TOWERS, LADDERS AND THE B. B. NEWMAN
SPELLING THEOREM

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Abstract

The Spelling Theorem of B. B. Newman states that for a one-relator group $\langle a_1, \dots \mid W^n \rangle$, any nontrivial word which represents the identity must contain a (cyclic) subword of $W^{\pm n}$ longer than W^{n-1}. We provide a new proof of the Spelling Theorem using towers of 2-complexes. We also give a geometric classification of reduced disc diagrams in one-relator groups with torsion. Either the disc diagram has three 2-cells which lie almost entirely along the boundary, or the disc diagram looks like a ladder. We use this ladder theorem to prove that a large class of one-relator groups with torsion are locally quasiconvex.

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1. Introduction

As explained by Howie in [3], some of the main results about one-relator groups can be proven using towers, a method first made explicit in the theory of 3-manifolds, in Papakyriakopolous's proof of Dehn's Lemma [12]. In fact, the tower method is implicit in Magnus's original solution of the word-problem and Freiheitssatz for one-relator groups, [7, 8]. The advantage of the tower method over the other methods of proof is that the arguments are more geometric and conceptual, rather than combinatorial.

The first main result of this paper is a tower proof in Section 4 of the following theorem (see [6, IV.5.5]).

THEOREM 1.1 (B. B. Newman Spelling Theorem). *Let U be a freely reduced word which represents the identity in $\langle a_1, \dots, a_r \mid W^n \rangle$, where W is freely and cyclically*

reduced. Then U *contains a subword* V *such that* V *is a subword of the cyclic word* $W^{\pm n}$ *and* $|V| > |W^{n-1}|$.

The statement of the B. B. Newman spelling theorem was first announced in [11]. We refer to [4] for a substantial generalization as well as for references to various other proofs of the theorem given by Gurevich, and others.

As usual for theorems about certain one-relator groups, the same proof works for the fundamental groups of certain staggered 2-complexes—in this case the staggered 2-complexes whose 2-cells are attached by proper powers. Theorem 1.1 is a special case of Theorem 4.6 which treats the general staggered case.

The second main result of the paper is Theorem 5.2 which provides a geometric classification of reduced disc diagrams in one-relator groups with torsion. Roughly speaking, we show that either the disc diagram has three 2-cells which lie almost entirely along the boundary, or the disc diagram looks like a ladder (see Figure 3). We use this to give the following application:

THEOREM 1.2. *Let* $G = \langle a_1, \ldots, a_r \mid W^n \rangle$, *where* $n \geq |W|$. *Then* G *is locally quasiconvex.*

In Section 6, we provide a tower proof of a Freiheitssatz for the fundamental groups of staggered 2-complexes. We also provide a tower proof of the combinatorial reducibility of staggered 2-complexes.

2. Towers and staggered 2-complexes

A map $X \to Y$ between CW-complexes is said to be *combinatorial* provided that its restriction to each cell of X is a homeomorphism onto a cell of Y. (Throughout this paper, the term *cell* will always mean *open cell*.) A CW-complex is *combinatorial* provided that the attaching map of each of its cells is a combinatorial map (after a suitable subdivision). All the spaces in this paper will be 2-dimensional combinatorial complexes, and all the maps between spaces will be combinatorial.

We now collect some background on towers which is due to Howie [2].

DEFINITION 2.1 (Tower). A map $A \to B$ of connected CW-complexes is a *tower* provided that it can be expressed as a composition

$$A = B_n \hookrightarrow \hat{B}_{n-1} \to B_{n-1} \hookrightarrow \cdots \hookrightarrow \hat{B}_2 \to B_2 \hookrightarrow \hat{B}_1 \to B_1 \hookrightarrow B,$$

where the maps are alternately inclusions of subcomplexes and infinite cyclic coverings. In other words, the covers are regular and connected with infinite cyclic covering transformation group.

Let $f : C \to B$ be a map of connected CW-complexes. A map $f' : C \to A$ is a *tower lift* of f provided that there is tower $g : A \to B$ such that the following diagram commutes:
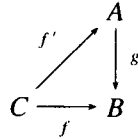
The tower lift f' is *maximal* if, for any tower lift $C \to D$ of f', the map $D \to A$ is an isomorphism.

LEMMA 2.2 ([2, Lemma 3.1]). *Let S be a compact CW-complex and $f : S \to K$ be a combinatorial map. Then f has a maximal tower lifting.*

DEFINITION 2.3 (Staggered). Let K be a 2-complex such that the attaching map of each 2-cell is locally injective. We say that K is *staggered* if there are linear orderings on the 2-cells and a subset O of the 1-cells (called the *ordered* 1-cells) such that

(1)   for each 2-cell α, at least one ordered 1-cell is in the image of the attaching map of α, and

(2)   for 2-cells α and β, if $\alpha < \beta$ then $(\min \alpha) < (\min \beta)$ and $(\max \alpha) < (\max \beta)$, where $(\min \alpha)$ and $(\max \alpha)$ are respectively the least and greatest ordered 1-cells in the attaching map of α.

A group presentation is *staggered* if its standard 2-complex is staggered.

The notion of *staggered* defined above appears in [6, page 152] and was implicit in Magnus' original proof of the Freiheitssatz [7]. Howie used a slightly less general definition in [3] to prove Lemma 2.6 and Lemma 2.4 below. His proofs work for the more general version of staggered that we use here, but we provide the details for the benefit of the reader.

LEMMA 2.4 ([3, Lemma 2]). *If $Y \to X$ is a tower, and X is a staggered 2-complex, then Y is staggered.*

PROOF. It is clear that a subcomplex of a staggered 2-complex is also staggered. Let $\phi : \hat{S} \to S$ be an infinite cyclic cover of a staggered 2-complex. We will show that $\hat{S}$ is staggered. Let γ be a generator of the covering transformation group. Let the set O of ordered 1-cells be the cells which project to ordered 1-cells in S. Linearly order O and the 2-cells of $\hat{S}$ by the rule that $\alpha < \beta$ provided that either $\phi(\alpha) < \phi(\beta)$ or $\beta = \gamma^n(\alpha)$ for some $n \in \mathbb{Z}$ with $n > 0$. Now let $\alpha < \beta$ be 2-cells of $\hat{S}$. If $\phi(\alpha) < \phi(\beta)$, then obviously $(\min \alpha) < (\min \beta)$. Otherwise $\beta = \gamma^n(\alpha)$ for some $n > 0$, and so $(\min \beta) = \gamma^n(\min \alpha)$. An identical argument holds for $\max \alpha$.   □

CONVENTION 2.5. In general, if X is staggered and $Y \to X$ is a tower, then Y may be staggered in several ways. However, the arguments of this paper will always employ the staggering of Y given in the proof of Lemma 2.4.

LEMMA 2.6 (Howie's Collapsing Lemma). *Let Y be a finite staggered 2-complex with at least one 2-cell and with $H^1(Y) = 0$. If the greatest 2-cell α of Y is not attached along a proper power in $\pi_1(Y^{(1)})$, then Y collapses across α with free edge $\max \alpha$.*

The following proof is identical to Howie's original proof which appeared as [3, Lemma 3].

PROOF. Note that, if some 2-cell γ of Y is attached along a proper power s^m say, then replacing γ by a 2-cell attached along s will not change $H^1(Y)$. Nor will this procedure affect the staggering of Y.

We argue by induction on the number of 2-cells in Y, which by hypothesis is at least one. If there is only one 2-cell, then the first Betti number of the 1-skeleton $Y^{(1)}$ is at most 1, since $H^1(Y) = 0$. On the other hand, since the attaching map P of the 2-cell is a cyclically reduced path of positive length, and not a proper power, it follows that $Y^{(1)}$ cannot be a tree, and P is the unique simple closed path in $Y^{(1)}$, whence the result.

For the inductive step, consider the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^1(Y) \longrightarrow H^1(Y - \alpha) \oplus H^1(D^2) \longrightarrow H^1(S^1) \longrightarrow \cdots$$

associated to the adjunction of α. From this we see that $H^1(Y - \alpha)$ is at most cyclic. If the subcomplex $Y' = Y - \{\alpha, \max \alpha\}$ is connected, then $H^1(Y - \alpha) = H^1(Y') \oplus \mathbb{Z}$, so $H^1(Y') = 0$. Otherwise, Y' has two components Y_1 and Y_2 say, and $H^1(Y - \alpha) = H^1(Y_1) \oplus H^1(Y_2)$. Without loss of generality we may assume $H(Y_1) = 0$. Also, in this case Y_1 cannot be a tree, since α is attached by a cyclically reduced closed path which meets Y_1.

Now apply the inductive hypothesis either to Y' or to Y_1, but with the staggering *opposite* to that inherited from Y (that is, the orderings of cells is opposite). Then the complex in question collapses across its least 2-cell β say (in the original ordering), with free edge $\min \beta$. But α does not involve $\min \beta$ since $\beta < \alpha$, so Y also collapses across β with free edge $\min \beta$. Let $Z = Y - \{\beta, \min \beta\}$ be the result of this collapse.

Then the inductive hypothesis applies to Z, so Z collapses across α with free edge $\max \alpha$. But β does not involve $\max \alpha$ since $\beta < \alpha$. Hence $Y = Z \cup \{\beta, \min \beta\}$ also collapses across α with free edge $\max \alpha$. □

LEMMA 2.7. *Let Y be a finite staggered 2-complex which has no infinite cyclic cover. Then the greatest 2-cell of Y is attached along a path U^n, where U is a closed*

path in Y that passes through $\max \alpha$ *exactly once. Furthermore, no other 2-cell is attached along* $\max \alpha$.

PROOF. Form a new 2-complex Y' from Y by replacing the greatest 2-cell α, which is attached along a path U^n (where U is not periodic), by a 2-cell α' which is attached along the word U. The obvious map $Y \to Y'$ is a π_1-surjection, and so $\mathsf{H}^1(Y') = 0$. Now by Lemma 2.6, we see that U passes through $\max \alpha'$ exactly once, and so the attaching path U^n of α passes through $\max \alpha$ exactly once as well.   □

We make one final observation.

REMARK 2.8. Let $\phi : Y \to X$ be a tower map. For any 2-cell α of Y, if $\phi(\alpha)$ is attached along a path U^n then α is attached along a path V^n. This is true for arbitrary towers because it holds for subcomplexes and infinite cyclic covers.

3. Background on disc diagrams and spherical diagrams

In this section we briefly review some background on disc diagrams and spherical diagrams. A more detailed account of this material can be found in [6].

A *disc diagram* D is a planar, simply connected 2-complex. Given a 2-complex X we will often study a map $D \to X$ and refer to this map as the disc diagram. Let $P \to X$ be a closed path which factors as $P \to D \to X$. We say that the disc diagram $D \to X$ is a disc diagram for $P \to X$ provided that P maps onto ∂D, and furthermore for each 1-cell e of ∂D, the preimage of e in P must consist of one or two 1-cells according as to whether or not e lies on the boundary of a 2-cell of D. The path P is said to be a *boundary path* for D.

If the words U and V represent the same element of $\langle A \mid R \rangle$, then by a disc diagram for $U = V$ we will mean a disc diagram mapping to the standard 2-complex of $\langle A \mid R \rangle$ for the path corresponding to the word $U^{-1} V$.

We will occasionally employ the similar notion of a *spherical diagram* $D \to X$ which is a combinatorial map from a 2-sphere to X.

A pair of (not necessarily distinct) 2-cells C_1, C_2 in a diagram D which meet along a 1-cell e of D, is a *cancelable pair* provided that the boundary cycles of C_1 and C_2 beginning with e (in the same direction) are sent to identical paths in X after composing with $D \to X$.

The disc [spherical] diagram $D \to X$ is said to be *reduced* provided that it has no cancelable pairs. It is a theorem of van Kampen, that a reduced disc diagram exists for any null-homotopic closed path (See [6, V.2.1]).

A *spur* in a disc diagram is the union of a valence one 0-cell together with the 1-cell incident with it.

We conclude this section with an easy lemma whose proof we leave to the reader.

LEMMA 3.1. *Let* $D \to Y$ *be a tower lift of a reduced disc [spherical] diagram* $D \to X$. *Then* $D \to Y$ *is itself a reduced disc [spherical] diagram.*

4. Tower proof of the B. B. Newman Spelling Theorem

In this section we prove Theorem 4.6 and Theorem 4.8 which are the first main results of the paper. The crucial geometric tool employed in the proof of these theorems is Lemma 4.5 which will also play a key role later in Section 5.

We begin the section by applying Lemma 2.7 to a maximal tower lift of a reduced disc [spherical] diagram. In this situation, the diagram has a useful geometric decomposition into smaller subdiagrams.

LEMMA 4.1 (On the Boundary). *Let* $D \to Y$ *be a maximal tower lift of a reduced disc [spherical] diagram* $D \to X$ *with* X *a staggered 2-complex. Let* A *be a 2-cell of* D *which maps to the greatest 2-cell* α *of* Y. *Let* n *be the positive number such that* α *is attached along a path* U^n, *where* U *is not homotopic to a periodic path in* $Y^{(1)}$. *Then* ∂A *contains exactly* n *1-cells* $E_1, \ldots, E_n$ *which map to* $\max \alpha$, *and each* E_i *lies in* ∂D.

Note that in the spherical case, ∂D is empty, and so the lemma asserts that no such reduced spherical diagram exists.

PROOF. Observe that Y has no infinite cyclic cover and $D \to Y$ is surjective, so Y is finite. By Lemma 2.7, the path U passes through $\max \alpha$ exactly once. Furthermore, no other 2-cell is attached along $\max \alpha$.

To see that each E_i is contained in ∂D note that if E_i were an interior 1-cell, then it would appear in two different ways in the boundary of a 2-cell. Since each such 2-cell would map to α, this would form a cancelable pair. This is impossible because $D \to X$ is reduced and therefore by Lemma 3.1 $D \to Y$ is reduced. It follows that each E_i is contained in ∂D. Since ∂A passes n times through maximal 1-cells, and ∂A passes through each exactly once, we see that there are n distinct maximal 1-cells on ∂A as claimed. □

DEFINITION 4.2 (Branches). Let A, D, Y, and $\{E_1, \ldots, E_n\}$ be as in Lemma 4.1 then the subcomplex $D - \{A, E_1, \ldots, E_n\}$ consists of n path components. These components are the *branches* at A with respect to $D \to Y$. A branch is *trivial* if it is contained in ∂A and *nontrivial* otherwise.

Figure 1 illustrates a disc diagram with a dark 2-cell A and four 1-cells mapping to e. There are four branches at A and exactly one of these branches is trivial.
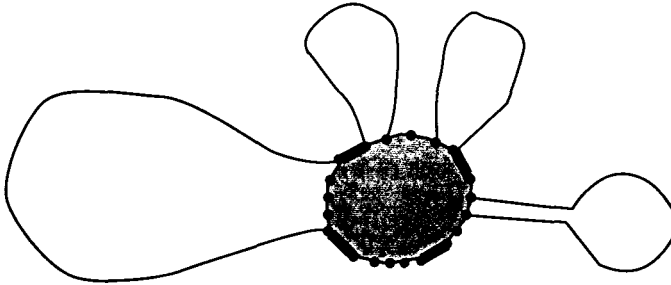
FIGURE 1.

DEFINITION 4.3 (λ-external). The 2-cell R of D is λ-*external* provided that there is a subpath Q of the attaching path of R such that:

(1) Q is a subpath of the boundary path of D.
(2) $|Q| > \lambda\,|\partial R|$.

We will sometimes refer to R as an *external* 2-cell when the value of λ is clear from the context.

LEMMA 4.4. *Let D and A be as in Lemma* 4.1, *let P be a branch at A, and let D' be the subdiagram $P \cup \overline{A}$. Suppose $B \neq A$ is a 2-cell of D'. If B is λ-external for the diagram D', then B is also λ-external for the diagram D.*

PROOF. By the definition of branch, the 2-cell B does not share any 1-cells with any 2-cell of $D - D'$. If a 1-cell E in ∂B lies in $\partial D'$, then it also lies in ∂D.   □

LEMMA 4.5 (Two external 2-cells). *Let X be a staggered 2-complex and let $D \to X$ be a reduced disc diagram. Suppose that D has no spurs, and D contains at least two 2-cells. Then D contains two distinct 2-cells C_1 and C_2 such that*

(1)   *C_i maps to a 2-cell γ_i in X whose attaching map is U^{n_i}, where U is not homotopic to a periodic path in $X^{(1)}$;*
(2)   *C_i is an $((n_i - 1)/n_i)$-external 2-cell of D.*

PROOF. We prove the statement by induction on the number of 2-cells in D. Lift $D \to X$ to a maximal tower:

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow & \downarrow \\
D & \longrightarrow & X
\end{array}
$$
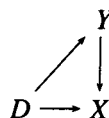
Suppose first that Y has a unique 2-cell α. Then every 2-cell of D maps to α, which is trivially the greatest 2-cell of Y. Let A_1 and A_2 be any two distinct 2-cells

of D. If they each have only one nontrivial branch, then they are external 2-cells of D and we are done. On the other hand, if one of these 2-cells (say A_1) has two nontrivial branches B_1 and B_2, then we can subdivide D into two strictly smaller disc diagrams D_1 and D_2 with $D_i = B_i \cup \overline{A_1}$. By induction, each of these subdiagrams has two external 2-cells, at least one of which is distinct from A. Thus there are two distinct 2-cells (different from A) $C_1 \subset D_1$ and $C_2 \subset D_2$ that are each external for the corresponding subdiagram. But then Lemma 4.4 implies that C_1 and C_2 are each external 2-cells for the original diagram D as well.

Now suppose that Y has more than one 2-cell. Let α and κ be respectively the greatest and least 2-cells of Y. Choose 2-cells A and K in D mapping to α and κ. We can define branches of K by reversing the ordering of the cells of Y. We now proceed as above, noting that A and K are distinct 2-cells. As before, if each of A and K has only one nontrivial branch, then we are done. Otherwise one of them splits the diagram into two strictly smaller subdiagrams, and the result follows from the inductive hypothesis.   □

At this point, we can easily prove the following form of the B. B. Newman Spelling Theorem.

THEOREM 4.6. *Let* $G = \langle x_1, \dots \mid R_1^{n_1}, \dots \rangle$ *be a staggered presentation. Let* U *be a freely and cyclically reduced word representing the identity of* G. *Then* U *contains a subword* S *such that* S *is a subword of the cyclic word* $R_j^{\pm n_j}$ *for some* j *and* $|S| > |R_j^{n_j-1}|$.

PROOF. Let X be the standard 2-complex for the presentation, and let $D \to X$ be a reduced disc diagram for U. It will be convenient to think of 1-cells of X as being oriented and labeled by their corresponding generators. This pulls back to an orientation and label for each 1-cell of D. Thus a path in D corresponds to a certain word in the generators and their inverses. To prove the theorem, we will find a certain 2-cell C of D mapping to an $R_j^{n_j}$ 2-cell of X. The boundary of C will have a subpath S corresponding to a subword of U satisfying $|S| > |R_j^{n_j-1}|$.

Either D has only one 2-cell C, in which case the result follows immediately, or we may apply Lemma 4.5 to D and conclude that D has two external 2-cells, C_1 and C_2. Notice that the path U may begin at a point in the interior of at most one of these external 2-cells, say C_1. Then C_2 maps to an $R_j^{n_j}$ 2-cell of X, and the external path of C_2 corresponds to a subword S of U with $|S| > |R_j^{n_j-1}|$ as required.   □

DEFINITION 4.7 (Intervals). Let $G = \langle x_1, \dots \mid R_1^{n_1}, \dots \rangle$ be a staggered presentation, and let W be a freely reduced word. The *interval spanned by* W, denoted I_W, is the convex closure of the set of ordered letters of W in the set of all ordered letters. In other words, I_W is the smallest interval containing all the ordered generators in W.
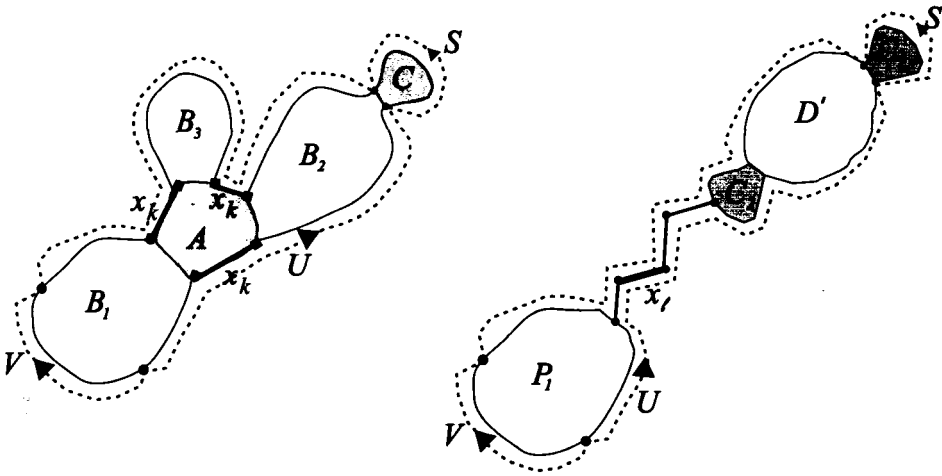
FIGURE 2. The diagram on the left corresponds to the case where x_k is the greatest 1-cell in X. The diagram on the right corresponds to the case where there is a greater 1-cell x_ℓ which is necessarily not on the boundary of any 2-cell. In both diagrams, the paths U and V begin and end at the bold 0-cells. Note that the path U wanders around quite a bit and contains the subpath S. Also note that on the left, B_2 is actually the union of the regions labeled by B_2 and C in the figure, while on the right, D' is actually the union of the regions labeled D', C_1, and C_2 in the figure.

The following generalization of the B. B. Newman Spelling Theorem was first proved in [4].

THEOREM 4.8. *Let $G = \langle x_1, \ldots \mid R_1^{n_1}, \ldots \rangle$ be a staggered presentation. Let U and V be two freely reduced words which represent the same element of G. Suppose that I_V omits a generator x_i which occurs in I_U. Then U contains a subword S such that S is a subword of the cyclic word $R_j^{\pm n_j}$ for some j and $|S| > |R_j^{n_j - 1}|$.*

PROOF. Let X be the standard 2-complex for the presentation, and let $D \to X$ be a reduced disc diagram for $U =_G V$. As in Theorem 4.6, we will find a certain 2-cell C of D mapping to an $R_j^{n_j}$ 2-cell of X. The boundary of C will have a subpath S corresponding to a subword of U^{-1} in the boundary path $U^{-1} V$ of D, and furthermore, $|S| > |R_j^{n_j - 1}|$.

Notice that we cannot directly apply Lemma 4.5 since D may have spurs, and we could not easily conclude that either of the long boundary paths produced by the lemma corresponds to a subword of U^{-1}. To proceed, we must find an appropriate subdiagram D' to which we apply Lemma 4.5.

Without loss of generality, we may assume that the generator x_i which is omitted from I_V is greater than the generators in I_V, since otherwise we can reverse the order

of the staggering. As before, we begin by lifting $D \to X$ to a maximal tower:

$$\begin{array}{ccc} & & Y \\ & \nearrow & \downarrow \\ D & \longrightarrow & X \end{array}$$
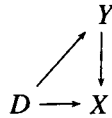
Let α be the greatest 2-cell of Y, and let A be a 2-cell of D which maps to α. Suppose that α maps to the $R_j^{n_j}$ 2-cell of X. Let e be the greatest 1-cell in $\partial \alpha$ and let x_k be the generator labeling the 1-cell it projects to in X. This generator x_k must be the greatest generator occurring in the boundary of a 2-cell in the image of $D \to X$. Indeed, if there were a greater generator on the boundary of a 2-cell R in the image $D \to X$, then as the staggering of Y was defined in Lemma 2.4, R would be greater than the image of α, and thus a 2-cell β of Y which maps to R would be greater than α which is a contradiction.

Suppose first that x_k is the greatest generator in the image of $D \to X$. Then ∂A contains 1-cells $E_1, \ldots, E_{n_j}$ which map to x_k in X, and by Lemma 4.1, these 1-cells lie in ∂D. It follows that x_k occurs in the word $U^{-1}V$, and since $x_k \geq x_i$ we see that x_k must occur in U, but not in V. So the V part of the boundary path $U^{-1}V$ of D must lie inside one of the branches of A. Let B_1 be the branch of A containing V. If A has no nontrivial branches other than B_1, then A is $((n_j - 1)/n_j)$-external for D. Otherwise, as illustrated on the left in Figure 2, A has a nontrivial branch $B_2 \neq B_1$. In this case, we can apply Lemma 4.5 to the subdiagram $D' = \overline{A} \cup B_2$ and conclude that D has an external 2-cell C as above.

Now suppose that x_k is not the greatest generator in the image of $D \to X$. Then the greatest generator x_ℓ appearing in the 1-skeleton of D is not in the boundary of any 2-cell of D, and so is not in the interior of D. Thus, as illustrated on the right in Figure 2, x_ℓ appears in the boundary path $U^{-1}V$ of D. And again, since $x_\ell \geq x_i$ we see that x_ℓ must occur in U but not in V. Consequently, some 1-cell of D with label x_ℓ separates D into two components P_1 and P_2. The V subpath of the boundary path of D lies entirely within one of these components (say P_1). Since U is freely reduced, P_2 contains at least one 2-cell. More precisely, P_2 is the union of a spurless 2-complex D' and a (possibly trivial) chain of 1-cells starting at a vertex v of D' and ending at a spur. (In the diagram on the right in Figure 2, P_2 is the union of D' and a chain of two 1-cells.) Applying Lemma 4.5 to D' provides 2-cells C_1 and C_2 which are external for D'. Since v can lie in the interior of at most one of the external paths of these 2-cells say C_2, we see that C_1 is the desired external 2-cell for D.   □

5. Ladders

The main result of this section is Theorem 5.2 which characterizes certain disc-diagrams. An application is given in Theorem 5.7 to the local quasiconvexity of

FIGURE 3.

certain one-relator groups with torsion. We begin with the following definition which is slightly different from the definition first introduced in [10] where ladders were studied in conjunction with small-cancellation theory.

DEFINITION 5.1 (Ladder). A disc diagram D is a *ladder* provided that it is a union $\bigcup_{1 \leq i \leq n} \overline{C_i}$ where each C_i is a 1-cell or 2-cell and:

(1) $\overline{C_i} \cap \overline{C_j}$ is nonempty if and only if i and j are consecutive.
(2) If $|i - j| = 1$, then $\overline{C_i} \cap \overline{C_j}$ is a (possibly trivial) interval.
(3) If $|i - j| = 1$ and C_i is a 1-cell, then $\overline{C_i} \cap \overline{C_j}$ is a 0-cell.
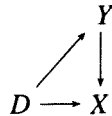
We refer the reader to Figure 3 for a picture of two ladders.

THEOREM 5.2 (Ladder). *Let X be a staggered 2-complex and suppose that there are constants n and k such that each 2-cell of X is attached along a path U^n, where U is a non-periodic path of length k.*

Let $D \to X$ be a reduced disc diagram with no spurs. Then either:

(1) *D consists of a single 0-cell.*
(2) *D is a ladder with an $((n-1)/n)$-external 2-cell at each end.*
(3) *D has at least three $((n-2)/n)$-external 2-cells.*

PROOF. We induct on the number of 2-cells in D. A disc diagram D with no spurs and a single 2-cell is a ladder. Now suppose D has at least two 2-cells. As usual, we begin by lifting $D \to X$ to a maximal tower:

$$\begin{array}{ccc} & & Y \\ & \nearrow & \downarrow \\ D & \longrightarrow & X \end{array}$$

Let α be the maximal 2-cell of Y, and let A be a 2-cell of D which maps to α. The argument splits into three cases corresponding to the number of branches at A.

Case ≥ 3: Suppose A has at least three nontrivial branches B_1, B_2, and B_3. Let D_i be the subdiagram $\overline{A} \cup B_i$. By Lemma 4.5, each D_i contains two $((n-1)/n)$-external 2-cells, at least one of which is distinct from A. So by Lemma 4.4, D has three 2-cells which are $((n-1)/n)$-external.

Case 2: Suppose A has exactly two nontrivial branches B_1 and B_2. Then we have two strictly smaller diagrams $D_1 = B_1 \cup \overline{A}$ and $D_2 = B_2 \cup \overline{A}$. Notice that A

FIGURE 3.

certain one-relator groups with torsion. We begin with the following definition which is slightly different from the definition first introduced in [10] where ladders were studied in conjunction with small-cancellation theory.

DEFINITION 5.1 (Ladder). A disc diagram D is a *ladder* provided that it is a union $\bigcup_{1 \le i \le n} \overline{C_i}$ where each C_i is a 1-cell or 2-cell and:

(1) $\overline{C_i} \cap \overline{C_j}$ is nonempty if and only if i and j are consecutive.
(2) If $|i - j| = 1$, then $\overline{C_i} \cap \overline{C_j}$ is a (possibly trivial) interval.
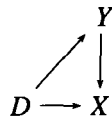(3) If $|i - j| = 1$ and C_i is a 1-cell, then $\overline{C_i} \cap \overline{C_j}$ is a 0-cell.

We refer the reader to Figure 3 for a picture of two ladders.

THEOREM 5.2 (Ladder). *Let X be a staggered 2-complex and suppose that there are constants n and k such that each 2-cell of X is attached along a path U^n, where U is a non-periodic path of length k.*

Let $D \to X$ be a reduced disc diagram with no spurs. Then either:

(1) *D consists of a single 0-cell.*
(2) *D is a ladder with an $((n-1)/n)$-external 2-cell at each end.*
(3) *D has at least three $((n-2)/n)$-external 2-cells.*

PROOF. We induct on the number of 2-cells in D. A disc diagram D with no spurs and a single 2-cell is a ladder. Now suppose D has at least two 2-cells. As usual, we begin by lifting $D \to X$ to a maximal tower:

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow & \downarrow \\
D & \longrightarrow & X
\end{array}
$$

Let α be the maximal 2-cell of Y, and let A be a 2-cell of D which maps to α. The argument splits into three cases corresponding to the number of branches at A.

Case ≥ 3: Suppose A has at least three nontrivial branches B_1, B_2, and B_3. Let D_i be the subdiagram $\overline{A} \cup B_i$. By Lemma 4.5, each D_i contains two $((n-1)/n)$-external 2-cells, at least one of which is distinct from A. So by Lemma 4.4, D has three 2-cells which are $((n-1)/n)$-external.

Case 2: Suppose A has exactly two nontrivial branches B_1 and B_2. Then we have two strictly smaller diagrams $D_1 = B_1 \cup \overline{A}$ and $D_2 = B_2 \cup \overline{A}$. Notice that A

is $((n-1)/n)$-external for both D_1 and D_2. By Lemma 4.5, each D_i contains two $((n-1)/n)$-external 2-cells. If one of the D_i (say D_1) has three $((n-2)/n)$-external 2-cells, then by Lemma 4.4, at least two of these 2-cells are $((n-2)/n)$-external for D. But D also has an $((n-1)/n)$-external 2-cell in B_2, so that D has a total of at least three 2-cells which are $((n-2)/n)$-external.

We can therefore assume that each of D_1 and D_2 has at most two $((n-2)/n)$-external 2-cells one of which must be A. Furthermore, since $D = D_1 \cup_{\overline{A}} D_2$ and D has no spurs, we see that neither D_1 nor D_2 has a spur. So D_1 and D_2 are each spurless reduced disc diagrams consisting of more than a single 0-cell. Thus by induction, the only possibility is that they are each ladders. Furthermore we can assume that A is the last cell of D_1 and that A is the first cell of D_2 because if A is not at the end then there would be three $((n-2)/n)$-external 2-cells.

It is easy to check that D is a ladder because D is the union of these two ladders along A, and thus D can be expressed as the union of a sequence of 1-cells and 2-cells by concatenating the sequences for D_1 and D_2. The intersection conditions in the definition of a ladder follow because $\overline{B}_1$ and $\overline{B}_2$ are disjoint.

Case 1: Now suppose that A has only one nontrivial branch B. Further suppose that D does not satisfy possibility (3) of the theorem. Notice that A is $((n-1)/n)$-external for D. Let D' be the subdiagram obtained from B by repeatedly removing spurs until no spurs remain. Since D has no spurs, D' must consist of more than a single 0-cell. We will now show that D' does not satisfy possibility (3) of the theorem, so that D' must be a ladder by induction.

If D' contains more than one 2-cell, then by Lemma 4.5, D' has two $((n-1)/n)$-external 2-cells C_1 and C_2. By the hypothesis that D does not satisfy possibility (3), only one of these, say C_1, can be $((n-2)/n)$-external for D. Let γ be a subpath of the boundary path of C_2 which is also a subpath of the boundary path of D' and which has more than $(n-1)k$ edges.

Define a subpath σ of the boundary path of D' as follows. If ∂A intersects $\partial D'$, then let σ be that intersection. Otherwise, A is connected to D' by a chain of isolated 1-cells. In this case, let σ be the trivial path at the 0-cell where this chain meets D'. Notice that the length of σ is less than k. Further, γ cannot contain a subpath with more than $(n-2)k$ edges which is also a subpath of the boundary path of D, since otherwise C_2 would be $((n-2)/n)$-external for D. So σ must lie in the interior of γ. Thus if E is any 2-cell of D' other than C_2, then E is λ-external for D' if and only if E is λ-external for D. So D' is a spurless reduced disc diagram which fails to satisfy possibilities (1) and (3) of the theorem, and, hence, D' must be a ladder with C_1 and C_2 at the two ends. Attaching A (and possibly an intermediate chain of isolated 1-cells) to D' along an interior subpath of γ, we see that D is, in fact, also a ladder.   □

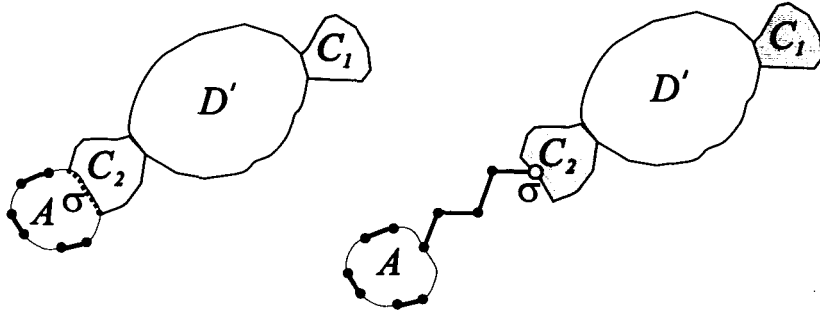REMARK 5.3. A more careful argument would provide a stronger version of Theo-

FIGURE 4. The diagram on the left corresponds to the case where ∂A intersects $\partial D'$. The diagram on the right is the case where A is connected to D' by a chain of isolated 1-cells. Note that in each case D' is the union of the regions labeled by D', C_1, and C_2 in the diagram.

rem 5.2 where we replace the third possibility in the conclusion by

3. D has at least three λ-external 2-cells, where $\lambda = ((n-2)k+1)/nk$.

The following useful variation allowing spurs can be deduced immediately from Theorem 5.2.

THEOREM 5.4. *Let X be a staggered 2-complex and suppose that there are constants n and k such that each 2-cell of X is attached along a path U^n, where U is a non-periodic path of length k.*
Let $D \to X$ be a reduced disc diagram. Then either:
(1) *D consists of a single 0-cell.*
(2) *D is a ladder with a spur or $((n-1)/n)$-external 2-cell at each end.*
(3) *D has at least three spurs and/or $((n-2)/n)$-external 2-cells.*

EXAMPLE 5.5. We now provide an example of a staggered 2-complex X and a reduced disc diagram $D \to X$ such that D is not a ladder, but D does not have three $((n-1)/n)$-external 2-cells.
Let X be the standard 2-complex of the one-relator presentation $\langle a, b \mid (ababa)^n \rangle$ and consider the word

$$W = ba(ababa)^{n-1}a(ababa)^{n-1}ab(ababa)^{-(n-1)}.$$

Then W is the boundary path of a reduced disc diagram $D \to X$ with three 2-cells, and only two of these are $((n-1)/n)$-external, but D is not a ladder. See Figure 5 for an illustration of this disc diagram in the case $n = 3$.

In conjunction with a result from [9], Theorem 5.2 yields a powerful result about subgroups of certain one-relator groups with torsion. We refer the reader to [13] for
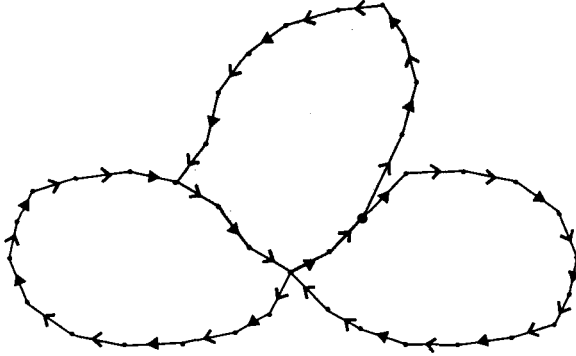
FIGURE 5.

the notion of a *quasiconvex* subgroup. We refer the reader to [9] for the notions of *weighted 2-complex, perimeter, sides,* and *weight of a 2-cell.*

We will apply the following result from [9].

THEOREM 5.6. *A compact weighted 2-complex X has a locally quasiconvex fundamental group provided that there exists K such that the following condition holds:*
Let $P \to \tilde{X}$ be any locally injective path and let $J \to \tilde{X}$ denote a geodesic with the same endpoints as P, and suppose that $|P| > K|J|$. Then there exists a 2-cell R of X attached along a path W^n such that W^n is a concatenation QS where Q is a subpath of $P \to X$ and Perimeter $(S) < n$ Weight (R).

THEOREM 5.7. *Let $G = \langle a_1, \ldots, a_r \mid W^n \rangle$. If $n \geq |W|$, then G is locally quasiconvex.*

PROOF. Without loss of generality we can assume that W is cyclically reduced. We can assume that $n \geq 4$ because otherwise G is virtually free and so the theorem is well-known [13]. Similarly, the theorem is true if $|W| = 1$ and so we will assume that at least two distinct letters occur in W. We will now apply Theorem 5.6 for an appropriate weighting on the sides of the 2-cells of the standard 2-complex X of the presentation for G. Let $K = n|W|$. Let P denote a path in $\tilde{X}$ and let J denote a geodesic with the same endpoints as P. Let D denote a disc diagram for $J = P$. By Theorem 5.4, either D has at least three spurs and/or $((n-2)/n)$-external 2-cells or D is a ladder with a spur or $((n-1)/n)$-external 2-cell at each end.

Observe that since P and J are locally injective, the only possible spurs are at their endpoints. Thus, if there are three spurs and/or $((n-2)/n)$-external 2-cells then at least one of the 2-cells must lie along P in the sense that it is a 2-cell R with a path Q in ∂R such that Q is a subpath of P and $|Q| > |W^{n-2}|$. This is because at most two of the spurs and/or external 2-cells lie at the endpoints of P and J, and since

$((n - 2)/n) \geq 1/2$, no such spur or 2-cell can lie along J because it would contradict that J is a geodesic. If D is a ladder then either $|P| \leq K |J|$ or for some 2-cell R at the end of D there is a path Q in ∂R such that Q is a subpath of P and $|Q| > |W^{n-1}|$.

For a 1-cell e of X that occurs in the attaching map of W^n, we weight the sides at e by $(1/\#(e))$ where $\#(e)$ is the number of times that W passes through e. It follows that Perimeter (e) is equal to n or 0 according as to whether or not e appears in the attaching map W^n of the unique 2-cell R of X. Let S be the path such that QS is a cyclic conjugate of $W^{\pm n}$ and observe that $|S| < 2 |W|$ and therefore Perimeter $(S) < 2 |W| n$. Next observe that Weight $(R) \geq 2n$ because Weight (R) is equal to n times the number of distinct letters occurring in W which we assumed to be at least 2. Finally, since by hypothesis $n \geq |W|$, we combine these inequalities to obtain

$$\text{Perimeter } (S) < 2 |W| n \leq 2nn \leq n \text{ Weight } (R)$$

and we are done.   □

6. Freiheitssatz and asphericity

In this section we provide tower proofs of the Freiheitssatz and asphericity theorems for staggered 2-complexes. We refer the reader to [6, III.9.5] for a combinatorial proof of the Freiheitssatz for staggered presentations. Our proof follows Howie's approach in [3] very closely.

THEOREM 6.1. *Let X be a staggered 2-complex. Let $Z \subset X$ be a connected subcomplex with the property that*

(1) *if C is a 2-cell of X with $\partial C \subset Z$ then $C \subset Z$, and*
(2) *the ordered 1-cells of X contained in Z form an interval,*

then $\pi_1 Z \to \pi_1 X$ is injective.

PROOF. Consider a reduced disc diagram $\phi: D \to X$ for a closed path $P \to Z$. We will show that D maps to Z and thus P is null-homotopic in Z.

If every ordered 1-cell in $\phi(D)$ is contained in Z, then by property 1, every 2-cell in D would map to Z. Arguing by contradiction, suppose that some ordered 1-cell of $\phi(D)$ is not contained in Z. By reversing the ordering of the 1-cells if necessary, we assume without loss of generality that $\phi(D)$ contains a greater 1-cell than Z does. Now consider a maximal tower lift

$$\begin{array}{ccc} & & Y \\ & \nearrow & \downarrow \\ D & \longrightarrow & X \end{array}$$
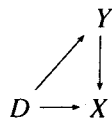
and observe that the top 1-cell e of Y does not map to Z. But ∂D maps to Z because P factors through ∂D and P maps to Z. So no preimage of e can lie in ∂D. It follows that since e is in the image of $D \to Y$, e must lie on the boundary of some 2-cell in Y. Consequently e is max α where α is the top 2-cell of Y. But this contradicts Lemma 4.1 which asserts that each preimage of e lies in ∂D. $\qquad\square$

The following special case of the above result is a direct generalization of the classical Freiheitssatz for one-relator groups.

COROLLARY 6.2. *Let X be a staggered 2-complex, and let $Y \subset X$ consist of the subgraph of unordered 1-cells, then for any choice of basepoints, $\pi_1 Y \to \pi_1 X$ is injective.*

DEFINITION 6.3. The 2-complex X is said to be *combinatorially reducible* provided that any spherical diagram $S \to X$ has a cancelable pair of 2-cells.

We refer the reader to Bogley and Pride [1, V.2.1] for a detailed account of the various asphericity notions that appear in the literature. Their account is not given in terms of diagrams, but rather in the equivalent language of *pictures*. Briefly, we note that if X is combinatorially reducible and if no 2-cell of X is attached by a proper power, and X has no pair of 2-cells with the same attaching map, then X is aspherical in the traditional sense that the universal cover $\tilde{X}$ is contractible.

The asphericity of the standard 2-complexes of cyclically reduced one-relator presentations where the relator is not a proper power was first determined by Lyndon in [5]. We also refer the reader to [6, III.9.7] for a combinatorial proof.

THEOREM 6.4. *Every staggered 2-complex is combinatorially reducible.*

PROOF. Let $S \to X$ be a reduced spherical diagram. Let $S \to Y$ be a maximal tower lift of $S \to X$. By Lemma 3.1, $S \to Y$ is reduced. To complete the proof, we apply Lemma 4.1 to see that there is a 1-cell on ∂S which is ridiculous. $\qquad\square$

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