\mathbb{Q}}\{v(a_i) + si\}$, and the corresponding multiplicative norm $|f|_s = \max_{i\in\mathbb{Q}}\{|a_i|p^{-si}\}$. We denote $\widetilde{\mathcal{R}}_K^{(0,r]}$ by $\widetilde{\mathcal{R}}_K^r$ for simplicity. We call the union $\widetilde{\mathcal{R}}_K = \bigcup_{r>0} \widetilde{\mathcal{R}}_K^r$ the *extended Robba ring* over K. We view $\widetilde{\mathcal{R}}_K$ as a difference algebra over K with the endomorphism $\varphi(f) = \sum_{i\in\mathbb{Q}} \varphi_K(a_i)u^{qi}$.

Remark 1.4.2. The definition of the extended Robba ring in [15] misses the second part of condition (2) of Definition 1.4.1.

Definition 1.4.3. For any $r > 0$, let $\widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$ be the subring of $\widetilde{\mathcal{R}}_K^r$ consisting of series with $\{v(a_i)\}_{i\in\mathbb{Q}}$ bounded below. Let $\widetilde{\mathcal{R}}_K^{\mathrm{bd}} = \bigcup_{r>0} \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$. We equip $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ with the valuation $w(f) = \min_{i\in\mathbb{Q}}\{v(a_i)\}$ and the corresponding multiplicative norm $|f| = \max_{i\in\mathbb{Q}}\{|a_i|\}$. Let $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$ be the valuation ring of $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$, and let $\widetilde{\mathcal{R}}_K^{\mathrm{int},r} = \widetilde{\mathcal{R}}_K^{\mathrm{int}} \cap \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$. We call $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ the *extended bounded Robba ring* over K.

Definition 1.4.4. Let $\widetilde{\mathcal{E}}_K$ be the ring of formal sums

$$f = \sum_{i\in\mathbb{Q}} a_i u^i$$

with coefficients in K satisfying the following conditions.

(1) For each $c > 0$, the set of $i \in \mathbb{Q}$ such that $|a_i| \geqslant c$ is well-ordered.

(2) The set $\{v(a_i)\}_{i\in\mathbb{Q}}$ is bounded below and $v(a_i) \to \infty$ as $i \to -\infty$.

We equip $\widetilde{\mathcal{E}}_K$ with the valuation $w(f) = \min_{i\in\mathbb{Q}}\{v(a_i)\}$ and the corresponding multiplicative norm $|f| = \max_{i\in\mathbb{Q}}\{|a_i|\}$. Let $\mathcal{O}_{\widetilde{\mathcal{E}}_K}$ be the valuation ring of $\widetilde{\mathcal{E}}_K$.

Remark 1.4.5. It is clear that $\widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$ is the subring of $\widetilde{\mathcal{E}}_K$ consisting of series such that $v(a_i) + ri \to \infty$ as $i \to -\infty$. The natural inclusion $\widetilde{\mathcal{R}}_K^{\mathrm{bd}} \to \widetilde{\mathcal{E}}_K$ is an isometry with respect

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to w, and identifies $\widetilde{\mathcal{E}}_K$ with the w-completion of $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$. The restriction of φ on $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ is an isometry with respect to w, and we still denote by φ its continuous extension to $\widetilde{\mathcal{E}}_K$. We view $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ and $\widetilde{\mathcal{E}}_K$ as difference fields with the endomorphism φ.

Definition 1.4.6. For any interval $I \subseteq (0, \infty]$, we equip $\widetilde{\mathcal{R}}_K^I$ with the Fréchet topology defined by $|\cdot|_s$ for all $s \in I$; $\widetilde{\mathcal{R}}_K^I$ is complete for this topology. If $I = [r_1, r_2]$ is a closed interval, then $\widetilde{\mathcal{R}}_K^I$ becomes a K-Banach algebra with norm $\max\{|\cdot|_{r_1}, |\cdot|_{r_2}\}$. We equip $\widetilde{\mathcal{R}}_K = \bigcup_{r>0} \widetilde{\mathcal{R}}_K^r$ with the locally convex inductive limit topology. For any $r > 0$, we equip $\widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$ with the norm $\max\{|\cdot|, |\cdot|_r\}$; it is a K-Banach algebra under this norm. We equip $\widetilde{\mathcal{R}}_K^{\mathrm{bd}} = \bigcup_{r>0} \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$ with the locally convex inductive limit topology.

Definition 1.4.7. For S a commutative ring, let $S((u^{\mathbb{Q}}))$ denote the Hahn–Malcev–Neumann algebra of generalized power series $\sum_{i\in\mathbb{Q}} c_i u^i$, where each $c_i \in S$ and the set of i with $c_i \neq 0$ is well-ordered; these series form a ring under formal series addition and multiplication.

Remark 1.4.8. It is clear that the residue fields of $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ and $\widetilde{\mathcal{E}}_K$ are isomorphic to $k((u^{\mathbb{Q}}))$.

Proposition 1.4.9. *The extended Robba ring* $\widetilde{\mathcal{R}}_K$ *is a Bézout domain, and the units of* $\widetilde{\mathcal{R}}_K$ *are precisely the non-zero elements of* $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$*. As a consequence,* $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ *(and hence* $\widetilde{\mathcal{E}}_K$*) is a discretely valued field.*

Proof. As explained in [15, Remark 2.2.5], $\widetilde{\mathcal{R}}_K$ is the analytic ring with residue field $k((u^{\mathbb{Q}}))$ in the sense of [14, §2.4], on taking φ_K to be an absolute Frobenius lift on K. The proposition then follows from [14, Theorem 2.9.6] and [14, Lemma 2.4.7]. □

Remark 1.4.10. It follows from [15, Proposition 2.2.6] that there is a φ-equivariant embedding $\tau_K : \mathcal{R}_K \to \widetilde{\mathcal{R}}_K$ such that for r sufficiently small, $\mathcal{R}_K^r$ maps to $\widetilde{\mathcal{R}}_K^r$ preserving w_r. It thus follows that τ_K maps $\mathcal{R}_K^{\mathrm{bd}}$ to $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ preserving w. Hence τ_K induces an φ-equivariant embedding from $\mathcal{E}_K$ to $\widetilde{\mathcal{E}}_K$ by taking the completion; we still denote this embedding by τ_K. In this way, we view $\mathcal{R}_K^{\mathrm{bd}}$, $\mathcal{R}_K$, $\mathcal{E}_K$ as difference subalgebras of $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$, $\widetilde{\mathcal{R}}_K$, $\widetilde{\mathcal{E}}_K$ respectively.

Proposition 1.4.11. *If* k *is strongly difference-closed, so is* $k((u^{\mathbb{Q}}))$.

Proof. This is [16, Proposition 2.5.5]. □

1.5. φ-modules over the extended Robba ring

Definition 1.5.1. For any $R \in \{\widetilde{\mathcal{R}}_K, \widetilde{\mathcal{R}}_K^{\mathrm{bd}}, \widetilde{\mathcal{E}}_K\}$, let M be a φ-module over R of rank $n > 0$. Let v be a generator of $\wedge^n M$, and suppose that $\varphi(v) = \lambda v$ for some $\lambda \in R^\times \subseteq \widetilde{\mathcal{E}}_K^\times$. We define the *degree* of M by setting $\deg(M) = -w(\lambda)$ which is independent of the choice of v because φ is an isometry on $\widetilde{\mathcal{E}}_K$, and we define the *slope* of M by setting $\mu(M) = \deg(M)/\mathrm{rank}\,(M)$. Define *stable*, *semistable*, *semistable filtration*, *slope multiset*, *slope polygon*, *HN filtration*, (c, d)*-pure model*, *pure of slope* s, *étale*, *étale* *model* for φ-modules over R by changing $\mathcal{E}_K$, $\mathcal{R}_K^{\mathrm{bd}}$, $\mathcal{R}_K$ to $\widetilde{\mathcal{E}}_K$, $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$, $\widetilde{\mathcal{R}}_K$ respectively in Definitions 1.2.10, 1.2.13 and 1.2.18.

Definition 1.5.2. By an *extension* of K, we mean a field extension L of K which is complete for a discrete valuation extending the one on K, and is equipped with an isometric field automorphism φ_L extending φ_K. The extension L is called *admissible* if it has the same value group as K.

Lemma 1.5.3. *The field K admits an admissible extension L such that its residue field k_L is strongly difference-closed with respect to the reduction of φ_L.*

Proof. By [15, Proposition 3.2.4], K admits an admissible extension L such that k_L is weakly difference-closed. (The condition that any étale φ-module over L is trivial is equivalent to the condition that k_L is weakly difference-closed). Since L is inversive, *a fortiori* k_L is inversive. □

Proposition 1.5.4. *Every pure φ-module over $\widetilde{\mathcal{R}}_K$ is semistable.*

Proof. The proposition follows from [15, Theorem 1.6.10(a)]. □

Proposition 1.5.5. *Every φ-module over $\widetilde{\mathcal{R}}_K$ admits a unique HN filtration.*

Proof. The proposition follows from [15, Proposition 1.4.15]. □

Proposition 1.5.6. *If M is a φ-module over $\mathcal{R}_K$, and if L is an extension of K, then the HN filtration of M, tensored up with $\mathcal{R}_L$ (resp. $\widetilde{\mathcal{R}}_L$), gives the HN filtration of $M\otimes_{\mathcal{R}_K}\mathcal{R}_L$ (resp. $M\otimes_{\mathcal{R}_K}\widetilde{\mathcal{R}}_L$).*

Proof. It reduces to showing that if M is semistable, then its base changes are also semistable. Note that M is pure by Theorem 1.2.20. Thus its base changes are also pure; hence they are semistable by Propositions 1.2.19 and 1.5.4. □

Definition 1.5.7. For any φ-module M over $\widetilde{\mathcal{R}}_K$, the slopes of the successive quotients and the slope polygon of the HN filtration of M are called the *slopes* and the *HN-polygon* of M respectively.

The following theorem is a combination of [15, Proposition 2.1.6, Theorem 2.1.8].

Theorem 1.5.8. *If k is strongly difference-closed, then every semistable φ-module M over $\widetilde{\mathcal{R}}_K$ is pure. Furthermore, it admits a Dieudonné–Manin decomposition $M=\bigoplus V_{\lambda,d}$, so each λ is a power of π.*

Lemma 1.5.9. *Let $\alpha=\sum_{i\in\mathbb{Q}}a_iu^i\in\widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$ and $n\in\mathbb{N}$. Suppose that $\beta=\sum_{i\in\mathbb{Q}}b_iu^i\in\widetilde{\mathcal{E}}_K$ satisfies*

$$\varphi(\beta)-\pi^n\beta=\alpha. \tag{1.5.9.1}$$

Then for any $i<0$, we have $v(b_i)\geqslant\min_{j\leqslant i}\{v(a_j)\}$. As a consequence, we have $\beta\in\widetilde{\mathcal{R}}_K^{\mathrm{bd},qr}$. Furthermore, if $w(\beta)\geqslant w(\alpha)$, then $w_r(\beta)\geqslant\min\{w(\alpha),w_r(\alpha)\}$, and if moreover $n>0$, then $w(\beta)=w(\alpha)$.

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Proof. Suppose that there exists some $i_0 < 0$ such that $v(b_{i_0}) < \min_{j \leqslant i_0}\{v(a_j)\}$. By (1.5.9.1), we have

$$\varphi(b_{i_0}) - \pi^n b_{qi_0} = a_{qi_0}$$

by comparing the coefficients of u^{qi_0}. Since $v(b_{i_0}) < v(a_{qi_0})$, we get

$$v(b_{qi_0}) = v(b_{i_0}) - n \leqslant v(b_{i_0}) < \min_{j \leqslant i_0}\{v(a_j)\} \leqslant \min_{j \leqslant qi_0}\{v(a_j)\}.$$

Iterating this argument, we get $v(b_{q^m i_0}) = v(b_{i_0}) - mn$ for any $m \in \mathbb{N}$. Thus we get an infinite descending sequence $q^m i_0$ with $v(b_{q^m i_0})$ decreasing which contradicts the condition that $v(b_i) \to \infty$ as $i \to -\infty$. This proves the first statement of the lemma. Hence $\varphi(\beta) = \pi^n \beta + \alpha$ belongs to $\widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$, yielding $\beta \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},qr}$.

Note that

$$\begin{aligned} w_r(\varphi(\beta)) &= \min_{i \in \mathbb{Q}}\{v(\varphi_K(b_i)) + rqi\} = \min_{i \in \mathbb{Q}}\{q(v(b_i) + ri) - (q-1)v(b_i)\} \\ &\leqslant \min_{i \in \mathbb{Q}}\{q(v(b_i) + ri)\} - (q-1)w(\beta) = qw_r(\beta) - (q-1)w(\beta). \end{aligned}$$

Thus if $w(\beta) \geqslant w(\alpha)$ and $w_r(\beta) < \min\{w(\alpha), w_r(\alpha)\}$, then

$$w_r(\varphi(\beta)) \leqslant qw_r(\beta) - (q-1)w(\beta) < w_r(\beta) < w_r(\alpha).$$

This contradicts the condition $w_r(\varphi(\beta)) \geqslant \min\{w_r(\pi^n\beta), w_r(\alpha)\} \geqslant \min\{w_r(\beta), w_r(\alpha)\}$. If $n > 0$, then $w(\pi^n\beta) > w(\varphi(\beta)) = w(\beta)$; hence $w(\alpha) = \min\{w(\varphi(\beta)), w(\pi^n\beta)\} = w(\beta)$. □

For any difference algebra R with an automorphism φ, we set the twisted powers $a^{\{m\}}$ for any $m \in \mathbb{Z}$ and $a \in R$ through the two-way recurrence

$$a^{\{0\}} = 1, \quad a^{\{m+1\}} = \varphi(a^{\{m\}})a.$$

Lemma 1.5.10. *Suppose that k is strongly difference-closed. Then the following are true.*

(1) *Let $\alpha \in \widetilde{\mathcal{E}}_K$. If $n \neq 0$, then (1.5.9.1) admits a unique solution $\beta \in \widetilde{\mathcal{E}}_K$ which is*

$$\beta = -\sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(\alpha) \tag{1.5.10.1}$$

if $n < 0$, or

$$\beta = \sum_{m=0}^{\infty} (\pi^{-n})^{\{-m\}} \varphi^{-m-1}(\alpha) \tag{1.5.10.2}$$

if $n > 0$. Furthermore, if $n > 0$, then $w(\beta) = w(\alpha)$, and if $n < 0$, then $w(\beta) = w(\alpha) - n$. If $n = 0$, then (1.5.9.1) admits a solution $\beta \in \widetilde{\mathcal{E}}_K$ with $w(\beta) = w(\alpha)$.

(2) *Let $\alpha \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$. If $n > 0$, then (1.5.10.2) provides the unique solution $\beta \in \widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ of (1.5.9.1). Furthermore, we have $\beta \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},qr}$, $w(\beta) = w(\alpha)$ and $w_r(\beta) \geqslant \min\{w(\alpha), w_r(\alpha)\}$. If $n = 0$, then (1.5.9.1) admits a solution $\beta \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},qr}$ with $w(\beta) = w(\alpha)$, $w_s(\beta) \geqslant w_s(\alpha)$ for any $0 < s \leqslant r$.*

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(3) *If* $\alpha \in \widetilde{\mathcal{R}}_K^r$ *and* $n \geqslant 0$, *then* (*1.5.9.1*) *admits a solution* $\beta \in \widetilde{\mathcal{R}}_K^{\mathrm{qr}}$ *with* $w_r(\beta) \geqslant w_r(\alpha) - n$.

Proof. We first prove (1). Suppose that $n \neq 0$. Then it is clear that (1.5.10.1) and (1.5.10.2) provide a solution of (1.5.9.1). The uniqueness is obvious since φ preserves w. For $n = 0$, since k is strongly difference-closed, $k((u^{\mathbb{Q}}))$ is strongly difference-closed by Proposition 1.4.11. Therefore there exists $\beta \in \widetilde{\mathcal{E}}_K$ with $w(\beta) = w(\alpha)$ such that $\varphi(\beta) - \beta = \alpha$ by Lemma 1.3.2.

For (2), the case $n > 0$ follows from (1) and Lemma 1.5.9. Now suppose that $n = 0$ and $\alpha = \sum_{i\in\mathbb{Q}} a_i u^i \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$. Put $\alpha^+ = \sum_{i>0} a_i u^i \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$ and $\alpha^- = \sum_{i<0} a_i u^i \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$. Note that the infinite sum $\beta^+ = -\sum_{m=0}^{\infty} \varphi^m(\alpha^+)$ is convergent in $\widetilde{\mathcal{R}}_K^r$ and has bounded coefficients; hence $\beta^+ \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$. It is clear that $w(\beta^+) \geqslant w(\alpha^+) \geqslant w(\alpha)$. Furthermore, since α^+ has only positive powers of u, we get that $w_s(\beta^+) \geqslant w_s(\alpha^+) \geqslant w_s(\alpha)$ for any $0 < s \leqslant r$. Choose $b_0 \in K$ with $v(b_0) = v(a_0)$ such that $\varphi_K(b_0) - b_0 = a_0$. Choose $\beta^- \in \widetilde{\mathcal{E}}_K$ such that $w(\beta^-) = w(\alpha^-)$ and $\varphi(\beta^-) - \beta^- = \alpha^-$. We may suppose that β^- only has negative powers of u by dropping the non-negative powers. Write $\beta^- = \sum_{i<0} b_i u^i$. Then $v(b_i) \geqslant \min_{j\leqslant i}\{v(a_j)\}$ by Lemma 1.5.9, yielding $w(\beta^-) \geqslant w(\alpha^-) \geqslant w(\alpha)$ and $w_s(\beta^-) \geqslant w_s(\alpha^-) \geqslant w_s(\alpha)$ for any $0 < s \leqslant r$. Then $\beta = \beta^+ + \beta^- + b_0$ is a desired solution and satisfies $w_s(\beta) \geqslant w_s(\alpha)$ for any $0 < s \leqslant r$.

For (3), if $n = 0$ and $\alpha \in \widetilde{\mathcal{R}}_K^r$, write $\alpha = \sum_{i=1}^{\infty} \alpha_i$ such that $\alpha_i \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$, $w_s(\alpha_i) \geqslant w_s(\alpha)$ for each $i \geqslant 1$ and $w_s(\alpha_i) \to \infty$ as $i \to \infty$ for any $0 < s \leqslant r$. For each α_i, by (2), choose $\beta_i \in \widetilde{\mathcal{R}}_K^{\mathrm{bd},r}$ such that $\beta_i - \varphi(\beta_i) = \alpha_i$ and $w_s(\beta_i) \geqslant w_s(\alpha_i)$ for any $0 < s \leqslant r$. Then $\sum_{i=1}^{\infty} \beta_i$ converges to a desired solution $\beta \in \widetilde{\mathcal{R}}_K^r$ of (1.5.9.1). Furthermore, since $\varphi(\beta) = \beta + \alpha \in \widetilde{\mathcal{R}}_K^r$, we get that $\beta \in \widetilde{\mathcal{R}}_K^{\mathrm{qr}}$.

Now suppose that $n > 0$. For $\alpha = \sum_{i\in\mathbb{Q}} a_i u^i \in \widetilde{\mathcal{R}}_K^r$, write $\alpha = \alpha_1 + \alpha_2$ where

$$\alpha_1 = \sum_{i\geqslant n/r} a_i u^i, \quad \alpha_2 = \sum_{i<n/r} a_i u^i.$$

Then a short computation shows that for any $m \geqslant 0$ and $0 < s \leqslant r$,

$$w_s((\pi^{-n})^{\{m+1\}}\varphi^m(\alpha_1)) \geqslant -n(m+1) + w_s(\alpha) + (q^m - 1)ns/r,$$

and

$$w_s((\pi^{-n})^{\{-m\}}\varphi^{-m-1}(\alpha_2)) \geqslant mn + w_s(\alpha) - \left(1 - \frac{1}{q^{m+1}}\right) ns/r.$$

This implies that $w_s((\pi^{-n})^{\{m+1\}}\varphi^m(\alpha_1))$ and $w_s((\pi^{-n})^{\{-m\}}\varphi^{-m-1}(\alpha_2))$ approach infinity as $m \to \infty$. Furthermore, we get

$$w_r((\pi^{-n})^{\{m+1\}}\varphi^m(\alpha_1)) \geqslant -n(m+1) + w_r(\alpha) + (q^m - 1)n \geqslant w_r(\alpha) - n,$$
$$w_r((\pi^{-n})^{\{-m\}}\varphi^{-m-1}(\alpha_2)) \geqslant mn + w_r(\alpha) - \left(1 - \frac{1}{q^{m+1}}\right) n \geqslant w_r(\alpha) - n.$$

Hence the sums

$$\beta_1 = \sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}}\varphi^m(\alpha_1), \quad \beta_2 = \sum_{m=0}^{\infty} (\pi^{-n})^{\{-m\}}\varphi^{-m-1}(\alpha_2)$$

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converge in $\widetilde{\mathcal{R}}_K^r$, and satisfy $w_r(\beta_i) \geqslant w_r(x) - n$ for $i = 1, 2$. It is clear that $\varphi(\beta_1) - \pi^n\beta_1 = -\alpha_1$ and $\varphi(\beta_2) - \pi^n\beta_2 = \alpha_2$. Hence $\beta = (\beta_2 - \beta_1)$ is a solution of (1.5.9.1); the condition $\varphi(\beta) = \pi^n\beta + \alpha \in \widetilde{\mathcal{R}}_K^r$ implies $\beta \in \widetilde{\mathcal{R}}_K^{\mathrm{qr}}$. Hence β is a desired solution. □

Proposition 1.5.11. *Suppose that k is strongly difference-closed. Let $\lambda_1, \lambda_2 \in \widetilde{\mathcal{R}}_K^\times$. Then $\mathrm{Ext}^1_{\varphi,\widetilde{\mathcal{R}}_K}(V_{\lambda_1,d_1}, V_{\lambda_2,d_2}) = 0$ if $\frac{w(\lambda_2)}{d_2} \leqslant \frac{w(\lambda_1)}{d_1}$. In particular, we have $\mathrm{H}^1(\widetilde{\mathcal{R}}_K(n)) = 0$ if $n \geqslant 0$.*

Proof. If we equip $\widetilde{\mathcal{R}}_K$ with the endomorphism $\varphi^{d_1d_2}$, then V_{λ_1,d_1} and V_{λ_2,d_2} become direct sums of rank 1 $\varphi^{d_1d_2}$-modules with slopes $-d_2w(\lambda_1)$ and $-d_1w(\lambda_2)$ respectively. Hence $V = V^\vee_{\lambda_1,d_1} \otimes V_{\lambda_2,d_2}$ is a direct sum of rank 1 $\varphi^{d_1d_2}$-modules with slopes $d_2w(\lambda_1) - d_1w(\lambda_2) \geqslant 0$. By Theorem 1.5.8, every rank 1 $\varphi^{d_1d_2}$-module is of the form $\widetilde{\mathcal{R}}_K(n)$ for some integer n. It thus follows from Lemma 1.5.10(3) that $V/(\varphi^{d_1d_2} - 1)V = 0$, yielding $V/(\varphi - 1)V = 0$. Hence $\mathrm{Ext}^1_{\varphi,\widetilde{\mathcal{R}}_K}(V_{\lambda_1,d_1}, V_{\lambda_2,d_2}) = \mathrm{H}^1(V) = 0$. □

Proposition 1.5.12. *Suppose that k is strongly difference-closed. Then every φ-module over $\widetilde{\mathcal{R}}_K$ admits a Dieudonné–Manin decomposition.*

Proof. By Theorem 1.5.8, every semistable φ-module over $\widetilde{\mathcal{R}}_K$ admits a Dieudonné–Manin decomposition. We therefore deduce from Proposition 1.5.11 that HN filtrations for φ-modules over $\widetilde{\mathcal{R}}_K$ are split. This yields the desired result. □

Proposition 1.5.13. *Suppose that k is strongly difference-closed. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ be a semistable filtration of a φ-module M over $\widetilde{\mathcal{R}}_K$. If the slope polygon of this filtration coincides with the HN-polygon of M, then the filtration splits.*

Proof. The analogue of the proposition for φ-modules over $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$ is [14, Corollary 4.7.4] which is proved by using the formal properties of HN filtrations and the other two facts about φ-modules over $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$. Namely, every φ-module over $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$ admits a Dieudonné–Manin decomposition, and

$$\mathrm{Ext}^1_{\varphi,\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}}(V_{\pi^{c_1},d_1}, V_{\pi^{c_2},d_2}) = 0$$

if $\frac{c_2}{d_2} \leqslant \frac{c_1}{d_1}$. In our case the analogues of these two facts are Propositions 1.5.11 and 1.5.12. Therefore we can establish the proposition in the same way as [14, Corollary 4.7.4]. □

1.6. Slope decomposition and reverse slope filtration

Proposition 1.6.1. *A φ-module M over $\mathcal{E}_K$ (resp. $\widetilde{\mathcal{E}}_K$) is semistable of slope s if and only if it is pure of spectral norm p^s in the sense of difference modules. Every φ-module over $\mathcal{E}_K$ (resp. $\widetilde{\mathcal{E}}_K$) admits a unique HN filtration. Furthermore, for any φ-module M over $\mathcal{E}_K$ (resp. $\widetilde{\mathcal{E}}_K$) and an extension L of K, the HN filtration of M, tensored up to $\widetilde{\mathcal{E}}_L$, gives the HN filtration of $M \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L$ (resp. $M \otimes_{\widetilde{\mathcal{E}}_K} \widetilde{\mathcal{E}}_L$).*

Proof. Granting the first assertion, the second one then follows from [16, Theorem 14.4.15]. Note that if M is irreducible, then M is clearly semistable. The ‘if’ part of

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the first assertion thus follows from the fact that any extension of two semistable φ-modules which have the same slope is still semistable with the same slope. Conversely, if M is semistable of slope s, by [16, Theorem 14.4.15], there exists a unique filtration $0 = M_0 \subset M_1 \cdots \subset M_l = M$ such that each successive quotient M_i/M_{i-1} is pure of spectral norm p^{s_i} with $s_1 > \cdots > s_l$. Since $\mu(M) \geqslant s_i$ for every i and $\mu(M)$ is the weighted average of these s_i, we must have $l = 1$, yielding that M is pure of spectral norm p^s. The last assertion follows from [16, Proposition 14.4.8]. □

Definition 1.6.2. For any φ-module M over $\mathcal{E}_K$ or $\widetilde{\mathcal{E}}_K$, the slopes of the successive quotients and the slope polygon of the HN filtration of M are called the *slopes* and the *HN-polygon* of M respectively.

Proposition 1.6.3. *If k is strongly difference-closed, then any exact sequence of φ-modules over $\widetilde{\mathcal{E}}_K$ splits.*

Proof. We first have that the residue field $k((u^{\mathbb{Q}}))$ of $\widetilde{\mathcal{E}}_K$ is strongly difference-closed by Proposition 1.4.11. The proposition then follows immediately from [16, Corollary 14.6.6]. □

Proposition 1.6.4. *Suppose that k is strongly difference-closed. Then for any φ-module M over $\widetilde{\mathcal{E}}_K$, its HN filtration splits uniquely, i.e. there exists a unique direct sum decomposition $M = \oplus_{1 \leqslant i \leqslant l} M_{s_i}$ of φ-modules, in which each M_{s_i} is a semistable submodule of slope s_i. Moreover, each M_{s_i} admits a Dieudonné–Manin decomposition. Furthermore, for each $V_{\lambda,d}$ in the decomposition, we may force λ to be a power of π.*

Proof. Note that the residue field $k((u^{\mathbb{Q}}))$ of $\widetilde{\mathcal{E}}_K$ is strongly difference-closed by Proposition 1.4.11. The first assertion then follows from [16, Theorem 14.4.13] and Proposition 1.6.1, and the second assertion follows from [16, Theorem 14.6.3]. □

Corollary 1.6.5. *If $0 \to M_1 \to M \to M_2$ is an exact sequence of φ-modules over $\mathcal{E}_K$, then the slope multiset of the HN filtration of M is the union of the slope multisets of the HN filtrations of M_1 and M_2.*

Proof. Let L be an admissible extension of K with strongly difference-closed residue field. By Proposition 1.6.3, the exact sequence

$$0 \to M_1 \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L \to M \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L \to M_2 \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L \to 0$$

splits. We then deduce from Proposition 1.6.4 that the slope multiset of the HN filtration of $M \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L$ is the union of the slope multisets of the HN filtrations of $M_1 \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L$ and $M_2 \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L$. It follows from Proposition 1.6.1 that the slope multisets of the HN filtrations of M, M_1, M_2 are equal to the slope multisets of the HN filtrations of $M \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L, M_1 \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L, M_2 \otimes_{\mathcal{E}_K} \widetilde{\mathcal{E}}_L$ respectively. The proposition then follows. □

Definition 1.6.6. Suppose that k is strongly difference-closed. For any φ-module M over $\widetilde{\mathcal{E}}_K$, we call the decomposition $M = \oplus_{1 \leqslant i \leqslant l} M_{s_i}$ given by Proposition 1.6.4 the *slope*

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decomposition of M. Moreover, suppose that $s_1 > \cdots > s_l$, and put $M_i^{\mathrm{rev}} = \oplus_{j=l-i+1}^{l} M_{s_j}$ for $1 \leqslant i \leqslant l$. We call

$$0 = M_0^{\mathrm{rev}} \subset M_1^{\mathrm{rev}} \subset \cdots \subset M_l^{\mathrm{rev}} = M$$

the *reverse filtration* of M.

Lemma 1.6.7. *Suppose that k is strongly difference-closed. If N is a φ-module over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ such that $M = N \otimes_{\widetilde{\mathcal{E}}_K^{\mathrm{bd}}} \widetilde{\mathcal{E}}_K$ has non-positive slopes, then N admits a φ-stable $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$-lattice.*

Proof. By Proposition 1.6.4, we know that M is a direct sum of some modules V_{d_i,λ_i}, where each $w(\lambda_i)$ is non-negative. We fix a standard basis for each V_{d_i,λ_i}. Then the $\mathcal{O}_{\widetilde{\mathcal{E}}_K}$-lattice L of M generated by these standard bases is stable under φ. Choose an $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$-lattice Q of N; then there exist integers $m \geqslant n$ such that

$$\pi^m Q \otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{int}}} \mathcal{O}_{\widetilde{\mathcal{E}}_K} \subseteq L \subseteq \pi^n Q \otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{int}}} \mathcal{O}_{\widetilde{\mathcal{E}}_K}$$

since $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$ is the valuation ring of $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$. Let $P = L \cap N$. Note that $(\pi^i Q \otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{int}}} \mathcal{O}_{\widetilde{\mathcal{E}}_K}) \cap N = \pi^i Q$ for any $i \in \mathbb{Z}$ because $\pi^i \mathcal{O}_{\widetilde{\mathcal{E}}_K} \cap \widetilde{\mathcal{R}}_K^{\mathrm{int}} = \pi^i \widetilde{\mathcal{R}}_K^{\mathrm{int}}$. Hence $\pi^m Q \subseteq P \subseteq \pi^n Q$. This yields that P is an $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$-lattice of N, and is stable under φ. □

Lemma 1.6.8. *Suppose that k is strongly difference-closed. Let N be a φ-module over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ such that $M = N \otimes_{\widetilde{\mathcal{E}}_K^{\mathrm{bd}}} \widetilde{\mathcal{E}}_K$ has non-negative slopes. Let $v \in M$ satisfying $\varphi(v) = \lambda v$ for some $\lambda \in \widetilde{\mathcal{R}}_K^{\mathrm{int}}$. Then $v \in N$.*

Proof. Applying Lemma 1.6.7 to the dual of N, we may choose an $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$-lattice P of N which is stable under φ^{-1}. Choose an $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$-basis $\mathrm{e} = \{e_1, \ldots, e_n\}$ of P, and let F be the matrix of φ under e; then F^{-1} has entries in $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$. Write $v = \mathrm{e}\mathbf{v}$ for some column vector $\mathbf{v}$ over $\widetilde{\mathcal{E}}_K$. Then $\varphi(v) = \lambda v$ implies $F\varphi(\mathbf{v}) = \lambda \mathbf{v}$; hence $F^{-1}\lambda\mathbf{v} = \varphi(\mathbf{v})$. By [15, Proposition 2.5.8], we get that $\mathbf{v}$ has entries in $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$. So $v \in N$. □

The following proposition establishes the existence of de Jong's 'reverse filtration' ([9, Proposition 5.8]) for φ-modules over extended bounded Robba rings with a relative Frobenius lift.

Proposition 1.6.9. *Suppose that k is strongly difference-closed. Then for any φ-module N over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$, the reverse filtration of $M = N \otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}} \widetilde{\mathcal{E}}_K$ descends uniquely to a filtration of N. Furthermore, if M is semistable, then its Dieudonné–Manin decompositions descend to Dieudonné–Manin decompositions of N.*

Proof. Let $0 = M_0^{\mathrm{rev}} \subset M_1^{\mathrm{rev}} \cdots \subset M_l^{\mathrm{rev}} = M$ be the reverse filtration of M. Replacing φ with φ^a for a suitable positive integer a, we may suppose that the slopes of M are integral. It then suffices to show that M_1^{rev} and its Dieudonné–Manin decompositions descend to N. By twisting, we reduce to the case where $\mu(M_1^{\mathrm{rev}}) = 0$. Then the slopes of M are all non-negative. We fix a Dieudonné–Manin decomposition of M_1^{rev}. If e is part of a standard basis of some $V_{\lambda,d}$ in this decomposition, then $\varphi^d(e) = \varphi^i(\lambda)e$ for some $0 \leqslant i \leqslant d-1$. We then deduce that $e \in N$ by Lemma 1.6.8. Hence $V_{\lambda,d}$ descends to N.

This implies that M_1^{rev} together with this Dieudonné–Manin decomposition descends to a φ-submodule N_1^{rev} of N. It is clear that $N_1^{\mathrm{rev}} = M_1^{\mathrm{rev}} \cap N$, yielding the uniqueness. □

We call this filtration the *reverse filtration* of N.

Lemma 1.6.10. *Suppose that k is strongly difference-closed. Let $\lambda \in K$, and let d be a positive integer. If $n = v_K(\lambda)/d \in \mathbb{Z}$, then $V_{\lambda,d}$ is isomorphic to the direct sum of d copies of $V_{\pi^n,1}$.*

Proof. By [16, Corollary 14.4.9], the φ-module $V_{\lambda,d} \otimes_K V_{\pi^{-n},1}$ is pure of norm 1. Hence it is trivial by [16, Proposition 14.4.16] and Lemma 1.3.4. This yields the lemma. □

Lemma 1.6.11. *Suppose that k is strongly difference-closed. Let D be an $n \times n$ diagonal matrix such that all the diagonal entries are powers of π. If F is an $n \times n$ matrix over $\widetilde{\mathcal{E}}_K$ satisfying $w(FD^{-1} - I_n) > 0$, then there exists an invertible $n \times n$ matrix U over $\widetilde{\mathcal{E}}_K$ with $w(U - I_n) > 0$ and $U^{-1}F\varphi(U) = D$.*

Proof. We follow the proof of [13, Proposition 5.9]. Suppose that $w(FD^{-1} - I_n) = c_0$. We will inductively construct a sequence of invertible $n \times n$ matrices $\{U_i\}_{i\in\mathbb{N}}$ over $\mathcal{O}_{\widetilde{\mathcal{E}}_K}$ satisfying

$$\min\{w(U_{i+1} - U_i), w(U_i^{-1}F\varphi(U_i)D^{-1} - I_n)\} \geqslant (i+1)c_0$$

as follows. Put $U_0 = I_n$. Given U_i, by Lemma 1.5.10(1), there exists an $n \times n$ matrix X_i over $\widetilde{\mathcal{E}}_K$ with

$$X_i - D\varphi(X_i)D^{-1} = U_i^{-1}F\varphi(U_i)D^{-1} - I_n$$

and

$$\min\{w(X_i), w(D\varphi(X_i)D^{-1})\} = w(U_i^{-1}F\varphi(U_i)D^{-1} - I_n).$$

Put $U_{i+1} = U_i(I_n + X_i)$; then $w(U_{i+1} - U_i) = w(X_i) \geqslant (i+1)c_0$ and

$$\begin{aligned}
&U_{i+1}^{-1}F\varphi(U_{i+1})D^{-1} - I_n \\
&\quad = (I_n - X_i + X_i^2 - \cdots)(I_n + (U_i^{-1}F\varphi(U_i)D^{-1} - I_n))(I_n + D\varphi(X_i)D^{-1}) - I_n.
\end{aligned}$$

It follows that $w(U_{i+1}^{-1}F\varphi(U_{i+1})D^{-1} - I_n) \geqslant 2(i+1)c_0 \geqslant (i+2)c_0$. Then $U = \lim_{i\to\infty} U_i$ satisfies the desired properties. □

Corollary 1.6.12. *Let M be a φ-module over $\mathcal{E}_K$, and let F be the matrix of φ under some basis of M. Then there exists $N = N(F) > 0$ such that for any φ-module M' over $\mathcal{E}_K$ with the same rank as M, if M' has a basis under which the matrix F' of φ satisfies $w(F - F') \geqslant N$, then the HN-polygons of M' and M coincide.*

Proof. Replacing φ with φ^a for some suitable positive integer a, we may suppose that the slopes of M are integral. Choose an admissible extension L of K with a strongly difference-closed residue field. By Proposition 1.6.4 and Lemma 1.6.10, there exists an invertible matrix U over $\widetilde{\mathcal{E}}_L$ such that $D = U^{-1}F\varphi(U)$ is a diagonal matrix with all

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diagonal entries being powers of π, and their valuations are the slopes of $M\otimes_{\mathcal{E}_K}\widetilde{\mathcal{E}}_L$. Let $N=1-w(U^{-1})-w(U)-w(D^{-1})$. If $w(F-F')\geqslant N$, then

$$w(U^{-1}F'\varphi(U)D^{-1}-I_n)=w((U^{-1}(F'-F)\varphi(U)D^{-1}))\geqslant 1.$$

By Lemma 1.6.11, we get that there exists an invertible matrix U' over $\widetilde{\mathcal{E}}_L$ such that $U'^{-1}F'\varphi(U')=D$. Hence the slopes of $M'\otimes_{\mathcal{E}_K}\widetilde{\mathcal{E}}_L$ are the same as those of $M\otimes_{\mathcal{E}_K}\widetilde{\mathcal{E}}_L$. This implies that the slopes of M' are the same as those of M by Proposition 1.6.1. □

1.7. Comparison of HN-polygons

Definition 1.7.1. For a φ-module N over $\mathcal{R}_K^{\mathrm{bd}}$ (resp. $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$), the *generic slope filtration* of N is the HN filtration of $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\mathcal{E}_K$ (resp. $N\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{E}}_K$); the slope polygon of the generic slope filtration is called the *generic HN-polygon* of N. The *special slope filtration* of N is the HN filtration of $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\mathcal{R}_K$ (resp. $N\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$); the slope polygon of the special slope filtration is called the *special HN-polygon* of N.

Proposition 1.7.2. *If N is a φ-module over $\mathcal{R}_K^{\mathrm{bd}}$ or $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$, the special HN-polygon of N lies above the generic HN-polygon of N with the same endpoint.*

Proof. By base change, it suffices to treat the case where N is over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ and k is strongly difference-closed. Let $M=N\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$. Suppose that $0=N_0\subset N_1\cdots\subset N_l=N$ is the reverse filtration of N, and we denote by

$$0=M_0\subset M_1\cdots\subset M_l=M \tag{1.7.2.1}$$

the base change of the reverse filtration. It follows from Proposition 1.6.9 that each quotient N_i/N_{i-1} admits a Dieudonné–Manin decomposition. This yields that each successive quotient M_i/M_{i-1} is a pure φ-module over $\widetilde{\mathcal{R}}_K$; hence it is semistable by Proposition 1.5.4. Hence (1.7.2.1) is a semistable filtration of M. We thus deduce the desired result by Proposition 1.2.17. □

Lemma 1.7.3. *If N is a φ-module over $\mathcal{R}_K^{\mathrm{bd}}$ (resp. $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$) whose generic slopes are all non-positive, then the natural map $\mathrm{H}^1(N)\to\mathrm{H}^1(N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\mathcal{R}_K)$ (resp. $\mathrm{H}^1(N)\to\mathrm{H}^1(N\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K)$) is injective.*

Proof. Let $M=N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\mathcal{R}_K$ (resp. $N\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$). It suffices to show that for any $m\in M$, if $(\varphi-1)m\in N$, then $m\in N$. Note that $(N\otimes\widetilde{\mathcal{R}}_L^{\mathrm{bd}})\cap M=N$ for any extension L of K. Hence it suffices to show the lemma in the case where N is over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$ and k is strongly difference-closed. Therefore by Lemma 1.6.7, N admits a φ-stable $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$-lattice; let $\mathrm{e}=\{e_1,\ldots,e_n\}$ be a basis of this lattice, and write $\varphi(\mathrm{e})=\mathrm{e}F$ for some $n\times n$ matrix F over $\widetilde{\mathcal{R}}_K^{\mathrm{int}}$. Suppose that $m\in M$ satisfies $(\varphi-1)m\in N$. Write $m=\mathrm{e}\boldsymbol{m}$ for some column vector $\boldsymbol{m}$ over $\widetilde{\mathcal{R}}_K$. Then $F\varphi(\boldsymbol{m})-\boldsymbol{m}$ is over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$. By [15, Proposition 2.2.8], we have that $\boldsymbol{m}$ is over $\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$. Hence $m\in N$. □

The following proposition generalizes [15, Theorem 5.5.2] to the relative Frobenius lift case.

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Proposition 1.7.4. *Suppose that k is strongly difference-closed. Let N be a φ-module over $\mathcal{R}_K^{\mathrm{bd}}$ whose generic and special HN-polygons coincide. Then the HN filtrations of $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{E}}_K$ and $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$, respectively, are obtained by base change from a filtration of N.*

Proof. We follow the proof of [14, Theorem 5.5.2]. It suffices to show that the first steps of the generic and special HN filtrations of $\widetilde{N}$ descend to N and coincide. Let $0 \subset \widetilde{N}_1 \subset \cdots \subset \widetilde{N}_{l-1} \subset \widetilde{N}_l = \widetilde{N}$ be the reverse filtration of $\widetilde{N} = N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K^{\mathrm{bd}}$. As shown in the proof of Proposition 1.7.2, the filtration

$$0 \subset \widetilde{N}_1\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K \subset \cdots \subset \widetilde{N}_{l-1}\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K \subset \widetilde{N}\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$$

is semistable. Since the slope polygon of this filtration is the same as the HN filtration of $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$, it is split by Proposition 1.5.13, yielding that the exact sequence

$$0 \to \widetilde{N}_{l-1} \to \widetilde{N} \to \widetilde{N}/\widetilde{N}_{l-1} \to 0$$

is split by Lemma 1.7.3. Let $\widetilde{N}'$ be a φ-submodule of $\widetilde{N}$ lifting $\widetilde{N}/\widetilde{N}_{l-1}$. It follows that $\widetilde{N}'\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{E}}_K$ is isomorphic to the first step of the generic HN filtration. Thus they coincide by the uniqueness of HN filtration. Similarly, we also have that $\widetilde{N}'\otimes_{\widetilde{\mathcal{R}}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$ coincides with the first step of the HN filtration of $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$. Hence both the first steps of the HN filtrations of $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{E}}_K$ and $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{R}}_K$ descend to a φ-submodule $\widetilde{N}'$ of $\widetilde{N}$. To show that $\widetilde{N}'$ can be further descended to a φ-submodule of N, by [14, Lemma 3.6.2], it suffices to treat the case where $\operatorname{rank}\widetilde{N}' = 1$. Choose a basis $\mathrm{e} = \{e_1, \dots, e_n\}$ of N. Let $v = \sum_{i=1}^n a_i e_i$ be a generator of $\widetilde{N}'$, and suppose that $a_1 \neq 0$. By Proposition 1.6.1, the first step of the HN filtration of $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\widetilde{\mathcal{E}}_K$ descends to $N\otimes_{\mathcal{R}_K^{\mathrm{bd}}}\mathcal{E}_K$. Hence $a_i/a_1 \in \mathcal{E}_K$ for $1 \leqslant i \leqslant n$. Thus $a_i/a_1 \in \mathcal{E}_K \cap \widetilde{\mathcal{R}}_K^{\mathrm{bd}} = \mathcal{R}_K^{\mathrm{bd}}$ for each i, yielding $v/a_1 \in N$. □

2. Variation of slopes

In this section, we consider families of φ-modules (over $\mathcal{R}_K$) over affinoid spaces. All affinoid algebras are equipped with the spectral norm, and we fix a reduced affinoid space $M(A)$ over $\mathbb{Q}_p$ as the base. For any Banach algebra B and p-adic field L, we assume that $|B|$ and $|L|$ are discrete, and set $B_L = B\widehat{\otimes}_{\mathbb{Q}_p}L$. We assume that K is a p-adic field, and that φ_K acts trivially on $\mathbb{Q}_p$. We adapt the normalization on the norm on K to $|p| = p^{-1}$ to fit the standard norm on $\mathbb{Q}_p$. We also set $v(b) = \log_{|\pi|}|b|$ for any $b \in B$. We caution that K, which is the 'base field' of the fibres of the families, is irrelevant to A.

2.1. Families of φ-modules

Definition 2.1.1. For any $\mathbb{Q}_p$-Banach algebra B, interval $I \subset (0, \infty]$, $s \in I$ and $r > 0$, define the rings

$$\mathcal{E}_B, \widetilde{\mathcal{E}}_B, \mathcal{R}_B^{\mathrm{int},r}, \widetilde{\mathcal{R}}_B^{\mathrm{int},r}, \mathcal{R}_B^{\mathrm{int}}, \widetilde{\mathcal{R}}_B^{\mathrm{int}}, \mathcal{R}_B^{\mathrm{bd},r}, \widetilde{\mathcal{R}}_B^{\mathrm{bd},r}, \mathcal{R}_B^{\mathrm{bd}}, \widetilde{\mathcal{R}}_B^{\mathrm{bd}}, \mathcal{R}_B^I, \widetilde{\mathcal{R}}_B^I, \mathcal{R}_B^r, \widetilde{\mathcal{R}}_B^r, \mathcal{R}_B, \widetilde{\mathcal{R}}_B$$

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and w_s, w, and equip these rings with certain topologies by changing K to B in Definitions 1.1.1, 1.1.2, 1.1.4, 1.1.7, 1.4.1, 1.4.3, 1.4.4 and 1.4.6. We set $|\cdot|_s = |\pi|^{w(\cdot)}$ and $|\cdot| = |\pi|^{w(\cdot)}$. We call $\mathcal{R}_B$ (resp. $\mathcal{R}_B^{\mathrm{bd}}$) the *Robba ring over B* (resp. *bounded Robba ring over B*) and $\widetilde{\mathcal{R}}_B$ (resp. $\widetilde{\mathcal{R}}_B^{\mathrm{bd}}$) the *extended Robba ring over B* (resp. *extended bounded Robba ring over B*). Note that for general B we only have $w_s(fg) \geqslant w_s(f) + w_s(g)$, $w(fg) \geqslant w(f) + w(g)$ and $|fg|_s \leqslant |f|_s|g|_s$, $|fg| \leqslant |f||g|$.

Proposition 2.1.2. *For any $\mathbb{Q}_p$-Banach algebra B, we have $\lim_{r\to 0^+} w_r(f) = w(f)$ for any $f = \sum_{i\in\mathbb{Q}} a_i u^i \in \widetilde{\mathcal{R}}_B^{\mathrm{bd}}$.*

Proof. The proof is similar to the proof of Proposition 1.1.3. Suppose that $w(f) = v(a_{i_0})$ for some $i_0 \in \mathbb{Q}$. For any $\epsilon > 0$, set $r_0 = \frac{\epsilon}{|2i_0|+1}$. We may suppose that $f \in \widetilde{\mathcal{R}}_B^{\mathrm{bd},r_0}$ by shrinking ϵ. It thus follows that for any $r \in (0, r_0]$, $w_r(f) \leqslant ri_0 + v(a_{i_0}) < w(f) + \epsilon/2$. On the other hand, choose some positive integer N such that $r_0 i + v(a_i) \geqslant w(f)$ for any $i \leqslant -N$. Let $r_1 = \min\{r_0, \frac{\epsilon}{N}\}$. It follows that for $0 < r \leqslant r_1$, if $i \leqslant -N$, then $ri + v(a_i) \geqslant r_0 i + v(a_i) \geqslant w(f)$; if $i > -N$, then $ri + v(a_i) \geqslant w(f) - rN \geqslant w(f) - \epsilon$. We thus deduce that $|w_r(f) - w(f)| \leqslant \epsilon$ for any $r \in (0, r_1]$, proving the proposition. □

Definition 2.1.3. Let L be a p-adic field, and let V be an L-Banach space. A *Schauder basis* of V is a sequence $\{v_i\}_{i\in I}$ of elements of V for a countable index set I such that for every element $v \in V$ there exists a unique sequence $\{\lambda_i\}_{i\in I}$ of elements of L such that

$$v = \sum_{i\in I} \lambda_i v_i.$$

It is further called an *orthogonal basis* if

$$|v| = \max_{i\in I}\{|\lambda_i||v_i|\}$$

for any $v \in V$.

Lemma 2.1.4. *Let L be a p-adic field. If V is an L-Banach space of countable type with $|V|$ discrete, then V admits an orthogonal basis.*

Proof. Since $|V|$ is discrete, there is a finite sequence $1 \leqslant c_1 < c_2 < \cdots < c_m < p$ such that

$$|V| = \{p^n c_j | n \in \mathbb{Z}, 1 \leqslant j \leqslant m\}.$$

Put $h = \min\{c_2/c_1, \ldots, c_m/c_{m-1}, p/c_m\}$. Choose some $h' \in (1, h)$. By [4, 2.7.2/3], V admits a Schauder basis $\{v_i\}_{i\in I}$ such that

$$h'\left|\sum_{i\in I} \lambda_i v_i\right| \geqslant \max_{i\in I}\{|\lambda_i||v_i|\}$$

for any convergent sum $\sum_{i\in I} \lambda_i v_i$. However, since $|\sum_{i\in I} \lambda_i v_i| \leqslant \max_{i\in I}\{|\lambda_i||v_i|\}$, if they are not equal, we must have

$$h\left|\sum_{i\in I} \lambda_i v_i\right| \leqslant \max_{i\in I}\{|\lambda_i||v_i|\};$$

this yields a contradiction. Hence $|\sum_{i\in I}\lambda_i v_i| = \max_{i\in I}\{|\lambda_i||v_i|\}$, yielding that $\{v_i\}_{i\in I}$ is an orthogonal basis. □

Remark 2.1.5. Note that A_L is an affinoid algebra over L. Hence it is of countable type as an L-Banach space, and $|A_L|$ is discrete. Thus Lemma 2.1.4 implies that A_L admits an orthogonal basis over L.

Lemma 2.1.6. *Let L be a p-adic field, and let B be a $\mathbb{Q}_p$-Banach algebra. For $R \in \{\mathcal{E}, \mathcal{R}^{\mathrm{bd},r}, \mathcal{R}^I\}$ and $\widetilde{R} \in \{\widetilde{\mathcal{E}}, \widetilde{\mathcal{R}}^{\mathrm{bd},r}, \widetilde{\mathcal{R}}^I\}$ where $I \subset (0,\infty]$ is a closed interval, the natural maps*

$$i : B\otimes_{\mathbb{Q}_p} R_L \to R_{B_L}, \quad \widetilde{i} : B\otimes_{\mathbb{Q}_p} \widetilde{R}_L \to \widetilde{R}_{B_L}$$

are isometric embeddings of L-Banach algebras. For $R = \mathcal{R}^r$ and $\widetilde{R} = \widetilde{\mathcal{R}}^r$, the natural maps

$$i : B\otimes_{\mathbb{Q}_p} R_L \to R_{B_L}, \quad \widetilde{i} : B\otimes_{\mathbb{Q}_p} \widetilde{R}_L \to \widetilde{R}_{B_L}$$

are isometric embeddings of L-Fréchet spaces. Furthermore, i always has dense image. Hence i induces an isomorphism $B\widehat{\otimes}_{\mathbb{Q}_p} R_L \cong R_{B_L}$ for any $R \in \{\mathcal{E}, \mathcal{R}^{\mathrm{bd},r}, \mathcal{R}^I, \mathcal{R}^r\}$, and $\widetilde{i}$ induces an isometric embedding $B\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{R}_L \hookrightarrow \widetilde{R}_{B_L}$ for any $\widetilde{R} \in \{\widetilde{\mathcal{E}}, \widetilde{\mathcal{R}}^{\mathrm{bd},r}, \widetilde{\mathcal{R}}^I, \widetilde{\mathcal{R}}^r\}$.

Proof. For $R = \mathcal{E}, \mathcal{R}^{\mathrm{bd},r}, \mathcal{R}^I$ and $\widetilde{R} = \widetilde{\mathcal{E}}, \widetilde{\mathcal{R}}^{\mathrm{bd},r}, \widetilde{\mathcal{R}}^I$, we denote by $|\cdot|_1$ the tensor product norms on Banach algebras $B\widehat{\otimes}_{\mathbb{Q}_p} R_L$ and $B\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{R}_L$, and denote by $|\cdot|_2$ the norms of the Banach algebras R_B and $\widetilde{R}_B$. Fix some $s \in (0,r]$. For $R = \mathcal{R}^r$ and $\widetilde{R} = \widetilde{\mathcal{R}}^r$, we denote $|f|_s$ by $|f|$ for any $f \in R_L$ and $f \in \widetilde{R}_L$. We denote by $|\cdot|_1$ the tensor products of the norm on B and $|\cdot|_s$ on R_L and $\widetilde{R}_L$, and denote by $|\cdot|_2$ the norms $|\cdot|_s$ on R_{B_L} and $\widetilde{R}_{B_L}$.

For any $f \in B\otimes_{\mathbb{Q}_p} \widetilde{R}_L$, if we write $f = \sum_{j=1}^n b_j \otimes f_j$, then it is clear that

$$|\widetilde{i}(f)|_2 = \left|\sum_{j=1}^n b_j \otimes f_j\right| \leqslant \max\{|b_j||f_j|\};$$

hence $|\widetilde{i}(f)|_2 \leqslant |f|_1$ by the definition of tensor product norms. On the other hand, let V be the $\mathbb{Q}_p$-subspace of B generated by $b_1, \ldots, b_n$. By Lemma 2.1.4, V admits an orthogonal basis $\{v_1, \ldots, v_m\}$. We may rewrite $f = \sum_{j=1}^m v_j \otimes f'_j$ for some $f'_j \in \widetilde{R}_L$. For any $i \in \mathbb{Q}$, let c_i and c_{ij} be the ith coefficients of f and f'_j; then $c_i = \sum_{j=1}^m v_j c_{ij}$. Hence $|c_i| = \max\{|v_j||c_{ij}|\}$. This implies that $|\widetilde{i}(f)|_2 = \max\{|v_j||f'_j|\}$. This yields $|\widetilde{i}(f)|_2 \geqslant |f|_1$. Hence $|\widetilde{i}(f)|_2 = |f|_1$. The proof for i is similar. The rest of the lemma is obvious. □

Henceforth for any $\mathbb{Q}_p$-Banach algebra B and $\widetilde{R} \in \{\widetilde{\mathcal{E}}, \widetilde{\mathcal{R}}^{\mathrm{bd},r}, \widetilde{\mathcal{R}}^I, \widetilde{\mathcal{R}}^r\}$, we view $B\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{R}_L$ as a subalgebra of $\widetilde{R}_{B_L}$ via $\widetilde{i}$

Definition 2.1.7. For any $\mathbb{Q}_p$-Banach algebra B, p-adic field L and $\widetilde{R} \in \{\widetilde{\mathcal{R}}^{\mathrm{bd}}, \widetilde{\mathcal{R}}\}$, we set

$$B\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{R}_L = \bigcup_{r>0} B\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{R}^r_L.$$

which is a subalgebra of $\widetilde{R}_{B_L}$.

Lemma 2.1.8. *Let B be a $\mathbb{Q}_p$-Banach algebra of countable type, and let L be a p-adic field. Suppose that S is a closed subspace of $\widetilde{\mathcal{E}}_L$ and put $S' = S \cap \widetilde{\mathcal{R}}_L^{\mathrm{bd},r}$, which is a closed subspace of $\widetilde{\mathcal{R}}_L^{\mathrm{bd},r}$. Then*

$$(B\widehat{\otimes}_{\mathbb{Q}_p} S) \cap \widetilde{\mathcal{R}}_{B_L}^{\mathrm{bd},r} = B\widehat{\otimes}_{\mathbb{Q}_p} S'.$$

Proof. We only need to show $(B\widehat{\otimes}_{\mathbb{Q}_p} S) \cap \widetilde{\mathcal{R}}_B^{\mathrm{bd},r} \subseteq B\widehat{\otimes}_{\mathbb{Q}_p} S'$. Since B is of countable type, by Lemma 2.1.4, B admits an orthogonal basis $\{v_j\}_{j\in J}$ over $\mathbb{Q}_p$; then $\{v_j\}_{j\in J}$ is also an orthogonal basis of $B\widehat{\otimes}_{\mathbb{Q}_p} L$ over L.

Now suppose that $f = \sum_{i\in\mathbb{Q}} a_i u^i \in (B\widehat{\otimes}_{\mathbb{Q}_p} S) \cap \widetilde{\mathcal{R}}_{B_L}^{\mathrm{bd},r}$. We may write f as a convergent sum $f = \sum_{j\in\mathbb{N}} v_j \otimes f_j$ in $B\widehat{\otimes}_{\mathbb{Q}_p} S$ where each $f_j \in S$. Since $f \in \widetilde{\mathcal{R}}_{B_L}^{\mathrm{bd},r}$, it follows that each $f_j \in \widetilde{\mathcal{R}}_L^{\mathrm{bd},r}$ and satisfies $|v_j||f_j| \leqslant |f|$, $|v_j||f_j|_r \leqslant |f|_r$. Hence all f_j belong to $S' = S \cap \widetilde{\mathcal{R}}_L^{\mathrm{bd},r}$. It remains to show that the sum $\sum_{j\in\mathbb{N}} v_j \otimes f_j$ is convergent in $B\widehat{\otimes}_{\mathbb{Q}_p} S'$. For any $\epsilon > 0$, choose $N < 0$ such that $\max\{|a_i|, |a_i u^i|_r)\} < \epsilon$ if $i < N$. Choose $m \in \mathbb{N}$ such that $|v_j \otimes f_j| < \epsilon|\pi|^{-Nr}$ if $j \geqslant m$. We claim that

$$\max\{|v_j \otimes f_j|, |v_j \otimes f_j|_r\} < \epsilon$$

for each $j \geqslant m$. In fact, for any $j \geqslant m$, if we write $v_j \otimes f_j = \sum_{i\in\mathbb{Q}} b_{ji} u^i$ where $b_{ji} \in B$, then

$$\max\{|b_{ji}|, |b_{ji} u^i|_r\} \leqslant \max\{|a_i|, |a_i u^i|_r\} < \epsilon$$

if $i < N$, and

$$\max\{|b_{ji}|, |b_{ji} u^i|_r\} < \max\{\epsilon|\pi|^{-Nr}, \epsilon|\pi|^{(i-N)r}\} \leqslant \epsilon$$

if $i \geqslant N$. This yields the claim. Hence $f \in B\widehat{\otimes}_{\mathbb{Q}_p} S'$. □

Definition 2.1.9. Let $\varphi_A : A_K \to A_K$ be the continuous extension of $\mathrm{id} \otimes \varphi_K$ on $A\otimes_{\mathbb{Q}_p} K$. We set the φ-action on $\mathcal{R}_{A_K}$ as the continuous extension of $\mathrm{id} \otimes \varphi$ on $A\otimes_{\mathbb{Q}_p} \mathcal{R}_K$. We set the φ-action on $\widetilde{\mathcal{R}}_{A_K}$ as

$$\varphi\left(\sum_{i\in\mathbb{Q}} a_i u^i\right) = \sum_{i\in\mathbb{Q}} \varphi_A(a_i) u^{qi}.$$

Remark 2.1.10. The embedding $\tau_K : \mathcal{R}_K \to \widetilde{\mathcal{R}}_K$ induces an embedding

$$\tau_A : \mathcal{R}_{A_K} = \bigcup_{r>0} A\widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}_K^r \to A\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{R}}_K = \bigcup_{r>0} A\widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{R}}_K^r$$

by tensoring with the identity on A and taking completion; then τ_A is φ-equivariant because τ_K is φ-equivariant. By Lemma 2.1.6, τ_A further induces a φ-equivariant embedding $\mathcal{R}_{A_K} \to \widetilde{\mathcal{R}}_{A_K}$ which we again denote by τ_A.

Definition 2.1.11. By a *vector bundle* over $\mathcal{R}_{A_K}^r \cong A_K\widehat{\otimes}_K \mathcal{R}_K^r$, we mean a locally free coherent sheaf over the product of the annulus $0 < v_p(T) \leqslant r$ over K with $M(A_K)$ in the category of rigid analytic spaces over K. If A_K is disconnected, we require that the rank

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be constant. By a vector bundle over $\mathcal{R}_{A_K}$, we will mean an object in the direct limit as $r \to 0$ of the categories of vector bundles over $\mathcal{R}^r_{A_K}$. For any morphism of affinoid algebras $A \to B$ and a vector bundle M_A over $\mathcal{R}_{A_K}$, we denote by $M_A \otimes_{\mathcal{R}_{A_K}} \mathcal{R}_{B_K}$ the base change of M_A to a vector bundle over $\mathcal{R}_{B_K}$.

Definition 2.1.12. By *a family of φ-modules* over $\mathcal{R}^{\mathrm{bd}}_{A_K}$ (resp. $\mathcal{R}_{A_K}$), we mean a finite locally free module N_A over $\mathcal{R}^{\mathrm{bd}}_{A_K}$ (resp. a vector bundle M_A over $\mathcal{R}_{A_K}$) equipped with an isomorphism $\varphi^* N_A \to N_A$ (resp. $\varphi^* M_A \to M_A$), viewed as a semilinear action φ on N_A (resp. M_A). If A_K is disconnected, we require that the rank be constant.

Remark 2.1.13. For $A = \mathbb{Q}_p$, every vector bundle over $\mathcal{R}_K$ is represented by a finite free $\mathcal{R}_K$-module by the Bézout property of $\mathcal{R}_K$ ([14, Theorem 2.8.4]). Hence the category of families of φ-modules over $\mathcal{R}_K$ coincides with the category of φ-modules over $\mathcal{R}_K$. For general A, it is only known that any family of φ-modules over $\mathcal{R}_{A_K}$ is A-locally free (Corollary 2.2.10).

Definition 2.1.14. Let N_A (resp. M_A) be a family of φ-modules over $\mathcal{R}^{\mathrm{bd}}_{A_K}$ (resp. $\mathcal{R}_{A_K}$). For any $x \in M(A)$, N_A (resp. M_A) specializes to a φ-module

$$N_x = N_A \otimes_{\mathcal{R}^{\mathrm{bd}}_{A_K}} (k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}^{\mathrm{bd}}_K)$$

over $k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}^{\mathrm{bd}}_K$ (resp. $M_x = M_A \otimes_{\mathcal{R}_{A_K}} (k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}_K)$ over $k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}_K$). We denote by p_x the natural projection map $N_A \to N_x$ (resp. $M_A \to M_x$).

Definition 2.1.15. For a family of φ-modules M_A over $\mathcal{R}_{A_K}$, a *model* of M_A is a subfamily of φ-modules N_A over $\mathcal{R}^{\mathrm{bd}}_{A_K}$ such that $N_A \otimes_{\mathcal{R}^{\mathrm{bd}}_{A_K}} \mathcal{R}_{A_K} = M_A$.

Definition 2.1.16. Let N_A be a family of φ-modules over $\mathcal{R}^{\mathrm{bd}}_{A_K}$. For $c, d \in \mathbb{Z}$ with $d > 0$, a (c,d)-*pure model* of N_A is a finite locally free sub-$\mathcal{R}^{\mathrm{int}}_{A_K}$-module N'_A of N_A with $N'_A \otimes_{\mathcal{R}^{\mathrm{int}}_{A_K}} \mathcal{R}^{\mathrm{bd}}_{A_K} = N_A$, so that the φ-action on N'_A induces an isomorphism $\pi^c (\varphi^d)^* N'_A \cong N'_A$. For a family of φ-modules M_A over $\mathcal{R}_{A_K}$, a (c,d)-*pure model* of M_A is a (c,d)-pure model of a model of M_A. For $s \in \mathbb{Q}$, we say that N_A (resp. M_A) is *globally pure of slope* s if N_A (resp. M_A) admits a (c,d)-pure model for some (and hence any) $c, d \in \mathbb{Z}$ with $d > 0$ and $s = c/d$. If $s = 0$, we also say that N_A (resp. M_A) is *globally étale*, and a $(0,1)$-pure model is also called an *étale model*.

Proposition 2.1.17. *Let M_A (resp. N_A) be a family of φ-modules over $\mathcal{R}_{A_K}$ (resp. $\mathcal{R}^{\mathrm{bd}}_{A_K}$), and let $x \in M(A)$. Suppose that*

$$k(x) \otimes_{\mathbb{Q}_p} K \cong \oplus_{i=1}^n K_i$$

where each K_i is a finite field extension of K. Then the following are true.

(1) *The induced φ-action on each K_i is an automorphism.*

(2) *Let $M_{x,i} = M_x \otimes_{k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}_K} \mathcal{R}_{K_i}$ (resp. $N_{x,i} = N_x \otimes_{k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}^{\mathrm{bd}}_K} \mathcal{R}^{\mathrm{bd}}_{K_i}$) for $1 \leqslant i \leqslant n$. Then the HN-polygons of all $M_{x,i}$ (generic HN-polygons of all $N_{x,i}$) coincide.*

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Proof. Since the φ-action is an automorphism on $k(x)\otimes_{\mathbb{Q}_p}K$, it is an automorphism on each K_i. This yields (1). By Proposition 1.5.6 (resp. Proposition 1.6.1), we see that HN-polygons of φ-modules over Robba rings (generic HN-polygons of φ-modules over bounded Robba rings) are stable under base change. By passing to normal closure of the field extension $k(x)/\mathbb{Q}_p$, we may suppose that $k(x)$ is Galois over $\mathbb{Q}_p$. In this case, $\mathrm{Gal}(k(x)/\mathbb{Q}_p)$ acts transitively on the set $\{K_i\}_{1\leqslant i\leqslant n}$; hence it acts transitively on $\{M_{x,i}\}_{1\leqslant i\leqslant n}$ (resp. $\{N_{x,i}\}_{1\leqslant i\leqslant n}$). Furthermore, this action commutes with φ. This implies that all $M_{x,i}$ (resp. $N_{x,i}$) have the same HN-polygon (generic HN-polygon), yielding (2). □

In the situation of Proposition 2.1.17, it is clear that M_x (resp. N_x) is isomorphic to the direct sum of all $M_{x,i}$ (resp. $N_{x,i}$). We call each $M_{x,i}$ (resp. $N_{x,i}$) a *component* of M_x (resp. N_x). We set the *slopes* and *HN-polygon* of M_x (resp. *generic slopes* and *generic HN-polygon* of N_x) as the slopes and HN-polygon of $M_{x,i}$ (resp. generic slopes and generic HN-polygon of $N_{x,i}$). We set the *HN filtration* of M_x (resp. *generic HN filtration* of N_x) as the direct sum of the HN filtrations of all $M_{x,i}$ (generic HN filtrations of all $N_{x,i}$).

Definition 2.1.18. Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$, and let N_A be a model of it. We call N_A a *good model* if for every $x\in M(A)$, the generic and special HN-polygons of N_x coincide, i.e. the generic HN-polygon of N_x coincides with the HN-polygon of M_x.

2.2. Semicontinuity of HN-polygons

Convention 2.2.1. Let r_φ be as in Lemma 1.2.8. It follows that for $0<r<r_\varphi$ and $a\in\mathcal{R}_K$, if $\varphi(a)\in\mathcal{R}_K^{r/q}$, then $a\in\mathcal{R}_K^r$. Furthermore, by Remark 1.2.2, we may shrink r_φ in such a way that φ maps $\mathcal{R}_K^r$ to $\mathcal{R}_K^{r/q}$ for $0<r<r_\varphi$. Hence for $0<r<r_\varphi$, we have that $\varphi(a)\in\mathcal{R}_K^{r/q}$ if and only if $a\in\mathcal{R}_K^r$, and that $w_{r/q}(\varphi(a))=w_r(a)$ for any $a\in\mathcal{R}_K^r$.

Proposition 2.2.2. *For any $\mathbb{Q}_p$-Banach algebra S and $x\in M(A)$, the natural projection map*

$$\rho_x : A\widehat{\otimes}_{\mathbb{Q}_p}S\to k(x)\otimes_{\mathbb{Q}_p}S$$

is surjective and $\ker(\rho_x)=\mathfrak{m}_x(A\widehat{\otimes}_{\mathbb{Q}_p}S)$ where $\mathfrak{m}_x$ is the maximal ideal of A corresponding to x. Furthermore, for any $\lambda>0$, there exists a Weierstrass subdomain $M(B)$ of $M(A)$ containing x such that if $f\in\ker(\rho_x)$, then the norm of f in $B\widehat{\otimes}_{\mathbb{Q}_p}S$ is no more than λ times the norm of f in $A\widehat{\otimes}_{\mathbb{Q}_p}S$.

Proof. By the Hahn–Banach theorem for Banach spaces over discretely valued fields ([22, Proposition 10.5]), the exact sequence

$$0\to\mathfrak{m}_x\to A\to k(x)\to 0$$

splits as $\mathbb{Q}_p$-Banach spaces. This yields the exact sequence

$$0\to\mathfrak{m}_x\widehat{\otimes}_{\mathbb{Q}_p}S\to A\widehat{\otimes}_{\mathbb{Q}_p}S\to k(x)\otimes_{\mathbb{Q}_p}S\to 0.$$

This shows that ρ_x is surjective.

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Choose a finite set of generators $b_1, \ldots, b_m$ of $\mathfrak{m}_x$ as an A-module. By the open mapping theorem for Banach spaces over discretely valued fields ([22, Proposition 8.6]), the surjective map of $\mathbb{Q}_p$-Banach spaces $A^m \to \mathfrak{m}_x$ defined by $(a_1, \ldots, a_m) \mapsto \sum_{i=1}^m a_i b_i$ is open. Hence there exists $c > 0$ such that for any $a \in \mathfrak{m}_x$, there exist $a_1, \ldots, a_m \in A$ with $|a_i| \leqslant c|a|$ such that $a = \sum_{i=1}^m a_i b_i$. Choose some non-zero $z \in \mathbb{Q}_p$ with $|z| = \lambda' \leqslant \lambda/c$. Set

$$B = A\langle X_1, \ldots, X_m\rangle/(zX_1 - b_1, \ldots, zX_m - b_m);$$

then $M(B) = \{y \in M(A) | |b_i(y)| \leqslant \lambda', 1 \leqslant i \leqslant m\}$ is a Weierstrass subdomain containing x. Let $\{v_i\}_{i\in I}$ be an orthogonal basis of $\mathfrak{m}_x$ over $\mathbb{Q}_p$. Now if $f \in \ker(\rho_x)$, write $f = \sum_{i\in I} v_i \otimes g_i$ with $g_i \in S$; then $|f| = \max_{i\in I}\{|v_i||g_i|\}$. For each $i \in I$, choose $a_{1i}, \ldots, a_{mi} \in A$ such that $\sum_{j=1}^m a_{ji} b_j = v_i$ with $|a_{ji}| \leqslant c|v_i|$ for $1 \leqslant j \leqslant m$. Put $f_j = \sum_{i\in I} a_{ji} g_i$ for $1 \leqslant j \leqslant m$. It then follows that

$$|f_j| \leqslant \max_{i\in I}\{|a_{ji}||g_i|\} \leqslant c \max_{i\in I}\{|v_i||g_i|\} = c|f|$$

and $f = \sum_{j=1}^m b_j f_j$. This implies that $f \in \mathfrak{m}_x(A\widehat{\otimes}_{\mathbb{Q}_p} S)$. Furthermore, since the norms of the b_j in B are no more than λ', the norm of f in $B\widehat{\otimes}_{\mathbb{Q}_p} S$ is no more than $\lambda' c$, which is no more than λ times the norm of f in $A\widehat{\otimes}_{\mathbb{Q}_p} S$. □

Corollary 2.2.3. *Let S be a $\mathbb{Q}_p$-Banach algebra. Let $x \in M(A)$, and let F_x be an invertible matrix over $k(x)\otimes_{\mathbb{Q}_p} S$. Let F be a matrix over $A\widehat{\otimes}_{\mathbb{Q}_p} S$ lifting F_x. Then there exists a Weierstrass subdomain $M(B)$ of $M(A)$ containing x such that F is invertible over $B\widehat{\otimes}_{\mathbb{Q}_p} S$.*

Proof. Using the first part of Proposition 2.2.2, we lift F_x^{-1} to a matrix F' over $A\widehat{\otimes}_{\mathbb{Q}_p} S$. Note that $F'F - I$ vanishes at x. It therefore follows from the second part of Proposition 2.2.2 that there exists a Weierstrass subdomain $M(B)$ containing x such that the norm of $F'F - I$, viewed as a matrix over $B\widehat{\otimes}_{\mathbb{Q}_p} S$, is less than 1. This implies that $F'F$ is invertible over $B\widehat{\otimes}_{\mathbb{Q}_p} S$; hence F is invertible over $B\widehat{\otimes}_{\mathbb{Q}_p} S$. □

Lemma 2.2.4. *Let $0 < r < r_\varphi$, and let M_A^r be a vector bundle over $\mathcal{R}_{A_K}^r$ equipped with an isomorphism $\varphi^* M_A^r \cong M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{r/q}$ as vector bundles over $\mathcal{R}_{A_K}^{r/q}$. Suppose that there exists a basis $e_1, \ldots, e_n$ of $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q,r]}$ over $\mathcal{R}_{A_K}^{[r/q,r]}$ on which φ acts via an invertible matrix F over $\mathcal{R}_{A_K}^{r/q}$; then $e_1, \ldots, e_n$ extends to a basis of M_A^r.*

Proof. We will proceed by induction on l to show that one can extend $e_1, \ldots, e_n$ to a basis of $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^l,r]}$ for each $l \geqslant 1$. The initial case is already known by assumption. Suppose that the claim is true for some $l - 1 \geqslant 1$. Write $\mathrm{e} = (e_1, \ldots, e_n)$. Since $\varphi(\mathrm{e})$ is equal to $\mathrm{e}F$ in $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q,r/q]}$, they are equal in $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^{l-1},r/q]}$ *a priori*. Then using the relation $\mathrm{e} = \varphi(\mathrm{e})F^{-1}$, we extend e to $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^l,r]}$ by gluing e and $\varphi(\mathrm{e})F^{-1}$. It remains to prove that e generates $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^l,r]}$. Let M' be the coherent subsheaf of $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^l,r]}$ generated by e. Note that $\varphi(\mathrm{e})$ is a basis of $M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^l,r/q]}$ by the

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isomorphism $\varphi^*M_A^r \cong M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{r/q}$. It therefore follows that

$$M'|_{M(\mathcal{R}_{A_K}^{[r/q^{l-1},r]})} = M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^{l-1},r]}, \quad M'|_{M(\mathcal{R}_{A_K}^{[r/q^l,r/q]})} = M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^l,r/q]}.$$

Hence $M' = M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q^l,r]}$. □

Lemma 2.2.5. *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$ such that it is represented by a vector bundle M_A^r over $\mathcal{R}_{A_K}^r$ for some $0 < r < r_\varphi$. Let $x \in M(A)$, and let e_x be a basis of*

$$M_x^{[r/q,r]} = M_A^r \otimes_{\mathcal{R}_{A_K}^r} (k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}_K^{[r/q,r]})$$

over $k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}_K^{[r/q,r]}$. Suppose that e is a lift of e_x in $M_A^{[r/q,r]} = M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{A_K}^{[r/q,r]}$. Then there exists a Weierstrass subdomain $M(B)$ containing x such that e is a basis of $M_B^{[r/q,r]} = M_A^r \otimes_{\mathcal{R}_{A_K}^r} \mathcal{R}_{B_K}^{[r/q,r]}$ over $\mathcal{R}_{B_K}^{[r/q,r]}$.

Proof. Since $M_A^{[r/q,r]}$ is a coherent sheaf over $M(\mathcal{R}_{A_K}^{[r/q,r]})$, we choose a finite set of generators $\mathrm{v} = (v_1, \ldots, v_m)$ of it. We lift the transformation matrix between the image of v in $M_x^{[r/q,r]}$ and e_x to a matrix U over $\mathcal{R}_{A_K}^{[r/q,r]}$. It is clear that the image of $\mathrm{e}U - \mathrm{v}$ in $M_x^{[r/q,r]}$ vanishes. Since $M_A^{[r/q,r]}$ is a finite locally free $\mathcal{R}_{A_K}^{[r/q,r]}$-module, by Proposition 2.2.2, we deduce that $\mathrm{e}U - \mathrm{v} \in \mathfrak{m}_x M_A^{[r/q,r]}$; thus there is a square matrix W over $\mathfrak{m}_x \mathcal{R}_{A_K}^{[r/q,r]}$ such that $\mathrm{e}U - \mathrm{v} = \mathrm{v}W$. By Proposition 2.2.2, we choose a Weierstrass subdomain $M(B)$ containing x such that $\min\{w_{r/q}(W), w_r(W)\} > 0$ over $\mathcal{R}_{B_K}^{[r/q,r]}$. This implies that $I + W$ is invertible over $\mathcal{R}_{B_K}^{[r/q,r]}$. Hence $\mathrm{e}U(I + W)^{-1} = \mathrm{v}$, yielding that e generates $M_B^{[r/q,r]}$. Since the number of entries of e is equal to the rank of $M_B^{[r/q,r]}$, we get that e is a basis of $M_B^{[r/q,r]}$ over $\mathcal{R}_{B_K}^{[r/q,r]}$. □

The following lemma is based on [14, Lemma 6.1.1].

Lemma 2.2.6. *For $r \in (0, r_\varphi/q)$, let D be an invertible $n \times n$ matrix over $\mathcal{R}_{A_K}^{[r,r]}$, and put $h = -w_r(D) - w_r(D^{-1})$. Let F be an $n \times n$ matrix over $\mathcal{R}_{A_K}^{[r,r]}$ such that $w_r(FD^{-1} - I_n) \geqslant c + h/(q-1)$ for a positive number c. Then for any positive integer k satisfying $2(q-1)k \leqslant c$, there exists an invertible $n \times n$ matrix U over $\mathcal{R}_{A_K}^{[r,\mathrm{qr}]}$ such that $U^{-1}F\varphi(U)D^{-1} - I_n$ has entries in $\pi^k \mathcal{R}_{A_K}^{\mathrm{int},r}$ and $w_r(U^{-1}F\varphi(U)D^{-1} - I_n) \geqslant c + h/(q-1)$.*

Proof. For any $i \in \{v(a) | a \in A_K\}$, $r > 0$, $f = \sum_{j=-\infty}^{+\infty} a_j T^j \in \mathcal{R}_{A_K}$, we set $v_i(f) = \min\{j : v(a_j) \leqslant i\}$ and $v_{i,r}(f) = rv_i(f) + i$. (If $A = \mathbb{Q}_p$, they are v_i^{naive}, $v_{i,r}^{\mathrm{naive}}$, introduced in [14, p. 458].) It is clear that

$$v_{i,r}(f) = rv_i(f) + i \geqslant rv_i(f) + v(a_{v_i(f)}) \geqslant w_r(f).$$

Furthermore, we claim that $w_r(f) = \min_i\{v_{i,r}(f)\}$. In fact, suppose that $w_r(f) = v(a_{j_0}) + rj_0$ for some j_0. Let $i_0 = v(a_{j_0})$. It follows that $v_{i_0}(f) \leqslant j_0$. This implies that $v_{i_0,r}(f) \leqslant w_r(f)$, yielding the claim.

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We define a sequence of invertible matrices $U_0, U_1, \dots$ over $\mathcal{R}_{A_K}^{[r,\mathrm{qr}]}$ and a sequence of matrices $F_0, F_1, \dots$ over $\mathcal{R}_{A_K}^{[r,r]}$ as follows. Set $U_0 = I_n$. Given U_l, put $F_l = U_l^{-1}F\varphi(U_l)$. Suppose that $F_lD^{-1} - I_n = \sum_{m=-\infty}^{\infty} V_mT^m$ where the V_m are $n \times n$ matrices over A_K. Let $X_l = \sum_{v(V_m)\leqslant k} V_mT^m$, and put $U_{l+1} = U_l(I_n + X_l)$. Set

$$c_l = \min_{i\leqslant k}\{v_{i,r}(F_lD^{-1} - I_n) - h/(q-1)\}.$$

By the construction of X_l we get

$$w_r(X_l) = \min_i v_{i,r}(X_l) = \min_{i\leqslant k} v_{i,r}(X_l) = \min_{i\leqslant k} v_{i,r}(F_lD^{-1} - I_n) = c_l + h/(q-1).$$

We now prove by induction that $c_l \geqslant \max\{c, \frac{l+1}{2}c\}$, $w_r(F_lD^{-1} - I_n) \geqslant c + h/(q-1)$ and U_l is invertible over $\mathcal{R}_{A_K}^{[r,\mathrm{qr}]}$ for any $l \geqslant 0$. For $l = 0$, by assumption, it is clear that

$$c_0 \geqslant w_r(FD^{-1} - I_n) - h/(q-1) \geqslant c.$$

Suppose that the claim is true for some $l \geqslant 0$. Note that for any $s \in [r, \mathrm{qr}]$ and $m \in \mathbb{Z}$,

$$\begin{aligned}(s/r)(v(V_m) + rm) &= v(V_m) + sm + (s/r - 1)v(V_m)\\ &\leqslant v(V_m) + sm + (s/r - 1)k.\end{aligned}$$

Hence $(s/r)w_r(X_l) \leqslant w_s(X_l) + (s/r - 1)k$. Since $c_l \geqslant \frac{l+1}{2}c \geqslant (q-1)k$, we therefore deduce that

$$\begin{aligned}w_s(X_l) &\geqslant (s/r)w_r(X_l) - (s/r - 1)k\\ &= (s/r)(c_l + h/(q-1)) - (s/r - 1)k\\ &> 0\end{aligned}$$

for any $s \in [r, \mathrm{qr}]$. It follows that U_{l+1} is invertible over $\mathcal{R}_{A_K}^{[r,\mathrm{qr}]}$. Furthermore, we have

$$\begin{aligned}w_r(D\varphi(X_l)D^{-1}) &\geqslant w_r(D) + w_r(\varphi(X_l)) + w_r(D^{-1})\\ &= w_{qr}(X_l) - h\\ &\geqslant q(c_l + h/(q-1)) - (q-1)k - h\\ &= qc_l + h/(q-1) - (q-1)k\\ &\geqslant c_l + \frac{1}{2}c + h/(q-1) + \left(\frac{1}{2}c - (q-1)k\right)\\ &\geqslant \frac{(l+2)}{2}c + h/(q-1)\end{aligned}$$

since $c_l \geqslant c$. Note that

$$\begin{aligned}F_{l+1}D^{-1} - I_n &= (I_n + X_l)^{-1}F_lD^{-1}(I_n + D\varphi(X_l)D^{-1}) - I_n\\ &= ((I_n + X_l)^{-1}F_lD^{-1} - I_n) + (I_n + X_l)^{-1}(F_lD^{-1})D\varphi(X_l)D^{-1}.\end{aligned}$$

Since $w_r(F_lD^{-1}) = w_r((I_n + X_l)^{-1}) = 0$, for $i \leqslant k$, we have

$$\begin{aligned}v_{i,r}((I_n + X_l)^{-1}(F_lD^{-1})D\varphi(X_l)D^{-1}) &\geqslant w_r(D\varphi(X_l)D^{-1})\\ &\geqslant \frac{(l+2)}{2}c + h/(q-1).\end{aligned}$$

Write

$$(I_n + X_l)^{-1} F_l D^{-1} - I_n = (I_n + X_l)^{-1}(F_l D^{-1} - I_n - X_l)$$
$$= \sum_{j=0}^{\infty} (-X_l)^j (F_l D^{-1} - I_n - X_l).$$

By the definition of X_l, we have $v_i(F_l D^{-1} - I_n - X_l) = \infty$ for $i \leqslant k$ and

$$w_r(F_l D^{-1} - I_n - X_l) \geqslant w_r(F_l D^{-1} - I_n) \geqslant c + h/(q-1).$$

Thus $v_{i,r}(F_l D^{-1} - I_n - X_l) = \infty$ for $i \leqslant k$, and for $j \geqslant 1$ and $i \leqslant k$, we have

$$\begin{aligned} v_{i,r}((-X_l)^j(F_l D^{-1} - I_n - X_l)) &\geqslant w_r((-X_l)^j(F_l D^{-1} - I_n - X_l)) \\ &\geqslant j w_r(X_l) + c + h/(q-1) \\ &= j(c_l + h/(q-1)) + c + h/(q-1) \\ &\geqslant c + c_l + 2h/(q-1) \\ &> \frac{l+2}{2}c + h/(q-1). \end{aligned}$$

Putting all of these together, we get

$$v_{i,r}(F_{l+1}D^{-1} - I_n) \geqslant \frac{l+2}{2}c + h/(q-1)$$

for any $i \leqslant k$ and $w_r(F_{l+1}D^{-1} - I_n) \geqslant c + h/(q-1)$; this yields $c_{l+1} \geqslant \frac{l+2}{2}c$. The induction step is finished.

Now since $w_s(X_l) \geqslant (s/r)(c_l + h/(q-1)) - (s/r-1)k$ for $s \in [r, \mathrm{qr}]$, and $c_l \to \infty$ as $l \to \infty$, the sequence U_l converges to a limit U, which is an invertible $n \times n$ matrix over $\mathcal{R}_{A_K}^{[r,\mathrm{qr}]}$ satisfying $w_r(U^{-1}F\varphi(U)D^{-1} - I_n) \geqslant c + h/(q-1)$. Furthermore, we have

$$v_{i,r}(U^{-1}F\varphi(U)D^{-1} - I_n) = \lim_{l\to\infty} v_{i,r}(U_l^{-1}F\varphi(U_l)D^{-1} - I_n) = \lim_{l\to\infty} v_{i,r}(F_{l+1}D^{-1} - I_n) = \infty$$

for any $i \leqslant k$. Therefore $U^{-1}F\varphi(U)D^{-1} - I_n$ has entries in $\pi^k \mathcal{R}_{A_K}^{\mathrm{int},r}$. □

Lemma 2.2.7. *For any free φ-modules N_1, N_2 over $\mathcal{R}_{A_K}^{\mathrm{bd}}$, the natural map*

$$\mathrm{Ext}^1_{\varphi,\mathcal{R}_{A_K}^{\mathrm{bd}}}(N_1, N_2) \to \mathrm{Ext}^1_{\varphi,\mathcal{R}_{A_K}}(N_1 \otimes_{\mathcal{R}_{A_K}^{\mathrm{bd}}} \mathcal{R}_{A_K}, N_2 \otimes_{\mathcal{R}_{A_K}^{\mathrm{bd}}} \mathcal{R}_{A_K})$$

is surjective. Here $\mathrm{Ext}^1_{\varphi,\mathcal{R}_{A_K}^{\mathrm{bd}}}$ and $\mathrm{Ext}^1_{\varphi,\mathcal{R}_{A_K}}$ denote the set of extensions in the category of φ-modules over $\mathcal{R}_{A_K}^{\mathrm{bd}}$ and $\mathcal{R}_{A_K}$ respectively.

Proof. Let M be any extension of $N_2 \otimes_{\mathcal{R}_{A_K}^{\mathrm{bd}}} \mathcal{R}_{A_K}$ by $N_1 \otimes_{\mathcal{R}_{A_K}^{\mathrm{bd}}} \mathcal{R}_{A_K}$ in the category of φ-modules over $\mathcal{R}_{A_K}$. We pick an $\mathcal{R}_{A_K}$-basis $\mathrm{e} = \{e_1, e_2, \ldots, e_{n_1+n_2}\}$ of M such that $\{e_1, e_2, \ldots, e_{n_1}\}$ is a basis of N_1 and $\{e_{n_1+1}, e_{n_1+2}, \ldots, e_{n_1+n_2}\}$ is a lift of a basis of N_2. The matrix of φ under e is then of the form

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix},$$

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where F_{11} is the matrix of φ under $\{e_1, e_2, \ldots, e_{n_1}\}$ and F_{22} is the matrix of φ under the image of $\{e_{n_1+1}, e_{n_2+2}, \ldots, e_{n_1+n_2}\}$. It is clear that for any $\lambda \in K^\times$, the matrix of φ under $\{e_1, e_2, \ldots, e_{n_1}, \lambda e_{n_1+1}, \ldots, \lambda e_{n_1+n_2}\}$ is

$$\begin{pmatrix} F_{11} & \varphi(\lambda)F_{12} \\ 0 & (\varphi(\lambda)/\lambda)F_{22} \end{pmatrix}.$$

Suppose that F is invertible over $\mathcal{R}^r_{A_K}$ for some $0 < r < r_\varphi/q$. Put

$$D = \begin{pmatrix} F_{11} & 0 \\ 0 & (\varphi(\lambda)/\lambda)F_{22} \end{pmatrix},$$

and $h = -w_r(D) - w_r(D^{-1})$ which is independent of λ. We choose a positive integer k and λ such that

$$w_r(FD^{-1} - I_{n_1+n_2}) = w_r(\lambda F_{12}F_{22}^{-1}) \geqslant 2k(q-1) + h/(q-1).$$

Then applying Lemma 2.2.6, we obtain an $(n_1+n_2) \times (n_1+n_2)$ invertible matrix U over $\mathcal{R}^{[r,\mathrm{qr}]}_{A_K}$ such that $U^{-1}F\varphi(U)D^{-1} - I_{n_1+n_2}$ lies in $\pi^k\mathcal{R}^{\mathrm{int},r}_{A_K}$ and

$$w_r(U^{-1}F\varphi(U)D^{-1} - I_{n_1+n_2}) \geqslant 2k(q-1) + h/(q-1) > 0.$$

This implies that $U^{-1}F\varphi(U)D^{-1}$, which is the matrix of φ under $\mathrm{e}U$, is invertible over $\mathcal{R}^{\mathrm{bd},r}_{A_K}$. It follows from Lemma 2.2.4 that $\mathrm{e}U$ extends to a basis of M. Moreover, following the construction of U given in Lemma 2.2.6, we see that each U_l is of the form

$$U_l = \begin{pmatrix} I_{n_1} & * \\ 0 & I_{n_2} \end{pmatrix}.$$

Hence so is U. Therefore the model N of M generated by $\mathrm{e}U$ is an extension of N_2 by N_1. □

Proposition 2.2.8. *Every φ-module over $\mathcal{R}_K$ admits a good model.*

Proof. Each pure φ-module over $\mathcal{R}_K$ has a unique good model. The general case then follows from Theorem 1.2.20, Lemma 2.2.7 and Corollary 1.6.5. □

Proposition 2.2.9. *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$. Then for any $x \in M(A)$ and a model N_x of M_x, there exists a Weierstrass subdomain $M(B)$ containing x such that $M_B = M_A \otimes_{\mathcal{R}_{A_K}} \mathcal{R}_{B_K}$ admits a finite free model N_B which lifts N_x. Furthermore, if $k(x) \subset A$, we can choose $M(B)$ such that N_y has constant generic HN-polygons for any $y \in M(B)$.*

Proof. Let e_x be a basis of N_x. By Lemma 2.2.5, after shrinking $M(A)$, we may lift e_x to a basis e_A of $M^{[r/q,r]}_A$ for some $0 < r < r_\varphi$. Let F be the matrix of φ under e_A. Then F is invertible over $\mathcal{R}^{[r/q,r/q]}_{A_K}$. Since $F(x)$ is the matrix of φ under e_x, we get that it is invertible over $k(x)\otimes_{\mathbb{Q}_p}\mathcal{R}^{\mathrm{bd},r/q}_K$. By Corollary 2.2.3, we may lift $F(x)^{-1}$ to an invertible matrix F' over $\mathcal{R}^{\mathrm{bd},r/q}_{A_K}$ by shrinking $M(A)$. Put

$$h = -w_{r/q}(F') - w_{r/q}((F')^{-1}).$$

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Since $FF' - I$ vanishes at x, by Proposition 2.2.2, we choose a positive integer k and a Weierstrass subdomain $M(B)$ containing x such that

$$w_{r/q}(FF' - I) \geqslant 2k(q-1) + h/(q-1)$$

in $\mathcal{R}_{B_K}^{[r/q,r/q]}$. Put $h' = -w_{r/q}(F') - w_{r/q}((F')^{-1})$ in $\mathcal{R}_{B_K}^{[r/q,r/q]}$; then $h' \leqslant h$. Let $c = w_{r/q}(FF' - I) - h'/(q-1)$; then $c \geqslant 2k(q-1)$. We therefore deduce from Lemma 2.2.6 that there exists an invertible matrix U over $\mathcal{R}_{B_K}^{[r/q,r]}$ such that $U^{-1}F\varphi(U)F' - I_n$ has entries in $\pi^k\mathcal{R}_{B_K}^{\mathrm{int},r/q}$ and satisfies $w_{r/q}(U^{-1}F\varphi(U)F' - I_n) > 0$. This implies that $U^{-1}F\varphi(U)F'$ is invertible over $\mathcal{R}_{B_K}^{\mathrm{int},r/q}$, yielding that $U^{-1}F\varphi(U)$, which is the matrix of φ under the basis $\mathrm{e}_A U$, is invertible over $\mathcal{R}_{B_K}^{\mathrm{bd},r/q}$. It therefore follows from Lemma 2.2.4 that $\mathrm{e}_A U$ extends to a basis of M_B. Furthermore, following the construction of U given in Lemma 2.2.6, we have $X_l(x) = 0$, $U_l(x) = I$ for each l. Hence $U(x) = I$. Therefore the $\mathcal{R}_{B_K}^{\mathrm{bd}}$-submodule N_B generated by this basis is a finite free model of M_B lifting N_x.

Finally, suppose that $k(x) \subset A$. By Corollary 1.6.12, if we further shrink $M(B)$ such that $|U^{-1}F\varphi(U) - F(x)|$ is sufficiently small in $\mathcal{E}_{B_K}$, then the generic HN-polygon of N_y is the same as the generic HN-polygon of N_x for any $y \in M(B)$. □

Corollary 2.2.10. *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$, and let $x \in M(A)$. Then there exists a Weierstrass subdomain $M(B)$ containing x such that the vector bundle $M_B = M_A \otimes_{\mathcal{R}_{A_K}} \mathcal{R}_{B_K}$ is freely generated over $\mathcal{R}_{B_K}$.*

Proof. This follows immediately from Proposition 2.2.8 and the first part of Proposition 2.2.9. □

Proposition 2.2.11. *Let M_A be a global pure family of φ-modules over $\mathcal{R}_{A_K}$. If N'_A and N''_A are two finite free pure models of M_A, then they generate the same finite free good model of M_A.*

Proof. Let $L_1, \ldots, L_n$ be the residue fields of the generic points of A_K; then the natural map $A_K \to L = L_1 \times \cdots \times L_n$ is a closed embedding (see [3] for more details about generic points of affinoid algebras and the embedding). Let U be the matrix of transformation between some bases of N'_A and N''_A. Note that the base changes $N'_A \otimes_{\mathcal{R}_{A_K}^{\mathrm{int}}} \mathcal{R}_L^{\mathrm{int}}$ and $N''_A \otimes_{\mathcal{R}_{A_K}^{\mathrm{int}}} \mathcal{R}_L^{\mathrm{int}}$ are pure models of $M_A \otimes_{\mathcal{R}_{A_K}} \mathcal{R}_L$. We therefore deduce from Proposition 1.2.21 that U is invertible over $\mathcal{R}_L^{\mathrm{bd}}$. Hence U is invertible over $\mathcal{R}_L^{\mathrm{bd}} \cap \mathcal{R}_{A_K} = \mathcal{R}_{A_K}^{\mathrm{bd}}$, yielding the proposition. □

Theorem 2.2.12. *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$. Suppose that M_x is pure of slope s for some $x \in M(A)$ with $k(x) \subset A$; then there exists a Weierstrass subdomain $M(B)$ containing x such that $M_B = M_A \otimes_{\mathcal{R}_{A_K}} \mathcal{R}_{B_K}$ admits a finite free (c,d)-pure model N_B where $d > 0$, $(c,d) = 1$ and $c/d = s$. In particular, M_B is globally pure of slope s.*

Proof. We first prove the proposition for M as a φ^d-module. By tensoring with $\mathcal{R}_{A_K}(-c)$, we may assume that $s = 0$. Let N_x be an étale model of M_x, and let e_x be a basis of N_x. By Proposition 2.2.9, for some Weierstrass subdomain $M(B)$ containing x,

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we can lift e'_x to a basis e_B of M_B which generates a finite free good model of M_B. Let F be the matrix of φ^d under e_B. Note that $F(x)$, which is the matrix of φ under e_x, is invertible over $k(x)\otimes_{\mathbb{Q}_p}\mathcal{R}_K^{\mathrm{int},r}$ for some $r>0$. We may suppose that F is invertible over $\mathcal{R}_{B_K}^{\mathrm{bd},r}$ by shrinking r. By Proposition 2.2.2, we may further shrink $M(B)$ such that $\min\{w(F-F(x)), w(F^{-1}-(F(x))^{-1})\}>0$ over $\mathcal{R}_{B_K}^{\mathrm{bd},r}$. This implies $F, F^{-1}\in\mathcal{R}_{B_K}^{\mathrm{int},r}$. Hence the $\mathcal{R}_{B_K}^{\mathrm{int}}$-submodule N_B of M_B generated by e_B is a finite free étale model of M_B.

Note that if N_B is a finite free pure (c,d)-model of M_B as a φ^d-module, so is φ^*N_B. Thus N_B and φ^*N_B generate the same finite free good model of M_B by Proposition 2.2.11, yielding that this model is stable under φ. This implies that N_B is a finite free pure (c,d)-model of M_B as a φ-module. □

2.3. Global slope filtration

We set L and $\widetilde{\mathcal{R}}$ (resp. $\widetilde{\mathcal{R}}^{\mathrm{int}}$, $\widetilde{\mathcal{R}}^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}$) separately in the following two cases.

(AF) (the absolute Frobenius case) If K is an unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ and φ is a q-power absolute Frobenius lift, let $L=\widehat{\mathbb{Q}_p^{\mathrm{ur}}}$, and let $\widetilde{\mathcal{R}}$, $\widetilde{\mathcal{R}}^{\mathrm{int}}$, $\widetilde{\mathcal{R}}^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}$ denote $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, $\bigcup_{r>0}\widetilde{\mathbf{A}}^{(0,r]}$, $\widetilde{\mathbf{B}}^{\dagger}$, $\widetilde{\mathbf{B}}$ respectively. (See [6] for more details about the constructions of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, $\widetilde{\mathbf{A}}^{(0,r]}$, $\widetilde{\mathbf{B}}^{\dagger}$ and $\widetilde{\mathbf{B}}$). The latter can be identified with $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$, $\Gamma^{\mathrm{alg}}_{\mathrm{con}}$, $\Gamma^{\mathrm{alg}}_{\mathrm{con}}[\pi^{-1}]$ and $\Gamma^{\mathrm{alg}}[\pi^{-1}]$ respectively. (See [1, §1.1] for more explanations about these identifications.) Here the latter are different kinds of basic rings associated with the residue field $\widehat{\mathbb{F}_p((u))^{\mathrm{alg}}}$ (where the completion is taken for the u-adic topology) introduced by Kedlaya in [14, §2]. On the other hand, $\widetilde{\mathcal{R}}_L$, $\widetilde{\mathcal{R}}_L^{\mathrm{int}}$, $\widetilde{\mathcal{R}}_L^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}_L$ are basic rings associated with $\overline{\mathbb{F}}_p((u^{\mathbb{Q}}))$. By [12, Theorem 8], $\widehat{\mathbb{F}_p((u))^{\mathrm{alg}}}$ is a proper closed subfield of $\overline{\mathbb{F}}_p((u^{\mathbb{Q}}))$. This leads to natural embeddings $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\subset\widetilde{\mathcal{R}}_L$, $\bigcup_{r>0}\widetilde{\mathbf{A}}^{(0,r]}\subset\widetilde{\mathcal{R}}_L^{\mathrm{int}}$, $\widetilde{\mathbf{B}}^{\dagger}\subset\widetilde{\mathcal{R}}_L^{\mathrm{bd}}$, $\widetilde{\mathbf{B}}\subset\widetilde{\mathcal{E}}_L$, which respect Frobenius actions, following [14, §2].

(RF) (the relative Frobenius case) For general K, let L be some admissible extension of K with strongly difference-closed residue field k_L, and let $\widetilde{\mathcal{R}}$, $\widetilde{\mathcal{R}}^{\mathrm{int}}$, $\widetilde{\mathcal{R}}^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}$ denote $\widetilde{\mathcal{R}}_L$, $\widetilde{\mathcal{R}}_L^{\mathrm{int}}$, $\widetilde{\mathcal{R}}_L^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}_L$ respectively.

Remark 2.3.1. In both cases, L are admissible extensions of K with strongly difference-closed residue fields. In the AF case, $\mathcal{R}_K$, $\mathcal{R}_K^{\mathrm{bd}}$, $\mathcal{E}_K$ are the basic rings associated with $k((T))$ ([14, §2.3]), and the φ-equivariant embeddings $\mathcal{R}_K\to\widetilde{\mathcal{R}}_L$, $\mathcal{R}_K^{\mathrm{bd}}\to\widetilde{\mathcal{R}}_L^{\mathrm{bd}}$, $\mathcal{E}_K\to\widetilde{\mathcal{E}}_L$ given in Remark 1.4.10 are induced by the natural embedding $k((T))\to\overline{\mathbb{F}}_p((u^{\mathbb{Q}}))$ defined as $\sum_{i>-\infty}a_iT^i\mapsto\sum_{i>-\infty}a_iu^i$; this embedding factors through $\widehat{\mathbb{F}_p((u))^{\mathrm{alg}}}$. Thus the embeddings $\mathcal{R}_K\to\widetilde{\mathcal{R}}_L$, $\mathcal{R}_K^{\mathrm{bd}}\to\widetilde{\mathcal{R}}_L^{\mathrm{bd}}$, $\mathcal{E}_K\to\widetilde{\mathcal{E}}_L$ factor through $\widetilde{\mathcal{R}}$, $\widetilde{\mathcal{R}}^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}$ respectively.

Remark 2.3.2. In the AF case, the slope theory for φ-modules over $\widetilde{\mathcal{R}}$, $\widetilde{\mathcal{R}}^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}$ has the same properties as the slope theory for φ-modules over $\widetilde{\mathcal{R}}_L$, $\widetilde{\mathcal{R}}_L^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}_L$. In fact, all the results of §§ 1.5–1.7 for φ-modules over the extended base rings are motivated by their counterparts for φ-modules over $\widetilde{\mathcal{R}}$, $\widetilde{\mathcal{R}}^{\mathrm{bd}}$, $\widetilde{\mathcal{E}}$ developed in [14].

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Definition 2.3.3. Let M_A (resp. N_A) be a family of φ-modules over $\mathcal{R}_{A_K}$ (resp. $\mathcal{R}^{\mathrm{bd}}_{A_K}$). For any $x \in M(A)$, we set

$$\widetilde{M}_x = M_A \otimes_{\mathcal{R}_{A_K}} (k(x) \otimes_{\mathbb{Q}_p} \widetilde{\mathcal{R}}) \quad (\text{resp. } \widetilde{N}_x = N_A \otimes_{\mathcal{R}^{\mathrm{bd}}_{A_K}} (k(x) \otimes_{\mathbb{Q}_p} \widetilde{\mathcal{R}}^{\mathrm{bd}}))$$

which is a φ-module over $k(x) \otimes_{\mathbb{Q}_p} \widetilde{\mathcal{R}}^{\mathrm{bd}}$ (resp. $k(x) \otimes_{\mathbb{Q}_p} \widetilde{\mathcal{R}}$).

Proposition 2.3.4. *Let M_A (resp. N_A) be a family of φ-modules over $\mathcal{R}_{A_K}$ (resp. $\mathcal{R}^{\mathrm{bd}}_{A_K}$), and let $x \in M(A)$. Suppose that*

$$k(x) \otimes_{\mathbb{Q}_p} L = \oplus_{i=1}^n L_i$$

where each L_i is a finite field extension of L. Then the following are true.

(1) *The residue field k_{L_i} of L_i is strongly difference-closed for each $1 \leqslant i \leqslant n$.*

(2) *Let $\widetilde{M}_{x,i} = M_x \otimes_{k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}_K} (L_i \otimes_L \widetilde{\mathcal{R}})$ (resp. $\widetilde{N}_{x,i} = N_x \otimes_{k(x) \otimes_{\mathbb{Q}_p} \mathcal{R}^{\mathrm{bd}}_K} (L_i \otimes_L \widetilde{\mathcal{R}}^{\mathrm{bd}})$) for $1 \leqslant i \leqslant n$. Then the HN-polygon of each $\widetilde{M}_{x,i}$ (resp. generic HN-polygon of each $\widetilde{N}_{x,i}$) is the same as that of M_x (resp. that of N_x).*

Proof. By Proposition 2.1.17(1), the φ-action on each L_i is an automorphism. Hence the residue field k_{L_i} is inversive. For the rest of (1), it reduces to showing that any dualizable difference module P over k_{L_i} is trivial. Since k_{L_i} is inversive, the φ-action on P is therefore bijective. Hence P is also dualizable as a difference module over k_L. This implies that P a φ-invariant basis over k_L. Hence it admits a φ-invariant basis over k_{L_i}. For (2), it is clear that each $\widetilde{M}_{x,i}$ (resp. $\widetilde{N}_{x,i}$) is a base change of some component of M_x (resp. N_x). Hence the HN-polygon of $\widetilde{M}_{x,i}$ (resp. generic HN-polygon of $\widetilde{N}_{x,i}$) is the same as that of M_x (resp. that of N_x) by Proposition 1.5.6 (for the RF case) and [14, Theorem 6.4.1] (for the AF case) (resp. Proposition 1.6.1 (for the RF case) and [14, Proposition 5.3.1] (for the AF case)), yielding the proposition. □

In the situation of Proposition 2.3.4, it is clear that $\widetilde{M}_x$ (resp. $\widetilde{N}_x$) is isomorphic to the direct sum of all $\widetilde{M}_{x,i}$ (resp. $\widetilde{N}_{x,i}$). We call each $\widetilde{M}_{x,i}$ (resp. $\widetilde{N}_{x,i}$) a *component* of $\widetilde{M}_x$. We set the *slopes* and *HN-polygon* of $\widetilde{M}_x$ (resp. *generic slopes* and *generic HN-polygon* of $\widetilde{N}_x$) as the slopes and HN-polygon of $\widetilde{M}_{x,i}$ (resp. generic slopes and generic HN-polygon of $\widetilde{N}_{x,i}$). We set the *HN filtration* of $\widetilde{M}_x$ (resp. *generic HN filtration* of $\widetilde{N}_x$) as the direct sum of the HN filtrations of all $\widetilde{M}_{x,i}$ (generic HN filtrations of all $\widetilde{N}_{x,i}$).

Lemma 2.3.5. *Consider the Frobenius equation*

$$\varphi(\beta) - \pi^n \beta = \alpha. \tag{2.3.5.1}$$

(1) *Let $\alpha \in A \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{E}}$. If $n \neq 0$, then (2.3.5.1) admits a unique solution $\beta \in A \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{E}}$ which is*

$$\beta = -\sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(\alpha) \tag{2.3.5.2}$$

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if $n<0$, *or*

$$\beta=\sum_{m=0}^{\infty}(\pi^{-n})^{\{-m\}}\varphi^{-m-1}(\alpha) \tag{2.3.5.3}$$

if $n>0$. *Furthermore, if* $n>0$, *then* $w(\beta)=w(\alpha)$, *and if* $n<0$, *then* $w(\beta)=w(\alpha)-n$. *If* $n=0$, *then* (2.3.5.1) *admits a solution* $\beta\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{E}}$ *with* $w(\beta)=w(\alpha)$.

(2) *Let* $\alpha\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},r}$. *If* $n>0$, *then* (2.3.5.3) *provides the unique solution* $\beta\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$ *of* (2.3.5.1). *Furthermore, we have* $\beta\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$, $w(\beta)=w(\alpha)$ *and* $w_r(\beta)\geqslant\min\{w(\alpha),w_r(\alpha)\}$. *If* $n=0$, *then* (2.3.5.1) *admits a solution* $\beta\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$ *with* $w(\beta)=w(\alpha)$ *and* $w_r(\beta)\geqslant w_r(\alpha)$.

(3) *Let* $\alpha\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},r}$, *and write* $\alpha=\sum_{i\in\mathbb{Q}}a_iu^i$ *as an element of* $\widetilde{\mathcal{R}}^{\mathrm{bd},r}_{A_L}$. *If* $n<0$, *then* (2.3.5.1) *admits at most one solution* $\beta\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$, *and it has a solution if and only if*

$$\sum_{m\in\mathbb{Z}}(\pi^{-n})^{\{m+1\}}\varphi^m(a_{iq^{-m}})=0 \tag{2.3.5.4}$$

for every $i<0$. *Furthermore, if* β *is a solution of* (2.3.5.1), *then it belongs to* $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$ *and satisfies* $w(\beta)=w(\alpha)-n$, $w_r(\beta)\geqslant w_r(\alpha)-C(q,r,n)$ *where* $C(q,r,n)$ *is some constant which only depends on* q,r,n. *As a consequence of (1), we see that* β *is given by* (2.3.5.2).

Proof. The uniqueness parts of (1) and (2) follow from the fact that w is preserved by φ. By Lemma 2.1.4, A admits an orthogonal basis over $\mathbb{Q}_p$. Since φ acts trivially on A, using an orthogonal basis, the rest of (1) and (2) reduces to the case $A=\mathbb{Q}_p$. For (1), if $n\neq 0$, it is clear that the series (2.3.5.2) and (2.3.5.3) converge in $\widetilde{\mathcal{E}}$, and give a solution of (2.3.5.1). For $n=0$, we apply Lemma 1.3.2. For (2), it follows from Lemma 1.5.10(2) (for the RF case), [14, Proposition 3.3.7(c)] (for the AF case and $n>0$) and [17, Lemma 5.1] (for the AF case and $n=0$).

For (3), suppose that $\beta=\sum_{i\in\mathbb{Q}}b_iu^i\in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$ is a solution of (2.3.5.1). Comparing the coefficients of both sides of (2.3.5.1), we get $\varphi(b_{i/q})-\pi^nb_i=a_i$ for every $i\in\mathbb{Q}$; hence $b_i=\pi^{-n}\varphi(b_{i/q})-\pi^{-n}a_i$. Since $n<0$ and $\{a_i\}_{i\in\mathbb{Q}}$ is bounded, we get

$$b_i=-\sum_{m=0}^{\infty}(\pi^{-n})^{\{m+1\}}\varphi^m(a_{iq^{-m}}) \tag{2.3.5.5}$$

by iteration. Thus β is uniquely determined by α and belongs to $\widetilde{\mathcal{R}}^{\mathrm{bd}}_{A_L}$. Furthermore, it follows that there exist $C\in\mathbb{R}$ and $s>0$ such that

$$v\left(\sum_{m=0}^{\infty}(\pi^{-n})^{\{m+1\}}\varphi^m(a_{iq^{-m}})\right)\geqslant C-si$$

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for any $i \in \mathbb{Q}$. Since $((\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}}))^{\{k\}} = (\pi^{-n})^{\{m+k+1\}} \varphi^{m+k}(a_{q^{-m}}))$ for any $k \in \mathbb{Z}$, it follows that

$$\left(\sum_{m=-k}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{(iq^{-m})})^{\{k\}}\right) = \sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{k-m}}) = \sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{(iq^k)q^{-m}}).$$

Hence

$$v\left(\sum_{m=-k}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})\right) = v\left(\sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{(iq^k)q^{-m}})\right) + nk \geqslant C - siq^k + nk.$$

Therefore, if $i < 0$, then $v(\sum_{m=-k}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})) \to +\infty$ as $k \to +\infty$, yielding $\sum_{m\in\mathbb{Z}} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}}) = 0$. This proves the 'only if' part of (3).

To prove the 'if' part, for any $f = \sum_{i\in\mathbb{Q}} a_i u^i \in \widetilde{\mathcal{R}}^r_{A_L}$ and $c \in \mathbb{R}$, we set

$$w_r^{c,-}(f) = \min_{i\leqslant c}\{v(a_i) + ri\}.$$

It is clear that $w_r^{c,-}(f) \to \infty$ as $c \to -\infty$. Now suppose that $\sum_{m\in\mathbb{Z}} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}}) = 0$ for every $i < 0$. If $i \leqslant -1$, then for each $m \leqslant -1$,

$$\begin{aligned} v((\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})) &= (v(a_{iq^{-m}}) + riq^{-m}) - riq^{-m} - n(m+1) \\ &\geqslant w_r^{i,-}(\alpha) - riq^{-m} - n(m+1) \\ &\geqslant (w_r^{i,-}(\alpha) - ri) - C_1(q, r, n) \end{aligned}$$

for some constant $C_1(q, r, n)$. Hence

$$\begin{aligned} w_r\left(\left(\sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})\right) u^i\right) &= v\left(-\sum_{m=-1}^{-\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})\right) + ri \\ &\geqslant w_r^{i,-}(\alpha) - C_1(q, r, n) \end{aligned} \tag{2.3.5.6}$$

for $i \leqslant -1$. If $i > -1$, then for any $m \geqslant 0$,

$$v((\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})) \geqslant w_r(\alpha) - riq^{-m} - n(m+1) \geqslant (w_r(\alpha) - ri) - C_2(q, r, n)$$

for some constant $C_2(q, r, n)$, yielding

$$w_r\left(\left(\sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})\right) u^i\right) \geqslant w_r(\alpha) - C_2(q, r, n). \tag{2.3.5.7}$$

Now suppose that β is given by (1.5.10.1). Since the series is convergent in $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{E}}$ (hence in $\widetilde{\mathcal{E}}_{A_L}$), a short computation shows that the ith coefficient of β is just b_i given by (2.3.5.5). Hence

$$\beta = -\sum_{i\in\mathbb{Q}} \left(\sum_{m=0}^{\infty} (\pi^{-n})^{\{m+1\}} \varphi^m(a_{iq^{-m}})\right) u^i. \tag{2.3.5.8}$$

We claim that $\beta \in \widetilde{\mathcal{R}}^{\mathrm{bd},r}_{A_L}$. Since $\beta \in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{E}}$, it satisfies (1) of Definition 1.4.1. By (2.3.5.6) and (2.3.5.7), we see that β satisfies (2) of Definition 1.4.1; hence $\beta \in \widetilde{\mathcal{R}}^{\mathrm{bd},r}_{A_L}$, and it satisfies $w_r(\beta) \geqslant w_r(\alpha) - C(q, r, n)$ for

$$C(q, r, n) = \max\{C_1(q, r, n), C_2(q, r, n)\}.$$

Furthermore, $\varphi(\beta) = \alpha + \pi^n\beta \in \widetilde{\mathcal{R}}^{\mathrm{bd},r}_{A_L}$ implies $\beta \in \widetilde{\mathcal{R}}^{\mathrm{bd},qr}_{A_L}$. Since $\widetilde{\mathcal{E}}$, which is complete with respect to w, is a closed subspace of $\widetilde{\mathcal{E}}_L$, and $\widetilde{\mathcal{E}} \cap \widetilde{\mathcal{R}}^{\mathrm{bd},qr}_L = \widetilde{\mathcal{R}}^{\mathrm{bd},qr}$, we deduce that

$$\beta \in \widetilde{\mathcal{R}}^{\mathrm{bd},qr}_{A_L} \cap (A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{E}}) = A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$$

by Lemma 2.1.8. □

In the situation of Lemma 2.3.5(3), we call the ideal of A_L generated by the left-hand side of (2.3.5.4) for all $i < 0$ the *obstruction* of the (2.3.5.1).

Lemma 2.3.6. *Let L' be a p-adic field, and let $a \in A_{L'}$. Then the set*

$$\{x \in M(A) | a(x) = 0\}$$

is a Zariski closed subset of $M(A)$.

Proof. Let $\{e_i\}_{i\in I}$ be an orthogonal basis of L' over $\mathbb{Q}_p$, and write $a = \sum_{i\in I} a_i e_i$ with $a_i \in A$. It is then clear that $a(x) = 0$ if and only if $a_i(x) = 0$ for all $i \in I$. This yields the lemma. □

Lemma 2.3.7. *Keep the notation as in Lemma 2.3.5(3). Then the set S of $x \in M(A)$ for which the specialization of (2.3.5.1) admits a solution in $k(x)\otimes_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$ forms a Zariski closed subset $M(B)$ of $M(A)$. Furthermore, (2.3.5.1) admits a unique solution in $B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$.*

Proof. By Lemma 2.3.5(3), we see that S is just the set of points for which the image of the obstruction in $k(x)\otimes_{\mathbb{Q}_p}L$ is the zero ideal. Hence S is a Zariski closed subset $M(B)$ of $M(A)$ by Lemma 2.3.6. Furthermore, since the obstruction vanishes in B_L, we get the rest of the lemma by Lemma 2.3.5(3) again. □

The following lemma is based on [14, Proposition 5.4.5].

Lemma 2.3.8. *Let D be an $n \times n$ diagonal matrix with entries $D_{ii} = \pi^{a_i}$ satisfying $a_1 \geqslant \cdots \geqslant a_n$, and let $F \in \mathrm{GL}_n(A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},r})$ for some $r > 0$. If $w(FD^{-1} - I_n) > 0$ and $w_r(FD^{-1} - I_n) > 0$, then there exists $U \in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$ with $w(U - I_n) > 0$ and $w_r(U - I_n) > 0$, such that $U^{-1}F\varphi(U)D^{-1} - I_n$ is upper triangular nilpotent.*

Proof. Put $c_0 = \min\{w(FD^{-1} - I_n), w_r(FD^{-1} - I_n)\}$ and $U_0 = I_n$. We will inductively construct a sequence $U_1, U_2 \cdots \in \mathrm{GL}_n(A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr})$ satisfying

$$\min\{w(U_l^{-1}F\varphi(U_l)D^{-1} - I_n), w_r(U_l^{-1}F\varphi(U_l)D^{-1} - I_n)\} \geqslant c_0,$$
$$w(U_l - I_n) \geqslant c_0, \min\{w(U_{l+1} - U_l), w_{qr}(U_{l+1} - U_l)\} \geqslant (l+1)c_0,$$

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and the lower triangular part of $U_l^{-1}F\varphi(U_l)D^{-1} - I_n$ has both w and w_r valuations $\geqslant (l+1)c_0$ for $l \geqslant 1$. Given U_l, put $F_l = U_l^{-1}F\varphi(U_l)$, and write $F_lD^{-1} - I_n = B_l + C_l$ where B_l is upper triangular nilpotent and C_l is lower triangular. Then $\min\{w(C_l), w_r(C_l)\} \geqslant (l+1)c_0$. We claim that there exists an $n \times n$ lower triangular matrix X_l over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$ satisfying $C_l + X_l = D\varphi(X_l)D^{-1}$ and

$$\min\{w(X_l), w(D\varphi(X_l)D^{-1}), w_{qr}(X_l), w_r(X_l), w_r(D\varphi(X_l)D^{-1})\} \geqslant (l+1)c_0.$$

In fact, since $a_i \geqslant a_j$ as $i \leqslant j$, this amounts to solving a system of equations of the form

$$c + x = \pi^m\varphi(x), \qquad c \in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},r}, m \leqslant 0. \tag{2.3.8.1}$$

By Lemma 2.3.5(2), (2.3.8.1) has a solution $x \in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$ with $w(x) \geqslant w(c)$ and $w_r(x) \geqslant \min\{w(c), w_r(c)\}$. Hence $w(\pi^m\varphi(x)) \geqslant \min\{w(c), w(x)\} \geqslant w(c)$ and

$$w_{qr}(x) = w_r(\varphi(x)) \geqslant w_r(\pi^m\varphi(x)) \geqslant \min\{w_r(c), w_r(x)\} \geqslant \min\{w(c), w_r(c)\}.$$

This yields the claim. Put $U_{l+1} = U_l(I_n - X_l)$; then

$$w(U_{l+1} - I_n) \geqslant c_0, \quad w_{qr}(U_{l+1} - U_l) = w_{qr}(X_l) \geqslant (l+1)c_0,$$
$$w(U_{l+1} - U_l) = w(X_l) \geqslant (l+1)c_0.$$

We have

$$U_{l+1}^{-1}F\varphi(U_{l+1})D^{-1} - I_n = (I_n + X_l + \cdots)(I_n + B_l + C_l)(I_n - D\varphi(X_l)D^{-1}) - I_n.$$

It follows that

$$\min\{w(U_{l+1}^{-1}F\varphi(U_{l+1})D^{-1} - I_n), w_r(U_{l+1}^{-1}F\varphi(U_{l+1})D^{-1} - I_n)\} \geqslant c_0$$

and

$$\min\{w(U_{l+1}^{-1}F\varphi(U_{l+1})D^{-1} - I_n - B_l), w_r(U_{l+1}^{-1}F\varphi(U_{l+1})D^{-1} - I_n - B_l)\} \geqslant (l+1)c_0.$$

This yields the inductive step. Then $U = \lim_{l\to\infty} U_l$ satisfies the desired property. □

Proposition 2.3.9. *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$ of rank n, and let $x \in M(A)$. Let N_x be a model of M_x. Suppose that the generic slopes (counted with multiplicity) of N_x are*

$$a_1/a \geqslant \cdots \geqslant a_n/a$$

where a is a positive integer and $a_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant n$. Then there exists a Weierstrass subdomain $M(B)$ containing x such that $\widetilde{M}_B = M_A \otimes_{\mathcal{R}_{A_K}} (B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}})$ is finite free over $(B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}})$, and admits a basis under which the matrix of φ^a is an upper triangular matrix F over $B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$ with diagonal entries $F_{ii} = \pi^{-a_{n+1-i}}$.

Proof. It suffices to treat the case where $a = 1$ by replacing φ with φ^a. We claim that $\widetilde{N}_x$ admits a filtration such that the ith successive quotient is isomorphic to $V_{\pi^{-a_{n+1-i}},1}$. For the RF case, the claim follows from Propositions 1.6.4 and 1.6.9 and Lemma 1.6.10. For the AF case, the claim follows from [16, Theorem 14.6.3], [14, Theorem 6.3.3(b)] and

Lemma 1.6.10. It thus follows that $\widetilde{N}_x$ admits a basis e_x under which the matrix F_x of φ is upper triangular with diagonal entries $(F_x)_{ii} = \pi^{-a_{n+1-i}}$.

By Proposition 2.2.9, after shrinking $M(A)$, we may suppose that M_A admits a finite free model N_A with a basis e_A. Let U_x be the square matrix satisfying $p_x(e_A)U_x = e_x$. Then U_x is invertible over $\widetilde{\mathcal{R}}^{\mathrm{bd},r}$ for some $r > 0$. By Corollary 2.2.3, after shrinking $M(A)$, we lift U_x to an invertible matrix U over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},r}$. Let $F \in A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},r'}$ be the matrix of φ under the base $e = e_A U$, and let D be a lift of F_x over $\widetilde{\mathcal{R}}^{\mathrm{bd},r'}$ such that D is upper triangular and $D_{ii} = \pi^{a_i}$. Note that $(FD^{-1})(x) = I_n$. By Proposition 2.2.2, we choose a Weierstrass subdomain $M(B)$ containing x such that $\min\{w(FD^{-1} - I_n), w_{r'}(FD^{-1} - I_n)\} > 0$ over $B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},r'}$. Then by Lemma 2.3.8, there exists a matrix V over $B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd},qr}$ such that $V^{-1}F\varphi(V)D^{-1} - I_n$ is upper triangular nilpotent. It follows that the basis eV satisfies the desired property. □

Theorem 2.3.10. *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$. Then for any $x \in M(A)$, there is a Weierstrass subdomain $M(B)$ containing x such that the HN-polygon of M_y lies above the HN-polygon of M_x with the same endpoint for any $y \in M(B)$.*

Proof. By Proposition 2.2.8, we choose a good model N_x of M_x. We apply Proposition 2.3.9 to N_x. It then follows from Corollary 1.6.5 that for any $y \in M(B)$, the generic HN-polygon of N_y is the same as the generic HN-polygon of N_x; hence it is the same as the HN-polygon of M_x. We thus deduce that the HN-polygon of M_y lies above the HN-polygon of M_x with the same endpoint by Proposition 1.7.2. □

Definition 2.3.11. Let $\widetilde{M}_A$ be a φ-module over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$. For $c, d \in \mathbb{Z}$ with $d > 0$, a (c, d)-*pure model* of $\widetilde{M}_A$ is a finite free $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{int}}$-submodule $\widetilde{N}_A$ with

$$\widetilde{N}_A \otimes_{A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{int}}} (A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}) = \widetilde{M}_A$$

and so $\widetilde{N}_A \otimes_{A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{int}}} (A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}})$ is stable under φ and the φ-action induces an isomorphism $\pi^c (\varphi^d)^* \widetilde{N}_A \cong \widetilde{N}_A$. For $s \in \mathbb{Q}$, we say that $\widetilde{M}_A$ is *pure of slope* s if $\widetilde{M}_A$ admits a (c, d)-pure model for some (and hence any) $c, d \in \mathbb{Z}$ with $d > 0$ and $s = c/d$. If $s = 0$, we also say that $\widetilde{M}_A$ is *étale*, and a $(0, 1)$-pure model is also called an *étale model*. By a *slope filtration* of $\widetilde{M}_A$ we mean a finite filtration of φ-submodules of $\widetilde{M}_A$ such that the successive quotients are pure φ-modules over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$ with decreasing slopes.

Proposition 2.3.12. *Any φ-module over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$ admits at most one slope filtration.*

Proof. It suffices to show that for any two pure φ-modules $\widetilde{M}_A$ and $\widetilde{M}'_A$ over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$, if the slope of $\widetilde{M}_A$ is bigger than the slope of $\widetilde{M}'_A$, then there is no non-trivial morphism from $\widetilde{M}_A$ to $\widetilde{M}'_A$. For this, by replacing φ with a suitable power of it, and by identifying $\mathrm{Hom}(\widetilde{M}_A, \widetilde{M}'_A)$ with $(\widetilde{M}_A^{\vee} \otimes \widetilde{M}'_A)^{\varphi=1}$, it suffices to show that any étale φ-module over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$ does not have a rank 1 pure φ-submodule with positive integral slope. If the contrary is true, then there exists a non-zero column vector $\boldsymbol{v}$ over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^r$, an invertible matrix W over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{int},r}$ (in the AF case, set $\widetilde{\mathcal{R}}^{\mathrm{int},r}$ to be $\widetilde{\mathbf{A}}^{(0,r]}$) and some

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negative integer n such that $W\varphi(\boldsymbol{v}) = \pi^n \boldsymbol{v}$. By Proposition 2.1.2 and Lemma 2.1.6, we may suppose that $w_s(W) > n$ for any $0 < s \leqslant r$ by shrinking r. It therefore follows that

$$w_{s/q}(\boldsymbol{v}) = w_{s/q}(W\varphi(\boldsymbol{v})) - n \geqslant w_{s/q}(W) + w_s(\boldsymbol{v}) - n > w_s(\boldsymbol{v})$$

for any $0 < s \leqslant r$. This implies that $w_{s/q^n}(\boldsymbol{v}) > w_s(\boldsymbol{v})$ for any $n \geqslant 1$. We claim that $\boldsymbol{v}$ is over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$. In fact, if a_i is the ith coefficient of some entry of $\boldsymbol{v}$, we then have

$$v(a_i) + \frac{is}{q^n} \geqslant w_{s/q}(\boldsymbol{v}) > w_s(\boldsymbol{v})$$

for any $n \geqslant 1$. Hence $v(a_i) > w_s(\boldsymbol{v})$; thus v is over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$ and satisfies $w(\boldsymbol{v}) > w_s(\boldsymbol{v})$. It then follows that $w(\boldsymbol{v}) = w(\varphi(\boldsymbol{v})W) - n > w(\varphi(\boldsymbol{v})) + w(W) \geqslant w(\boldsymbol{v})$, yielding a contradiction. □

Lemma 2.3.13. *Let L' be an extension of K with strongly difference-closed residue field. Let M be a φ-module over $\widetilde{\mathcal{R}}_{L'}$. Suppose that the maximal slope m of M is integral. Then for any $s \in \mathbb{Q}$, $m \geqslant s$ if and only if M admits a non-zero eigenvector of φ with eigenvalue $\pi^{[-s]}$.*

Proof. If $\varphi(v) = \pi^{[-s]}v$ for some non-zero $v \in M$, it follows that $m \geqslant -[-s] \geqslant s$. Conversely, suppose that $m \geqslant s$. Using Theorem 1.5.8, let $V_{\lambda,d}$ with $\lambda \in L'$ be in a Dieudonné–Manin decomposition of the first step of the HN filtration of M. Since $v(\lambda)/d = -m$ is an integer, by Lemma 1.6.10, we deduce that $V_{\lambda,d}$ admits a non-zero φ-eigenvector v with eigenvalue π^{-m}. If $-m = [-s]$, then we are done. Otherwise, let $n = [-s] + m$, and put

$$f = \sum_{i \in \mathbb{Z}} (\pi^{-n})^{\{i\}} u^{q^i}.$$

Since $n > 0$, it is clear that f is a well-defined element of $\widetilde{\mathcal{R}}_{L'}$, and satisfies $\varphi(f) = \pi^n f$. It follows that fv is a non-zero φ-eigenvector of M with eigenvalue $\pi^{-m+n} = \pi^{[-s]}$. □

Remark 2.3.14. Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$. It is clear that if $\widetilde{M}_A = M_A \otimes_{\mathcal{R}_{A_K}} (A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}})$ is a φ-module over $A\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$ admitting a slope filtration, the HN-polygons of M_x are constant over $M(A)$. In § 2.4, we will construct a family of φ-modules such that its HN-polygons are not even locally constant over the base. Thus this family does not admit a slope filtration over the extended Robba ring.

Theorem 2.3.15. *Let M_A be a family of φ-modules over $\mathcal{R}_{A_K}$, and let $x \in M(A)$. Then there exists a Weierstrass subdomain $M(B)$ containing x such that the set of $y \in M(B)$ where the HN-polygon of M_y coincides with the HN-polygon of M_x forms a Zariski closed subset $M(C)$ of $M(B)$, and*

$$\widetilde{M}_C = M_A \otimes_{\mathcal{R}_{A_K}} (C\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}})$$

admits a unique slope filtration which lifts the HN filtration of $\widetilde{M}_x$.

Proof. Let a' be the least common multiple of the denominators of the slopes of M_x. Let $a = a'(2^n)!$ where n is the rank of M. We view M as a φ^a-module. Suppose that the slopes of M_x are $s_1 > \cdots > s_l$ where each s_j has multiplicity d_j. Then all s_j are integers. By Proposition 2.3.9, there exists a Weierstrass subdomain $M(B)$ containing x and a basis $\widetilde{e}_B$ of $\widetilde{M}_B = M_A \otimes_{\mathcal{R}_{A_K}} (B \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{R}})$ such that the matrix F of φ^a under $\widetilde{e}_B$ is $n \times n$ upper triangular over $B \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{R}}^{\mathrm{bd},r}$ for some $r > 0$ with diagonal entries $F_{ii} = \pi^{m_i}$ where $m_i = -s_j$ if $d_l + \cdots + d_{j+1} < i \leqslant d_l + \cdots + d_j$.

As explained in the proof of Theorem 2.3.10, the HN-polygon of M_y lies above the HN-polygon of M_x for any $y \in M(B)$. Therefore the HN-polygon of M_y coincides with the HN-polygon of M_x if and only if the former lies below the latter. Since HN-polygons are convex, by Proposition 1.2.22, we deduce that the HN-polygon of M_y lies above the HN-polygon of M_x if and only if the maximal slope of $\wedge^{d_1+\cdots+d_j} M_y$ is no less than $\sum_{i=1}^{j} s_i d_i$ for each $1 \leqslant j \leqslant l$. By Proposition 2.3.4(2), M_y and $\widetilde{M}_y$ have the same HN-polygons. Let n_j be the rank of $\wedge^{d_1+\cdots+d_j} \widetilde{M}_B$; then $n_j \leqslant 2^n$. By the construction of a, we see that all the slopes of $\wedge^{d_1+\cdots+d_j} \widetilde{M}_y$ are integral. Hence by Proposition 2.3.4(1), Lemma 2.3.13 (for the RF case) and [14, Proposition 3.3.2] (for the AF case), we conclude that the HN-polygon of M_y lies below the HN-polygon of M_x if and only if each component of $\wedge^{d_1+\cdots+d_j} \widetilde{M}_y$ admits a non-zero φ^a-eigenvector with eigenvalue $\pi^{-s_1 d_1 - \cdots - s_j d_j}$ for $1 \leqslant j \leqslant l$. Let S_j be the set of $y \in M(B)$ which satisfies this condition for j. Therefore, to prove the first part of the theorem, it suffices to show that each S_j is a Zariski closed subset of $M(B)$.

Note that under the basis $\wedge^{d_1+\cdots+d_j} \widetilde{e}_B$ of $\wedge^{d_1+\cdots+d_j} \widetilde{M}_B$, the matrix G for φ^a is upper triangular, and its diagonal entries are of the forms π^{-m} where m goes through all the sums of $d_1 + \cdots + d_j$ elements of the slope multiset of M_x. In particular, $G_{n_j,n_j} = \pi^{-s_1 d_1 - \cdots - s_j d_j}$ is the smallest power of π among diagonal entries. Using the basis $\wedge^{d_1+\cdots+d_j} \widetilde{e}_B$ and matrix G, a short computation shows that finding φ^a-eigenvectors with eigenvalues $\pi^{-s_1 d_1 - \cdots - s_j d_j}$ amounts to solving a series of equations $\varphi(\beta_i) - h_i \beta_i = \alpha_i$ for $1 \leqslant i \leqslant n_j$, where

$$h_i = \pi^{-s_1 d_1 - \cdots - s_j d_j} / G_{n_j+1-i, n_j+1-i},$$

and

$$\alpha_i = -G^{-1}_{n_j+1-i, n_j+1-i} (G_{n_j+1-i, n_j} \varphi(\beta_1) + \cdots G_{n_j+1-i, n_j+2-i} \varphi(\beta_{i-1})).$$

Note that $h_1 = 1, \alpha_1 = 0$, and that h_i is a negative power of π for each $i \geqslant 2$. Let $L' = L^{\varphi=1}$; it follows that $\beta_1 \in (B \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathcal{R}}^{\mathrm{bd}})^{\varphi} = B_{L'}$. Furthermore, by Lemma 2.3.5(3), for any initial value $\beta_1 \in B_{L'}$, this series of equation admits at most one solution. By its definition, we see that S_j is just the set $y \in M(B)$ for which the specialization of this series of equations admits a solution for the initial value 1 (and hence for any initial value) in $k(y) \otimes_{\mathbb{Q}_p} L'$. Applying Lemmas 2.3.5 and 2.3.7 inductively, we thus deduce that S_j is a Zariski closed subset of $M(B)$, yielding the first part of the theorem.

To show the rest of the theorem, we may suppose that $B = C$. Using the basis $\widetilde{e}_B$ and matrix F, a short computation shows that finding φ^a-eigenvectors with eigenvalues π^{-s_1}

in $\widetilde{M}_B$ amounts to solving a series of equations $\varphi(\beta_i) - h_i\beta_i = \alpha_i$ for $1 \leqslant i \leqslant n$, where

$$h_i = \pi^{-s_1}/F_{n+1-i,n+1-i},$$

and

$$\alpha_i = -F^{-1}_{n+1-i,n+1-i}(F_{n+1-i,n}\varphi(\beta_1) + \cdots F_{n+1-i,n+2-i}\varphi(\beta_{i-1})).$$

Note that $h_1 = \cdots = h_{d_1} = 1, \alpha_1 = 0$, and h_i is a negative power of π for each $d_1+1 \leqslant i \leqslant n$. Note that for $\alpha \in B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$, any two solutions of the equation $\varphi(\beta)-\beta = \alpha$ differ by an element of $B_{L'}$. We therefore deduce from Lemma 2.3.5(2) that $\beta_1, \ldots, \beta_{d_1}$ are of the forms

$$\beta_1 = c_1$$
$$\beta_2 = c_1\beta_{21} + c_2$$
$$\vdots$$
$$\beta_{d_1} = c_1\beta_{d_11} + c_2\beta_{d_12} + \cdots + c_{d_1}$$

where β_{ij} are fixed elements of $B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$ determined by the coefficients of $F_{n-c,n-d}$ for $0 \leqslant c \leqslant d_1 - 1$ and $0 \leqslant d \leqslant d_1 - 2$ and $c_1, \ldots, c_{d_1} \in B_{L'}$. Furthermore, by Lemma 2.3.5, we see that for any initial values $c_1, \ldots, c_d$, both this series of equations and its specialization at any $y \in M(B)$ admit at most one solution. On the other hand, by Theorem 1.5.8 (for the RF case), [14, Proposition 4.2.5] (for the AF case) and Lemma 1.6.10, we see that the set of φ^a-eigenvectors with eigenvalue π^{-s_1} in $\widetilde{M}_y$ is a free $k(y)\otimes_{\mathbb{Q}_p}L'$-module of rank d_1. We therefore deduce that the specialization of this series of equations at y admits a unique solution in $k(y)\otimes_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$ with the initial values $c_1(y), \ldots, c_{d_1}(y)$. Hence by Lemmas 2.3.5 and 2.3.7, we conclude that this series of equations admits a unique solution in $B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}^{\mathrm{bd}}$ for any initial values $c_1, \ldots, c_d$. As a consequence, the set of φ^a-eigenvectors with eigenvalue π^{-s_1} is a free $B_{L'}$-module generated by $\boldsymbol{v}_i = (0, \ldots, 0, 1, \ldots)^t$ where the first 1 lies in the ith entry for $1 \leqslant i \leqslant d_1$. Set $\widetilde{M}^1_B$ as the φ^a-submodule of $\widetilde{M}_B$ generated by $\boldsymbol{v}_i$ for all $1 \leqslant i \leqslant d_1$.

It is clear that $\widetilde{M}^1_B$ is free of rank d_1 and $\widetilde{M}_B/\widetilde{M}^1_B$ is free of rank $n - d_1$. Furthermore, $\widetilde{M}^1_B$ is pure of slope s_1 as a φ^a-module. Thus by induction on $\widetilde{M}_B/\widetilde{M}^1_B$, we deduce that as a φ^a-module, $\widetilde{M}_B$ admits a slope filtration

$$0 \subset \widetilde{M}^1_B \subset \cdots \subset \widetilde{M}^l_B = \widetilde{M}_B. \tag{2.3.15.1}$$

Note that the φ-pullback of (2.3.15.1) is also the slope filtration of $\widetilde{M}_B$. Hence (2.3.15.1) is stable under φ^* by Proposition 2.3.12, yielding that each $\widetilde{M}^j_B$ is also a φ-submodule of $\widetilde{M}_B$. It remains to show that the successive quotients are pure as φ-modules. By induction, we only need to show this for $\widetilde{M}^1_B$. Since k_L is strongly difference-closed, by [16, Lemma 14.3.3], we choose some $\lambda \in L^\times$ such that $\varphi^a(\lambda)/\lambda = \varphi(\pi^{s_i})/\pi^{s_i}$. A short computation shows that if $\boldsymbol{v}$ is a φ^a-eigenvector with eigenvalues π^{-s_i}, so is $\lambda\varphi(\boldsymbol{v})$. Hence the free $B_{L'}$-module generated by $\{\boldsymbol{v}_i\}_{1\leqslant i\leqslant d_1}$ is stable under φ. This implies that the finite free good model of $\widetilde{M}^1_B$ generated by $\{\boldsymbol{v}_i\}_{1\leqslant i\leqslant d_1}$ is stable under φ, yielding that $\widetilde{M}^1_B$ is pure of slope s_1 as a φ-module. We therefore conclude that (2.3.15.1) is the slope filtration of $\widetilde{M}_B$ as a φ-module over $B\widehat{\otimes}_{\mathbb{Q}_p}\widetilde{\mathcal{R}}$. □

2.4. An example

In this subsection we construct a family of φ-modules with non-constant HN-polygons. Our construction is inspired by the computation of H^1 of rank 1 (φ, Γ)-modules in [7].

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. We set the φ, Γ-actions on $\mathcal{R}_K$ as

$$\gamma(f(T)) = f((1+T)^{\chi(\gamma)} - 1), \quad \varphi(f(T)) = f((1+T)^p - 1), \qquad \gamma \in \Gamma, f(T) \in \mathcal{R}_K,$$

where χ is the p-adic cyclotomic character. Set the 'p-adic $2\pi i$' $t = \log(1+T)$ which satisfies $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$.

Definition 2.4.1. By a (φ, Γ)-module over $\mathcal{R}_K$ we mean a φ-module over $\mathcal{R}_K$ equipped with a continuous semilinear Γ-action which commutes with the φ-action.

Definition 2.4.2. For any continuous character $\delta : \mathbb{Q}_p^\times \to K^\times$, define the rank 1 (φ, Γ)-module $\mathcal{R}_K(\delta)$ by setting the φ, Γ-action as

$$\varphi(av) = \delta(p)\varphi(a)v, \quad \gamma(av) = \delta(\chi(\gamma))\gamma(a)v, \quad a \in R_K \tag{2.4.2.1}$$

for some $\mathcal{R}_K$-basis v. For $\delta(x) = x^{-n}$ for some $n \in \mathbb{Z}$, we denote $\mathcal{R}_K(\delta)$ by $\mathcal{R}_K(n)$. For any (φ, Γ)-module M over $\mathcal{R}_K$, set the (φ, Γ)-module $M(\delta) = M \otimes_{\mathcal{R}_K} \mathcal{R}_K(\delta)$.

Remark 2.4.3. We set $\mathcal{R}_K(n)$ as $\mathcal{R}_K(x^{-n})$ in order to match our sign convention of slopes. We caution that our sign convention is opposite to the one used in [7].

If $p > 2$, then Γ is topologically procyclic. We fix a topological generator γ of Γ.

Definition 2.4.4. Suppose that $p > 2$. For a (φ, Γ)-module M over $\mathcal{R}_K$, set the complex $C^\bullet_{\varphi,\gamma}(M)$ as

$$0 \longrightarrow M \xrightarrow{d_1} M \oplus M \xrightarrow{d_2} M \longrightarrow 0$$

with $d_1(x) = ((\gamma - 1)x, (\varphi - 1)x)$ and $d_2(x, y) = (\varphi - 1)x - (\gamma - 1)y$. Let $\mathrm{H}^\bullet(M)$ denote cohomology groups of this complex.

It is straightforward to see that $\mathrm{H}^1(M)$ classifies the extension of the trivial (φ, Γ)-module $\mathcal{R}_K$ by M.

Remark 2.4.5. If $p = 2$, Γ is no longer topologically procyclic; we need to modify the definition of $C^\bullet_{\varphi,\gamma}(M)$. See [20, §2.1] for more details.

Lemma 2.4.6. *For a (φ, Γ)-module M over $\mathcal{R}_K$, the slope filtration of M is a filtration of (φ, Γ)-submodules.*

Proof. Since the slope filtration of M is unique and φ, Γ-actions commute, it is stable under the Γ-action. This yields the lemma. □

Lemma 2.4.7. *If M is a (φ, Γ)-module over $\mathcal{R}_K$ satisfying the exact sequence*

$$0 \longrightarrow \mathcal{R}_K(-1) \longrightarrow M \longrightarrow \mathcal{R}_K(1) \longrightarrow 0.$$

then M is étale if and only if the exact sequence is non-split.

Proof. The 'only if' part is trivial. Suppose that the exact sequence is non-split. We first have $\deg(M) = 0$. If M is not étale, by Lemma 2.4.6, it has a rank 1 (φ, Γ)-submodule N with positive slope. Since $\mathcal{R}_K(-1)$ is a saturated (φ, Γ)-submodule of M with negative slope, we have $N \cap \mathcal{R}_K(-1) = 0$ by Corollary 1.2.12. This implies that N maps isomorphically to a (φ, Γ)-submodule of $\mathcal{R}_K(1)$. The (φ, Γ)-submodules of $\mathcal{R}_K(1)$ are of the forms $t^k\mathcal{R}_K(1)$ for $k \in \mathbb{N}$ (see [7, Lemme 3.2, Remarque 3.3] for a proof (the proof works for our general K)). Hence $N \cong t^k\mathcal{R}_K(1)$ for some $k \geqslant 1$ as the exact sequence is non-split. It therefore follows that $\deg(N) \leqslant 0$, yielding a contradiction. □

Lemma 2.4.8. *Suppose that $p > 2$. Then the natural map $\mathrm{H}^1(\mathcal{R}_{\mathbb{Q}_p}(\delta)) \to \mathrm{H}^1(\mathcal{R}_L(\delta))$ is injective for any continuous character $\delta : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ such that $\delta(p) \neq p^{-n}$ for any $n \in \mathbb{N}$.*

Proof. Suppose that (a, b) represents an element in the kernel of this map; then

$$a = (\delta(\chi(\gamma))\gamma - 1)f, \quad b = (\delta(p)\varphi - 1)f$$

for some $f = \sum_{i\in\mathbb{Z}} a_iT^i \in \mathcal{R}_L$. We claim that f belongs to $\mathcal{R}_{\mathbb{Q}_p}$. We proceed by induction on j to show that $a_{-j}, a_j \in \mathbb{Q}_p$ for each $j \in \mathbb{N}$. The constant term of $(\delta(p)\varphi - 1)f$ is $(\delta(p)-1)a_0$. Since $\delta(p) \neq 1$, we get $a_0 \in \mathbb{Q}_p$. Now suppose that $a_{1-j}, \ldots, a_0, \ldots, a_{j-1} \in \mathbb{Q}_p$ for some $j \geqslant 1$. Note that the coefficient of T^j in $(\delta(p)\varphi - 1)f$ is the sum of $(\delta(p)p^j - 1)a_j$ and a $\mathbb{Q}_p$-linear combination of $a_1, \ldots, a_{j-1}$. Since $\delta(p)p^j - 1 \neq 0$, we get $a_j \in \mathbb{Q}_p$. Note that for any positive integer k, we have

$$\varphi\left(\frac{1}{T^k}\right) = \frac{1}{((1+T)^p - 1)^k} = \frac{1}{T^{pk}} \cdot \frac{1}{((1+\frac{1}{T})^p - \frac{1}{T^p})^k} = \frac{1}{T^{pk}}\left(1 - \frac{pk}{T} - \cdots\right).$$

Hence the coefficient of T^{-pj} in $(\delta(p)\varphi - 1)f$ is the sum of $(\delta(p) - 1)a_{-j}$ and a linear combination of $a_{-1}, \ldots, a_{-j+1}$, yielding $a_{-j} \in \mathbb{Q}_p$. Hence $f \in \mathcal{R}_{\mathbb{Q}_p}$. This yields $(a, b) = 0$ in $\mathrm{H}^1(\mathcal{R}_{\mathbb{Q}_p}(\delta))$. □

Example 2.4.9. Let $A = \mathbb{Q}_p\langle x\rangle$. By [7, Theorem 2.9], $\mathrm{H}^1(\mathcal{R}_{\mathbb{Q}_p}(-2))$ is a one-dimensional $\mathbb{Q}_p$-vector space. Choose a representative (a, b) of a non-zero element of $\mathrm{H}^1(\mathcal{R}_{\mathbb{Q}_p}(-2))$. Let M_A be the rank 2 φ-module over $\mathcal{R}_{A_K} = \mathcal{R}_{K\langle x\rangle}$ such that the φ-action is defined by the matrix

$$\begin{pmatrix} p^2 & xb \\ 0 & 1 \end{pmatrix}$$

for some basis. Using the same basis, we equip a Γ-action on M_A by the matrix

$$\begin{pmatrix} \chi^2(\gamma) & xa \\ 0 & 1 \end{pmatrix}$$

for any $\gamma \in \Gamma$. It is clear that at each $y \in M(A)$, M_y is an extension of $k(y)\otimes_{\mathbb{Q}_p}\mathcal{R}_K$ by $k(y)\otimes_{\mathbb{Q}_p}\mathcal{R}_K(-2)$ defined by $(x(y)a, x(y)b)$. It follows from Lemmas 2.4.8 and 2.4.7 that $M_y(1)$ is étale if and only if y is not the origin. Hence the HN-polygons of M_y are not locally constant around the origin.

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References

1. Laurent Berger, Construction de (φ, Γ)-modules: représentations p-adiques et B-paires, *Algebra Number Theory* **2**(1) (2008), 91–120.
2. Laurent Berger and Pierre Colmez, Familles de représentations de de Rham et monodromie p-adique, *Astérisque* **319** (2008), 303–337.
3. Vladimir Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, Volume 33. p. 169 (American Mathematical Society, Providence, RI, 1990).
4. Siegfried Bosch, Ulrich Güntzer and Reinhold Remmert, *Non-Archimedean analysis*, Grundlehren der Math. Wiss., Volume 261 (Springer-Verlag, Berlin, 1984).
5. Nicolas Bourbaki, *Espaces Vectoriels Topologiques*, reprint of the 1981 original (Springer, Berlin, 2007).
6. Pierre Colmez, Espaces Vectoriels de dimension finie et représentations de de Rham, *Astérisque* **319** (2008), 117–186.
7. Pierre Colmez, Représentations triangulines de dimension 2, *Astérisque* **319** (2008), 213–258.
8. Pierre Colmez, Fonctions d'une Variable p-adique, *Astérisque* **330** (2010), 13–59.
9. Aise Johan de Jong, Homomorphisms of Barsotti–Tate groups and crystals in positive characteristic, *Invent. Math.* **134** (1998), 301–333.
10. Laurent Fargues and Jean-Marc Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p-adique, in preparation.
11. Eugen Hellmann, On arithmetic families of filtered φ-modules and crystalline representations, preprint, Bonn, 2011.
12. Kiran S. Kedlaya, The algebraic closure of the power series field in positive characteristic, *Proc. Amer. Math. Soc.* **129** (2001), 3461–3470.
13. Kiran S. Kedlaya, A p-adic local monodromy theorem, *Ann. of Math. (2)* **160** (2004), 93–184.
14. Kiran S. Kedlaya, Slope filtrations revisited, *Doc. Math.* **10** (2005), 447–525.
15. Kiran S. Kedlaya, Slope filtrations for relative Frobenius, *Astérisque* **319** (2008), 259–301.
16. Kiran S. Kedlaya, *p-adic differential equations*, Cambridge Studies in Advanced Mathematics, Volume 125 (Cambridge University Press, 2010).
17. Kiran S. Kedlaya and Ruochuan Liu, On families of (φ, Γ)-modules, *Algebra Number Theory* **4** (2010), 943–967.
18. Kiran S. Kedlaya and Ruochuan Liu, Relative p-adic Hodge theory, I: *Foundations*, preprint 2011.
19. Kiran S. Kedlaya and Ruochuan Liu, Relative p-adic Hodge theory, II: (φ, Γ)-modules, preprint 2011.

20. Ruochuan Liu, Cohomology and duality for (φ, Γ)-modules over the Robba ring, *Int. Math. Res. Not. 2008* **3**; Art ID. rnm150.

21. Michael Lazard, Les zéros des fonctions analytiques d'une variable sur un corps valué complet, *Publ. Math. IHÉS* **14** (1962), 47–75.

22. Peter Schneider, *Non-Archimedean functional analysis.* (Springer-Verlag, Berlin, 2002).