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AN APPLICATION OF RECURSION THEORY TO ANALYSIS

LIANG YU

Abstract. Mauldin [15] proved that there is an analytic set, which cannot be represented by $B \cup X$ for some Borel set B and a subset X of a $\mathbf{\Sigma}^0_2$-null set, answering a question by Johnson [10]. We reprove Mauldin's answer by a recursion-theoretical method. We also give a characterization of the Borel generated σ-ideals having approximation property under the assumption that every real is constructible, answering Mauldin's question raised in [15].

§1. Introduction. A set is *Jordan measurable* if its characteristic function is Riemann integrable. For example, a null Cantor set is Jordan measurable but the set of rational numbers in $[0,1]$ is not. Actually, if $A \subseteq \mathbb{R}$ is bounded, then A is Jordan measurable if and only if its boundary is null. We use J to denote the collection of Jordan measurable sets and $\sigma(J)$ to denote the σ-algebra generated J. A natural question is what does $\sigma(J)$ look like? In [10], Johnson proves the following nice result.

Theorem 1.1 (Johnson [10]). *A set A belongs to $\sigma(J)$ if and only if there is a Borel set $B \subseteq A$ and a null $\mathbf{\Sigma}^0_2$-set X so that $A \subseteq B \cup X$.*

In the same paper, he raised the following question.

Question 1.2 (Johnson [10]). *Where does $\sigma(J)$ stand relative to $\mathbf{\Sigma}^1_1$-sets in* $[0,1]$*?*[1]

It is clear that there is a non-$\mathbf{\Sigma}^1_1$ set belonging to $\sigma(J)$. Thus, the question essentially asks whether there is a $\mathbf{\Sigma}^1_1$-set not in $\sigma(J)$. It was answered by Mauldin in [15] by a set theoretical method; that is, the following theorem.

Theorem 1.3 (Mauldin [15]). *There is an $\mathbf{\Sigma}^1_1$-set A for which there is no Borel B such that $A \setminus B$ is a subset of a $\mathbf{\Sigma}^0_2$ set with Lebesgue measure zero.*

Actually Johnson's question can be put in a more general background as noted by several set theorists (see [15], [1], and [11]).

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[1] The question can also be found in [16] (Question 1.6).

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DEFINITION 1.4. A collection $\mathcal{I} \neq \emptyset$ of sets of reals is a σ-*ideal* if

- for any $X \in \mathcal{I}$, $Y \subseteq X$ implies $Y \in \mathcal{I}$; and
- if $\{X_n\}_n$ is a sequence sets from $\mathcal{I}$, then $\bigcup_n X_n \in \mathcal{I}$.

A σ-ideal is usually considered as a collection of "small" sets. For example, the collection of Lebesgue null sets is a σ-ideal.

Borel sets are considered to be "well behaved." If a set A can be approximated by a Borel set modulo, an element in a σ-ideal, then A can also be considered as "well behaved." For example, a set A can be identified as a Borel set modulo, a $\mathbf{\Pi}^0_2$-null set, if and only if A is Lebesgue measurable. To formulate the idea, we need the following definition.

DEFINITION 1.5. Given a σ-ideal $\mathcal{I}$.

(1) A set A is *approximable* by $\mathcal{I}$ if there is a Borel set B and set $X \in \mathcal{I}$ so that $(A \setminus B) \cup (B \setminus A) \subseteq X$.
(2) A set A is *inner approximable* by $\mathcal{I}$ if there is a Borel set $B \subseteq A$ and set $X \in \mathcal{I}$ so that $A \setminus B \subseteq X$.
(3) $\mathcal{I}$ has *approximation property* if every $\mathbf{\Sigma}^1_1$-set is approximable by $\mathcal{I}$.
(4) $\mathcal{I}$ has *inner approximation property* if every $\mathbf{\Sigma}^1_1$-set is inner approximable by $\mathcal{I}$.

σ-ideals can be fairly wild. Thus, we will restrict our attention to more "tamed" σ-ideals, that is, those that are Borel-generated.

DEFINITION 1.6. Given a collection Γ of sets of reals. A σ-ideal $\mathcal{I}$ is said to be *generated by* Γ if for any $X \in \mathcal{I}$, there is a $Y \in \Gamma \cap \mathcal{I}$ so that $X \subseteq Y$.

Clearly, the collection of Lebesgue null sets is a $\mathbf{\Pi}^0_2$-generated σ-ideal.

The following fact is well known. We give a detailed proof for the completeness.

LEMMA 1.7. (1) *For any σ-ideal $\mathcal{I}$, the set $\mathcal{X} = \{X \mid X$ is approximable by $\mathcal{I}\}$ is a σ-algebra.*
(2) *For any Borel generated σ-ideal $\mathcal{I}$, a set X is inner approximable by $\mathcal{I}$ if and only if it is approximable by $\mathcal{I}$.*

PROOF. (1). Suppose that $\mathcal{I}$ is a σ-ideal. If $X \in \mathcal{X}$, then there is a Borel set B and a set $C \in \mathcal{I}$ so that $(B \setminus X) \cup (X \setminus B) \subseteq C$. Now let $B_1 = 2^\omega \setminus B$ be a Borel set and $Y = 2^\omega \setminus X$, then $Y \setminus B_1 = (2^\omega \setminus X) \cap B = B \setminus X$ and $B_1 \setminus Y = (2^\omega \setminus B) \cap X = X \setminus B$. So $(Y \setminus B_1) \cup (B_1 \setminus Y) = (B \setminus X) \cup (X \setminus B) \subseteq C \in \mathcal{I}$. In other words, $Y \in \mathcal{X}$.

If $\{X_i\}_{i\in\omega} \subseteq \mathcal{X}$, then there is a sequence of Borel sets $\{B_i\}_{i\in\omega}$ such that for any $i \in \omega$, $(B_i \setminus X_i) \cup (X_i \setminus B_i) \in \mathcal{I}$. Since $\mathcal{I}$ is a σ-ideal, we have that $\bigcup_{i\in\omega}(B_i \setminus X_i) \cup (X_i \setminus B_i) \in \mathcal{I}$. Now let $B = \bigcup_{i\in\omega} B_i$ be a Borel set, then $(B \setminus \bigcup_{i\in\omega} X_i) \cup (\bigcup_{i\in\omega} X_i \setminus B) \subseteq \bigcup_{i\in\omega}(B_i \setminus X_i) \cup (X_i \setminus B_i)$. So we have that $(B \setminus \bigcup_{i\in\omega} X_i) \cup (\bigcup_{i\in\omega} X_i \setminus B) \in \mathcal{I}$.

(2). Clearly, if X is inner approximable by $\mathcal{I}$, then it is approximable by $\mathcal{I}$. Now suppose that there is a Borel set B and another Borel set $C \in \mathcal{I}$ such that $(X \setminus B) \cup (B \setminus X) \subseteq C$. Then $B \setminus C \subseteq X$ is a Borel set. Moreover, $X \setminus (B \setminus C) \subseteq (X \setminus B) \cup C \subseteq C$. So X is inner approximable by $\mathcal{I}$. ⊣

So for a Borel generated σ-ideal $\mathcal{I}$, it has approximation property if and only if it has inner approximation property.

Let $\mathcal{I}_F$ be the σ-ideal generated by $\mathbf{\Sigma}^0_2$ null sets. Then Johnson's question can be reformulated as whether $\mathcal{I}_F$ has approximation property.

To generalize Johnson's question, Mauldin raised the following more general question.

QUESTION 1.8 (Mauldin [15]). *For what σ-ideals can one derive similar results?*

Kechris and Solecki [11] gave an answer to Question 1.8 for some "well-behaved" σ-ideals under AD, Axiom of Determinacy. For example, they prove the following theorem.

THEOREM 1.9 (Kechris and Solecki [11]). *Assume AD. If $\mathcal{I}$ is a Borel generated σ-ideal, then $\mathcal{I}$ has approximation property if and only if there is no $\mathbf{\Sigma}^1_1$-equivalence relation E with uncountably many equivalence classes whose all, but possibly countably many, equivalence classes are not in $\mathcal{I}$.*

The target of this paper is to give a natural answer to Question 1.2 under ZFC. Furthermore, we obtain a full answer to the generalized Johnson's Question 1.8 different than Kechris–Solecki's under the assumption that every real is constructible. Namely, we will prove that (see Corollary 3.8) if every real is constructible, then for any Borel generated ideal $\mathcal{I}$, $\mathcal{I}$ has approximation property if and only if for any x, there is some $X \in \mathcal{I}$ and $x_0 \geq_T x$ such that $\{z \mid z \oplus x \geq_h x_0\} \subseteq X$.

§2. An introduction to higher recursion theory. In this section, we give a brief introduction to higher recursion theory. The higher recursion theory facts in this section are fairly basic. Sometimes we use them without even mentioning. For more details, readers may refer to [18] and [3].

For convenience, we identify a real as an element in Cantor space.

For any real x, we use ω_1^x to denote the least non-x-recursive ordinal. A well-known result in higher recursion theory is that the reals in $L_{\omega_1^x}[x]$ are exactly the $\Delta^1_1(x)$-reals. We also call a Δ^1_1-real a hyperarithmetic real. We say that $x \leq_h y$, x is hyperarithmetically reducible to y, if x is Δ^1_1-definable in y. In other words, $x \in L_{\omega_1^y}[y]$. A hyperdegree is a $\equiv_h$-class. The following theorem gives a nice characterization of Borel sets.

THEOREM 2.1 (Suslin [20]; Kleene [13]). *A set of reals is Borel if and only if it is $\Delta^1_1(x)$ for some real x.*

Borel sets are considered as "well behaved" and have "regular properties." For example, if B is $\Delta^1_1(x)$ set without a perfect subset, then B must be countable. Actually for such B, every member is hyperarithmetic in x.

One of the central results in higher recursion theory is the following so-called Spector–Gandy's theorem.

THEOREM 2.2 (Spector [19]; Gandy [7]). *Given a set X of reals, the following are equivalent:*

(1) *X is Π^1_1.*
(2) *There is a Σ_1-formula φ in set theory language so that $\forall x(x \in X \leftrightarrow L_{\omega_1^x}[x] \models \varphi(x))$.*
(3) *There is an arithmetical set $A \subseteq 2^\omega$ so that $\forall x(x \in X \leftrightarrow \exists y \in \Delta^1_1(x \oplus y \in A))$, where $x \oplus y$ is a real z so that $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for every n.*

By Theorem 2.2, roughly speaking, a Π^1_1 set can be viewed as a "recursively enumerable set" over the "inner" model L. Another conclusion of Theorem 2.2 is that a Borel subset Y of a Π^1_1-set X is precisely a "finite subset" of X in the sense that all the elements of Y are the ones enumerated into X up to a "bounded stage" (or a fixed stage less than ω_1). This analogy phenomenon opens a door to enable recursion theorists to enter (or pollute) set theory.

The "halting problem" in the hyperarithmetic theory is Kleene's $\mathcal{O}$, a Π^1_1-complete set. We use $\mathcal{O}^x$ to denote Kleene's $\mathcal{O}$ relative to x.

Definition 2.3. Let $\mathrm{C} = \{x \mid x \in L_{\omega_1^x}\}$ and $\mathrm{C}(x) = \{z \mid z \in L_{\omega_1^{x\oplus z}}[x]\}$. Let $\mathrm{D} = \{x \oplus y \mid y \in \mathrm{C}(x)\}$.

C is the largest Π^1_1 thin set or the largest Π^1_1-set without a perfect subset. D is also a Π^1_1 set. $\mathrm{C}(x)$ is a $\Pi^1_1(x)$-set, a Π^1_1-set relative to x. Moreover, $\mathrm{C} = \{x \mid \forall z(\omega_1^z \geq \omega_1^x \rightarrow z \geq_h x)\}$. It is also clear that for any countable set A, the set $\mathrm{D}_A = \{x \oplus y \mid x \in A \wedge y \in \mathrm{C}(x)\}$ has no perfect subset. Readers can find more details concerning these facts in [18] and [3].

Note that $\mathcal{O}^x \in \mathrm{C}(x)$ for any x.

The following Π^1_1-basis theorem is needed later.

Theorem 2.4 (Guaspari [8]). *Suppose that X is a nonempty Π^1_1-set. Then there is a real $x \in X \cap C$.*

For any real x, there is a real $y \geq_T x$ with $y \notin \mathrm{C}$. The following theorem will be used so frequently in this paper that we don't even mention it sometimes.

Theorem 2.5 (Boolos and Putnam [2]; Jensen [9]). *For any ordinal α for which $L_{\alpha+1} \setminus L_\alpha$ contains a real, there is a real $y \in L_{\alpha+1} \setminus L_\alpha$ so that y Turing computes a well ordering of ω with order type α, and all the reals in L_α. In particular, for any constructible real x, there is a real $y \geq_T x$ so that $y \in C$.*[2]

§3. On σ-ideals having approximation property. We first introduce a "lightface version" of approximation property.

Definition 3.1. $\mathcal{I}$ has *lightface approximation property* if every Σ^1_1-set is approximable by $\mathcal{I}$.

Approximation property will be separated from lightface approximation property by Proposition 3.10.

The following theorem by Martin is quite important to our paper.

[2]For the proofs, see Proposition 4.3.4 in [3].

THEOREM 3.2 (Martin [14]). *Suppose that A is a Δ^1_1-set of reals with a nonhyperarithmetic member, then A ranges over an upper cone of hyperdegrees. Or there is some real x so that for any $y \geq_h x$, there is some $z \in A$ with $z \equiv_h y$.*

A more recursion-theoretical flavored proof of the theorem can be found in [21].

THEOREM 3.3. *Suppose that every real is constructible.*

(1) *For any Borel generated σ-ideal $\mathcal{I}$, if there is some $X \in \mathcal{I}$ containing an upper cone of hyperdegrees, then $\mathcal{I}$ has lightface approximation property.*
(2) *For any Borel generated σ-ideal $\mathcal{I}$, if $\mathcal{C}$ is approximable by $\mathcal{I}$, then there is some $X \in \mathcal{I}$ containing an upper cone of hyperdegrees.*

Before proceeding the formal proof, we give an intuitive description why the theorem is a clear fact to a (higher-) recursion theorist. By Lemma 1.7, the class of the sets approximable by $\mathcal{I}$ is a σ-algebra. So it is sufficient to consider whether every Π^1_1-set is approximable by $\mathcal{I}$. As said before, a Π^1_1 set Y can be viewed as a recursively enumerable set. And its Borel subsets can be viewed as those "finite" subsets enumerated up to "a bounded stage." So if $\mathcal{I}$ contains an element X "cutting off" those members which will be enumerated into Y quite late, then Y can be decomposed into a "finite" (and so Borel) subset of Y and a subset of X. Conversely, if there is no such "cutting off" element X in $\mathcal{I}$, then Y can never be approximated by a Borel set modulo a member in $\mathcal{I}$. Keeping in mind the idea would be quite helpful to understand most proofs throughout the paper.

We may turn to the formal proof now.

PROOF. (1). By Lemma 1.7, it is sufficient to prove that every Π^1_1-set is approximable by $\mathcal{I}$. Now fix a Π^1_1-set Y. Let $X \in \mathcal{I}$ contain an upper cone of hyperdegrees. Then, by Theorem 2.2, there is a Σ_1-formula φ so that for any z,

$$z \in Y \Leftrightarrow L_{\omega_1^z}[z] \models \varphi(z).$$

By Theorem 2.5 and the assumption that every real is constructible, for every real z, there must be some real $z_0 \in \mathcal{C}$ with $z_0 \geq_T z$. By the assumption on X, we may let x_0 be in $\mathcal{C}$ so that $\{z \mid z \geq_h x_0\} \subseteq X$. Then, $\{z \mid \omega_1^z \geq \omega_1^{x_0}\} = \{z \mid z \geq_h x_0\} \subseteq X$. Now let $B = \{z \mid \omega_1^z \leq \omega_1^{x_0} \wedge L_{\omega_1^z} \models \varphi(z)\} \subseteq Y$. By Lemma 3.7.7 in [3], it is clear that B is $\Pi^1_1(\mathcal{O}^{x_0})$. Note that if $\omega_1^z \leq \omega_1^{x_0}$, then $L_{\omega_1^z} \models \varphi(z)$ is equivalent to the fact that there is a real s coding an ordinal α isomorphic to an initial segment of $\omega_1^{x_0}$ so that for any $\beta \leq \alpha$, $L_\beta[z]$ is not admissible and $L_\alpha[z] \models \varphi(z)$. So it is also $\Sigma^1_1(\mathcal{O}^{x_0})$ and so a Borel set. Moreover, $Y \setminus B \subseteq X$.

(2). Now suppose that there is a Borel set $B \subseteq \mathcal{C}$ and a Borel set $X \in \mathcal{I}$ so that $\mathcal{C} \setminus B \subseteq X$. By the assumption that every real is constructible, we may fix a real x so that both B and X are $\Delta^1_1(x)$ and $x \in L_{\omega_1^x}$. Let $\alpha < \omega_1^x$ be the ordinal so that $x \in L_{\alpha+1} \setminus L_\alpha$ and fix a real $r \leq_T x$ coding a well ordering of order type $\alpha + 1$. Now suppose that X does not contain an upper cone of hyperdegrees. Then the $\Pi^1_1(x)$-set $\{z \mid z \geq_h \mathcal{O}^x \wedge z \notin X\}$ is nonempty. By the $\Pi^1_1(x)$-basis Theorem 2.4, there is some real $x_0 \notin X$ so that $x_0 \in L_{\omega_1^{x_0\oplus x}}[x] = L_{\omega_1^{x_0}}$ and $x_0 >_h \mathcal{O}^x$. Fix an oracle Turing function Φ so that Φ^{x_0} codes a well ordering of ω of order type $\alpha+1$. Since $\mathrm{C}\setminus B \subseteq X$, we have that $x_0 \in B$. Define $B_0 = \{x\oplus z \mid z \in B \wedge \Phi^z \cong r\}$. Clearly B_0 is $\Sigma^1_1(r)$. Since r codes a well ordering, if there exists an $f : \Phi^z \cong r$, then the f must be unique and so $\Delta^1_1(y\oplus r)$. Thus, by Theorem 2.2 and the fact that $r \leq_T x$, B_0 is $\Delta^1_1(x)$. Then, by Theorem 3.2, B_0 ranges an upper cone of hyperdegrees. But for any $z \in B_0$, we have that $\omega_1^z \geq \omega_1^{x_0} > \omega_1^x$ and so $x \in L_{\omega_1^z}[z]$. Thus, the set $\{z \mid z \in B \wedge \Phi^z \cong r\}$ ranges over an upper cone of hyperdegrees and so $B \not\subseteq \mathrm{C}$, a contradiction. ⊣

Now we may obtain a characterization of the σ-ideals having lightface approximation property under the assumption that every real is constructible.

Corollary 3.4. *Suppose that every real is constructible. Then for any Borel generated σ-ideal $\mathcal{I}$, the following are equivalent:*

(1) *$\mathcal{I}$ has lightface approximation property.*
(2) *C is approximable by $\mathcal{I}$.*
(3) *There is some $X \in \mathcal{I}$ containing an upper cone of hyperdegrees.*

Let

$$\geq_h x = \{z \mid z \geq_h x\} \text{ and } \mathcal{F}_{\geq_h} = \{X \mid (\exists x)(\geq_h x \subseteq X)\}.$$

Then $\mathcal{F}_{\geq_h}$ is a filter which is usually considered as a collection of "large" sets. Then Corollary 3.4 says that every σ-ideal having approximation property must contain a "large member" under the assumption that every real is constructible.

For the general approximation property, we need to relativize the proof of Theorem 3.3. The following lemma is a partial realization of Martin's Theorem 3.2.

Lemma 3.5. *Suppose that $x \in L_{\omega_1^x}$ and B is a $\Delta^1_1(x)$ set of reals in which there is a member $z >_h x$, then B ranges over an upper cone of hyperdegrees.*

Proof. Fix $x \in L_{\omega_1^x}$ and B to be a $\Delta^1_1(x)$ set of reals in which there is a member $z >_h x$. Fix such a $z \in B$. Since $x \in L_{\omega_1^x}$, we may let $\alpha < \omega_1^x$ so that $x \in L_{\alpha+1}\setminus L_\alpha$. So if y is a real so that $\omega_1^y > \alpha$, then $x \in L_{\omega_1^y}$ and so $x \leq_h y$. In other words, for any y, $\omega_1^y > \alpha$ if and only if $y \geq_h x$. So we may fix an oracle Turing function Φ so that Φ^z codes a well ordering of ω of order type $\alpha+1$. Also let $r \leq_T x$ code a well ordering of ω of order type $\alpha+1$. Define $A = \{y \mid \Phi^y \cong r \wedge y \in B\}$ and observe that A is $\Delta^1_1(r)$. Since $r \leq_T x$, we have that A is a $\Delta^1_1(x)$ set containing $z \not\leq_h x$. So, by Theorem 3.2 relative to x, $\{x\oplus y \mid y \in A\}$ ranges over an upper cone of hyperdegrees. Since every $y \in A$ is hyperarithmetically above x by the fact that $x \in \mathrm{C}$ and the same reason as above, we have that A, and so B, ranges over an upper cone of hyperdegrees. ⊣

Remark. It is an open problem whether Lemma 3.5 holds for arbitrary x.

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THEOREM 3.6. *For any Borel generated σ-ideal $\mathcal{I}$,*

(1) *If for any real x, there is some $X \in \mathcal{I}$ and a countable ordinal α so that $\{z \mid \omega_1^{x\oplus z} \geq \alpha\} \subseteq X$, then $\mathcal{I}$ has approximation property.*
(2) *Suppose that D is approximable by $\mathcal{I}$. Then for any x, there is some $X \in \mathcal{I}$ and $x_0 \geq_T x$ so that $\{y \oplus z \mid y \equiv_h x_0 \wedge x_0 \oplus z \geq_h \mathcal{O}^{\mathcal{O}^{x_0}}\} \subseteq X$.*
(3) *Suppose that every real is constructible. Then for any x, if $C(x)$ is approximable by $\mathcal{I}$, then there is some $X \in \mathcal{I}$ and x_0 such that $\{z \mid x \oplus z \geq_h \mathcal{O}^{x_0}\} \subseteq X$.*

PROOF. (1). By Lemma 1.7, the class approximable by $\mathcal{I}$ is a σ-algebra. So to show that every $\mathbf{\Sigma}^1_1$-set is approximable by $\mathcal{I}$, it is sufficient to prove that for any real x, every $\Pi^1_1(x)$-set is approximable by $\mathcal{I}$. Now fix a real x and a $\Pi^1_1(x)$-set Y. Then, by Theorem 2.2 relative to x, there is a Σ_1-formula φ so that for any z,

$$z \in Y \Leftrightarrow L_{\omega_1^{x\oplus z}}[x \oplus z] \models \varphi(x,z).$$

Then fix a countable ordinal α and a set $X \in \mathcal{I}$ so that $\{z \mid \omega_1^{z\oplus x} \geq \alpha\} \subseteq X$. Let $B = \{z \mid \omega_1^{x\oplus z} \leq \alpha \wedge L_{\omega_1^{x\oplus z}}[x \oplus z] \models \varphi(x,z)\} \subseteq Y$. Fix any real r coding a well ordering of ω of order type α, then B is $\Delta^1_1(x \oplus r)$ and so Borel. Moreover, $Y \setminus B \subseteq X \in \mathcal{I}$. So $\mathcal{I}$ is an inner approximation.

(2). Suppose that there exists some $x_0 \geq_T x$, a $\Delta^1_1(x_0)$-set $B \subseteq \mathrm{D}$ and a $\Delta^1_1(x_0)$-set $X \in \mathcal{I}$ so that $\mathrm{D} \setminus B \subseteq X$. Note that the set

$$B_{x_0} = \{y_0 \oplus z \mid y_0 \equiv_h x_0 \wedge y_0 \oplus z \in B\} \subseteq B = \mathrm{D} \cap B$$

is a $\Delta^1_1(\mathcal{O}^{x_0})$-set. Since $B \subseteq \mathrm{D}$, B_{x_0} is a Borel set without a perfect subset. So we have that B_{x_0} is countable and only contains reals hyperarithmetic in $\mathcal{O}^{x_0}$. Now suppose that there are some z and $y \equiv_h x_0$ with $y \oplus z \geq_h \mathcal{O}^{\mathcal{O}^{x_0}}$ but $y \oplus z \notin X$. X is $\Delta^1_1(x_0)$, so is $2^\omega \setminus X$. Let

$$X_1 = \{y \oplus z \mid y \equiv_h x_0 \wedge y \oplus z \notin X \wedge y \oplus z \geq_h \mathcal{O}^{\mathcal{O}^{x_0}}\}$$

be a $\Pi^1_1(\mathcal{O}^{x_0})$-set. We just need to show that X_1 is empty. If X_1 is not empty, then by Theorem 2.4 relative to $\mathcal{O}^{x_0}$, we have some $y_0 \oplus z_0 \notin X$ with $y_0 \equiv_h x_0$, $y_0 \oplus z_0 \in L_{\omega_1^{\mathcal{O}^{x_0}\oplus y_0 \oplus z_0}}[\mathcal{O}^{x_0}]$, and $y_0 \oplus z_0 >_h \mathcal{O}^{x_0}$. Note that $y_0 \equiv_h x_0$ and $y_0 \oplus z_0 >_h \mathcal{O}^{x_0}$ imply that $L_{\omega_1^{\mathcal{O}^{x_0}\oplus y_0 \oplus z_0}}[\mathcal{O}^{x_0}] \subseteq L_{\omega_1^{y_0\oplus z_0}}[y_0]$. And $y_0 \oplus z_0 \in L_{\omega_1^{\mathcal{O}^{x_0}\oplus y_0 \oplus z_0}}[\mathcal{O}^{x_0}]$ implies that $L_{\omega_1^{y_0\oplus z_0}}[y_0] \subseteq L_{\omega_1^{\mathcal{O}^{x_0}\oplus y_0 \oplus z_0}}[\mathcal{O}^{x_0}]$. Hence $L_{\omega_1^{\mathcal{O}^{x_0}\oplus y_0 \oplus z_0}}[\mathcal{O}^{x_0}] = L_{\omega_1^{y_0\oplus z_0}}[y_0]$. In other words, $y_0 \oplus z_0 \notin B \cup X$ but $y_0 \oplus z_0 \in \mathrm{D}$, a contradiction to the fact that $\mathrm{D} \setminus B \subseteq X$.

(3). For any real x, fix a real $x_0 \geq_T x$ so that there is a $\Delta^1_1(x_0)$-set $B \subseteq \mathrm{C}(x)$ with $\mathrm{C}(x) \setminus B \subseteq X$ for some $\Delta^1_1(x_0)$-set $X \in \mathcal{I}$. Since every real is constructible, by Theorem 2.5, we may assume that $x_0 \in L_{\omega_1^{x_0}}$. Since B is a Borel set, it must be countable and so every member in B is hyperarithmetic in x_0. Now, for a contradiction, suppose that there is a real $y_0 \notin X$ with $x \oplus y_0 \geq_h \mathcal{O}^{x_0}$. Since $x_0 \in L_{\omega_1^{x_0}}$, there is a countable ordinal $\alpha < \omega_1^{x_0}$ so that $x_0 \in L_{\alpha+1} \setminus L_\alpha$. Fix a real $r \leq_T x_0$ coding a well ordering ω of order type $\alpha+1$. Also fix an oracle Turing function Φ so that $\Phi^{x\oplus y_0}$ codes a well ordering of order

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type $\alpha+1$. Let $X_1 = 2^\omega \setminus X$ be a $\Delta^1_1(x_0)$-set. Define $Y = \{z \in X_1 \mid \Phi^{x\oplus z} \cong r\}$ to be a $\Delta^1_1(x_0)$-set. Then for any $z \in Y$, $x \oplus z \geq_h x_0$ and so $x \oplus z \equiv_h x_0 \oplus z$. Since $y_0 \in Y$, by Lemma 3.5, there must be some real $x_1 >_h x_0$ so that for any $y \geq_h x_1$, there is some $z \in Y$ with $x_0 \oplus z \equiv_h y$. Since $z \in Y$, we have that $x \oplus z \equiv_h x_0 \oplus z$. Since every real is constructible, by Theorem 2.5 again, there must be some real $y_1 \geq_h x_1 \geq_h x$ with $y_1 \in L_{\omega_1^{y_1}}$. Then there is some $z_1 \in Y$ so that $x \oplus z_1 \equiv_h y_1$ and so $x \oplus z_1 \in L_{\omega_1^{x\oplus z_1}} \subseteq L_{\omega_1^{x\oplus z_1}}[x]$. Clearly, $z_1 \not\leq_h x_0$. So $z_1 \in \mathrm{C}(x) \setminus B \subseteq X$, a contradiction to the fact that $z_1 \in Y \subseteq X_1$. ⊣

Note that (1) and (2) of Theorem 3.6 give some criteria of having approximation property within ZFC. We may obtain a full description of having such property under the assumption that every real is constructible.

Lemma 3.7. *Suppose that every real is constructible. Then for any set of reals X and real x, there is a real x_0 so that $\{z \mid x \oplus z \geq_h x_0\} \subseteq X$ if and only if there is a countable ordinal α so that $\{z \mid \omega_1^{x\oplus z} \geq \alpha\} \subseteq X$.*

Proof. The direction from right to left is clear. Now suppose that for any real x, there is a real x_0 so that $\{z \mid x \oplus z \geq_h x_0\} \subseteq X$. Since every real is constructible, there is a real $x_1 \geq_h x_0$ so that $x_1 \in L_{\omega_1^{x_1}}$. Then $\{z \mid \omega_1^{x\oplus z} \geq \omega_1^{x_1}\} = \{z \mid x \oplus z \geq_h x_1\} \subseteq \{z \mid x \oplus z \geq_h x_0\} \subseteq X$. ⊣

The following corollary gives an answer to Question 1.8 for Borel generated σ-ideals under the assumption that every real is constructible.

Corollary 3.8. *Suppose that every real is constructible. Then for any Borel generated ideal $\mathcal{I}$, $\mathcal{I}$ has approximation property if and only if for any x, there is some $X \in \mathcal{I}$ and $x_0 \geq_T x$ such that $\{z \mid z \oplus x \geq_h x_0\} \subseteq X$.*

Proof. The direction from left to right follows from (3) of Theorem 3.6. Another direction follows from Lemma 3.7 and (1) of Theorem 3.6. ⊣

So Corollary 3.8 converts the set theoretical property, the approximation property, to be a recursion theoretical property under the assumption that every real is constructible.

Finally, we separate approximation property from lightface approximation property within ZFC.

Lemma 3.9. *For any real y with $\omega_1^y = \omega_1^{CK}$ and any real x with $(\omega_1)^{L[x\oplus y]} = (\omega_1)^{L[y]}$, there is a real z so that $\omega_1^{y\oplus z} = \omega_1^{CK}$, $z \in L_{\omega_1^{\mathcal{O}\oplus y\oplus z}}[\mathcal{O} \oplus y]$, and $z \not\leq_h x$.*

Proof. Fix the reals y and x as in the assumption. Note that the set $A_y = \{y \oplus z \mid \omega_1^{y\oplus z} = \omega_1^y\}$ is an uncountable $\Sigma^1_1(y)$-set and so there is a $\Delta^1_1(\mathcal{O}^y)$-perfect tree $T \subseteq 2^{<\omega}$ for which $[T] \subseteq A_y$. Since $(\omega_1)^{L[x\oplus y]} = (\omega_1)^{L[y]}$, there must be an $L[y]$-countable ordinal $\alpha > \omega_1^{x\oplus \mathcal{O}^y} \geq \omega_1^x$. Since T is a $\Delta^1_1(\mathcal{O}^y)$-perfect tree, the set $\{\mathcal{O}^y \oplus z \mid z \in [T]\}$ ranges over every hyperdegree greater or equal to $\mathcal{O}^y$. Then by Theorem 2.5 relative to $\mathcal{O}^y$, there is a real $z \in [T]$ so that $\omega_1^{\mathcal{O}^y\oplus z} > \alpha$ (and so $z \not\leq_h x$) and $z \in L_{\omega_1^{\mathcal{O}^y\oplus z}}[\mathcal{O}^y]$. Since $\omega_1^y = \omega_1^{\mathrm{CK}}$, we have that $\mathcal{O}^y \equiv_h \mathcal{O} \oplus y$. Hence, $\omega_1^{y\oplus z} = \omega_1^y = \omega_1^{\mathrm{CK}}$, $z \not\leq_h x$, and $z \in L_{\omega_1^{\mathcal{O}\oplus y\oplus z}}[\mathcal{O} \oplus y]$. ⊣

Let $\mathcal{I}_{>\omega_1^{\mathrm{CK}}} = \{X \mid X \subseteq \{x \mid \omega_1^x > \omega_1^{\mathrm{CK}}\}\}$. Then $\mathcal{I}_{>\omega_1^{\mathrm{CK}}}$ is a Borel generated σ-ideal, and contains the upper cone $\geq_h \mathcal{O}$. By the proof of (1) of Theorem 3.3,[3] $\mathcal{I}_{>\omega_1^{\mathrm{CK}}}$ has lightface approximation property.

Proposition 3.10. *$\mathcal{I}_{>\omega_1^{\mathrm{CK}}}$ has no approximation property.*

Proof. We prove that the set $\mathrm{D}(\mathcal{O}) = \{y \oplus z \mid z \in L_{\omega_1^{\mathcal{O}\oplus y \oplus z}}[z]\}$ is not approximable by $\mathcal{I}_{>\omega_1^{\mathrm{CK}}}$.

Otherwise, there is a real $y_0 \geq_h \mathcal{O}$ and a $\Delta^1_1(y_0)$-set $B \subseteq \mathrm{D}(\mathcal{O})$ so that

$$\mathrm{D}(\mathcal{O}) \subseteq B \cup \{x \mid \omega_1^x > \omega_1^{\mathrm{CK}}\}.$$

Pick a real y_1 with $\omega_1^{y_1} = \omega_1^{\mathrm{CK}}$ so that $\mathcal{O}^{y_1} \equiv_h \mathcal{O} \oplus y_1 \equiv_h y_0$.[4] Then the set $\{y_1 \oplus z \mid z \in L_{\omega_1^{\mathcal{O}\oplus y_1 \oplus z}}[z]\} \cap B = \{y_1 \oplus z \mid z \in 2^\omega\} \cap B$ is a thin, and so countable, $\Delta^1_1(y_0)$ set. So every member in $\{y_0 \oplus z \mid z \in L_{\omega_1^{\mathcal{O}\oplus y_0 \oplus z}}[z]\} \cap B$ is hyperarithmetic in y_0. Clearly, $(\omega_1)^{L[y_0 \oplus y_1]} = (\omega_1)^{L[y_1]}$. So by Lemma 3.9, there is a real z such that $\omega_1^{y_1 \oplus z} = \omega_1^{\mathrm{CK}}$, $z \in L_{\omega_1^{\mathcal{O}\oplus y_1 \oplus z}}[\mathcal{O} \oplus y_1]$, and $z \not\leq_h y_0$. In other words, $y_1 \oplus z \in \mathrm{D}(\mathcal{O}) \setminus (B \cup \{x \mid \omega_1^x > \omega_1^{\mathrm{CK}}\})$.

In summary, $\mathcal{I}_{>\omega_1^{\mathrm{CK}}}$ has no approximation property. ⊣

§4. A recursion theoretical solution to Johnson's question. Recall that $\mathcal{I}_F$ is the σ-ideal generated by $\boldsymbol{\Sigma}^0_2$ null sets. To answer Johnson's Question 1.2, we prove a slightly stronger result.

Theorem 4.1. *D is not approximable by $\mathcal{I}_F$.*

To prove Theorem 4.1, we need some results from algorithmic randomness theory. Somehow the area can be viewed as "effective measure theory." For more details concerning this area, please refer [4] and [17].

Definition 4.2. A real r is called *Kurtz-random* if r does not belong to any Π^0_1 null set.

The definition of Kurtz-randomness can be relativized to any real x. We have the following result.

Theorem 4.3 (Kjos-Hanssen et al. [12]). *For any reals x and $z \geq_T x'$, the Turing jump of x, there is an x-Kurtz random real $r \equiv_T z$.*

Note that if every real is constructible, then by Theorem 4.3 and Corollary 3.4, it is clear that C is not approximable by $\mathcal{I}_F$. And so, we may obtain a negative answer to Question 1.2 under the assumption that every real is constructible. To derive an answer without additional axioms, we need one more result from algorithmic randomness theory.

[3]To prove it within ZFC, just replace x_0 with $\mathcal{O}$ in the proof.

[4]For the existence of the real y_1. Note that the set $\{x \mid \omega_1^x = \omega_1^{\mathrm{CK}} \wedge x \not\leq_h \emptyset\}$ is an uncountable Σ^1_1-set. So it is not hard to build an $\mathcal{O}$-recursive perfect tree $T \subseteq 2^{<\omega}$ so that for any $z \in [T]$, $\omega_1^z = \omega_1^{\mathrm{CK}}$. Now it is a standard fact in higher recursion theory (see Corollary 2.4.10 in [3]) that for any such real z, $\mathcal{O}^z \equiv_h \mathcal{O} \oplus z$. So it is simple, by a zig-zag coding, to see that there is such a real y_1 so that $\mathcal{O} \oplus y_1 \equiv_h y_0$.

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THEOREM 4.4 (Franklin and Stephan [6]). *If r_0 is Kurtz random and r_1 is r_0-Kurtz random, then $r_0 \oplus r_1$ is Kurtz random.*

Now we are ready to prove Theorem 4.1.

PROOF OF THEOREM 4.1. By (2) of Theorem 3.6, it is sufficient to prove that for any real x_0 and $\Sigma_2^0(x_0)$-null set X, there are two reals $y \equiv_h x_0$ and z with $y \oplus z \geq_h \mathcal{O}^{\mathcal{O}^{x_0}}$ so that $y \oplus z \notin X$. By Theorem 4.3, we may let $y \equiv_T x_0'$ be an x_0-Kurtz random. By Theorem 4.3 again, there is a y-Kurtz random, and so $x_0 \oplus y$-Kurtz random, real $z \geq_T \mathcal{O}^{\mathcal{O}^{x_0}}$ and so $y \oplus z \geq_T \mathcal{O}^{\mathcal{O}^{x_0}}$. Then by Theorem 4.4 relative to x_0, $y \oplus z$ is x_0-Kurtz random and so $y \oplus z \notin X$.
Hence D is not approximable by $\mathcal{I}_F$. ⊣

§5. More results and questions.

We give some more applications of Theorem 3.6.

Let $I_C = \{X \mid \exists Y \in \mathbf{\Sigma}_2^0(Y \text{ is meager} \wedge X \subseteq Y)\}$ be a σ-ideal. By the results in [5] (also can be found in [18] and [3]), for any real x, there is some $X \in \mathcal{I}_C$ so that $\{z \mid \omega_1^{x\oplus z} \geq \omega_1^{\mathcal{O}^x}\} \subseteq X$. Then by (1) of Theorem 3.6, $\mathcal{I}_C$ has approximation property.

Let $\mathcal{I}_T = \{X \mid \exists x(X \subseteq \{z \mid z \not\geq_T x\})\}$. Clearly, $\mathcal{I}_T$ is a Borel generated σ-ideal. By (2) of Theorem 3.6, $\mathcal{I}_T$ does not even have lightface approximation property.

We enumerate some questions that we are quite interested in.

QUESTION 5.1. (1) *Under ZFC, is there a recursion theoretical characterization of σ-ideals having approximation property?*
(2) *For some "well behaved" σ-ideals, are there nice recursion theoretical characterizations?*

Concerning Question (2), we may consider those Borel generated σ-ideals $\mathcal{I}$ so that the corresponding set $\{x \mid x \text{ is a Borel code} \wedge B_x \in \mathcal{I}\}$ is $\mathbf{\Pi}_1^1$, where B_x is the Borel set coded by x.

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DEPARTMENT OF MATHEMATICS
NANJING UNIVERSITY
NANJING 210093, P.R. CHINA
E-mail: yuliang.nju@gmail.com