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ON GROUP DISTRIBUTIVELY GENERATED NEAR-RINGS

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Abstract

The group near-ring constructed from a right near-ring R and a group G is studied in the special case where the near-ring is distributively generated. In particular, results concerning homomorphisms of near-rings or of groups and the augmentation ideal are obtained which resemble closely those obtained for group rings.

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1. Introduction

By defining the group near-ring of a group G over an arbitrary near-ring R as a subnear-ring of $M(R^G)$, generated by certain functions of R^G into itself, Le Riche, Meldrum and van der Walt [2] developed a general theory of group near-rings which coincides with the usual notion when the base near-ring is a ring and so laid the foundation for further development of this subject. In this paper we study the group near-ring constructed in this way in the special case of our near-ring R being distributively generated. We first give alternative proofs of results in [2] concerning homomorphisms of near-rings or of groups which can more readily be restated in the dg case to resemble similar known results in group rings. We also devote some attention to the augmentation ideal which was defined in [2].

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2. Notation

Let G be a (multiplicatively written) group with identity e. R^G denotes the cartesian direct sum of $|G|$ copies of $(R, +)$ indexed by the elements of G. $M(R^G)$ is the right near-ring of all mappings of the group R^G into itself. Denote by $[r, g]$ the function of $M(R^G)$ defined by $([r, g](\mu))(h) = r\mu(hg)$, for all $\mu \in R^G$, $h \in G$. The set $\{[r, g] | r \in R, g \in G\}$ generates a subnear-ring of $M(R^G)$ which in [2] is denoted by $R[G]$, and called the *group near-ring* constructed from R and G.

Since $R[G]$ is a subnear-ring of $M(R^G)$ it follows that for all $A, B \in R[G]$, $\mu \in R^G$, $(A+B)\mu = A\mu + B\mu$ and $(AB)\mu = A(B\mu)$. This makes R^G into an $R[G]$-module. Moreover, R^G is a faithful $R[G]$-module, because $A\mu = 0$ for all $\mu \in R^G$ implies that $A = 0$.

3. Generating sequences

Let S be a non-empty subset of the right near-ring $(N, +, .)$. For an element A of $N(S)$, the subnear-ring of N generated by S, we first describe how A is constructed starting from elements of S. For our purposes this description must be such that it facilitates proofs by induction.

We introduce the notion of a *generating sequence* on S. A generating sequence of length m on S is a sequence $A_1, A_2, \ldots, A_m$ where each A_i for all $1 \leqq i \leqq n$ has either of the following two forms:

(i) A_i is an element of S;
(ii) $A_i = t_i$, where $t_i \in N(S)$ and $t_i = t_k *_i t_l$ with $1 \leqq k, l < i$, $*_i \in \{+, -, .\}$.

For example, let $s_1, s_2, s_3 \in S \subseteq N$ then $A_1, A_2, \ldots, A_9$ is a generating sequence on S where

$$A_1 = s_1, \quad A_2 = s_2, \quad A_3 = s_3, \quad A_4 = s_2^2, \quad A_5 = s_1 s_2^2, \quad A_6 = s_1 s_3,$$
$$A_7 = s_1 s_2^2 - s_1 s_3, \quad A_8 = s_3(s_1 s_2^2 - s_1 s_3), \quad A_9 = s_3(s_1 s_2^2 - s_1 s_3) + s_1.$$

Note that for all i, $4 \leqq i \leqq 9$, $A_i = A_k *_i A_l$ for some $1 \leqq k, l < i$ and $*_i \in \{+, -, .\}$.

Given any $A \in N(S)$ it is possible to construct a generating sequence for A. Let $A_1, \ldots, A_m$ be such a generating sequence on S with $A_m = A$ for some integer $m \geqq 1$. Then we say A is the *result* of the generating sequence. It is now possible to express $N(S)$ in terms of generating sequences in the following way.

PROPOSITION 3.1. *$N(S) = \{A|A$ is the result of some generating sequence on $S\}$.*

If $A_1, \dots, A_n$ is a generating sequence on S then clearly $A_1, \dots, A_i$ for each i, $1 \leqq i \leqq n$ is a generating sequence for $A_i \in N(S)$. Since $A_i = t_k *_i t_l$ with $1 \leqq k$, $l < i$, $*_i \in \{+, -, .\}$ for each i, $1 \leqq i \leqq n$, it follows that $A_i = A_k \pm A_l$ or $A_i = A_k A_l$ with $1 \leqq k, l < i$ or $A_i \in S$. The length of a generating sequence of minimal length for A will be called the *complexity* of A and denoted $c(A)$. Intuitively speaking, the complexity is an indication of how far A is from being an element of S. From the above, since A is the result of some generating sequence on S, it follows that $c(A) = 1$ if and only if $A \in S$ and if $c(A) > 1$ then $A = B + C$ or $A = BC$ where $B, C \in N(S)$ with $c(B)$, $c(C) < c(A)$.

REMARK. For $A, B \in N(S)$ it is possible to construct a generating sequence for $A \pm B$, AB in the following way. If $A_1, A_2, \dots, A_m = A$ and $B_1, B_2, \dots, B_n = B$ are generating sequences for A and B, respectively, then $C_1, C_2, \dots, C_m, C_{m+1}, \dots, C_{m+n+1}$ where

$$C_i = \begin{cases} A_i & \text{for } i = 1, \dots, m \\ B_{i-m} & \text{for } i = m+1, \dots, m+n \end{cases}$$

and $C_{m+n+1} = A * B$ with $* \in \{+, -, .\}$ is a generating sequence for $A * B$ where $* \in \{+, -, .\}$.

Now let N_1 and N_2 be right near-rings with non-empty subsets S_1 and S_2, respectively. Suppose that $\phi: S_1 \to S_2$ is a surjection and let $A_1, \dots, A_m$, $m \geqq 1$, be a generating sequence on S_1. Consider the generating sequence $B_1, B_2, B_3, \dots, B_m$ on S_2 where

$$B_i = \begin{cases} \phi(t_i) & \text{if } A_i = t_i \in S_1 \\ t'_k *'_i t'_l & \text{if } A_i = t_k *_i t_l, \text{ with } 1 \leqq k, l < i \end{cases}$$

with $t'_k *'_i t'_l \in N_2(S_2)$ where t'_k and t'_l are elements of $N_2(S_2)$ obtained from t_k and t_l, respectively, by replacing every occurrence of $s \in S_1$ by $\phi(s)$ and $*'_i$ is the corresponding operation in N_2. This defines a mapping Φ from the set of all generating sequences on S_1 onto the set of all generating sequences on S_2. We are interested in the case where Φ induces a mapping from $N_1(S_1)$ onto $N_2(S_2)$. We collect sufficient conditions for this to be the case in our next two results. We first introduce the following notation. Denote by $T = (A_1, \dots, A_m)$ the generating sequence $A_1, \dots, A_m$, of length m, $m \geqq 1$ on S_1 and by $\Phi(T)$ the generating sequence $T' = (B_1, \dots, B_m)$ on S_2 obtained from T in the way described above. Note that the result of T, denoted $r(T)$, is A_m.

Our next result gives the first of these sufficient conditions.

Theorem 3.2. *Let N_1, N_2 be right near-rings with faithful left modules H_1, H_2, respectively. Suppose $S_i \subseteq N_i$ are such that $N_i = N_i(S_i)$ for $i = 1, 2$. Let $\phi: S_1 \to S_2$ be a surjection. Let $\theta: H_2 \to H_1$ be a (group) monomorphism such that for every generating sequence T_1 on S_1 we have*

$$r(T_1)\theta(h_2) = \theta(r(\Phi(T_1))h_2) \text{ for all } h_2 \in H_2.$$

Then $\widetilde{\Phi}: r(T_1) \to r(\Phi(T_1))$ defines an epimorphism from N_1 onto N_2.

Proof. Suppose $\widetilde{\Phi}$ is not well defined. Then there are generating sequences T_1 and T_1' on S_1 such that $r(T_1) = r(T_1')$ but $r(\Phi(T_1)) \neq r(\Phi(T_1'))$. From T_1 and T_1' we can easily by concatenation and renumbering (as we indicated earlier) construct a generating sequence T for 0 such that $r(\Phi(T)) \neq 0$. Since $r(\Phi(T)) \neq 0$, there is $h_2 \in H_2$ such that $r(\Phi(T))(h_2) \neq 0$, since H_2 is a faithful N_2 module. But then $\theta(r(\Phi(T))h_2) \neq 0$ since θ is a monomorphism and so $r(T)\theta(h_2) \neq 0$ which contradicts $r(T) = 0$.

We have established that the mapping is well defined.

To show that $\widetilde{\Phi}: N_1 \to N_2$ is a homomorphism, let $A, B \in N_1$, then by Proposition 3.1 $A = r(T_1)$ and $B = r(T_1')$ for some generating sequences $T_1 = (A_1, \dots, A_m)$, $T_1' = (B_1, \dots, B_n)$ on S_1. We can find a generating sequence $T = (C_1, C_2, \dots, C_{m+n+1})$ on S_1 such that

$$r(T_1) + r(T_1') + r(T) = C_{m+n+1}$$

where

$$C_i = \begin{cases} A_i \text{ for } i = 1, 2, \dots, m \\ B_{i-m} \text{ for } i = m+1, \dots, m+n \end{cases}$$

$C_{m+n+1} = A_m + B_n$.

Then $\Phi(T) = (C_1', \dots, C_{m+n+1}')$, where for each $i = 1, 2, \dots, m+n+1$, C_i' are the elements of $N_2(S_2)$ obtained from $C_i \in N_1(S_1)$ in the way described earlier. In particular, $C_{m+n+1}' = A_m' + B_n'$, where we use the same symbol without ambiguity for the addition in N_1 and N_2. Therefore

$$\begin{aligned}\widetilde{\Phi}(A+B) &= \widetilde{\Phi}(r(T_1) + r(T_1')) = \widetilde{\Phi}(r(T)) = r(\Phi(T)) \\ &= A_m' + B_n' = r(\Phi(T_1)) + r(\Phi(T_1')) = \widetilde{\Phi}(r(T_1)) + \widetilde{\Phi}(r(T_1')).\end{aligned}$$

We can also find a generating sequence $T = (C_1, \dots, C_{m+n+1})$ on S_1 such that $AB = r(T_1)r(T_1') = r(T)$, where the C_i are defined in the same way as above for $i = 1, 2, \dots, m+n$ and $C_{m+n+1} = A_m B_n$. Define C_i' for $i = 1, 2, \dots, m+n$ as above and $C_{m+n+1}' = A_m' B_n'$, where we use the same symbol without ambiguity for multiplication in N_1 and N_2. Then

$$\begin{aligned}\widetilde{\Phi}(AB) &= \widetilde{\Phi}(r(T_1)r(T_1')) = \widetilde{\Phi}(r(T)) = r(\Phi(T)) \\ &= A_m' B_n' = r(\Phi(T_1))r(\Phi(T_1')) = \widetilde{\Phi}(r(T_1))\widetilde{\Phi}(r(T_1')).\end{aligned}$$

This completes the proof that $\tilde{\Phi}$ is a homomorphism.

To show that $\tilde{\Phi}$ is an epimorphism, take any $n_2 \in N_2$. Then $n_2 = r(T_2)$ where T_2 is a generating sequence on S_2, by Proposition 3.1. But then there is a generating sequence T_1 on S_1 such that $\Phi(T_1) = T_2$ and so

$$\tilde{\Phi}\colon r(T_1) \mapsto r(\Phi(T_1)) = r(T_2).$$

This implies that $\tilde{\Phi}$ is an epimorphism.

The last of these sufficient conditions which will be of interest to us is given next.

THEOREM 3.3. *Let N_1 and N_2 be right near-rings with faithful left modules H_1 and H_2, respectively. Suppose $S_i \subseteq N_i$ are such that $N_i = N_i(S_i)$ for $i = 1, 2$ and let $\bar{\phi}\colon S_1 \to S_2$ be a surjection. Let $\theta\colon H_1 \to H_2$ be a (group) epimorphism such that for every generating sequence T_1 on S_1 we have $\theta(r(T_1)h_1) = r(\Phi(T_1))\theta(h_1)$, for all $h_1 \in H_1$. Then $\tilde{\Phi}\colon r(T_1) \mapsto r(\Phi(T_1))$ is an epimorphism from N_1 onto N_2.*

PROOF. We shall only prove that the mapping is well defined. The remainder of the proof follows in exactly the same way as in Theorem 3.2. Suppose the mapping is not well defined. Then there are generating sequences T_1 and T_1' on S_1 such that $r(T_1) = r(T_1')$ but $r(\Phi(T_1)) \neq r(\Phi(T_1'))$. From T_1 and T_1' we can construct by concatenation and renumbering a generating sequence T for 0, such that $r(\Phi(T)) \neq 0$. Now since $r(\Phi(T)) \neq 0$, there is $h_2 \in H_2$ such that $r(\Phi(T))h_2 \neq 0$ since H_2 is a faithful N_2-module. Since θ is an epimorphism, there exists $h_1 \in H_1$ such that

$$0 \neq r(\Phi(T))\theta(h_1) = \theta(r(T)h_1).$$

This contradicts $r(T) = 0$. So we have established that the mapping is well defined.

4. Applications in group near-rings

It was shown in [2] that every near-ring epimorphism $\phi\colon R \to T$ induces an epimorphism $\tilde{\Phi}\colon R[G] \to T[G]$ on the corresponding group near-rings, where G is an arbitrary (multiplicatively written) group and every group epimorphism $\phi\colon G \to H$ induces an epimorphism $\tilde{\Phi}\colon R[G] \to R[H]$ on the corresponding group near-rings, where R is an arbitrary right near-ring. We show now that the results of the previous section yield alternative proofs for the above mentioned results. This facilitates the statement of similar results in the special case of our near-rings being distributively generated. This will be dealt with in our next section.

Let R be a right near-ring and let G and H be (multiplicatively written) groups. Let $\phi: G \to H$ be an epimorphism. As an application of Theorem 3.2 we want to show that ϕ induces an epimorphism $\widetilde{\Phi}: R[G] \to R[H]$ on the corresponding group near-rings.

Let $S_1 = \{[r, g] | r \in R, g \in G\}$, $S_2 = \{[r, h] | r \in R, h \in H\}$ and let $\overline{\phi}: S_1 \to S_2$ be the surjection defined by $\overline{\phi}: [r, g] \mapsto [r, \phi(g)]$. Let the mapping $\theta: R^H \to R^G$ be defined by $(\theta\mu)(g) = \mu(\phi(g))$, for all $\mu \in R^H$, $g \in G$. For any $\mu_1, \mu_2 \in R^H$,

$$\begin{aligned}(\theta(\mu_1 + \mu_2))(g) &= (\mu_1 + \mu_2)(\phi(g)) = \mu_1(\phi(g)) + \mu_2(\phi(g)) \\ &= (\theta\mu_1)(g) + (\theta\mu_2)(g) = (\theta\mu_1 + \theta\mu_2)(g),\end{aligned}$$

for all $g \in G$, and therefore,

$$\theta(\mu_1 + \mu_2) = \theta\mu_1 + \theta\mu_2.$$

This proves that θ is a homomorphism.

To show that θ is injective, let $\mu_1, \mu_2 \in R^H$ be such that $\theta\mu_1 = \theta\mu_2$. Then $(\theta\mu_1)(g) = (\theta\mu_2)(g)$, for all $g \in G$. Therefore, $\mu_1(\phi(g)) = \mu_2(\phi(g))$ and so $\mu_1 = \mu_2$ since ϕ is an epimorphism.

It only remains to show that $r(T)(\theta\mu) = \theta(r(\Phi(T))\mu)$ for all generating sequences T on S_1, $\mu \in R^H$. We do this by induction on the length of $r(T)$. If the length of $r(T)$ is one, then $r(T) = [r, g]$ for some $r \in R, g \in G$, and therefore $r(\Phi(T)) = [r, \phi(g)]$.

Now we have

$$r(T)(\theta\mu)(h) = [r, g](\theta\mu)(h) = r(\theta\mu)(hg) = r\mu(\phi(hg)),$$

for all $h \in G$. On the other hand,

$$\begin{aligned}\theta(r(\Phi(T))\mu)(h) &= \theta([r, \phi(g)]\mu)(h) \\ &= ([r, \phi(g)]\mu)(\phi(h)) = r\mu(\phi(h)\phi(g)) = r\mu(\phi(hg)),\end{aligned}$$

for all $h \in G$. So in this case we have $r(T)(\theta\mu) = \theta(r(\Phi(T))\mu)$, for all generating sequences T on S_1 of length one and for all $\mu \in R^H$.

For any generating sequence T on S_1, $r(T)$ is either of the form $r(T) = r(T_1) + r(T_1')$ or $r(T) = r(T_1)r(T_1')$ for some generating sequences T_1, T_1' on S_1 of shorter length. We assume the result is true for T_1 and T_1'. We consider the two cases separately.

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(1) $r(T) = r(T_1) + r(T_1')$. Then

$$\begin{aligned} r(T)(\theta\mu) &= (r(T_1) + r(T_1'))(\theta\mu) \\ &= r(T_1)(\theta\mu) + r(T_1')(\theta\mu) = \theta(r(\Phi(T_1))\mu) + \theta(r(\Phi(T_1'))\mu) \\ &= \theta((r(\Phi(T_1)) + r(\Phi(T_1')))\mu) = \theta(r(\Phi(T))\mu), \end{aligned}$$

for all $\mu \in R^G$.
(2) $r(T) = r(T_1)r(T_1')$. Then

$$\begin{aligned} (r(T_1)r(T_1'))(\theta\mu) &= r(T_1)(r(T_1')(\theta\mu)) = r(T_1)(\theta(r(\Phi(T_1'))\mu)) \\ &= \theta(r(\Phi(T_1))(r(\Phi(T_1'))\mu)) = \theta((r(\Phi(T_1))r(\Phi(T_1')))\mu) \\ &= \theta(r(\Phi(T))\mu). \end{aligned}$$

This completes the proof by induction. We have proved

THEOREM 4.1. *Let G and H be groups and let $\phi: G \to H$ be an epimorphism, then ϕ induces an epimorphism $\widetilde{\Phi}: R[G] \to R[H]$ of the corresponding group near-rings. Moreover, $R[G]/Ann_{R[G]}(\mathrm{Im}\theta) \simeq R[H]$ where*

$$\mathrm{Im}\theta = \{\mu \in R^G | \mu(g) = \beta(\phi(g)) \text{ for some } \beta \in R^H, \forall g \in G\}.$$

Observe that for any generating sequence T on $\{[r, g] | r \in R, g \in G\}$, $r(T) \in \ker\widetilde{\Phi}$ if and only if $r(\Phi(T)) = 0$ if and only if $r(\Phi(T))\mu = 0$ if and only if $\theta(r(\Phi(T))\mu) = 0$ if and only if $r(T)(\theta\mu) = 0$, for all $\mu \in R^H$. It therefore follows that $\ker\widetilde{\Phi} = \mathrm{Ann}_{R[G]}(\mathrm{Im}\theta)$, where

$$\begin{aligned} \mathrm{Im}\theta &= \{\mu \in R^G | \mu = \theta(\beta) \text{ for some } \beta \in R^H\} \\ &= \{\mu \in R^G | \mu(g) = (\theta\beta)(g) \text{ for some } \beta \in R^H, \text{ for all } g \in G\} \\ &= \{\mu \in R^G | \mu(g) = \beta(\phi(g)) \text{ for some } \beta \in R^H, \text{ for all } g \in G\}. \end{aligned}$$

The last part of Theorem 4.1 therefore follows from the fundamental homomorphism theorem.

COROLLARY 4.2. *Let H be a normal subgroup of G. Then*

$$R[G]/\mathrm{Ann}_{R[G]}R_H^G \simeq R[G/H]$$

where

$$R_H^G = \mathrm{Im}\theta = \{\mu \in R^G | Hx = Hy \text{ imply } \mu(x) = \mu(y) \text{ for all } x, y \in G\}$$

and where $\theta: R^{G/H} \to R^G$ is defined by

$$(\theta\overline{\mu})(g) = \overline{\mu}(Hg) \text{ for all } g \in G, \overline{\mu} \in R^{G/H}.$$

Let $N_1 = R$, $N_2 = T$ be right near-rings and let $\phi\colon R \to T$ be a near-ring epimorphism. As an application of Theorem 3.3 we wish to show that ϕ induces an epimorphism $\tilde{\Phi}\colon R[G] \to T[G]$, where G is a (multiplicatively written) group. Let $S_1 = \{[r, g] | r \in R, g \in G\}$, $S_2 = \{[t, g] | t \in T, g \in G\}$ and let $\bar{\phi}\colon S_1 \to S_2$ be the surjection defined by $\bar{\phi}\colon [r, g] \mapsto [\phi(r), g]$. Define a mapping $\theta\colon R^G \to T^G$ by $(\theta\mu)(g) = \phi(\mu(g))$ for all $\mu \in R^G$, $g \in G$. Let $\mu_1, \mu_2 \in R^G$, then for all $g \in G$,

$$\begin{aligned}(\theta(\mu_1 + \mu_2))(g) &= \phi((\mu_1 + \mu_2)(g)) = \phi(\mu_1(g) + \mu_2(g)) \\ &= \phi(\mu_1(g)) + \phi(\mu_2(g)) = (\theta\mu_1)(g) + (\theta\mu_2)(g) \\ &= (\theta\mu_1 + \theta\mu_2)(g), \quad \text{for all } g \in G.\end{aligned}$$

Hence $\theta(\mu_1 + \mu_2) = \theta\mu_1 + \theta\mu_2$ and so θ is a homomorphism. To show that θ is an epimorphism, let $\beta \in T^G$ and define $\mu \in R^G$ by $\mu(g) = \tilde{\phi}(\beta(g))$, where $\tilde{\phi}\colon T \to R$ is such that $\phi\tilde{\phi}(t) = t$ for all $t \in T$. Then

$$(\theta\mu)(g) = \phi(\mu(g)) = \phi(\tilde{\phi}(\beta(g))) = \beta(g), \quad \text{for all } g \in G.$$

Hence $\theta\mu = \beta$.

It only remains to show that for any generating sequence T_1 on S_1,

$$\theta(r(T_1)\mu) = r(\Phi(T_1))(\theta\mu),$$

for all $\mu \in R^G$. We do this by induction on the length of $r(T_1)$. If $r(T_1)$ is of length one, then $r(T_1) = [r, g]$ for some $r \in R$, $g \in G$. Then $r(\Phi(T_1)) = [\phi(r), g]$. Now we have

$$\theta([r, g]\mu)(h) = \phi(([r, g]\mu)(h)) = \phi(r\mu(hg)) = \phi(r)\phi(\mu(hg)),$$

for all $h \in G$. On the other hand,

$$r(\Phi(T_1))(\theta\mu)(h) = [\phi(r), g](\theta\mu)(h) = \phi(r)(\theta\mu)(hg) = \phi(r)\phi(\mu(hg)),$$

for all $h \in G$. So in this case we have $\theta(r(T_1)\mu) = r(\Phi(T_1))(\theta\mu)$, for all $\mu \in R^G$ and for all generating sequences of length 1.

For any T on S_1, $r(T)$ is either of the form $r(T) = r(T_1) + r(T_1')$ or $r(T) = r(T_1)r(T_1')$ for some generating sequences T_1 and T_1' on S_1. We assume the result is true for T_1 and T_1'. We consider the two cases separately.
(1) $r(T) = r(T_1) + r(T_1')$. Then

$$\begin{aligned}\theta(r(T)\mu) &= \theta((r(T_1) + r(T_1'))\mu) = \theta(r(T_1)\mu + r(T_1')\mu) \\ &= \theta(r(T_1)\mu) + \theta(r(T_1')\mu) = r(\Phi(T_1))(\theta\mu) + r(\Phi(T_1'))(\theta\mu) \\ &= (r(\Phi(T_1)) + r(\Phi(T_1')))(\theta\mu) = r(\Phi(T))(\theta\mu), \quad \text{for all } \mu \in R^G.\end{aligned}$$

(2) $r(T) = r(T_1)r(T_1')$. Then

$$\begin{aligned}\theta(r(T)\mu) &= \theta((r(T_1)r(T_1'))\mu) = \theta(r(T_1)(r(T_1')\mu)) = r(\Phi(T_1))\theta(r(T_1')\mu)\\ &= r(\Phi(T_1))(r(\Phi(T_1'))(\theta\mu)) = (r(\Phi(T_1))r(\Phi(T_1')))(\theta\mu)\\ &= r(\Phi(T))(\theta\mu).\end{aligned}$$

This completes the proof by induction. We have thus proved

THEOREM 4.3. *Let R and T be near-rings and G a (multiplicatively written) group and let $\phi\colon R \to T$ be a near-ring epimorphism, then ϕ induces an epimorphism $\tilde{\Phi}\colon R[G] \to T[G]$ of the corresponding group near-rings.*

Denote the kernel of ϕ by A. Then $\mu \in \ker\theta$ if and only if $\theta\mu = 0$ if and only if $(\theta\mu)(g) = 0$ if and only if $\phi(\mu(g)) = 0$ if and only if $\mu(g) \in A$, for all $g \in G$, if and only if $\mu \in A^G$. Therefore, $\ker\theta = A^G$.

COROLLARY 4.4. *Let R and T be near-rings, G a group and $\phi\colon R \to T$ an epimorphism of near-rings. Then $R[G]/A^* \simeq T[G]$, where $A^* = (A^G : R^G) := \{B| \in R[G] | B\mu \in A^G$ for all $\mu \in R^G\}$ and $A = \ker\phi$.*

5. The group distributively generated near-ring $(R[G], S[G])$

Recall that a distributively generated near-ring (hereafter written as dg near-ring) is a near-ring R such that $(R, +)$ is generated as an additive group by the subset S, which need not be the set of all distributive elements of R. We denote a dg near-ring by (R, S). We state as the first result of this section the following theorem, the proof of which can be found in [1].

THEOREM 5.1. *If (R, S) is a dg near-ring then*

$$R[G] = \left\{\sum_{i=1}^{m} \sigma_i[s_i, g_i] | m \in \mathbf{N}, \sigma_i = \pm 1, s_i \in S, g_i \in G\right\}.$$

If (R, S) is a dg near-ring we can now, in view of Theorem 5.1, redefine the complexity of $A \in R[G]$, which we also denote by $c(A)$, as the smallest natural number m such that

$$A = \sum_{i=1}^{m} \sigma_i[s_i, g_i]$$

for some $s_i \in S$, $g_i \in G$ and $\sigma_i = \pm 1$. It is now obvious that if (R, S) is a dg near-ring and $A \in R[G]$ with $c(A) \geqq 2$, then $A = A_1 + A_2$ with

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$c(A_1), c(A_2) < c(A)$ and $A_1, A_2 \in R[G]$. We shall henceforth always write $(R[G], S[G])$ for the group dg near-ring $R[G]$ if (R, S) is dg.

We would like to restate the results in section 4 in the special case of our near-rings being distributively generated in a way which will resemble similar results in group rings.

Let (R, S) be a dg near-ring and let $\phi: G \to H$ be an epimorphism of groups. Consider the dg near-rings $(R[G], S[G])$ and $(R[H], S[H])$ having generating sets of distributive elements $S_1 = \{[s, g] | s \in S, g \in G\}$ and $S_2 = \{[s, h] | s \in S, h \in H\}$, respectively. Let $\overline{\phi}$ and θ be the mappings defined in the proof of Theorem 4.1.

It can now easily be shown that

$$\left(\left(\sum_{i=1}^{n} \sigma_i[s_i, g_i]\right)(\theta\mu)\right)(g) = \theta\left(\left(\sum_{i=1}^{n} \sigma_i[s_i, \phi(g_i)]\right)\mu\right)(g)$$

for all $g \in G$, $\mu \in R^H$.

By Theorem 3.2 it now follows that the mapping

$$\Phi: \sum_{i=1}^{n} \sigma_i[s_i, g_i] \mapsto \sum_{i=1}^{n} \sigma_i[s_i, \phi(g_i)]$$

defines an epimorphism from $(R[G], S[G])$ onto $(R[H], S[H])$.

We state this result as follows.

THEOREM 5.2. *Let (R, S) be a dg near-ring and let $\phi: G \to H$ be an epimorphism of groups. Then $\Phi: (R[G], S[G]) \to (R[H], S[H])$ defined by*

$$\Phi\left(\sum_{i=1}^{n} \sigma_i[s_i, g_i]\right) = \sum_{i=1}^{n} \sigma_i[s_i, \phi(g_i)]$$

is an epimorphism of dg group near-rings.

Let (R, S) and (T, U) be dg near-rings and let $\phi: R \to T$ be a near-ring epimorphism. Consider the dg near-rings $(R[G], S[G])$ and $(T[G], U[G])$ with generating sets of distributive elements $S_1 = \{[s, g] | s \in S, g \in G\}$ and $S_2 = \{[u, g] | u \in U, g \in G\}$.

Let $\overline{\phi}$ and θ be the mappings defined in the proof of Theorem 4.3. It is easily shown that

$$\theta\left(\left(\sum_{i=1}^{n} \sigma_i[s_i, g_i]\right)\mu\right)(g) = \left(\left(\sum_{i=1}^{n} \sigma_i[\phi(s_i), g_i]\right)(\theta\mu)\right)(g),$$

for all $g \in G$, $\mu \in R^G$. Thus by Theorem 3.3 the mapping

$$\Phi: \sum_{i=1}^{n} \sigma_i[s_i, g_i] \mapsto \sum_{i=1}^{n} \sigma_i[\phi(s_i), g_i]$$

is an epimorphism from $(R[G], S[G])$ onto $(T[G], U[G])$.

We have thus proved

THEOREM 5.3. *Let (R, S) and (T, U) be two dg near-rings and G a group. Let $\phi: R \to T$ be a near-ring epimorphism, then the mapping $\Phi: (R[G], S[G]) \to (T[G], U[G])$ defined by*

$$\Phi: \sum_{i=1}^{n} \sigma_i[s_i, g_i] \mapsto \sum_{i=1}^{n} \sigma_i[\phi(s_i), g_i]$$

is a group dg near-ring epimorphism.

We can combine the results of Theorems 5.2 and 5.3 into a single result.

THEOREM 5.4. *Let (R, S) and (T, U) be two dg near-rings and let G and H be groups. Let $\phi: G \to H$ be an epimorphism of groups and let $\theta: R \to T$ be an epimorphism of near-rings. Then there is an epimorphism $\Phi: (R[G], S[G] \to (T[H], U[H])$ defined by*

$$\Phi\left(\sum_{i=1}^{n} \sigma_i[s_i, g_i]\right) = \sum_{i=1}^{n} \sigma_i[\theta(s_i), \phi(g_i)].$$

PROOF. By Theorem 5.2 the mapping $\phi^*: (R[G], S[G]) \to (R[H], S[H])$ defined by $\phi^*(\sum_{i=1}^{n} \sigma_i[s_i, g_i]) = \sum_{i=1}^{n} \sigma_i[s_i, \phi(g_i)]$ is an epimorphism.

By Theorem 5.3 the mapping $\theta^*: (R[H], S[H]) \to (T[H], U[H])$ defined by

$$\theta^*\left(\sum_{i=1}^{n} \sigma_i[s_i, h_i]\right) = \sum_{i=1}^{n} \sigma_i[\theta(s_i), h_i]$$

is an epimorphism. Then $\Phi = \theta^*\phi^*$ is the required epimorphism.

Before we leave this section we turn our attention to the ideals I^+ and I^* of the group near-ring. Recall that for an ideal I of the right near-ring R, the ideals $I^* = (I^G: R^G) = \{A \in R[G] | (A\mu)(g) \in I \text{ for all } g \in G, \mu \in R^G\}$ and $I^+ = \mathrm{id}\{[a, e] | a \in I\}$, the ideal of $R[G]$ generated by the subset $\{[a, e] | a \in I\}$, were defined in [2]. It was shown there that $I^+ \subseteq I^*$. Theorem 5.1 facilitates proving that if R is a dg near-ring and I is an ideal of R such that R is distributive over I, then $R[G]$ is distributive over I^*. We require the following preliminary results.

LEMMA 5.5. *Let R be a near-ring, I an ideal of R. If R is distributive over I, then*

$$[r, g]\mu_1 + [s, g']\mu_2 = [s, g']\mu_2 + [r, g]\mu_1,$$

where $r, s \in R$, $g, g' \in G$, $\mu_1, \mu_2 \in I^G$.

PROOF. Let $r, s \in R$, $x, y \in I$, then

$$(r+s)(x+y) = r(x+y) + s(x+y) = rx + ry + sx + sy.$$

Also since R is distributive over I,

$$(r+s)(x+y) = (r+s)x + (r+s)y = rx + sx + ry + sy.$$

Hence $ry + sx = sx + ry$. For every $h \in G$,

$$\begin{aligned}([r, g]\mu_1 + [s, g']\mu_2)(h) &= [r, g]\mu_1(h) + [s, g']\mu_2(h)\\ &= r\mu_1(hg) + s\mu_2(hg') = s\mu_2(hg') + r\mu_1(hg)\\ &= [s, g']\mu_2(h) + [r, g]\mu_1(h)\\ &= ([s, g']\mu_2 + [r, g]\mu_1)(h),\end{aligned}$$

for all $h \in G$. Hence the result follows.

LEMMA 5.6. *Let R be a dg near-ring and I an ideal of R. If R is distributive over I, then $A\mu_1 + B\mu_2 = B\mu_2 + A\mu_1$, where $A, B \in R[G]$, $\mu_1, \mu_2 \in I^G$.*

PROOF. By induction on $c(A)$ and $c(B)$. If $c(A) = c(B) = 1$ then $A = [r, g]$ and $B = [s, g']$ for some $r, s \in R$, $g, g' \in G$. By Lemma 5.5, the result is true in this case. Let $m, n \in \mathbf{N}$, $m, n \geqq 2$ and $c(A) = m$, $c(B) = n$. Then $A = A_1 + A_2$ and $B = B_1 + B_2$ where $c(A), c(A_2) < m$ and $c(B_1), c(B_2) < n$. Assume that for all A, B with $c(A) < m$, $c(B) < n$, $A\mu_1 + B\mu_2 = B\mu_2 + A\mu_1$.

Now we have

$$\begin{aligned}A\mu_1 + B\mu_2 &= (A_1 + A_2)\mu_1 + (B_1 + B_2)\mu_2\\ &= A_1\mu_1 + A_2\mu_1 + B_1\mu_2 + B_2\mu_2 = A_1\mu_1 + B_1\mu_2 + A_2\mu_1 + B_2\mu_2\\ &= B_1\mu_2 + A_1\mu_1 + B_2\mu_2 + A_2\mu_1\\ &= B_1\mu_2 + B_2\mu_2 + A_1\mu_1 + A_2\mu_1\\ &= (B_1 + B_2)\mu_2 + (A_1 + A_2)\mu_1 = B\mu_2 + A\mu_1,\end{aligned}$$

by the induction hypothesis. It remains to consider the case when $c(A) = 1$ and $c(B) > 1$ or $c(A) > 1$ and $c(B) = 1$. If $c(A) = 1$ and $c(B) > 1$ then $A = [s, g]$ for some $s \in S$, $g \in G$ and $B = B_1 + B_2$ where $c(B_1), c(B_2) <$

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$c(B)$. The required result follows immediately from our induction hypothesis that

$$[s, g]\mu_1 + C\mu_2 = C\mu_2 + [s, g]\mu_1$$

for all C with $c(C) < c(B)$. The case $c(A) > 1$ and $c(B) = 1$ follows similarly. This completes the proof.

LEMMA 5.7. *If R is distributive over I, then*

$$[r, g](\mu_1 + \mu_2) = [r, g]\mu_1 + [r, g]\mu_2,$$

where $r \in R$, $g \in G$, $\mu_1, \mu_2 \in I^G$.

PROOF. For all $h \in G$,

$$\begin{aligned}[r, g](\mu_1 + \mu_2)(h) &= r((\mu_1 + \mu_2)(hg)) \\ &= r(\mu_1(hg) + \mu_2(hg)) = r\mu_1(hg) + r\mu_2(hg) \\ &= [r, g]\mu_1(h) + [r, g]\mu_2(h) = ([r, g]\mu_1 + [r, g]\mu_2)(h).\end{aligned}$$

Hence the result follows.

LEMMA 5.8. *If R is dg and distributive over I, then $R[G]$ is distributive over I^G.*

PROOF. We must show that for all $A \in R[G]$, $\mu_1, \mu_2 \in I^G$, $A(\mu_1 + \mu_2) = A\mu_1 + A\mu_2$. We do this by induction on $c(A)$. If $c(A) = 1$, then $A = [r, g]$ for some $r \in R$, $g \in G$. By Lemma 5.7 the result is true in this case. Let $m \in \mathbf{N}$, $m \geqq 2$ and let $c(A) = m$. Then $A = A_1 + A_2$ where $c(A_1), c(A_2) < m$. We assume for all $B \in R[G]$ with $c(B) < m$, $B(\mu_1 + \mu_2) = B\mu_1 + B\mu_2$ for all $\mu_1, \mu_2 \in I^G$. Now

$$\begin{aligned}A(\mu_1 + \mu_2) &= (A_1 + A_2)(\mu_1 + \mu_2) = A_1(\mu_1 + \mu_2) + A_2(\mu_1 + \mu_2) \\ &= A_1\mu_1 + A_1\mu_2 + A_2\mu_1 + A_2\mu_2 = A_1\mu_1 + A_2\mu_1 + A_1\mu_2 + A_2\mu_2 \\ &= (A_1 + A_2)\mu_1 + (A_1 + A_2)\mu_2 = A\mu_1 + A\mu_2,\end{aligned}$$

the third last step being a result of Lemma 5.6.

We can now prove the result which we mentioned earlier.

THEOREM 5.9. *If R is dg and distributive over I, then $R[G]$ is distributive over I^*.*

PROOF. We must show that $A(B + C) = AB + AC$ for all $A \in R[G]$, $B, C \in I^*$. For all $\mu \in R^G$,

$$(A(B + C))\mu = A(B\mu + C\mu) = A(B\mu) + A(C\mu),$$

by Lemma 5.8 since $B\mu$, $C\mu \in I^G$. Hence

$$(A(B+C))\mu = A(B\mu) + A(C\mu) = (AB)\mu + (AC)\mu = (AB+AC)\mu,$$

for all $\mu \in R^G$. Hence the result follows.

An immediate consequence of this result is

COROLLARY 5.10. *If R is dg and distributive over an ideal I of R, then $R[G]$ is distributive over I^+.*

PROOF. This follows immediately from Theorem 5.9 and the fact that $I^+ \subseteq I^*$.

6. The augmentation ideal $\Delta = \Delta(R[G], S[G])$

It was shown in [2] that the augmentation ideal of the group near-ring $R[G]$ is generated as an ideal by the set $\{[1, g]-[1, e] | g \in G\}$. If (R, S) is a dg near-ring we would like to give a set of generators for $\Delta = \Delta(R[G], S[G])$ as a normal subgroup of $(R[G], +)$. In order to do this we make use of the following result, the proof of which can be found in [4, Lemma 13.10].

LEMMA 6.1. *Let (R, S) be a dg near-ring and let $X \subseteq R$. Then the ideal of (R, S) generated by X is the normal subgroup of $(R, +)$ generated by*

$$SXR := \{sxr, sx, xr, x | x \in X, r \in R, s \in S\}.$$

We now restate Lemma 6.1 in the context of the group dg near-ring $(R[G], S[G])$.

LEMMA 6.2. *Let (R, S) be a dg near-ring and let $X \subseteq R[G]$. Then the ideal of $(R[G], S[G])$ generated by X is the normal subgroup of $(R[G], +)$ generated by*

$$S[G]XR[G] := \{[s, h]xr', [s, h]x, xr', x | x \in X, r' \in R[G], s \in S, h \in G\}.$$

If we take $X = \{[1, g]-[1, e] | g \in G\} \subseteq R[G]$ in Lemma 6.2 and remembering that our near-rings have identity, we get the following result.

THEOREM 6.3. *Let (R, S) be a dg near-ring. Then the augmentation ideal of $(R[G], S[G])$ is the normal subgroup of $(R[G], +)$ generated by the set $\{[s, g]-[s, e] \mid s \in S, g \in G\}$.*

PROOF. By Lemma 6.2, $\Delta = \Delta(R[G], S[G])$ is the normal subgroup of $(R[G], +)$ generated by the set

$$\{[s, h]([1, g] - [1, e])r' | r' \in R[G], s \in S, g, h \in G\},$$

since (R, S) has an identity. We show that the elements in this set are a sum of conjugates of elements of the form $[s, e]([1, g] - [1, e])$ and their inverses. Consider the element of the form $[s, h]([1, g] - [1, e])r'$, where $r' \in R[G]$, $s \in S$, $g, h \in G$. Let $r' = \sum_{i=1}^{m} \sigma_i[s_i, g_i]$. Then

$$\begin{aligned}
[s, h]([1, g] - [1, e]) \sum_{i=1}^{m} \sigma_i[s_i, g_i] &= ([s, hg] - [s, h]) \sum_{i=1}^{m} \sigma_i[s_i, g_i] \\
&= [s, hg] \sum_{i=1}^{m} \sigma_i[s_i, g_i] - [s, h] \sum_{i=1}^{m} \sigma_i[s_i, g_i] \\
&= \sum_{i=1}^{m} \sigma_i[ss_i, hgg_i] - \sum_{i=1}^{m} \sigma_i[ss_i, hg_i],
\end{aligned}$$

by the fact that $[s, hg]$ and $[s, h]$ are distributive. We then have

$$\begin{aligned}
&\sum_{i=1}^{m} \sigma_i[ss_i, hgg_i] - \sum_{i=1}^{m} \sigma_i[ss_i, hg_i] \\
&\quad = \sum_{i=1}^{m-1} \sigma_i[ss_i, hgg_i] + \sigma_m[ss_m, hgg_m] - \sigma_m[ss_m, hg_m] - \sum_{i=1}^{m-1} \sigma_i[ss_i, hg_i].
\end{aligned}$$

Consider the element $\sigma_m[ss_m, hgg_m] - \sigma_m[ss_m, hg_m]$. If $\sigma_m = +1$, then this element can be written in the form

$$[ss_m, e]([1, hgg_m] - [1, e]) - [ss_m, e]([1, hg_m] - [1, e]).$$

If $\sigma_m = -1$ then it can be written in the form

$$\begin{aligned}
&-[ss_m, hg_m] + [ss_m, e]([1, hg_m] - [1, e]) \\
&\qquad - [ss_m, e]([1, hgg_m] - [1, e]) + [ss_m, hg_m].
\end{aligned}$$

It therefore follows that if

$$\begin{aligned}
\sigma_m = +1, &\ [s, h]([1, g] - [1, e])r' \\
&= \sum_{i=1}^{m-1} \sigma_i[ss_i, hgg_i] - \sum_{i=1}^{m-1} \sigma_i[ss_i, hg_i] + y + [ss_m, e]([1, hgg_m] - [1, e]) \\
&\quad - y + y - [ss_m, e]([1, hg_m] - [1, e]) - y,
\end{aligned}$$

where $y = \sum_{i=1}^{m-1} \sigma_i[ss_i, hg_i]$ and if

$$\begin{aligned}\sigma_m &= -1, [s, h]([1, g] - [1, e])r' \\ &= \sum_{i=1}^{m-1} \sigma_i[ss_i, hgg_i] - \sum_{i=1}^{m-1} \sigma_i[ss_i, hg_i] \\ &\quad + y + [ss_m, e]([1, hg_m] - [1, e]) - y \\ &\quad + y - [ss_m, e]([1, hgg_m] - [1, e]) - y,\end{aligned}$$

where $y = \sum_{i=1}^{m-1} \sigma_i[ss_i, hg_i] - [ss_m, hg_m]$. It then follows by an induction argument that $[s, h]([1, g] - [1, e])r'$ is a sum of conjugates of elements of the form $[s, e]([1, g] - [1, e])$ and their inverses. This completes the proof.

The next result is an element wise characterization of $\Delta = \Delta(R[G], S[G])$ analogous to the one in the ring-theoretic case.

THEOREM 6.4. $\Delta(R[G], S[G]) = \{\sum_{i=1}^{m} \sigma_i[s_i, g_i] \| \sum_{i=1}^{m} \sigma_i s_i = 0\}$.

PROOF. Let $T = \{\sum_{i=1}^{m} \sigma_i[s_i, g_i] \| \sum_{i=1}^{m} \sigma_i s_i = 0\}$. It is clear that $\sum_{i=1}^{m} \sigma_i s_i = 0$ if and only if $\sum_{i=1}^{m} \sigma_i[s_i, e] = 0$. By Theorem 6.3 $\Delta(R[G], S[G])$ is the normal subgroup of $(R[G], +)$ generated by the set $\{[s, g] - [s, e] \| s \in S, g \in G\}$. It is easy to see that sums and conjugates of elements of this type lie in T. Hence the augmentation ideal is a subset of T. To show that T is a subset of the augmentation ideal, let $X = \sum_{i=1}^{m} \sigma_i[s_i, g_i]$ be an element in T. Then

$$\begin{aligned}\sum_{i=1}^{m} \sigma_i[s_i, g_i] &= \sum_{i=1}^{m} \sigma_i[s_i, g_i] - \sum_{i=1}^{m} \sigma_i[s_i, e] \\ &= \sum_{i=1}^{m-1} \sigma_i[s_i, g_i] + \sigma_m[s_m, g_m] - \sigma_m[s_m, e] - \sum_{i=1}^{m-1} \sigma_i[s_i, e].\end{aligned}$$

So if $\sigma_m = +1$, then

$$X = \sum_{i=1}^{m-1} \sigma_i[s_i, g_i] - \sum_{i=1}^{m-1} \sigma_i[s_i, e] + y + [s_m, e]([1, g_m] - [1, e]) - y$$

where $y = \sum_{i=1}^{m-1} \sigma_i[s_i, e]$. If $\sigma_m = -1$ then

$$X = \sum_{i=1}^{m-1} \sigma_i[s_i, g_i] - \sum_{i=1}^{m-1} \sigma_i[s_i, e] + y - [s_m, e]([1, g_m] - [1, e]) - y,$$

where $y = \sum_{i=1}^{m-1} \sigma_i[s_i, g_i] - [s_m, e]$. It now follows by an induction argument that X is a sum of conjugates of elements of the form $[s, e]([1, g] -$

$[1, e])$ and their inverses and hence X is an element of $\Delta(R[G], S[G])$. This completes the proof.

We conclude this section with a few remarks concerning the group dg near-ring $(R[G], S[G])$. From Theorem 6.4 it is immediate that the augmentation ideal $\Delta(R[G], S[G])$ is the kernel of the dg near-ring homomorphism $\phi: (R[G], S[G]) \to (R, S)$ defined by

$$\phi\left(\sum_{i=1}^{n} \sigma_i[s_i, g_i]\right) = \sum_{i=1}^{n} \sigma_i s_i.$$

It is shown in [1] and [2] that $(R[G], S[G])$ is an epimorphic image of the group dg near-ring $(R(G), SG)$ defined by Meldrum [3]. Many of the results obtained by Meldrum [3] carry over to the group dg near-ring $(R[G], S[G])$, some of which are immediate consequences of the epimorphism existing between the two group dg near-rings.

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