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Fixed points of multivalued maps under local Lipschitz conditions and applications

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In this work we are concerned with the existence of fixed points for multivalued maps defined on Banach spaces. Using the Banach spaces scale concept, we establish the existence of a fixed point of a multivalued map in a vector subspace where the map is only locally Lipschitz continuous. We apply our results to the existence of mild solutions and asymptotically almost periodic solutions of an abstract Cauchy problem governed by a first-order differential inclusion. Our results are obtained by using fixed point theory for the measure of noncompactness.

Keywords: fixed points; set-valued maps; condensing mappings; k-set contractions; differential inclusions; mild solutions; asymptotically almost periodic functions

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Secondary 47H08; 34A60; 34C27

1. Introduction

In this paper we are concerned with the existence of fixed points of multivalued maps defined on Banach spaces. We apply our results to establish the existence of asymptotically almost periodic mild solutions for a class of abstract Cauchy problem governed by a first-order differential inclusion.

To describe the problem, throughout this work we denote by X a Banach space provided with a norm $\|\cdot\|$. We assume that $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $(T(t))_{t\geqslant 0}$ on X. We will consider the abstract first-order differential inclusion

$$x'(t) - Ax(t) \in f(t, x(t)), \quad t \geqslant 0, \tag{1.1}$$

$$x(0) = x_0 \in X, \tag{1.2}$$

where $x(t) \in X$ and f is a set valued map defined on $[0, \infty) \times X$ whose properties will be specified later.

As a model, we consider a general heat equation described by a first-order differential inclusion

$$\frac{\partial u(t,\xi)}{\partial t} - \frac{\partial^2 u(t,\xi)}{\partial \xi^2} \in f(t, u(t,\cdot)), \tag{1.3}$$

$$u(t,0) = u(t,\pi) = 0, \tag{1.4}$$

$$u(0,\xi) = \varphi(\xi), \tag{1.5}$$

for $t \geqslant 0$ and $\xi \in (0,\pi)$. In this system we assume that f is a multivalued map, and the inclusion indicated in (1.3) will be explained in § 4. Moreover, φ is an appropriate function.

Here we briefly discuss the context in which our work is inserted. We do not intend to make an exhaustive list of references but just mention those most recent and directly related to the topic of this paper. Differential inclusions are used to describe many phenomena arising from different fields as physics, chemistry, population dynamics, etc. For this reason, last years several researchers have studied various aspects of the theory. We mention here [**1**, **2**, **6**, **8**, **12**, **13**, **20**, **21**, **24**, **26**, **29**] and references in these works for the motivations of the theory.

In this paper, we establish a general result of fixed point in scales of Banach spaces, and we combine this result with the theory of measure of noncompactness to establish the existence of solutions to the problem (1.1)–(1.2). It is important to mention here that several authors have studied strongly nonlinear problems, that is, problems of type (1.1)–(1.2) in which A is a m-dissipative operator. The reader can see [**28**, **30**–**32**] and the references in these works. However, between those works and ours there are important differences that we will describe in § 4.

This paper has four sections. In § 2 we develop some properties about the measure of noncompactness, and multivalued analysis which are needed to establish our results. In § 3 we discuss the existence of fixed points. Finally, in § 4 we apply our results to establish the existence of solutions and the existence of asymptotically almost periodic solutions to problem (1.1)–(1.2).

2. Preliminaries

In this section we will present the basic concepts and properties of the abstract Cauchy problem and the theory of multivalued functions on which this work is based. The terminology and notations that will be used throughout the text are those generally used in functional analysis. In particular, if $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are Banach spaces, we denote by $\mathcal{L}(Y,Z)$ the Banach space of bounded linear operators from Y into Z and, we abbreviate this notation to $\mathcal{L}(Y)$ whenever $Z = Y$. If $y \in Y$, then $B_r(y,Y)$ denotes the closed ball with centre at y and radius $r > 0$. When the space Y is clear from the context, we abbreviate this notation to $B_r(y)$. Moreover, for a compact interval $J \subseteq \mathbb{R}$, we denote by $C(J;Y)$ the space of continuous functions from J into Y endowed with the norm of uniform convergence. Similarly, $C_b([0,\infty);Y)$ is the space of bounded continuous functions from $[0,\infty)$ into Y provided with the norm of uniform convergence and $C_0([0,\infty);Y)$ is the subspace of $C_b([0,\infty);Y)$ consisting of functions that vanishes at infinite. Furthermore,

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$L^p(J;Y)$, $1 \leqslant p \leqslant \infty$, denotes the space of p-integrable functions in the Bochner sense from J into Y.

2.1. The first-order abstract Cauchy problem

In this subsection we collect the main facts concerning the existence of solutions for first-order abstract differential equations. For the theory of strongly continuous semigroup of operators we refer to [**4**, **25**]. We next only mention a few concepts and properties relative to the first-order abstract Cauchy problem. Throughout this paper, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ on the Banach space X. We denote by $M \geqslant 1$ and $\omega_1 \in \mathbb{R}$ some constants such that $\|T(t)\| \leqslant Me^{\omega_1 t}$ for $t \geqslant 0$.

The existence of solutions of the first-order abstract Cauchy problem

$$x'(t) = Ax(t) + f(t), \quad t \geqslant 0, \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where $f : [0,\infty) \to X$ is a locally integrable function, and the existence of solutions for the semilinear first-order abstract Cauchy have been discussed in many works [**4**, **25**]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)\mathrm{d}s, \quad t \geqslant 0, \tag{2.3}$$

is called mild solution of (2.1)–(2.2).

In § 4 we will use the following uniqueness property.

Remark 2.1. Let $f \in L^1([0,a];X)$ be a function that satisfies $\int_0^t T(t-s)f(s)\,\mathrm{d}s = 0$ for all $0 \leqslant t \leqslant a$. Then $f(t) = 0$ a.e. $t \in [0,a]$.

2.2. Multivalued maps

In this subsection we recall some facts concerning multivalued analysis, which will be used later. Let (Ω, d) be a metric space. Throughout this paper $\mathcal{P}(\Omega)$ denotes the collection of all nonempty subsets of Ω, $\mathcal{P}_b(\Omega)$ (respectively, $\mathcal{P}_c(\Omega)$) stands for the collection of all bounded (respectively, closed) nonempty subsets of Ω, and $\mathcal{P}_{cb}(\Omega)$ denotes the collection of all closed bounded nonempty subsets of Ω. The Hausdorff metric d_H on $\mathcal{P}_{cb}(\Omega)$ is given by

$$d_H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$.

Let $F : \Omega \to \mathcal{P}(\Omega)$ be a multivalued map. A point $x \in \Omega$ is said to be a fixed point of F if $x \in F(x)$. We denote $Fix(F)$ the set consisting of fixed points of F.

Let (Ω^1, d^1) be a metric space. A multivalued map $F : \Omega \to \mathcal{P}_{cb}(\Omega^1)$ is said to be k-contraction, where $0 \leqslant k < 1$, if

$$d_H^1(Fx, Fy) \leqslant kd(x,y), \quad \forall x, \quad y \in \Omega.$$

The following result relates these concepts and extends the Banach principle to multivalued mappings [**14**, theorem I.2.3.1].

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THEOREM 2.2. *Let* (Ω, d) *be a complete metric space and let* $F : \Omega \to \mathcal{P}_{cb}(\Omega)$ *be a* k*-contraction map. Then* F *has a fixed point.*

We also need to relate the notion of fixed point with the concept of measure of noncompactness. For this reason, we next recall a few properties of this concept. For general information the reader can see [**3**, **5**, **7**, **15**, **20**]. In this paper, we use the notion of Hausdorff measure of noncompactness on the corresponding working space.

DEFINITION 2.3. *Let* B *be a bounded subset of a metric space* Ω*. The Hausdorff measure of noncompactness of* B *is defined by*

$$\eta(B) = \inf\{\varepsilon > 0 : B \textit{ has a finite cover by closed balls of radius } < \varepsilon\}.$$

REMARK 2.4. Let $B, B_1, B_2 \subseteq \Omega$ be bounded sets. The Hausdorff measure of noncompactness has the following properties.

(a) If $B_1 \subseteq B_2$, then $\eta(B_1) \leqslant \eta(B_2)$.

(b) $\eta(B) = \eta(\overline{B})$.

(c) $\eta(B) = 0$ if and only if B is totally bounded.

(d) $\eta(B_1 \cup B_2) = \max\{\eta(B_1), \eta(B_2)\}$.

(e) The function $\eta : \mathcal{P}_{cb}(\Omega) \to [0, \infty)$ is d_H-continuous.

In what follows, we assume that Y is a normed space. For a set $B \subseteq Y$, we denote by $\overline{co}(B)$ the closed convex hull of the set B.

REMARK 2.5. Let $B, B_1, B_2 \subseteq Y$ be bounded sets. The following properties hold.

(a) For $\lambda \in \mathbb{R}$, $\eta(\lambda B) = |\lambda|\eta(B)$.

(b) $\eta(B_1 + B_2) \leqslant \eta(B_1) + \eta(B_2)$, where $B_1 + B_2 = \{b_1 + b_2 : b_1 \in B_1, b_2 \in B_2\}$.

(c) $\eta(B) = \eta(\overline{co}(B))$.

For the proof of these properties we refer the reader to the already mentioned references. Moreover, in these references the reader will find the development of the abstract concept of measure of noncompactness as well as numerous concrete examples of measure of noncompactness.

Henceforth we use the notations $\upsilon(Y)$ and $\mathcal{K}\upsilon(Y)$ to denote the following sets

(s1) $\upsilon(Y) = \{D \in \mathcal{P}(Y) : \text{D is convex}\}$.

(s2) $\mathcal{K}\upsilon(Y) = \{D \in \upsilon(Y) : \text{D is compact}\}$.

Moreover, for a multivalued map $F : \Omega \to \mathcal{P}(Y)$, we denote

$$F^{-1}(V) = \{w \in \Omega : F(w) \subseteq V\},$$

$$F_+^{-1}(V) = \{w \in \Omega : F(w) \cap V \neq \emptyset\}.$$

Definition 2.6. *Let Ω be a metric space. A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is said to be:*

(i) *Upper semi-continuous (u.s.c. for short) if $F^{-1}(V)$ is an open subset of Ω for all open set $V \subseteq Y$.*

(ii) *Closed if its graph $G_F = \{(w, y) : y \in F(w)\}$ is a closed subset of $\Omega \times Y$.*

(iii) *Compact if its range $F(\Omega)$ is relatively compact in Y.*

(iv) *Lower semi-continuous (l.s.c. for short) if $F_+^{-1}(V)$ is an open subset of Ω for all open set $V \subseteq Y$.*

Definition 2.7. *Let $(\Omega, \mathcal{A})$ be a measurable space and let Y be a Banach space. A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is said to be measurable if $F_+^{-1}(V) \in \mathcal{A}$ for all open set $V \subseteq Y$.*

It is clear that if Ω is a metric space and $\mathcal{A}$ is the Borel σ-algebra in Ω, then every l.s.c. map $F : \Omega \to \mathcal{P}(Y)$ is measurable.

In what follows, we assume that Y is a Banach space and Ω is a closed subset of Y. We denote by η any measure of noncompactness on Y that satisfies the properties mentioned in remarks 2.4 and 2.5. The following concept is taken from [**20**, definition 2.2.6].

Definition 2.8. *A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is said to be a condensing map with respect to η (abbreviated, η-condensing) if for every set $D \subset \Omega$ that is not relatively compact we have that $\eta(F(D)) \ngeq \eta(D)$.*

The next result is essential for the development of the rest of our work. We point out that if $F : \Omega \to K\upsilon(Y)$ is u.s.c., then F is closed. This allows us to establish the following version of the fixed point theorem [**20**, corollary 3.3.1].

Theorem 2.9. *Let M be a convex closed subset of Y, and let $F : M \to K\upsilon(M)$ be a u.s.c. η-condensing multivalued map. Then $Fix(F)$ is a nonempty compact set.*

Next we study some properties of the measure of noncompactness on a space of functions with values in X. To establish some properties, in what follows we denote by χ the Hausdorff measure of noncompactness in X, and by β the Hausdorff measure of noncompactness in a space of continuous functions with values in X. We next collect some properties of measure β which are needed to establish our results. Let $J = [0, a]$.

Lemma 2.10. *Let $G : J \to \mathcal{L}(X)$ be a strongly continuous operator valued map. Let $D \subset X$ be a bounded set. Then $\beta(\{G(\cdot)x : x \in D\}) \leqslant \sup_{0 \leqslant t \leqslant a} \|G(t)\| \, \chi(D)$.*

The proof of this property is an immediate consequence of definition 2.3.

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Lemma 2.11 ([5]). *Let $W \subseteq C(J;X)$ be a bounded set. Then $\chi(W(t)) \leqslant \beta(W)$ for all $t \in J$. Furthermore, if W is equicontinuous on J, then $\chi(W(\cdot))$ is continuous on J, and*

$$\beta(W) = \sup\{\chi(W(t)) \ : \ t \in J\}.$$

Lemma 2.12 ([19, Lemma 2.9]). *Let $W \subseteq C(J;X)$ be a bounded set. Then there exists a countable set $W_0 \subseteq W$ such that $\beta(W_0) = \beta(W)$.*

A set $W \subseteq L^1(J;X)$ is said to be uniformly integrable if there exists a positive function $\mu \in L^1(J)$ such that $\|w(t)\| \leqslant \mu(t)$ a.e. for $t \in J$ and all $w \in W$.

Lemma 2.13. *Let $G : J \to \mathcal{L}(X)$ be a strongly continuous operator valued map such that G is continuous for the norm of operators on $(0, a)$, and $\Lambda : L^1(J;X) \to C(J;X)$ be the map defined by*

$$\Lambda(u)(t) = \int_0^t G(t-s)u(s)\mathrm{d}s.$$

Let $W \subset L^1(J;X)$ be a uniformly integrable set. Assume that there is a positive function $q \in L^1(J)$ such that $\chi(W(t)) \leqslant q(t)$ for a.e. $t \in J$. Then

$$\beta(\Lambda(W)) \leqslant 2 \sup_{0 \leqslant t \leqslant a} \|G(t)\| \int_0^a q(t)\mathrm{d}t.$$

Proof. Applying lemma 2.10 and [20, theorem 4.2.2], we can affirm that

$$\chi(\Lambda(W)(t)) \leqslant 2 \sup_{0 \leqslant t \leqslant a} \|G(t)\| \int_0^t q(s)\mathrm{d}s$$

for every $t \in J$. Moreover, we will prove that $\Lambda(W)$ is an equicontinuous set of continuous functions. Let $M \geqslant 0$ be a constant such that $\|G(t)\| \leqslant M$ for all $t \in J$, and $\mu \in L^1(J)$ such that $\|w(t)\| \leqslant \mu(t)$ a.e. for $t \in J$ and all $w \in W$.

Initially we study the equicontinuity of $\Lambda(W)$ at $t = 0$. For $u \in W$, we estimate

$$\begin{aligned}\|\Lambda(u)(h)\| &= \left\| \int_0^h G(t-s)u(s)\mathrm{d}s \right\| \\ &\leqslant M \int_0^h \mu(s)\mathrm{d}s \to 0, \quad h \to 0,\end{aligned}$$

independent of $u \in W$, which implies that $\Lambda(W)$ is equicontinuous at $t = 0$.

We next study the equicontinuity of $\Lambda(W)$ at $t > 0$. Let $\varepsilon > 0$. We select $0 < \delta_1 < t/2$. Then $G : [\delta_1, a] \to \mathcal{L}(X)$ is uniformly continuous for the norm of operators. Consequently there exists $0 < \delta < \delta_1$ such that

$$\|G(\xi + h) - G(\xi)\| \leqslant \varepsilon,$$

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for all $\xi \in [2\delta_1, a]$ and $|h| < \delta$ such that $\xi + h \leqslant a$. To simplify the writing, we consider $h \geqslant 0$. We have that

$$\Lambda(u)(t+h) - \Lambda(u)(t) = \int_0^{t+h} G(t+h-s)u(s)\mathrm{d}s - \int_0^t G(t-s)u(s)\mathrm{d}s$$

$$= \int_0^t [G(t+h-s) - G(t-s)]u(s)\mathrm{d}s + \int_t^{t+h} G(t+h-s)u(s)\mathrm{d}s$$

$$= \int_0^{t-2\delta_1} [G(t+h-s) - G(t-s)]u(s)\mathrm{d}s + \int_{t-2\delta_1}^t [G(t+h-s) - G(t-s)]u(s)\mathrm{d}s$$

$$+ \int_t^{t+h} G(t+h-s)u(s)\mathrm{d}s.$$

From this decomposition, we can estimate

$$\|\Lambda(u)(t+h) - \Lambda(u)(t)\| \leqslant \varepsilon \int_0^{t-2\delta_1} \mu(s)\mathrm{d}s + 2M \int_{t-2\delta_1}^t \mu(s)\mathrm{d}s + M \int_t^{t+h} \mu(s)\mathrm{d}s,$$

which shows that $\Lambda(u)(t+h) - \Lambda(u)(t) \to 0$ as $h \to 0$ independent of $u \in W$, which implies that $\Lambda(W)$ is equicontinuous at t.

Using now lemma 2.11 we obtain the assertion. □

3. Existence of fixed points

Let $(Y, \|\cdot\|)$ be a Banach space and let $F : Y \to \mathcal{P}_b(Y)$ be a map. Let Z be a closed vector subspace of Y which is invariant under F, that is to say, $F : Z \to \mathcal{P}_b(Z)$. In this section we establish the existence of fixed points of F in Z. We assume that F only satisfies certain local conditions on Y. To represent the idea of local conditions, we assume that there exists a scale of Banach spaces

$$(Y, \|\cdot\|) \ldots \hookrightarrow \ldots (Y_n, \|\cdot\|_n) \overset{R_{n-1,n}}{\hookrightarrow} (Y_{n-1}, \|\cdot\|_{n-1}) \hookrightarrow \ldots (Y_1, \|\cdot\|_1),$$

where $(Y_n, \|\cdot\|_n)$ are Banach spaces for $n \in \mathbb{N}$, $R_{n-1,n} : (Y_n, \|\cdot\|_n) \to (Y_{n-1}, \|\cdot\|_{n-1})$ are bounded surjective linear maps, and there exist bounded surjective linear maps $R_n : (Y, \|\cdot\|) \to (Y_n, \|\cdot\|_n)$, and u.s.c. maps $F_n : Y_n \to \mathcal{P}_{cb}(Y_n)$ for all $n \in \mathbb{N}$. We assume that F, F_n, $R_{n-1,n}$ and R_n are related as follows.

(H1) Uniqueness property. Let $y, z \in Y$ such that $R_n y \in F_n(R_n z)$ for all $n \in \mathbb{N}$, then $y \in F(z)$.

(H2) Extension property. For every $n \in \mathbb{N}$, and for every $y \in Y_{n+1}$ such that $y^n = R_{n,n+1} y \in F_n(y^n)$ there exists $z \in F_{n+1}(y)$ such that $R_{n,n+1} y = R_{n,n+1} z$.

(H3) Inclusion property. If $(y^n)_n$ is a sequence such that $y^n \in Y_n$, $y^n = R_{n,n+1} y^{n+1}$, and $\{\|y^n\|_n : n \in \mathbb{N}\}$ is a bounded set, then there exists $y \in Y$ such that $y^n = R_n y$ for all $n \in \mathbb{N}$.

(H4) Concatenation property. For every $n \in \mathbb{N}$,

$$R_{n,n+1} F_{n+1} \subseteq F_n R_{n,n+1}.$$

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REMARK 3.1. It follows from (H4) that if $u^{n+1} \in Fix(F_{n+1})$, then $R_{n,n+1}u^{n+1} \in Fix(F_n)$. In fact,

$$R_{n,n+1}u^{n+1} \in R_{n,n+1}F_{n+1}u^{n+1} \subseteq F_n R_{n,n+1}u^{n+1},$$

which shows that $R_{n,n+1}u^{n+1} \in Fix(F_n)$.

In what follows, to abbreviate the text, we represent by β a generic measure of noncompactness on Y_n that satisfies the properties mentioned in remarks 2.4 and 2.5.

LEMMA 3.2. *Assume that $F_n : Y_n \to Kv(Y_n)$, $n \in \mathbb{N}$, satisfy the conditions (H1)–(H4). Assume further that F_n is an u.s.c. β-condensing multivalued map for all $n \in \mathbb{N}$. If $y^n \in Fix(F_n)$, then there exists $y^{n+1} \in Fix(F_{n+1})$ such that $y^n = R_{n,n+1}y^{n+1}$.*

Proof. Let $C_{n+1} = \{y \in Y_{n+1} : y^n = R_{n,n+1}y\}$. Since $R_{n,n+1}$ is a linear bounded surjective map, we infer that C_{n+1} is a nonempty closed convex set. We define the map G_{n+1} by

$$G_{n+1}(y) = \{z \in F_{n+1}(y) : y^n = R_{n,n+1}z\}, \quad y \in Y_{n+1}.$$

It follows from (H2) that $G_{n+1}y \neq \emptyset$, and $G_{n+1} : C_{n+1} \to Kv(C_{n+1})$ is an u.s.c. β-condensing multivalued map. It follows from theorem 2.9 that $Fix(G_{n+1})$ is a nonempty compact set. Therefore, there exists $y^{n+1} \in G_{n+1}(y^{n+1})$. This implies that $y^{n+1} \in F_{n+1}(y^{n+1})$ and $y^n = R_{n,n+1}y^{n+1}$. □

THEOREM 3.3. *Assume that $F : Y \to \mathcal{P}_b(Y)$, $F_n : Y_n \to Kv(Y_n)$, $n \in \mathbb{N}$, satisfy the conditions (H1)–(H4), and $\{\|z\|_n : z \in F_n(y), y \in Y_n, n \in \mathbb{N}\}$ is a bounded set. Assume further that F_n is an u.s.c. β-condensing multivalued map for all $n \in \mathbb{N}$. Then $Fix(F)$ is a nonempty set.*

Proof. It follows from theorem 2.9 that $Fix(F_n)$ is a nonempty compact set. Proceeding inductively by using lemma 3.2, we can construct a sequence $(y^n)_n$ such that $y^n \in Fix(F_n)$ and $y^n = R_{n,n+1}y^{n+1}$. It follows from our hypotheses that $\{\|y^n\|_n : n \in \mathbb{N}\}$ is a bounded set. Applying condition (H3), we infer that there exists $y \in Y$ such that $y^n = R_n y$ for all $n \in \mathbb{N}$. Since

$$y^n = R_n y \in F_n(R_n y),$$

for all $n \in \mathbb{N}$, using now (H1), we obtain that $y \in F(y)$. □

We next maintain the notation y for the fixed point of F whose existence was established in theorem 3.3.

COROLLARY 3.4. *Assume that $F : Y \to \mathcal{P}_b(Y)$ and $F_n : Y_n \to Kv(Y_n)$, $n \in \mathbb{N}$, satisfy the conditions of theorem 3.3. Let Z be a closed vector subspace of Y such that $F : Z \to \mathcal{P}_c(Z)$. Assume further that F satisfies the local Lipschitz condition*

$$d_H(Fx_2, Fx_1) \leqslant L(r, x)\|x_2 - x_1\|,$$

for all $x \in Y$, $r > 0$, and $x_1, x_2 \in B_r(x)$. If there exists $r_0 > 0$ such that $B_{r_0}(y) \cap Z \neq \emptyset$ and $L(r_0, y) < 1$, then F has a fixed point in Z.

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Proof. Let $x \in B_{r_0}(y)$. Then

$$d(y, F(x)) \leqslant d_H(F(x), F(y)) \leqslant L(r_0, y)\|x - y\| < \|x - y\|,$$

which implies that $F(x) \subseteq B_{r_0}(y)$. Consequently, $F : B_{r_0}(y) \cap Z \to \mathcal{P}_c(B_{r_0}(y) \cap Z)$ and F is a $L(r_0, y)$-contraction. It follows from theorem 2.2 that F has a fixed point in $B_{r_0}(y) \cap Z$. □

COROLLARY 3.5. *Assume that $F : Y \to \mathcal{P}_b(Y)$ and $F_n : Y_n \to K\upsilon(Y_n)$, $n \in \mathbb{N}$, satisfy the conditions of theorem 3.3. Let Z be a closed vector subspace of Y such that $F : Z \to \mathcal{P}_c(Z)$. Assume further that there exists $r > 0$ such that $B_r(y) \cap Z \neq \emptyset$ and $F : B_r(y) \to \mathcal{K}\upsilon(B_r(y))$ is β-condensing. Then F has a fixed point in Z.*

Proof. Since $F : B_r(y) \cap Z \to \mathcal{K}\upsilon(B_r(y) \cap Z)$ is β-condensing follows from theorem 2.9 that F has a fixed point in $B_r(y) \cap Z$. □

4. Applications

In this section we establish some results of existence of mild solutions of problem (1.1)–(1.2). We have mentioned in § 1 that some authors have studied differential inclusions in which the operator A is m-dissipative [**28**, **30**–**32**]. We wish to emphasize that these works and our work present important differences.

(i) Our aim is to establish the existence of asymptotically almost periodic solutions. For this reason, we need to guarantee the existence of global solutions. We will use the properties of scale of Banach spaces developed in § 3 to obtain existence of solutions defined in $[0, \infty)$.

(ii) Because in our case A is a linear operator, this allows us to decompose f into the form $f = f_1 + f_2$ and obtain the existence of solutions under different conditions in f_1 and f_2. Specifically, we will show that it is sufficient for f_1 to verify a local Lipschitz condition, while f_2 must verify a compactness property, established in terms of the measure of noncompactness, of global type, that is, in $[0, \infty)$.

(iii) Our results do not require that the semigroup $(T(t))_{t\geqslant 0}$ to be compact. We only need the semigroup $(T(t))_{t\geqslant 0}$ to be continuous in the norm of operators in $(0, \infty)$. The class of semigroups is very wide, including the differentiable, analytic and compact semigroups, etc. [**11**], which are the semigroups that frequently arise in applications.

Initially we will establish the general framework of conditions under which we will study this problem. Throughout this section, χ denotes the Hausdorff measure of noncompactness in X and β denotes the Hausdorff measure of noncompactness in any space $C([0, a]; X)$ for $a > 0$. We assume that the semigroup $(T(t))_{t\geqslant 0}$ is uniformly asymptotically stable. That is, there exist constants $M \geqslant 1$ and $\omega > 0$

such that

$$\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}, \quad t \geqslant 0. \tag{4.1}$$

Moreover, in what follows we assume that $f = f_1 + f_2$, where $f_1 : [0,\infty) \times X \to X$ and f_2 is a multivalued map from $[0,\infty) \times X$ into $\upsilon(X)$. We assume that f_1 satisfies the following conditions.

(F1) The function $f_1(\cdot, x) : [0,\infty) \to X$ is strongly measurable for each $x \in X$ and the function $f_1(\cdot, 0)$ is bounded on $[0,\infty)$.

(F2) For each $t \geqslant 0$, the function $f_1(t, \cdot) : X \to X$ is continuous.

(F3) There is a function $\nu \in L^1_{loc}([0,\infty))$ such that

$$\|f_1(t, x_2) - f_1(t, x_1)\| \leqslant \nu(t)\|x_2 - x_1\|, \ a.e.\ t \geqslant 0,$$

for all $x_2, x_1 \in X$.

4.1. Existence under compactness conditions

In this subsection we will study the case characterized by $f_2 : [0,\infty) \times X \to \mathcal{K}\upsilon(X)$. We assume that f_2 satisfies the following properties:

(F4) The function $f_2(\cdot, x) : [0,\infty) \to \mathcal{K}\upsilon(X)$ admits a strongly measurable selection for each $x \in X$.

(F5) For each $t \geqslant 0$, the function $f_2(t, \cdot) : X \to \mathcal{K}\upsilon(X)$ is u.s.c.

(F6) For each $r > 0$, there is a function $\mu_r \in L^1_{loc}([0,\infty))$ such that $\sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}\mu_r(s)\mathrm{d}s < \infty$ and

$$\|f_2(t,x)\| := \sup\{\|v\| : v \in f_2(t,x)\} \leqslant \mu_r(t), \ a.e.\ t \geqslant 0,$$

for all $x \in X$ with $\|x\| \leqslant r$.

(F7) There exists a positive function $k \in L^1([0,\infty))$ such that

$$\chi(f_2(t,\Omega)) \leqslant k(t)\chi(\Omega), \ a.e.\ t \geqslant 0,$$

for all bounded set $\Omega \subseteq X$.

Remark 4.1. Let $x(\cdot) \in C_b([0,\infty); X)$. From conditions (F4)–(F6), and applying [**20**, theorem 1.3.5] we infer that the function $f_2(\cdot, x(\cdot)) : [0,\infty) \to \mathcal{K}\upsilon(X)$, $t \mapsto f_2(t, x(t))$, admits a Bochner locally integrable selection. As a consequence, the set

$$\mathcal{S}_{f_2,x} = \{u \in L^1_{loc}([0,\infty); X) : u(t) \in f_2(t, x(t)),\ t \in [0,\infty)\} \neq \emptyset,$$

and $\mathcal{S}_{f_2,x}$ is convex.

Motivated by expression (2.3), we introduce the following concept of mild solution to problem (1.1)–(1.2).

DEFINITION 4.2. *A function $x(\cdot) \in C_b([0,\infty);X)$ is said to be a mild solution of problem (1.1)–(1.2) if the integral equation*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,x(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s$$

is satisfied for $u \in \mathcal{S}_{f_2,x}$ and all $t \in [0,\infty)$.

Without possibility of confusion, we denote by $(L^1_{loc}([0,\infty);X), |||\cdot|||)$ the Banach space consisting of the equivalence classes of locally integrable functions $u : [0,\infty) \to X$ such that $\sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}\|u(s)\|\mathrm{d}s < \infty$, endowed with the norm

$$|||u||| = \sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}\|u(s)\|\mathrm{d}s.$$

We introduce now the operator $\Lambda : L^1_{loc}([0,\infty);X) \to C_b([0,\infty);X)$ given by

$$\Lambda u(t) = \int_0^t T(t-s)u(s)\mathrm{d}s, \quad t \geqslant 0. \tag{4.2}$$

It is clear that Λ is a bounded linear operator. Using Λ we can construct the multivalued map $\widetilde{\Lambda} : C_b([0,\infty);X) \to \upsilon(C_b([0,\infty);X))$ given by

$$\widetilde{\Lambda}x = \Lambda(\mathcal{S}_{f_2,x}).$$

We next define the solution map for problem (1.1)–(1.2) as follows. Let $x \in C_b([0,\infty);X)$. We define $\Gamma(x)$ to be the set formed by all functions v given by

$$v(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,x(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s, \quad t \geqslant 0,$$

for $u \in \mathcal{S}_{f_2,x}$. It follows from our hypotheses that $v \in C_b([0,\infty);X)$. Hence, Γ is a multivalued map from $C_b([0,\infty);X)$ into $\mathcal{P}(C_b([0,\infty);X))$. Furthermore, it is clear that $x(\cdot)$ is a mild solution of problem (1.1)–(1.2) if and only if $x(\cdot)$ is a fixed point of Γ.

In order to apply our results of § 3, we take $Y = C_b([0,\infty);X)$ and $Y_n = C([0,n];X)$ for $n \in \mathbb{N}$. The maps $R_n : Y \to Y_n$ are defined by $R_n y = y|_{[0,n]}$, and $R_{n,n+1} : Y_{n+1} \to Y_n$ are defined by $R_{n,n+1}y = y|_{[0,n]}$ for $n \in \mathbb{N}$. Proceeding as above, for $n \in \mathbb{N}$ and $x \in C([0,n];X)$, we define

$$\mathcal{S}^n_{f_2,x} = \{u \in L^1([0,n];X) : u(t) \in f_2(t,x(t)),\ t \in [0,n]\} \neq \emptyset.$$

We introduce the operator $\Lambda_n : L^1([0,n];X) \to C([0,n];X)$ given by

$$\Lambda_n u(t) = \int_0^t T(t-s)u(s)\mathrm{d}s, \quad 0 \leqslant t \leqslant n. \tag{4.3}$$

It is clear that Λ_n is a bounded linear operator. Using Λ_n we can construct the multivalued map $\widetilde{\Lambda_n} : C([0,n];X) \to \upsilon(C([0,n];X))$ given by

$$\widetilde{\Lambda_n}x = \Lambda_n(\mathcal{S}^n_{f_2,x}).$$

Since $T(\cdot)$ is a strongly continuous operator valued function, the assertion in [20, lemma 4.2.1] remains valid for Λ_n. Hence, combining our previous remarks with [20, corollary 5.1.2, theorem 5.1.2] we have the following property.

LEMMA 4.3. *Let $f_2 : [0, \infty) \times X \to \mathcal{K}v(X)$ be a multivalued map satisfying conditions (F4)–(F7). Then $\widetilde{\Lambda_n}$ is an u.s.c. map with convex compact values.*

Proceeding in a similar way, we can establish the following property.

LEMMA 4.4. *Let $f_2 : [0, \infty) \times X \to \mathcal{K}v(X)$ be a multivalued map satisfying conditions (F4)–(F7). Then $\widetilde{\Lambda}$ is an u.s.c. map with convex closed values.*

Proof. Following the proof of lemma 4.3 in [20], we note that the assertion is consequence of the fact that $\mathcal{S}^n_{f_2,x}$ is weakly compact. Since the Lebesgue's measure of $[0, \infty)$ is σ-finite, using a standard diagonal selection process and [10, corollary 2.6] we conclude that $\mathcal{S}_{f_2,x}$ is weakly compact. □

For $x \in C([0, n]; X)$, we define $\Gamma_n(x)$ as the set formed by all functions v given by

$$v(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s, \quad t \in [0, n],$$

for $u \in \mathcal{S}^n_{f_2,x}$.

PROPOSITION 4.5. *Assume conditions (F1)–(F7) are satisfied. Then the scheme $(\Gamma, \Gamma_n, R_n, R_{n,n+1})_{n\in\mathbb{N}}$ satisfies conditions (H1)–(H4) of § 3, and $\Gamma_n : C([0, n]; X) \to Kv(C([0, n]; X))$ is an u.s.c. map.*

Proof. (i) To prove (H1), we consider $y, z \in C_b([0, \infty); X)$ such that $R_n y \in \Gamma_n(R_n z)$ for all $n \in \mathbb{N}$. This means that

$$y(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, z(s))\mathrm{d}s + \int_0^t T(t-s)u^n(s)\mathrm{d}s,\ t \in [0, n],\ n \in \mathbb{N},$$

where $u^n \in \mathcal{S}^n_{f_2,z}$. Applying remark 2.1, we have that $u^n = u^{n+1}|_{[0,n]}$. This allows us to define $u(t) = u^n(t)$ for $0 \leqslant t \leqslant n$. Hence,

$$y(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, z(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s, \quad t \geqslant 0.$$

Moreover, from (F6) it follows that $u \in (L^1_{loc}([0, \infty); X), ||| \cdot |||)$, and combining this assertion with the previous expression we conclude that $y \in \Gamma(z)$.

(ii) We now consider $n \in \mathbb{N}$ and $y \in C([0, n+1]; X)$ such that $y^n = R_{n,n+1} y \in \Gamma_n(y^n)$. This implies that

$$y(t) = T(t)x_0 + \int_0^t T(t-s) f_1(s, y(s))\mathrm{d}s + \int_0^t T(t-s) u^n(s)\mathrm{d}s, \quad t \in [0, n],$$

for some $u^n \in \mathcal{S}^n_{f_2, y^n}$. It follows from remark 4.1 that there exists a locally integrable function $\tilde{u}$ defined on $[0, \infty)$ such that $\tilde{u} \in \mathcal{S}_{f_2, y}$. Defining

$$v(t) = \begin{cases} u^n(t), & 0 \leqslant t \leqslant n, \\ \tilde{u}(t), & n < t \leqslant n+1, \end{cases}$$

we obtain that $v(\cdot)$ is an extension of u^n such that $v \in \mathcal{S}^{n+1}_{f_2, y}$.

We define $z(\cdot)$ by

$$z(t) = T(t)x_0 + \int_0^t T(t-s) f_1(s, y(s))\mathrm{d}s + \int_0^t T(t-s) v(s)\mathrm{d}s, \quad t \in [0, n+1].$$

It is clear that $z \in \Gamma_{n+1}(y)$ and $y(t) = z(t)$ for all $t \in [0, n]$. This shows that (H2) is fulfilled.

(iii) To prove that (H3) hold, we take a sequence $(y^n)_n$ such that $y^n \in C([0, n]; X)$, $y^n = R_{n,n+1} y^{n+1}$, and $\|y^n\| \leqslant r$ for some $r \geqslant 0$ and all $n \in \mathbb{N}$. This allows us to define $y(t) = y^n(t)$ for $0 \leqslant t \leqslant n$. It is clear that $\|y(t)\| \leqslant r$ for all $t \geqslant 0$, which implies that $y \in C_b([0, \infty); X)$ and $y^n = R_n y$ for all $n \in \mathbb{N}$.

(iv) Condition (H4) arises easily from the construction.

Finally, that $\Gamma_n : C([0, n]; X) \to Kv(C([0, n]; X))$ is an u.s.c. map is a direct consequence of lemma 4.3 □

We are now in a position to prove our first result of this section. A strongly continuous semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ is said to be immediately norm continuous if the function $T : (0, \infty) \to \mathcal{L}(X)$ is continuous for the norm of operators in $\mathcal{L}(X)$ ([**11**, definition II.4.17]).

Theorem 4.6. *Let $(T(t))_{t\geqslant 0}$ be an immediately norm continuous semigroup. Assume conditions (F1)–(F7) and (4.1) hold. Assume further the following conditions are fulfilled:*

$$M \sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)} \nu(s)\mathrm{d}s + M \liminf_{r>0} \sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)} \frac{\mu_r(s)}{r}\mathrm{d}s < 1, \tag{4.4}$$

$$M \left[\sup_{0\leqslant t<\infty} \int_0^t \mathrm{e}^{-\omega(t-s)} \nu(s)\mathrm{d}s + 2\int_0^\infty k(t)\mathrm{d}t \right] < 1. \tag{4.5}$$

Then there exists a mild solution $y(\cdot)$ of problem (1.1)–(1.2).

Proof. Let $n \in \mathbb{N}$. It follows from our hypotheses and proposition 4.5 that Γ_n is an u.s.c. multivalued map with convex compact values.

Using (4.4) we can prove that there exists $R > 0$ such that $\Gamma_n(B_R(0, Y_n)) \subseteq B_R(0, Y_n)$. In fact, it follows from (4.4) that there exists $R > 0$ large enough such that

$$M\left(\frac{\|x_0\|}{R} + \frac{1}{R}\sup_{t\geqslant 0}\int_0^t \mathrm{e}^{-\omega(t-s)}\|f_1(s,0)\|\mathrm{d}s + \sup_{t\geqslant 0}\int_0^t \mathrm{e}^{-\omega(t-s)}\nu(s)\mathrm{d}s + \sup_{t\geqslant 0}\int_0^t \mathrm{e}^{-\omega(t-s)}\frac{\mu_R(s)}{R}\mathrm{d}s\right) < 1.$$

Let $x \in B_R(0, Y_n)$ and $v \in \Gamma_n(x)$. This implies that

$$v(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s$$

for $u \in \mathcal{S}^n_{f_2,x}$. Hence

$$\begin{aligned}\|v(t)\| &\leqslant M\mathrm{e}^{-\omega t}\|x_0\| + M\int_0^t \mathrm{e}^{-\omega(t-s)}\|f_1(s,0)\|\mathrm{d}s + M\int_0^t \mathrm{e}^{-\omega(t-s)}\nu(s)\mathrm{d}sR \\ &\quad + M\int_0^t \mathrm{e}^{-\omega(t-s)}\mu_R(s)\mathrm{d}s \\ &\leqslant R,\end{aligned}$$

for all $t \geqslant 0$.

Next we show that Γ_n is β-condensing on $B_R(0, Y_n)$. Let $\Omega \subset B_R(0, Y_n)$. It follows from lemma 2.12 that there exists a sequence $(v_k)_k$ in $\Gamma_n(\Omega)$ such that $\beta(\Gamma_n(\Omega)) = \beta(\{v_k : k \in \mathbb{N}\})$. We can write $v_k \in \Gamma_n(x_k)$ for some $x_k \in \Omega$. Using (4.3) we can write

$$v_k(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x_k(s))\mathrm{d}s + \Lambda_n(u_k)(t), \quad 0 \leqslant t \leqslant n, \tag{4.6}$$

for $u_k \in \mathcal{S}^n_{f_2,x_k}$.

Using lemma 2.10, we obtain

$$\begin{aligned}\beta(\{v_k(\cdot) : k \in \mathbb{N}\}) \leqslant M \max_{0\leqslant t\leqslant n}\int_0^t \mathrm{e}^{-\omega(t-s)}\nu(s)\mathrm{d}s\beta(\{x_k(\cdot) : k \in \mathbb{N}\}) \\ + \beta(\{\Lambda_n(u_k)(\cdot) : k \in \mathbb{N}\}).\end{aligned} \tag{4.7}$$

On the other hand, since $u_k \in \mathcal{S}^n_{f_2,x_k}$, for $t \in [0, n]$ we have that $u_k(t) \in f_2(t, x_k(t))$ a.e. This implies that $\{u_k : k \in \mathbb{N}\}$ is uniformly integrable and, applying condition (F7), we obtain

$$\chi(\{u_k(t) : k \in \mathbb{N}\}) \leqslant k(t)\chi(\{x_k(t) : k \in \mathbb{N}\}),\ a.e.\ t \in [0, n].$$

Combining this estimate with lemma 2.13 we infer that

$$\beta(\{\Lambda_n(u_k)(\cdot) : k \in \mathbb{N}\}) \leqslant 2M\beta(\{x_k : k \in \mathbb{N}\})\int_0^n k(t)\mathrm{d}t.$$

Substituting in (4.7), and using (4.5), we obtain

$$\begin{aligned}\beta(\{v_k(\cdot) : k \in \mathbb{N}\}) \leqslant M \max_{0 \leqslant t \leqslant n} \int_0^t \mathrm{e}^{-\omega(t-s)} \nu(s) \mathrm{d}s \beta(\{x_k(\cdot) : k \in \mathbb{N}\}) \\ + 2M\beta(\{x_k : k \in \mathbb{N}\}) \int_0^n k(t)\mathrm{d}t \\ \leqslant M \left[\max_{0 \leqslant t \leqslant n} \int_0^t \mathrm{e}^{-\omega(t-s)} \nu(s) \mathrm{d}s + 2 \int_0^n k(t)\mathrm{d}t \right] \beta(\Omega),\end{aligned}$$

which implies that Γ_n is a β-condensing map.

Finally, taking $Z = C_b([0, \infty); X)$ and applying theorem 3.3, we infer the existence of a mild solution $y(\cdot)$ of problem (1.1)–(1.2). □

We point out that the constant $R > 0$ defined in the proof of theorem 4.6 is independent of $n \in \mathbb{N}$.

Corollary 4.7. *Assume that X is a reflexive space. Let $(T(t))_{t \geqslant 0}$ be a compact semigroup. Assume conditions (F1)–(F6), (4.1) and (4.4) hold. Then there exists a mild solution $y(\cdot)$ of problem (1.1)–(1.2).*

Proof. Using the reflexivity of X and the compactness of Λ_n, we can establish that the assertions in lemma 4.3 hold. Let $R > 0$ be the constant defined in the proof of theorem 4.6. Using that $T(t)$ is a compact operator for all $t > 0$, we can show that $\Gamma_n(B_R(0, Y_n))$ is a relatively compact set. We conclude the proof arguing as in the proof of theorem 4.6. □

4.2. Existence under measurability conditions

We can avoid the condition that f_2 has compact values used in the theorem 4.6 applying the Kuratowski–Ryll–Nardzewski theorem [**13**, theorem 19.7].

Initially, we recall the concept of Lipschitzian map ([**27**]). Let (Ω, d) be a metric space.

Definition 4.8. *A multivalued map $f : \Omega \to \mathcal{P}(X)$ is said to be:*

(a) *Lipschitzian if there exists $L \geqslant 0$ such that*

$$f(\omega_1) \subseteq f(\omega_2) + Ld(\omega_1, \omega_2)B_1(0, X), \tag{4.8}$$

for all $\omega_1, \omega_2 \in \Omega$.

(b) *Locally Lipschitzian if for every $\omega_0 \in \Omega$ there exist $\varepsilon > 0$ and $L \geqslant 0$ such that (4.8) holds for all $\omega_1, \omega_2 \in B_\varepsilon(\omega_0)$.*

In particular, we can adapt the previous concept for maps defined on product spaces.

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Definition 4.9. *We say that a multivalued map* $f : [0, d] \times X \to \mathcal{P}(X)$ *is locally Lipschitzian if for every* $(t_0, x_0) \in [0, d] \times X$ *there exist* $\varepsilon > 0$ *and* $L_1, L_2 \geqslant 0$ *such that*

$$f(t_2, x_2) \subseteq f(t_1, x_1) + (L_1|t_2 - t_1| + L_2\|x_2 - x_1\|)B_1(0, X),$$

for all $|t_i - t_0| \leqslant \varepsilon$ *and* $\|x_i - x_0\| \leqslant \varepsilon$ *for* $i = 1, 2$.

We next denote by m the Lebesgue measure on $[0, d]$. We say that a function $u : [0, d] \to X$ is m-measurable if it is strongly measurable in the Bochner sense. The reader can see [**17**, **22**] for properties of m-measurable functions.

Proposition 4.10. *Assume that* X *is a separable Banach space. Let* $f_2 : [0, d] \times X \to \mathcal{P}(X)$ *be a locally Lipschitzian map with closed values. Let* $x : [0, d] \to X$ *be a continuous function. Then there exists a* m*-measurable function* $u : [0, d] \to X$ *such that* $u(t) \in f_2(t, x(t))$ *for all* $t \in [0, d]$.

Proof. Let $g : [0, d] \to \mathcal{P}(X)$ be given by $g(t) = f_2(t, x(t))$. We first show that g is l.s.c. In fact, let $V \subseteq X$ be an open set and $I = \{t \in [0, d] : g(t) \cap V \neq \emptyset\}$. If $t_0 \in I$, then there exists $u_0 \in g(t_0) \cap V$. Since V is an open set, there exists $\varepsilon > 0$ such that $B_\varepsilon(u_0) \subset V$. Moreover, there exists $\varepsilon_1 > 0$ such that

$$u_0 \in f_2(t_0, x(t_0)) \subseteq f_2(t, x(t)) + (L_1|t - t_0| + L_2\|x(t) - x(t_0)\|)B_1(0, X),$$

for all t such that $|t - t_0| \leqslant \varepsilon_1$ and $\|x(t) - x(t_0)\| \leqslant \varepsilon_1$. We can take $\varepsilon_1 > 0$ small enough so that $(L_1 + L_2)\varepsilon_1 < \varepsilon$. Since $x(\cdot)$ is continuous, there exists $0 < \delta < \varepsilon_1$ such that $\|x(t) - x(t_0)\| < \varepsilon_1$ for $|t - t_0| < \delta$. Combining these assertions, if $|t - t_0| < \delta$, we infer that there exists $u \in f_2(t, x(t))$ that satisfies

$$u_0 = u + \alpha B_1(0, X),$$

where $0 \leqslant \alpha < \varepsilon$. Consequently, $u \in B_\varepsilon(u_0) \subset V$ and $g(t) \cap V \neq \emptyset$. It follows that $t \in I$ and I is open.

It follows from the Kuratowski–Ryll–Nardzewski theorem [**13**, theorem 19.7] that there exists a measurable function $u : [0, d] \to X$ such that $u(t) \in f_2(t, x(t))$ for all $t \in [0, d]$. Applying now [**22**, proposition 2.2.6], we infer that u is m-measurable. □

We next weaken the concept of u.s.c map.

Definition 4.11. *Let* Ω *be a metric space and let* Y *be a Banach space. A multivalued map* $F : \Omega \to \mathcal{P}(Y)$ *is said to be weakly upper semi-continuous (w.u.s.c. for short) if* $F^{-1}(V)$ *is an open subset of* Ω *for all weakly open set* $V \subseteq Y$.

It is clear from definition 4.11 that $F : \Omega \to \mathcal{P}(Y)$ is w.u.s.c. if and only if $F_+^{-1}(V)$ is a closed subset of Ω for all weakly closed set $V \subseteq Y$. We use this equivalence in the proof of the next Proposition.

Proposition 4.12. *Assume* X *is a reflexive space. Let* $F : X \to \upsilon(X)$ *be a locally Lipschitzian map with closed values that takes bounded sets into bounded sets. Then* F *is w.u.s.c.*

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Proof. Let $V \subset X$ be a weakly closed set and $x_n \in F_+^{-1}(V)$, $n \in \mathbb{N}$, such that $x_n \to x$ as $n \to \infty$. Let $y_n \in F(x_n) \cap V$. Since $\{y_n : n \in \mathbb{N}\}$ is a bounded set, there is a subsequence $(y_{n_k})_k$ of $(y_n)_n$ which converges to y weakly. Since

$$F(x_{n_k}) \subseteq F(x) + w_k$$

where $w_k \to 0$ as $k \to \infty$, we can affirm that there exists $z_k \in F(x)$ such that $y_{n_k} = z_k + w_k$ and $z_k \to y$ as $k \to \infty$ weakly. Since $F(x)$ is a closed convex set, we infer that $y \in F(x)$. Hence $y \in F(x) \cap V$, which implies that $x \in F_+^{-1}(V)$. □

Example 4.13. Let $X = L^2([0, \pi])$, $a, b : \mathbb{R} \to \mathbb{R}$, $a \leqslant b$, functions that satisfy the Lipschitz conditions

$$|a(x_2) - a(x_1)| \leqslant a_0|x_2 - x_1|,$$
$$|b(x_2) - b(x_1)| \leqslant b_0|x_2 - x_1|,$$

for some positive constants a_0, b_0, and all $x_1, x_1 \in \mathbb{R}$. Let $f : X \to \upsilon(X)$ be the multivalued function given by

$$f(x) = \{u \in X : a(x(\xi)) \leqslant u(\xi) \leqslant b(x(\xi)),\ 0 \leqslant \xi \leqslant \pi\}.$$

Then f is a w.u.s.c. Lipschitzian map with closed bounded values.

Proof. It follows from [**23**, Chapter 5] that $a(x(\cdot)), b(x(\cdot)) \in X$ for all $x \in X$. This implies that $f(x) \neq \emptyset$. Moreover, it is easy to see that $f(x)$ is a closed bounded set in X. Let $L = \max\{a_0, b_0\}$. Then

$$f(x_2) \subset f(x_1) + L\|x_2 - x_1\| B_1(0, X),$$

for all $x_1, x_2 \in X$. In fact, if $u \in f(x_2)$, for every $\xi \in [0, \pi]$, we have

$$\begin{aligned} u(\xi) &\in [a(x_2(\xi)), b(x_2(\xi))] \\ &\subseteq [a(x_1(\xi)) - a_0|x_2(\xi) - x_1(\xi)|, b(x_1(\xi)) + b_0|x_2(\xi) - x_1(\xi)|] \\ &\subseteq [a(x_1(\xi)) - L|x_2(\xi) - x_1(\xi)|, b(x_1(\xi)) + L|x_2(\xi) - x_1(\xi)|]. \end{aligned}$$

We define the sets

$$\begin{aligned} E_1 &= \{\xi \in [0, \pi] : u(\xi) < a(x_1(\xi))\}, \\ E_2 &= \{\xi \in [0, \pi] : a(x_1(\xi)) \leqslant u(\xi) \leqslant b(x_1(\xi))\}, \\ E_3 &= \{\xi \in [0, \pi] : b(x_1(\xi)) < u(\xi)\}. \end{aligned}$$

Let $v(\xi) = u(\xi) + L|x_2(\xi) - x_1(\xi)|(\chi_{E_1} - \chi_{E_3})$, where χ_E denotes the characteristic function of E. Let $w(\xi) = L|x_2(\xi) - x_1(\xi)|(\chi_{E_1} - \chi_{E_3})$. It follows from this construction that $v \in f(x_1)$, $w \in X$, and

$$\|w\| = \left(\int_0^\pi |w(\xi)|^2 \mathrm{d}\xi\right)^{1/2} \leqslant L\|x_2 - x_1\|,$$

which shows that f is a Lipschitzian map.

Finally, we infer from proposition 4.12 that f is w.u.s.c. □

We next modify slightly conditions (F6–F7).

(F6') For each $r > 0$, there is a function $\mu_r \in L^2_{loc}([0, \infty))$ such that

$$\|f_2(t, x)\| := \sup\{\|v\| : v \in f_2(t, x)\} \leqslant \mu_r(t), \ a.e. \ t \geqslant 0,$$

for all $x \in X$ with $\|x\| \leqslant r$.

(F7') There exists a positive function $k \in L^2([0, \infty))$ such that

$$\chi(f_2(t, \Omega)) \leqslant k(t)\chi(\Omega), \ a.e. \ t \geqslant 0,$$

for all bounded set $\Omega \subseteq X$.

The following consequence is immediate.

Corollary 4.14. *Assume that the hypotheses of proposition 4.10 are fulfilled, and that condition(F6') holds. Let u be the function whose existence was established in proposition 4.10. Then $u \in L^2([0, d]; X)$.*

Next we assume that the hypotheses of corollary 4.14 hold. Proceeding as in the previous part, for $d > 0$ and $x \in C([0, d]; X)$, we define

$$\mathcal{S}^d_{f_2,x} = \{u \in L^2([0, d]; X) : u(t) \in f_2(t, x(t)), \ t \in [0, d]\} \neq \emptyset.$$

We consider Λ_d given by (4.3) on $L^2([0, d]; X)$. In a similar way, for $x \in C_b([0, \infty); X)$ we define

$$\mathcal{S}_{f_2,x} = \{u \in L^2_{loc}([0, \infty); X) : u(t) \in f_2(t, x(t)), \ t \in [0, \infty)\} \neq \emptyset.$$

Without possibility of confusion, in this case we denote by $(L^2_{loc}([0, \infty); X), |||\cdot|||)$ the Banach space consisting of the equivalence classes of locally integrable functions $u : [0, \infty) \to X$ that satisfy the condition $\sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}\|u(s)\|^2 \mathrm{d}s < \infty$, endowed with the norm

$$|||u||| = \sup_{t\geqslant 0} \left(\int_0^t \mathrm{e}^{-\omega(t-s)}\|u(s)\|^2 \mathrm{d}s\right)^{1/2}.$$

We introduce now the operator $\Lambda : L^2_{loc}([0, \infty); X) \to C_b([0, \infty); X)$ given by (4.2). It is clear that Λ is a bounded linear operator.

Remark 4.15. The assertion in [**20**, lemma 5.1.1] remains valid for w.u.s.c. maps. Specifically, let $F : [0, d] \times X \to v(X)$ be a map with closed values such that $F(t, \cdot)$ is w.u.s.c. for each $t \in [0, d]$. Assume that $x_n \in C([0, d]; X)$, $x_n \to x_0$, $n \to \infty$, and $u_n \in \mathcal{S}^d_{F,x_n}$ is a sequence that converges weakly to u_0. Then $u_0 \in \mathcal{S}^d_{F,x_0}$.

Proposition 4.16. *Assume that X is a separable reflexive Banach space and that $(T(t))_{t\geqslant 0}$ is an immediately norm continuous semigroup. Let $f_2 : [0, \infty) \times X \to$*

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$\upsilon(X)$ be a locally Lipschitzian map with closed values such that (F6') holds. Then $\widetilde{\Lambda}_d : C([0,d];X) \to \mathcal{K}\upsilon(C([0,d];X))$ is an u.s.c. map.

Proof. We separate the proof in three steps.

(i) Initially we prove that $\widetilde{\Lambda}_d(x) = \Lambda_d(\mathcal{S}^d_{f_2,x})$ is closed for any $x \in C([0,d];X)$. Let $u_n \in \mathcal{S}^d_{f_2,x}$ such that $y_n = \Lambda_d(u_n) \to y$ as $n \to \infty$. Since $L^2([0,d];X)$ is a reflexive space, there exists a subsequence $(u_{n_k})_k$ that converges weakly to some function u. Moreover, $\mathcal{S}^d_{f_2,x}$ is a convex and norm closed set. This implies that $\mathcal{S}^d_{f_2,x}$ is a weakly closed set and $u \in \mathcal{S}^d_{f_2,x}$. Since Λ_d is norm continuous, it is also weakly continuous, and $\Lambda_d(u_{n_k}) \to \Lambda_d(u) = y$ as $k \to \infty$.

(ii) We now prove that $\widetilde{\Lambda}_d(x)$ is a relatively compact set. Let $(u_n)_n$ be a sequence in $\mathcal{S}^d_{f_2,x}$. Proceeding as in (i), there exists a subsequence $(u_{n_k})_k$ that converges weakly to some function u and $\Lambda_d(u_{n_k}) \to \Lambda_d(u)$ as $k \to \infty$ weakly in $C([0,d];X)$. This implies that $\Lambda_d(u_{n_k})(t) \to \Lambda_d(u)(t)$ as $k \to \infty$ for all $t \in [0,d]$. Moreover, proceeding as in the proof of lemma 2.13 we obtain that $\{\Lambda_d(u_n) : n \in \mathbb{N}\}$ is an equicontinuous set.

(iii) We can argue as in [**20**, corollary 5.1.2] to conclude that $\widetilde{\Lambda}_d$ is u.s.c. Specifically, let V be a closed subset of

$C([0,d];X)$ and $x_n \in C([0,d];X)$ such that $x_n \to x$ as $n \to \infty$ and $\widetilde{\Lambda}_d(x_n) \cap V \neq \emptyset$. We take $y_n \in \widetilde{\Lambda}_d(x_n) \cap V$ and $u_n \in \mathcal{S}^d_{f_2,x_n}$ such that $y_n = \Lambda_d(u_n)$. Since $\{u_n : n \in \mathbb{N}\}$ is a bounded set in $L^2([0,d];X)$, there exists a subsequence $(u_{n_k})_k$ that converges weakly to some function u. This implies that $y_{n_k} \to y = \Lambda_d(u)$, $k \to \infty$, weakly. Furthermore, proceeding as in (ii), we can see that the set $\cup_{n=1}^{\infty} \widetilde{\Lambda}_d(x_n)$ is relatively compact. Therefore, there is a subsequence of $(y_{n_k})_k$ which converges uniformly. This implies that $y \in V$. Using now remark 4.15 we obtain that $u \in \mathcal{S}^d_{f_2,x}$, which in turn implies that $y \in \widetilde{\Lambda}_d(x)$. □

Using proposition 4.16 in the space $C([0,n];X)$ for $n \in \mathbb{N}$, we can argue as in the proofs carried out in proposition 4.5 and theorem 4.6 in order to establish the following property.

Theorem 4.17. *Assume that X is a separable reflexive Banach space and that $(T(t))_{t\geqslant 0}$ is an immediately norm continuous semigroup. Let $f_1 : [0,\infty) \times X \to X$ be a function that satisfies (F1)–(F3), and let $f_2 : [0,\infty) \times X \to \upsilon(X)$ be a locally Lipschitzian map with closed values such that conditions (F6')–(F7') are fulfilled. Assume further that conditions* (4.1)*,* (4.4) *and* (4.5) *hold. Then there exists a mild solution $y(\cdot)$ of problem* (1.1)–(1.2).

In similar way, modifying slightly the proof of corollary 4.7, by using theorem 4.17 instead of theorem 4.6, we obtain the property.

Corollary 4.18. *Assume that X is a separable reflexive Banach space and that $(T(t))_{t\geqslant 0}$ is a compact semigroup. Let $f_1 : [0,\infty) \times X \to X$ be a function that satisfies (F1)–(F3), and let $f_2 : [0,\infty) \times X \to \upsilon(X)$ be a locally Lipschitzian map with* *closed values such that the condition (F6') is fulfilled. Assume further that conditions* (4.1) *and* (4.4) *hold. Then there exists a mild solution* $y(\cdot)$ *of problem* (1.1)–(1.2).

In what follows, we will reserve the notation $y(\cdot)$ to denote the solution of problem (1.1)–(1.2) established in any of theorems 4.6, 4.17, corollaries 4.7 or 4.18. It follows from theorem 3.3 and its corollary that $\|y\|_\infty \leqslant R$. From the definition of solutions we have that

$$y(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, y(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s, \quad 0 \leqslant t < \infty, \tag{4.9}$$

for some $u \in \mathcal{S}_{f_2,y}$. Moreover, combining with remark 2.1 we can affirm that there is a unique $u \in \mathcal{S}_{f_2,y}$ that verifies (4.9).

We next study the existence of asymptotically almost periodic solutions of (1.1)–(1.2). For general properties of almost periodic and asymptotically almost periodic functions with values in abstract spaces we refer the reader to [**9**, **16**, **33**]. We only recall here the basic definitions. In the first place, we remember that a set P is called relatively dense in $\mathbb{R}$ (respectively, in $[0, \infty)$) if there exists $L > 0$ so that for any interval $I \subset \mathbb{R}$ (respectively, $I \subset [0, \infty)$) with length greater than or equal to L we have $I \cap P \neq \emptyset$.

Definition 4.19. *A function* $x \in C(\mathbb{R}; X)$ *is called almost periodic (in short, a.p.) if for every* $\varepsilon > 0$ *there exists a relatively dense subset* P_ε *of* $\mathbb{R}$ *such that*

$$\|x(t+\tau) - x(t)\| \leqslant \varepsilon, \quad t \in \mathbb{R}, \quad \tau \in P_\varepsilon.$$

Definition 4.20. *A function* $z \in C_b([0, \infty); X)$ *is called asymptotically almost periodic (abbreviated, a.a.p.) if there exists* $w \in C_0([0, \infty); X)$ *and an almost periodic function* $x(\cdot)$ *such that* $z(t) = x(t) + w(t)$ *for all* $t \geqslant 0$.

Remark 4.21 ([**33**, Theorem 5.5]). A function $f \in C([0, \infty); X)$ is asymptotically almost periodic if and only if for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ and a relatively dense subset P_ε of $[0, \infty)$ such that

$$\|f(t+\xi) - f(t)\| \leqslant \varepsilon,$$

for all $t \geqslant t_\varepsilon$ and $\xi \in P_\varepsilon$.

In this paper, $AP(X)$ and $AAP(X)$ denote the spaces consisting of a.p. (respectively, a.a.p.) functions endowed with the norm of the uniform convergence. The following property is well known ([**18**, lemma 3.1]).

Lemma 4.22. *Let* $(T(t))_{t \geqslant 0}$ *be a uniformly asymptotically stable* C_0*-semigroup on* X*, and let* $u : [0, \infty) \to X$ *be an a.a.p. function. Then the function* $v : [0, \infty) \to X$ *given by*

$$v(t) = \int_0^t T(t-s)u(s)\mathrm{d}s, \quad t \geqslant 0,$$

also is a.a.p.

Definition 4.23. *A function $f \in C([0,\infty) \times X; X)$ is called uniformly asymptotically almost periodic (abbreviated, u.a.a.p.) on compact sets if for every $\varepsilon > 0$ and every compact $K \subset X$ there exists a relatively dense subset $P_{K,\varepsilon}$ in $[0,\infty)$ and $t_{K,\varepsilon} > 0$ such that*

$$\|f(t+\tau, x) - f(t,x)\| \leqslant \varepsilon,\ t \geqslant t_{K,\varepsilon},\ (\tau, x) \in P_{K,\varepsilon} \times K.$$

To establish our results we need some properties of a.a.p. functions. We begin with the following remark.

Remark 4.24.

(a) Assume that $(T(t))_{t\geqslant 0}$ is a uniformly asymptotically stable C_0-semigroup on X. Let $f_1 \in C([0,\infty) \times X; X)$ be a function that satisfies (F3) and $f_1(\cdot, 0)$ is bounded on $[0,\infty)$. Let $K \subset X$ be a compact set. Then

$$\int_a^t T(s) f_1(t-s, z) \mathrm{d}s \to 0, \quad a \to \infty,$$

uniformly for $t \geqslant a$ and $z \in K$.

Proof. We can assume that $(T(t))_{t\geqslant 0}$ satisfies (4.1). This implies that

$$\begin{aligned}\left\|\int_a^t T(s) f_1(t-s,z)\mathrm{d}s\right\| &\leqslant M \int_a^t \mathrm{e}^{-\omega s}\nu(t-s)\|z\|\mathrm{d}s + M\int_a^t \mathrm{e}^{-\omega s}\|f_1(t-s,0)\|\mathrm{d}s \\ &\leqslant M\|z\| \int_0^\infty \nu(\xi)\mathrm{d}\xi \mathrm{e}^{-\omega a} + \frac{M}{\omega}\sup_{t\geqslant 0}\|f_1(t,0)\|\mathrm{e}^{-\omega a} \\ &\to 0,\ a \to \infty,\end{aligned}$$

uniformly for $z \in K$ and $t \geqslant a$. □

(b) Let $x : [0,\infty) \to X$ be an a.a.p. function. Then the range of $x(\cdot)$ is a relatively compact set in X.

This is a consequence of the fact that both the range of an a.p. function, and the range of a function that vanishes at infinity are relatively compact sets. The reader can see [**9**, proposition 3.9] or [**33**, proposition 5.3].

Using remark 4.24 we can establish an important property of a.a.p. functions.

Lemma 4.25. *Let $f_1 \in C([0,\infty) \times X; X)$ be a uniformly asymptotically almost periodic on compact sets function that satisfies condition (F3). Let $x : [0,\infty) \to X$ be an a.a.p. function. Then the function $v : [0,\infty) \to X$ given by*

$$v(t) = \int_0^t T(t-s) f_1(s, x(s))\mathrm{d}s, \quad t \geqslant 0,$$

also is a.a.p.

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Proof. Since the range $Im(x)$ of $x(\cdot)$ is a relatively compact set, for every $\varepsilon > 0$ there exist a set P relatively dense in $[0, \infty)$ and $t_\varepsilon^1 > 0$ such that

$$\|x(t+\tau) - x(t)\| \leqslant \varepsilon, \tag{4.10}$$

$$\|f_1(t+\tau, z) - f_1(t, z)\| \leqslant \varepsilon, \tag{4.11}$$

for all $\tau \in P$, $t \geqslant t_\varepsilon^1$ and $z \in K = Im(x)$.

Let $a_\varepsilon > 0$. For $t \geqslant a_\varepsilon$, we can write

$$\begin{aligned} v(t+\tau) - v(t) &= \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t+\tau-s)) - f_1(t+\tau-s, x(t-s))]\mathrm{d}s \\ &\quad + \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t-s)) - f_1(t-s, x(t-s))]\mathrm{d}s \\ &\quad + \int_{a_\varepsilon}^{t+\tau} T(s) f_1(t+\tau-s, x(t+\tau-s))\mathrm{d}s \\ &\quad - \int_{a_\varepsilon}^{t} T(s) f_1(t-s, x(t-s))\mathrm{d}s. \end{aligned}$$

Now we estimate each term on the right-hand side of the above expression separately. We select $a_\varepsilon > 0$ appropriately as follows. For the third and fourth terms, using remark 4.24 we can assume that

$$\left\| \int_{a_\varepsilon}^{t+\tau} T(s) f_1(t+\tau-s, x(t+\tau-s))\mathrm{d}s \right\| \leqslant \varepsilon,$$

$$\left\| \int_{a_\varepsilon}^{t} T(s) f_1(t-s, x(t-s))\mathrm{d}s \right\| \leqslant \varepsilon,$$

for all $t \geqslant a_\varepsilon$ and $\tau \in P$. Let $t_\varepsilon = t_\varepsilon^1 + a_\varepsilon$. For $t \geqslant t_\varepsilon$, using (4.10) we can estimate the first term as

$$\begin{aligned} &\left\| \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t+\tau-s)) - f_1(t+\tau-s, x(t-s))]\mathrm{d}s \right\| \\ &\quad \leqslant M \int_0^{a_\varepsilon} \mathrm{e}^{-\omega s} \nu(t+\tau-s) \|x(t+\tau-s) - x(t-s)\| \mathrm{d}s \\ &\quad \leqslant M \int_0^{\infty} \nu(\xi)\mathrm{d}\xi \varepsilon. \end{aligned}$$

Proceeding in similar way, using (4.11) instead of (4.10), the second term yields

$$\begin{aligned} &\left\| \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t-s)) - f_1(t-s, x(t-s))]\mathrm{d}s \right\| \\ &\quad \leqslant M \int_0^{a_\varepsilon} \mathrm{e}^{-\omega s} \mathrm{d}s \varepsilon \leqslant \frac{M}{\omega} \varepsilon. \end{aligned}$$

Combining these estimates, we obtain that

$$\|v(t+\tau) - v(t)\| \leqslant \left(2 + M \int_0^{\infty} \nu(\xi)\mathrm{d}\xi + \frac{M}{\omega} \right) \varepsilon$$

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for all $\tau \in P$, $t \geqslant t_\varepsilon$. Using remark 4.21 we can affirm that $v(\cdot)$ is an a.a.p. function. □

In the following statement, assuming that the hypotheses of theorems 4.6, 4.17, corollaries 4.7 or 4.18 are fulfilled, we denote

$$r = M \sup_{t \geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}[\nu(s)R + \mu_R(s)]\mathrm{d}s.$$

THEOREM 4.26. *Assume the hypotheses of theorems 4.6, 4.17, corollaries 4.7 or 4.18 hold. Assume further that the following conditions are satisfied:*

(i) *The function f_1 is uniformly asymptotically almost periodic on compact sets.*

(ii) *For every $\delta > 0$ there exists a measurable function $\sigma_\delta : [0,\infty) \to [0,\infty)$ such that $\sigma_{\delta_1} \leqslant \sigma_{\delta_2}$ for $\delta_1 \leqslant \delta_2$ and having the following property: for every $x_1, x_2 \in C_b([0,\infty); X)$ with $\|x_2 - x_1\|_\infty \leqslant \delta$, for every $u_2 \in \mathcal{S}_{f_2,x_2}$, there exists $u_1 \in \mathcal{S}_{f_2,x_1}$ such that*

$$\|u_2(t) - u_1(t)\| \leqslant \sigma_\delta(t)\|x_2(t) - x_1(t)\|, \; t \geqslant 0.$$

(iii) *For every $x \in AAP(X)$ the set $\widetilde{\mathcal{S}}_{f_2,x} = \mathcal{S}_{f_2,x} \cap AAP(X) \neq \emptyset$.*

If

$$M \sup_{t \geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}[\nu(s) + \sigma_{2r}(s)]\mathrm{d}s < 1, \tag{4.12}$$

then there exists an a.a.p. mild solution of problem (1.1)–(1.2).

Proof. As was previously explained in this section, there is a fixed point y of Γ with $\|y\|_\infty \leqslant R$.

On the other hand, for $x \in AAP(X)$, we define $\widetilde{\Gamma}(x)$ as the set formed by all functions v given by

$$v(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s, \quad t \geqslant 0,$$

with $u \in \widetilde{\mathcal{S}}_{f_2,x}$.

Since $\widetilde{\Gamma}$ is a restriction of Γ on $AAP(X)$, we infer that $\widetilde{\Gamma} : AAP(X) \to \mathcal{K}v(C_b([0,\infty), X))$. Moreover, $\widetilde{\Gamma}(AAP(X)) \subseteq \mathcal{K}v(AAP(X))$. In fact, this is an immediate consequence of lemmas 4.22 and 4.25. As a consequence, we can affirm that $\widetilde{\Gamma}(AAP(X) \cap B_R(0)) \subseteq \mathcal{K}v(AAP(X) \cap B_R(0))$.

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We next estimate $d(y, AAP(X))$. Using (4.9) we have that

$$\begin{aligned} y(t) &= T(t)x_0 + \int_0^t T(t-s)f_1(s, y(s))\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s \\ &= T(t)x_0 + \int_0^t T(t-s)[f_1(s, y(s)) - f_1(s, 0)]\mathrm{d}s + \int_0^t T(t-s)f_1(s, 0)\mathrm{d}s \\ &\quad + \int_0^t T(t-s)u(s)\mathrm{d}s \end{aligned}$$

for $u \in \mathcal{S}_{f_2,y}$. Since the function $z(\cdot)$ given by

$$z(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, 0)\mathrm{d}s, \quad t \geqslant 0,$$

is a.a.p., we obtain that

$$\begin{aligned} d(y, AAP(X)) &\leqslant \left\| \int_0^t T(t-s)[f_1(s, y(s)) - f_1(s, 0)]\mathrm{d}s + \int_0^t T(t-s)u(s)\mathrm{d}s \right\|_\infty \\ &\leqslant M \sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}[\nu(s)\|y(s)\| + \mu_R(s)]\mathrm{d}s \\ &\leqslant M \sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-\omega(t-s)}[\nu(s)R + \mu_R(s)]\mathrm{d}s \\ &= r. \end{aligned}$$

Consequently, $AAP(X) \cap B_r(y)$ is a nonempty closed set.

Let $x_i \in AAP(X) \cap B_r(y)$, $i = 1, 2$. We now estimate $d_H(\widetilde{\Gamma}(x_2), \widetilde{\Gamma}(x_1))$. Let $z_2 \in \widetilde{\Gamma}(x_2)$ and $u_2 \in \widetilde{\mathcal{S}}_{f_2,x_2}$ be such that

$$z_2(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x_2(s))\mathrm{d}s + \int_0^t T(t-s)u_2(s)\mathrm{d}s, \quad t \geqslant 0.$$

Using (ii), we can take $u_1 \in \mathcal{S}_{f_2,x_1}$ such that

$$\|u_2(t) - u_1(t)\| \leqslant \sigma_{2r}(t)\|x_2(t) - x_1(t)\|, \quad t \geqslant 0,$$

and define $z_1(\cdot)$ by

$$z_1(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x_1(s))\mathrm{d}s + \int_0^t T(t-s)u_1(s)\mathrm{d}s, \quad t \geqslant 0.$$

We obtain

$$\begin{aligned}\|z_2(t)-z_1(t)\| &\leqslant M\int_0^t e^{-\omega(t-s)}\nu(s)\|x_2(s)-x_1(s)\|ds\\ &\quad+M\int_0^t e^{-\omega(t-s)}\|u_2(s)-u_1(s)\|ds\\ &\leqslant M\int_0^t e^{-\omega(t-s)}\nu(s)\|x_2(s)-x_1(s)\|ds\\ &\quad+M\int_0^t e^{-\omega(t-s)}\sigma_{2r}(s)\|x_2(s)-x_1(s)\|ds\\ &\leqslant M\int_0^t e^{-\omega(t-s)}[\nu(s)+\sigma_{2r}(s)]ds \max_{0\leqslant s\leqslant t}\|x_2(s)-x_1(s)\|\end{aligned}$$

for $t \geqslant 0$. This implies that

$$d(z_2,\widetilde{\Gamma}(x_1)) \leqslant M\sup_{t\geqslant 0}\int_0^t e^{-\omega(t-s)}[\nu(s)+\sigma_{2r}(s)]ds\|x_2-x_1\|_\infty.$$

Since the right-hand side of the above inequality only depends on x_1 and x_2, we conclude that

$$d_H(\widetilde{\Gamma}(x_2),\widetilde{\Gamma}(x_1)) \leqslant M\sup_{t\geqslant 0}\int_0^t e^{-\omega(t-s)}[\nu(s)+\sigma_{2r}(s)]ds\|x_2-x_1\|_\infty.$$

In addition, $\widetilde{\Gamma}(AAP(X)\cap B_r(y)) \subseteq AAP(X)\cap B_r(y)$. In fact, if $x \in AAP(X)\cap B_r(y)$ and $z \in \widetilde{\Gamma}(x)$, using (4.9) and proceeding as above, we have that

$$\begin{aligned}\|z-y\|_\infty &\leqslant M\sup_{t\geqslant 0}\int_0^t e^{-\omega(t-s)}[\nu(s)+\sigma_r(s)]ds\|x-y\|_\infty\\ &\leqslant M\sup_{t\geqslant 0}\int_0^t e^{-\omega(t-s)}[\nu(s)+\sigma_{2r}(s)]ds r\\ &\leqslant r\end{aligned}$$

which shows that $z \in B_r(y)$.

Combining (4.12) and theorem 2.2 we infer that $\widetilde{\Gamma}$ has a fixed point x in $AAP(X)\cap B_r(y)$. It is clear that $x(\cdot)$ is an a.a.p. mild solution of (1.1)–(1.2). □

Next we use our previous results to study the existence of solutions of problem (1.3)–(1.5). To model this problem in the abstract form (1.1)–1.2, we consider the space $X = L^2([0,\pi])$. We define the operator $A : D(A) \subset X \to X$ by

$$Az(\xi) = z''(\xi), \quad 0 \leqslant \xi \leqslant \pi,$$

on $D(A) = \{z \in X : z'' \in X, z(0) = z(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a compact semigroup $(T(t))_{t\geqslant 0}$ that satisfies the estimate (4.1) with $M = \omega = 1$. We assume that $\varphi \in X$.

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EXAMPLE 4.27. Let $f_1 : [0,\infty) \times X \to X$ be a function given by $f_1(t,x)(\xi) = \tilde{f}_1(t,x(\xi))$ for $0 \leqslant \xi \leqslant \pi$, where $\tilde{f}_1 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a function that satisfies the Caratheódory conditions [**23**] and

$$|\tilde{f}_1(t,w_2) - \tilde{f}_1(t,w_1)| \leqslant \nu(t)|w_2 - w_1|, \quad t \geqslant 0, \quad w_1, w_2 \in \mathbb{R},$$

where $\nu \in L^1_{loc}([0,\infty))$.

This implies that f_1 satisfies conditions (F1)–(F3). Let $f_2 : X \to \upsilon(X)$ be the map given by

$$f_2(x) = \{u \in X : a(x(\xi)) \leqslant u(\xi) \leqslant b(x(\xi)),\ 0 \leqslant \xi \leqslant \pi\},$$

where a, b are functions that satisfy the conditions considered in example 4.13.

Under these conditions, problem (1.3)–(1.4) is modelled as (1.1)–(1.2) with $x_0 = \varphi$. Moreover, for $u \in f_2(x)$ we obtain

$$\|u\| \leqslant c_0 + c_1\|x\|,$$

where $c_0 = \sqrt{\pi}\max\{|a(0)|, |b(0)|\}$ and $c_1 = \max\{a_0, b_0\}$. Which shows that f_2 satisfies (F6') with $\mu_r(t) = c_1 r + c_0$.

We assume that

$$\sup_{t\geqslant 0} \int_0^t \mathrm{e}^{-(t-s)}\nu(s)\mathrm{d}s + c_1 < 1. \tag{4.13}$$

Consequently, it follows from corollary 4.18 that there exists a mild solution $u(\cdot)$ of problem (1.3)–(1.5).

We assume further that $\tilde{f}_1$ satisfies the condition

$$|\tilde{f}_1(t,x) - \tilde{f}_1(s,x)| \leqslant |q(t) - q(s)||x|,\ s,t \geqslant 0,\ x \in \mathbb{R}, \tag{4.14}$$

where $q : [0,\infty) \to \mathbb{R}$ is an a.a.p. function. Hence we infer that f_1 is uniformly a.a.p. on compact sets, which implies that condition (i) of theorem 4.26 is fulfilled.

On the other hand, for every $x_1, x_2 \in C_b([0,\infty); X)$ and $u_2 \in \mathcal{S}_{f_2,x_2}$, proceeding as in the proof of example 4.13 we can select $u_1 \in \mathcal{S}_{f_2,x_1}$ such that

$$|u_2(t,\xi) - u_1(t,\xi)| \leqslant c_1|x_2(t,\xi) - x_1(t,\xi)|\ t \geqslant 0,\ \xi \in [0,\pi],$$

which implies that condition (ii) of theorem 4.26 is satisfied with $\sigma_\delta = c_1$.

Finally, as the function $a(\cdot)$ satisfies a Lipschitz condition, for every $x \in AAP(X)$ the function $\tilde{a} : [0,\infty) \to X$ given by $\tilde{a}(t)(\xi) = a(x(t,\xi))$ also is a.a.p., which shows that $\mathcal{S}_{f_2,x} \cap AAP(X) \neq \emptyset$. Combining with (4.13), and applying theorem 4.26, we infer that there exists an a.a.p. mild solution $u(\cdot)$ of problem (1.3)–(1.5).

EXAMPLE 4.28. Let $f_1 : [0,\infty) \times X \to X$ be a function as in example 4.27, and let $\tilde{f}_2 : X \to X$ be the map given by

$$\tilde{f}_2(x)(\xi) = a(x(\xi)), \quad 0 \leqslant \xi \leqslant \pi,$$

where a is a function that satisfies the conditions considered in example 4.13, and for every $s_0 \in \mathbb{R}$ there exist $\delta, k > 0$ such that

$$|a(s) - a(s_0)| \geqslant k|s - s_0| \tag{4.15}$$

for all $s \in \mathbb{R}$ such that $|s - s_0| < \delta$.

For fixed $\varepsilon > 0$, we define

$$f_2(x) = \tilde{f}_2(x + C_\varepsilon(0)) = \{\tilde{f}_2(x+z) : z \in C_\varepsilon(0)\},$$

where $C_\varepsilon(0) = \{z \in X : |z(\xi)| \leqslant \varepsilon, \text{ a.e. } 0 \leqslant \xi \leqslant \pi\}$. It is clear that $C_\varepsilon(0)$ is a closed convex subset of X. Under these conditions, problem (1.3)–(1.4) is modelled as (1.1)–(1.2) with $x_0 = \varphi$.

As a consequence of the intermediate value theorem, the map f_2 has convex values. Moreover, it follows from (4.15) that $f_2(x)$ is closed for all $x \in X$. In addition, for $u \in f_2(x)$ we obtain

$$\|u\| \leqslant c_0 + c_1\|x\|,$$

where $c_0 = \sqrt{\pi}|a(0)| + a_0\varepsilon$ and $c_1 = a_0$. This shows that f_2 satisfies (F6') with $\mu_r(t) = c_1 r + c_0$. We assume that (4.13) holds. It is also clear that f_2 is a Lipschitz continuous map. Collecting these assertions, we can apply again corollary 4.18 to conclude that there exists a mild solution $u(\cdot)$ of problem (1.3)–(1.5).

On the other hand, assume further that f_1 satisfies (4.14). In a similar way, using that $a(\cdot)$ is Lipschitz continuous, we obtain that condition (ii) of theorem 4.26 is satisfied with $\sigma_\delta = c_1$, and proceeding as in example 4.27, we infer that $\mathcal{S}_{f_2,x} \cap AAP(X) \neq \emptyset$. Combining with (4.13), and applying theorem 4.26, we infer that there exists an a.a.p. mild solution $u(\cdot)$ of problem (1.3)–(1.5).

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