On Equivalent Truth-Tables of Many-Valued Logics

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Many-valued or non-Aristotelian calculi of propositions (logics) were originally introduced by generalisation of the truth-table method. It was known by the end of the nineteenth century that ordinary "binary" formulae of the calculus of propositions, such as

$$p \rightarrow p \tag{1}$$

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) \tag{2}$$

could be verified directly by means of the truth-table:

$$A_2 \equiv \begin{array}{r|ll} p \rightarrow q & 1 & 2 \quad q \\ \hline {}^*1 & 1 & 2 \\ 2 & 1 & 1 \\ p & & \end{array} \tag{3}$$

although the terminology and symbolism used were different.[1]

To decide, for example, whether (2) is a formula of binary logic, the numbers 1 and 2 are substituted in all possible ways for the "variables" p, q, r of (2) and the resulting expressions are contracted by writing

number 2 for $1 \rightarrow 2$

and number 1 for $1 \rightarrow 1$, $2 \rightarrow 1$, and $2 \rightarrow 2$

in accordance with table (3). If (2) contracts to 1 for all possible substitutions, then (2) is a "theorem" of the binary logic of implication; otherwise it is not.

In (3), the asterisk placed next to the value 1 indicates the special significance of the value 1. The inner meanings attached to the symbols 1 and 2 may be "truth" and "falsity" respectively, and (3) expresses some intuitions suggested by an analysis of implication.

J. Łukasiewicz [2] and E. Post [3] have formed many-valued logics by

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[1] See [8].

[2] See [5], [7].

[3] See [9].

extending the method. For example, Łukasiewicz proposes the table

$p \to q$	1	2	3	q
*1	1	2	3	
2	1	1	1	
3	1	3	1	
p				

(4)

as the table for the implication of his "ternary logic of implication." [1] To decide whether (2) is a theorem of this logic, all possible permutations (repeated elements being allowed) of the numbers 1, 2, 3 are substituted for p, q, r, and the resulting expression contracted according to table (4). If (2) contracts to 1 for all substitutions, then (2) is a theorem of the ternary logic; if (2) contracts into 2 or 3 for at least one substitution, then (2) is not a theorem of that logic.

The author has succeeded in proving that, for every $n > m$ and any m-valued truth-table, there is an n-valued truth-table such that the two corresponding m-valued and n-valued logics are the same.[2, 3] The author has discussed also the problems of mutual relationships between logics formed by different truth-tables with finite and infinite numbers of values.[4] But no method seems to be known for deciding whether two given truth-tables form the same set of theorems or not (i.e. are "equivalent" or not).

In this note a method is presented by which two truth-tables can, in certain cases, be proved equivalent. The method is demonstrated for the case of four particular truth-tables (called A_2, A_3, A_4 and A_5 below), but generalisation to cases of similar type presents no difficulty.

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[1] See [6]. Łukasiewicz writes 0 for 2, $\frac{1}{2}$ for 3, 1 for 1. "Possibility" is the inner meaning of $\frac{1}{2}$.

[2] See [3].

[3] To justify calling two logics generated by two truth-tables with different numbers of values the "same," it will be noted that many-valued logics are basically formalisations for which certain formulae are called "theorems" and others not. In the sense of this paper two formalisations are the "same" logics, irrespective of possible truth-table interpretations, if the set Ω of possible well-formed formulae is the same for the two formalisations and if the classification of each element of Ω into a "theorem" or "non-theorem" is the same for the two formalisations. (Cf. e.g. [10], [1], [7].)

[4] See [2], [4].

First. the equivalence of

$A_5 \equiv$

$p \to q$	1	2	3	4	5
*1	1	2	3	4	5
2	1	1	3	3	5
3	1	2	1	2	5
4	1	1	1	1	5
5	1	2	3	4	1

and

$A_2 \equiv$

$p \to q$	1	2
*1	1	2
2	1	1

will be proved. To do this it will be shown that

(i) every non-theorem of A_5 is not a theorem of A_2,

(ii) every non-theorem of A_2 is not a theorem of A_5.

(i) If an implicational formula α is not a theorem of A_5, there is a substitution of the values 1, 2, 3, 4, 5 for its variables which makes α contract to a value different from 1. The variables of α may be classified into five categories I, II, III, IV, V such that, when the substitution referred to above is made, the value 1 is substituted for the variables in category I, the value 2 is substituted for the variables in category II, and so on. (Some categories may be missing.) Suppose by this substitution α contracted to 5; then the process may be written

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = 5. \tag{5}$$

Now it will be noticed that the values 1, 2, 3, 4 in A_5 are such that, if value 1 is written in place of values 1, 2, 3, 4 (without changing the value 5), there results the table

$p \to q$	1	1	1	1	5
1	1	1	1	1	5
1	1	1	1	1	5
1	1	1	1	1	5
1	1	1	1	1	5
5	1	1	1	1	1

(6)

in which the first four columns and the first four rows are repetitions of the first column and the first row respectively (the question of asterisks being disregarded). In such cases, the values of 1, 2, 3, 4 will be called *similar*, written

$$1 \sim 2 \sim 3 \sim 4\ ;\quad 5 \sim 5.$$

Table (6) may then be rewritten as

$p \to q$	1	5	q
1	1	5	
5	1	1	
p			

(7)

Thus, substitution of value 1 instead of values 2, 3, 4, for the variables of categories II, III, IV of α gives, by table (7),

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/1,\ \mathrm{III}/1,\ \mathrm{IV}/\mathrm{I},\ \mathrm{V}/5) = 5. \tag{8}$$

Comparison of A_2 and table (7) gives at once

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/1,\ \mathrm{III}/1,\ \mathrm{IV}/1,\ \mathrm{V}/2) = 2 \tag{a}$$

by A_2. In other words, α is not a theorem of A_2.

Similarly, if, according to A_5,

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = 4 \tag{9}$$

then

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/1,\ \mathrm{III}/4,\ \mathrm{IV}/4,\ \mathrm{V}/1) = 4 \tag{10}$$

according to

$p \to q$	1	4	q
1	1	4	
4	1	1	
p			

(11)

since $1 \sim 2 \sim 5$; $3 \sim 4$ in A_5.

Comparison of A_2 and table (11) gives at once

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/1,\ \mathrm{III}/2,\ \mathrm{IV}/2,\ \mathrm{V}/1) = 2 \tag{b}$$

by A_2.

Further, if

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = 3 \tag{12}$$

by A_5, then

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/1,\ \mathrm{III}/3,\ \mathrm{IV}/3,\ \mathrm{V}/1) = 3 \tag{13}$$

according to

$p \to q$	1	3	q
1	1	3	
3	1	1	
p			

(14)

since $1 \sim 2 \sim 5$; $3 \sim 4$ in A_5, and hence

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/1,\ \mathrm{III}/2,\ \mathrm{IV}/2,\ \mathrm{V}/1) = 2 \tag{c}$$

by A_2.

Finally, if

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = 2 \tag{15}$$

by A_5, then

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/1,\ \mathrm{IV}/2,\ \mathrm{V}/1) = 2 \tag{d}$$

by A_2, since $1 \sim 3 \sim 5$; $2 \sim 4$ in A_5.

This completes the proof of (i).

(ii) If, now, α is not a theorem of A_2, then must

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/2) = 2 \tag{e}$$

according to A_2, and hence

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/5) = 5 \tag{16}$$

according to table (7), and consequently also according to A_5.

This completes the proof of (ii).

Thus A_2 and A_5 are equivalent. Moreover, the tables

$$A_3 \equiv \begin{array}{r|lll} p \rightarrow q & 1 & 2 & 3 \quad q \\ \hline {*1} & 1 & 2 & 3 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ p & & & \end{array} \quad \text{and} \quad A_4 \equiv \begin{array}{r|llll} p \rightarrow q & 1 & 2 & 3 & 4 \quad q \\ \hline {*1} & 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 3 & 3 \\ 3 & 1 & 2 & 1 & 2 \\ 4 & 1 & 1 & 1 & 1 \\ p & & & & \end{array}$$

are equivalent to A_2 and A_5, since every theorem of A_5 is also a theorem of A_4 and A_3, every theorem of A_4 and A_3 is a theorem of A_2, and A_2 and A_5 are equivalent. The result for A_3 has been obtained by the author by a different and more complicated method.[1]

It may be of interest to note that A_5, A_4, and A_3 are such that if according to them any formula takes a value different from 1 then it takes all the values of the particular table other than 1.

To prove this for A_5, it will be seen by the above that if

$$\alpha\ (\mathrm{I}/1,\ \mathrm{II}/2,\ \mathrm{III}/3,\ \mathrm{IV}/4,\ \mathrm{V}/5) = k \qquad (k = 2,\ 3,\ 4,\ 5)$$

then there are values

$$a_1\ a_2\ a_3\ a_4\ a_5 \qquad (a_i = 1,\ 2, \qquad i = 1,\ 2,\ 3,\ 4,\ 5)$$

such that

$$\alpha\ (\mathrm{I}/a_1,\ \mathrm{II}/a_2,\ \mathrm{III}/a_3,\ \mathrm{IV}/a_4,\ \mathrm{V}/a_5) = 2 \tag{f}$$

according to A_2.

Comparing A_2 with (7), (11), and (14), it will be seen that by substituting $b_i = 1$ for the variables of those categories for which $a_i = 1$ in (f), and $b_i = h$, where h is any one of the values 2, 3, 4, 5, for the variables of those categories for which $a_i = 2$ in (f), then

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[1] See [3].

$$a\,(\mathrm{I}/b_1,\ \mathrm{II}/b_2,\ \mathrm{III}/b_3,\ \mathrm{IV}/b_4,\ \mathrm{V}/b_5) = h.$$

Similar procedures can be applied to A_3 and A_4. For example, the formula

$$a \equiv (((s \to p) \to (q \to r)) \to s) \to (((q \to q) \to (p \to p) \to r))$$

takes the value 4 according to A_5 if 2, 3, 4, 5 are substituted for p, q, r, s respectively. Hence, to make a contract into 2 according to A_2, then by (c) the substitution

$$a\,(p/1,\ q/2,\ r/2,\ s/1) = 2$$

is made. Therefore to make a contract into 5, say, according to A_5, the substitution

$$a\,(p/1,\ q/5,\ r/5,\ s/1) = 5$$

is made. This may be verified directly by substitution in a and evaluation by A_5.

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