Annals of Actuarial Science (2022), **16**, pp. 151–182
doi:10.1017/S1748499521000154

PAPER

A robust random coefficient regression representation of the chain-ladder method

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(Received 20 July 2020; revised 30 March 2021; accepted 17 April 2021; first published online 9 June 2021)

Abstract

It is well known that the presence of outliers can mis-estimate (underestimate or overestimate) the overall reserve in the chain-ladder method, when we consider a linear regression model, based on the assumption that the coefficients are fixed and identical from one observation to another. By relaxing the usual regression assumptions and applying a regression with randomly varying coefficients, we have a similar phenomenon, i.e., mis-estimation of the overall reserves. The lack of robustness of loss reserving regression with random coefficients on incremental payment estimators leads to the development of this paper, aiming to apply robust statistical procedures to the loss reserving estimation when regression coefficients are random. Numerical results of the proposed method are illustrated and compared with the results that were obtained by linear regression with fixed coefficients.

Keywords: Loss reserving; random coefficient regression models; robust estimation; chain-ladder method

JEL classification: G22

1 Introduction

The estimation of outstanding claims and claim reserves is crucial for the regular operation of insurance companies in order to manage losses, pay all possible claims, avoid insolvencies and determine the profit. Usually, in a loss reserving process embedded with ordinary least squares estimation, it is assumed that the regression coefficients are fixed, which means that the claims run-off pattern remains the same for each accident year.

In practice, many times, the claims run-off pattern is not the same because of unobserved endogenous or exogenous factors, which affect accident years and cannot be incorporated into the model. Thus, the assumption of fixed coefficients can be relaxed by allowing the regression coefficients to vary across individual claims and a linear regression model with random coefficients is proposed. In this way, the chain-ladder technique is presented in a regression form, where coefficients of explanatory variables (in our model the explanatory variables are dummy variables, indicating the position of a cell in a run-off triangle) may be subject to random variations, which means that the development of claims differs over the accident years.

Verrall (1994) proposed a model, which adapts a chain-ladder technique by allowing the development factor to evolve over the accident years, as a special application of his general state-space model (Verrall 1989), where a recursive model for row parameters had been used. De Alba (2002) and Wüthrich (2007) incorporated Bayesian methods to claims reserving. Furthermore, De Jong & Zehnwirth (1983) described a state-space approach for claims reserving that focuses on the forecasting nature of the claims reserving problem accommodating both objective and subjective information.

It is also well known that the loss reserving estimation is very sensitive to outliers (large claims or catastrophic events), which can lead to an unsatisfactory behaviour of the chain-ladder technique. Even small changes in observed values, in certain regions of the run-off triangle, can result in a large change in the predicted values and consequently to a mis-estimation of ultimate reserves. Kremer (1997) incorporated the ideas of robust statistics into loss reserving techniques. Verdonck et al. (2009) created a technique for detecting outliers in a run-off triangle of claim amounts and solved the problem of the non-robustness of the chain ladder by replacing the mean by the median, while Busse *et al.* (2010) designed a filter for outliers and large jumps and proposed a robust version of Mack's variance estimator. Verdonck & Debruyne (2011) exploited the influence function approach to present a diagnostic tool for highlighting the influence of every individual claim on the classical chain-ladder estimations and obtained robust estimations of generalised linear models in a chain-ladder framework. Also, a detailed overview of research papers related to the use of state-space models in claims reserving, where robustness can be achieved by using kalman filtering is presented by Chukhrova & Johannssen (2017).

Regarding some other robust approaches within the chain-ladder technique, Hubert *et al.* (2017) proposed the "FastSUR" algorithm in order to robustify the general multivariate chain-ladder method of Zhang (2010), where the parameters are estimated using seemingly unrelated regression (SUR). Based on the MM estimators, Peremans *et al.* (2018) suggested an alternative, robust method to estimate the SUR parameters in a more outlier-resistant way. Pitselis *et al.* (2015) applied a class of robust estimators to the chain-ladder procedure, where data are in a log-linear form, including robust estimators that simultaneously attain maximum breakdown point and full asymptotic efficiency under error normality. The aim of the present paper is to incorporate robust statistical procedures to the loss reserving estimation when regression coefficients are random.

The remainder of this paper is organised as follows. In section 2, we describe the random coefficient regression (RCR) model, the parameter estimation and how the loss reserving model can be incorporated into the RCR model. Section 3 deals with the identification of outliers and robust M and MM estimators of RCR models are proposed in order to remedy the effects of outliers that might appear in the data. In the same section, robust algorithms of the RCR models are constructed and applied to the loss reserving estimation. Numerical examples of the proposed methods are illustrated in section 4 and our concluding remarks are given in section 5.

2 Random Coefficient Regression Models

In insurance loss reserving applications, the idea of considering constant regression coefficients in successive observations may be questioned. There are cases where the coefficients are random, due to the fact that the claims development vary with accident years. The model of random coefficients has frequently different implications for decision-making problems. While in the ordinary least squares model, a decision affects only the mean, and in the RCR model, it both affects the mean and the variance. Here, we relax the usual regression assumptions by allowing the development factors to be subject to random variations. We consider the model of Hildreth & Houck (1968), where the response coefficients in a general linear model are considered as random variables and the mean of the distribution of these coefficients can be estimated. Some other key references related to random coefficient models are those of Swamy (1971), Hsiao (2003) and Greene (2012).

The multiple linear regression model with random coefficients is given by

$$Y_i = \beta_{1i} + \sum_{k=2}^{p} \beta_{ki} X_{ki} + \upsilon_{0i} = (\beta_1 + \upsilon_{1i}) + \sum_{k=2}^{p} (\beta_k + \upsilon_{ki}) X_{ki} + \upsilon_{0i}, \tag{1}$$

where Y is the dependent variable and the X s are the explanatory variables. We assume that for $i = 1, \ldots, n$, the υ_{0i}'s are independently distributed with zero mean and constant variance σ_0^2, while β_k and υ_{ki} are the deterministic and random parts, respectively, with $E(\beta_{ki}) = \beta_k$, for all

https://doi.org/10.1017/S1748499521000154 Published online by Cambridge University Press

i, $\mathrm{Var}(\beta_{ki}) = \mathrm{Var}(\upsilon_{ki}) = \sigma_k^2$, for all i, and $\mathrm{Cov}(\beta_{ki}, \beta_{k'i'}) = \mathrm{Cov}(\upsilon_{ki}, \upsilon_{k'i'}) = 0$ for $i \neq i'$ and $k \neq k'$, where $i' = 1, ..., n$ and $k' = 1, ..., p$.

Let $u_i = \upsilon_{0i} + \upsilon_{1i} + \sum_{k=2}^{p} X_{ki}\upsilon_{ki}$ as the error term, with $E(u_i) = 0$, $\mathrm{Cov}(u_i, u_{i'}) = 0$ for $i \neq i'$ and $\mathrm{Var}(u_i) = (\sigma_0^2 + \sigma_1^2) + \sum_{k=2}^{p} \sigma_k^2 X_{ik}^2 = \lambda_{ii}$, then (1) becomes

$$Y_i = \beta_1 + \sum_{k=2}^{p} \beta_k X_{ki} + u_i. \tag{2}$$

For n observations, (2) can be written as

$$\begin{vmatrix} Y_1 \\ \vdots \\ Y_n \end{vmatrix} = \begin{vmatrix} 1 \; X_{11} \; ... \; X_{p-1,1} \\ \vdots \\ 1 \; X_{1n} \; ... \; X_{p-1,n} \end{vmatrix} \begin{vmatrix} \beta_1 \\ \vdots \\ \beta_p \end{vmatrix} + \begin{vmatrix} u_1 \\ \vdots \\ u_n \end{vmatrix},$$

or compactly as

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}, \tag{3}$$

where $\boldsymbol{Y} = (Y_1, ..., Y_n)'$ is a $n \times 1$ vector, $\boldsymbol{X}$ is a $n \times p$ matrix of explanatory variables (design matrix), $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)'$ is a $p \times 1$ vector of coefficients and $\boldsymbol{u} = (u_1, ..., u_n)'$ is a $n \times 1$ vector of errors with $E(\boldsymbol{u}) = 0$ and $\boldsymbol{\Lambda} = E(\boldsymbol{u}\boldsymbol{u}') = \mathrm{diag}(\lambda_{11}, ..., \lambda_{nn})$. The regression coefficients $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_p)'$ and their variance are estimated by

$$\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{\Lambda}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Lambda}^{-1}\boldsymbol{Y} \text{ and } \mathrm{Var}(\widehat{\boldsymbol{\beta}}) = (\boldsymbol{X}'\boldsymbol{\Lambda}^{-1}\boldsymbol{X})^{-1}. \tag{4}$$

But, $\boldsymbol{\Lambda}$ is also unknown since σ_k^2's are unknown. Thus, in practice, the estimated regression coefficients in (4) can be obtained by using an estimator of $\boldsymbol{\Lambda}$. Following Hildreth & Houck (1968) we consider the least square residuals $\boldsymbol{r} = (r_1, ..., r_n)' = \boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b} = \boldsymbol{M}\boldsymbol{Y} = \boldsymbol{M}\boldsymbol{u}$, where $\boldsymbol{b}$ is the least squares estimator of $\boldsymbol{\beta}$, i.e., $\boldsymbol{b} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$ and $\boldsymbol{M} = \{m_{ij}\}_{i,j=1,..n} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$. Then we have $E(r_i) = 0$ and $E(r_i^2) = \mathrm{Var}(r_i) = \sum_{j=1}^{n} m_{ij}\lambda_{ij}$, or in matrix form

$$E(\dot{\boldsymbol{r}}) = \dot{\boldsymbol{M}}\dot{\boldsymbol{X}}\dot{\boldsymbol{\sigma}}, \tag{5}$$

where $\dot{\boldsymbol{r}} = (r_1^2, ..., r_n^2)'$, $\dot{\boldsymbol{M}} = \boldsymbol{M} * \boldsymbol{M} = \{m_{ij}^2\}_{i,j=1,...,n}$ and $\dot{\boldsymbol{\sigma}} = (\sigma_1^2, ..., \sigma_p^2)'$. The symbol $*$ represents the Hadamard matrix product, i.e., for two matrices $\boldsymbol{A} = \{a_{ij}\}_{i,j=1,..n}$ and $\boldsymbol{B} = \{b_{ij}\}_{i,j=1,..n}$, $\boldsymbol{A} * \boldsymbol{B} = \{a_{ij}b_{ij}\}_{i,j=1,..n}$. Then (5) can be written as

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{M}}\dot{\boldsymbol{X}}\dot{\boldsymbol{\sigma}} + \boldsymbol{w} = \boldsymbol{G}\,\dot{\boldsymbol{\sigma}} + \boldsymbol{w}. \tag{6}$$

The formulation of (6) can be considered as a regression of the ordinary least squares residuals $\dot{\boldsymbol{r}}$ on $\dot{\boldsymbol{\sigma}}$, where $\boldsymbol{G} = \dot{\boldsymbol{M}}\dot{\boldsymbol{X}}$, $E(\boldsymbol{w}) = 0$ and variance–covariance matrix $\mathrm{Cov}(\boldsymbol{w}) = E(\boldsymbol{w}\boldsymbol{w}') = 2\dot{\boldsymbol{\Omega}}$, where $\dot{\boldsymbol{\Omega}} = \boldsymbol{\Omega} * \boldsymbol{\Omega}$, with $\boldsymbol{\Omega} = E(\boldsymbol{r}\boldsymbol{r}') = E(\boldsymbol{M}\boldsymbol{u}\boldsymbol{u}'\boldsymbol{M}) = \boldsymbol{M}\boldsymbol{\Lambda}\boldsymbol{M}$. Then, the generalised least squares estimator of $\dot{\boldsymbol{\sigma}}$ in (6) is given by (see Hildreth & Houck (1968))

$$\widehat{\dot{\boldsymbol{\sigma}}} = (\boldsymbol{G}^T\dot{\boldsymbol{\Omega}}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}^T\dot{\boldsymbol{\Omega}}^{-1}\dot{\boldsymbol{r}}, \tag{7}$$

where $\widehat{\dot{\boldsymbol{\Omega}}} = \widehat{\boldsymbol{\Omega}} * \widehat{\boldsymbol{\Omega}}$, with $\widehat{\boldsymbol{\Omega}} = \boldsymbol{M}\widehat{\boldsymbol{\Lambda}}\boldsymbol{M}$ and $\widehat{\boldsymbol{\Lambda}} = \mathrm{diag}(\hat{\lambda}_{11}, ..., \hat{\lambda}_{nn})$, with

$$\hat{\lambda}_{ii} = \hat{\sigma}_1^2 + \sum_{k}^{p} \hat{\sigma}_k^2 X_{ki}^2. \tag{8}$$

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2.1 Actual response coefficients in a purely random coefficient model

Sometimes, it is useful to predict the individual component of the actual response coefficients β_i that provides information about the behaviour of each individual claim. Griffiths (1972) derived the minimum variance and linear unbiased estimators for the actual response coefficients, and has shown that the best estimator of the actual response coefficients is not identical to the best estimator of the mean response coefficients.

The vector $\boldsymbol{\beta}$ in (3) contains the mean response coefficients and the actual response coefficients given by

$$\boldsymbol{b} = \boldsymbol{L}\boldsymbol{\beta} + \boldsymbol{v}, \tag{9}$$

where $\boldsymbol{L}^{'}$ is a $p \times np$ matrix, $\boldsymbol{b}^{'}$ and $\boldsymbol{v}^{'}$ are $1 \times np$ vectors defined, respectively, as

$$\boldsymbol{L}^{'} = \begin{vmatrix} 1\ 1..\ 1 & & & \\ & 1\ 1...\ 1 & & \\ & & \ddots & \\ & & & 1\ 1...\ 1 \end{vmatrix},$$

$$\boldsymbol{b}^{'} = (\boldsymbol{b}_1^{'}, \boldsymbol{b}_2^{'}, ..., \boldsymbol{b}_p^{'}), \text{ with } \boldsymbol{b}_k = (b_{1k}, b_{2k}, ..., b_{nk})^{'}$$

and

$$\boldsymbol{v}^{'} = (\boldsymbol{v}_1^{'}, \boldsymbol{v}_2^{'}, ..., \boldsymbol{v}_p^{'}), \text{ with } \boldsymbol{v}_k = (v_{1k}, v_{2k}, ..., v_{nk})^{'}.$$

Then, we obtain

$$\boldsymbol{V} = E(\boldsymbol{v}\boldsymbol{v}^{'}) = \begin{vmatrix} \sigma_{11}\boldsymbol{I} & & & \\ & \sigma_{22}\boldsymbol{I} & & \\ & & \ddots & \\ & & & \sigma_{pp}\boldsymbol{I} \end{vmatrix}.$$

If $\boldsymbol{X}_k = diag(X_{1k}, X_{2k}, ..., X_{nk})$ and $\boldsymbol{Z}$ a $n \times np$ matrix defined as $\boldsymbol{Z} = (\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_p)$, then

$$\boldsymbol{u} = \boldsymbol{Z}\boldsymbol{v} \text{ and } \boldsymbol{\Lambda} = E(\boldsymbol{u}\boldsymbol{u}^{'}) = \boldsymbol{Z}\boldsymbol{V}\boldsymbol{Z}^{'}.$$

Following Griffiths (1972), the estimates of the disturbances associated with the *kth* coefficient can be written as $\hat{\boldsymbol{v}}_k = \sigma_{kk}\boldsymbol{X}_k\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{u}}$, or compactly as

$$\hat{\boldsymbol{v}} = \boldsymbol{V}\boldsymbol{Z}_k\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{u}}.$$

Thus, we can obtain an estimator of $\boldsymbol{b}$ as

$$\hat{\boldsymbol{b}} = \boldsymbol{L}\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{v}} = \boldsymbol{L}(\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1}\boldsymbol{Y} + \boldsymbol{V}\boldsymbol{Z}^{'}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{u}},$$

which via $\hat{\boldsymbol{u}} = [\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1}]\boldsymbol{Y}$ leads to the following linear expression:

$$\hat{\boldsymbol{b}} = \boldsymbol{L}(\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1} + \boldsymbol{V}\boldsymbol{Z}^{'}\boldsymbol{\Lambda}^{-1}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{\Lambda}^{-1}]\boldsymbol{Y} = \boldsymbol{Q}\boldsymbol{Y} \tag{10}$$

and the variance of $\boldsymbol{b}$ as

$$Var(\hat{\boldsymbol{b}}) = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{'} - \boldsymbol{Q}\boldsymbol{Z}\boldsymbol{V} - \boldsymbol{V}\boldsymbol{Z}^{'}\boldsymbol{Q} + \boldsymbol{V}.$$

2.2 Random coefficients regression in loss reserving procedures

A run-off triangle can be divided into cells, where each cell is the corresponding payment arising from a specific accident year $i \in \{1, \cdots, n\}$ (rows) and a development year $j \in \{1, \cdots, n\}$ (columns). The accident year shows the losses that occurred during a specific period, while the development year specifies after how many years the reported claim is getting settled.

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Accident Year i	Development Year j 1	2	$\cdots$	j	$\cdots$	n-1	n
1	Y_{11}	Y_{12}	$\cdots$	Y_{1j}	$\cdots$	$Y_{1,n-1}$	Y_{1n}
2	Y_{21}	Y_{22}	$\cdots$	Y_{2j}	$\cdots$	$Y_{2,n-1}$	
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		
i	Y_{i1}	$\cdots$	$\cdots$	$Y_{i,n+1-i}$			
$\vdots$	$\cdots$	$\cdots$	$\cdots$				
n	Y_{n1}						

Figure 1. Run-off triangle of paid claims.

The calendar year k is the diagonal element of the triangle and is defined as $k = i + j$, with $k \in \{1, 2, \cdots, n\}$. Then, Y_{ij} is defined as the total incremental payments in accident year i with development year j, where $i + j \leq n$ as the calendar $i + j > n$ has not occurred yet. Some reserving methods use the cumulative payments $S_{in} = \sum_{j=1}^{n} Y_{ij}$. Let D_n denote the information available in calendar year n (the upper part of the triangle)

$$D_n = \{Y_{ij}, i + j \leq n\} = \{S_{ij}, i + j \leq n\}.$$

For accident year i, we want to get the best estimate for the total amount of payment, given the information D_n, i.e.,

$$\widehat{S}_{i\infty}^{n-i} = \lim_{j \to \infty} E[S_{ij}|D_n] = E[S_{i\infty}|D_n],$$

and the difference with the payment made in calendar year n gives the required reserve

$$\widehat{R}_i = \widehat{S}_{i\infty}^{n-i} - S_{i,n-i+1}.$$

Finally, an interesting quantity is the (ultimate) uncertainty, which is the variance of the total amount of payment $Var(S_{i\infty}|D_n)$ or $Var(\widehat{S}_{i\infty}^{n-i})$. In our examples, we make the assumption that all claims have a lifetime of n years. After n years all claims have been settled, which means that $Y_{ij} = 0$ for each $j = n + 1, n + 2, \ldots$ The RCR model defined in (1) can also be written, in connection with Figure 1, as

$$\begin{vmatrix} Y_{1,\leq n} \\ \vdots \\ Y_{n,\leq 1} \end{vmatrix} = \begin{vmatrix} 1 \ X_{11} \ \ldots \ X_{p-1,1} \\ \vdots \\ 1 \ X_{1n} \ \ldots \ X_{p-1,n} \end{vmatrix} \begin{vmatrix} \beta_1 \\ \vdots \\ \beta_p \end{vmatrix} + \begin{vmatrix} \boldsymbol{X}_1^{'} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \ldots & \boldsymbol{X}_n^{'} \end{vmatrix} \begin{vmatrix} \boldsymbol{\upsilon}_1 \\ \vdots \\ \boldsymbol{\upsilon}_n \end{vmatrix},$$

or compactly as

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{D}_x\boldsymbol{\upsilon}, \tag{11}$$

where

(i) $\boldsymbol{Y} = (Y_{1,\leq n}, Y_{2,\leq n-1}, \ldots, Y_{n,\leq 1})^{'}$ is the $[n(n+1)/2] \times 1$ vector of the past and present incremental claims in the upper triangle,

(ii) $\boldsymbol{X} = (\boldsymbol{X}_1^{'}, \ldots, \boldsymbol{X}_n^{'})^{'}$ is a $n(n+1)/2 \times p$ design matrix, with $p = 2n - 1$, reflecting the position of incremental claims in Table 1,

(iii) $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^{'}$, with $p = 2n - 1$ unknown parameters,

(iv) $\boldsymbol{D}_x$ is a block diagonal matrix of $\boldsymbol{X}_i^{'}s$, with $\boldsymbol{X}_i^{'}$ being the ith row of $\boldsymbol{X}$,

(v) $\boldsymbol{\upsilon} = (\boldsymbol{\upsilon}_1^{'}, \ldots, \boldsymbol{\upsilon}_n^{'})^{'}$ is a $[n(n+1)/2] \times 1$ vector, with $\boldsymbol{\upsilon}_i = (\upsilon_{1i}, \ldots, \upsilon_{ni})^{'}$.

Remark 1. *The magnitude of p depends on the choice of the appropriate design matrix that reflects the position of the claims in the run-off triangle (see Christofides (1997)).*

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3 Robust Estimation and Identification of Outliers

Most of the insurance data include claim size distributions with long tail. Robust regression techniques are complementary tools for the ordinary least squares estimation when the errors do not satisfy the normality condition due to large deviations or outlier observations.

Outliers are observations that may be caused by exceptional events (e.g. a catastrophic event). Therefore, outliers can be described as points, which do not follow the trend of the majority of data. The problem occurs when a trend (due to an outlier) that appeared in one of the first accident years of a chain-ladder model carried on for the next years, resulting in an overestimation or underestimation of the reserves. In order to remedy the effect of outliers, we can use robust statistics.

3.1 M estimators

A statistical procedure is called robust, if it is insensitive to the occurrence of gross errors in the data. M estimators are a generalisation of Maximum Likelihood (ML) estimators. Instead of minimising the sum of scores $\log f(x,\theta)$ as in the ML estimation, a more general function $\rho(x,\theta)$ is allowed (see Huber (1981)). M estimators as solutions of the following minimisation problem:

$$\min \sum_{i=1}^{n} \rho\left(\frac{r_i}{S}\right),$$

where $\rho(.)$ is a symmetric function with unique minimum at zero, r_i is the ith residual and S is a scale parameter. Differentiating this expression with respect to the regression coefficients $\boldsymbol{\beta}$ yields,

$$\sum_{i=1}^{n} \psi\left(\frac{r_i}{S}\right)\boldsymbol{x}_i = \mathbf{0},$$

where $\psi(.)$ is the derivative of $\rho(.)$ and $\boldsymbol{x}_i$ is the row vector of explanatory variables of the ith case. In practice, it is advisable to use $S = med|r_i|$ as an initial value. The advantage of M estimators is that they can be computed in much less time than other robust estimators. The disadvantage is that they are sensitive to high leverage points and they do not enjoy high breakdown point (see Rousseeuw & Leroy (1987)).

3.2 MM estimators

The MM estimators proposed by Yohai (1987) have high breakdown point and high efficiency when the errors are normally distributed. The MM estimators are defined in a three-stage procedure (see Maronna *et al.* (2006)) as follows:

(i) Compute an initial estimate $\widehat{\boldsymbol{\beta}}_0$ of $\boldsymbol{\beta}$ in (3). This estimate is consistent with high breakdown point and possibly low normal efficiency.

(ii) Compute an M estimate of the form

$$L(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n} \rho\left(\frac{r_i}{S}\right) = 0.5,$$

where $\rho(.)$ is as defined before.

(iii) Find a solution $\widehat{\boldsymbol{\beta}}$ of the relation

$$\frac{1}{n}\sum_{i=1}^{n} \psi\left(\frac{r_i}{S}\right)\boldsymbol{x}_i = \mathbf{0},$$

using an iterative procedure starting with the initial estimate $\widehat{\boldsymbol{\beta}}_0$.

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3.3 Robust functions

Robust functions can be constructed for location and scale parameters, and they can also be used in robust regression. In the following, we present some robust functions that are going to be used to robustify our models. These functions have the advantage of combining robustness with efficiency under the regression model with normal errors (see Huber (1981) and Hampel *et al.* (1986)).

3.3.1 Huber function

The family of Huber functions is defined as

$$\rho_k(x) = \begin{cases} \frac{1}{2}x^2, & \text{if } |x| \le k, \\ k\left(|x| - \frac{k}{2}\right), & \text{if } |x| > k, \end{cases}$$
$$\psi_k(x) = \begin{cases} x, & \text{if } |x| \le k, \\ k\, sign(x), & \text{if } |x| > k. \end{cases} \tag{12}$$

The value k is called a tuning constant. Note that smaller values of k produce more resistance to outliers, but at the expense of lower efficiency when the errors are normally distributed. The tuning constant is generally picked to give reasonably high efficiency in the normal case. In particular, $k = 1.345$ gives 95% efficiency at the normal model, and still offer protection against outliers.

3.3.2 Hampel function

The Hampel family of functions is defined as

$$\rho_{a,b,r}(x) = \begin{cases} \frac{1}{2}x^2/C, & \text{if } |x| \le a, \\ \left[\frac{1}{2}a^2 + a(|x| - a)\right] / \left[\frac{a}{2}(b - a + r)\right], & \text{if } a < |x| \le b, \\ \frac{a}{2}\left[2b - a + \left(|x| - b\right)\left(1 + \frac{r-|x|}{r-b}\right)\right] / \left[\frac{a}{2}(b - a + r)\right], & \text{if } b < |x| \le r, \\ 1, & \text{if } r < |x|, \end{cases}$$
$$\psi_{a,b,r}(x) = \begin{cases} |x|\, sign(x), & \text{if } |x| \le a, \\ a\, sign(x), & \text{if } a < |x| \le b, \\ a\, \frac{r-|x|}{r-b}\, sign(x), & \text{if } b < |x| \le r, \\ 0, & \text{if } r < |x|, \end{cases} \tag{13}$$

where the slope of the redescending part ($x \in (b, r]$) is set to $-1/2$, i.e., $r = 2a + b$. Then values $a = 2$, $b = 4$, $r = 8$ give the desired efficiency. Note that $\psi_{a,b,r}(x)$ can also be tuned to have a downward slope of $-1/3$ (see Koller and Stahel (2011)), where $a = 1.353$, $b = 3.157$, $r = 7.216$ give 95% efficiency at the normal.

3.3.3 Bisquare function

Tukey's bisquare family of functions is defined as

$$\rho_k(x) = \begin{cases} 1 - \left[1 - \left(\frac{x}{k}\right)^2\right]^3, & \text{if } |x| \le k, \\ 1, & \text{if } |x| > k, \end{cases} \tag{14}$$

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$$\psi_k(x) = \begin{cases} x\left[1-\left(\frac{x}{k}\right)^2\right]^2, & \text{if } |x| \le k, \\ 0, & \text{if } |x| > k. \end{cases} \tag{15}$$

For $k = 4.685$, the regression estimator produce 95% efficiency at the normal.

The aforementioned robust functions are included in various R packages such as the "MASS" (2002), the "robust" (2020) and the "robustbase" (2020) packages. More specifically with the rlm function of "MASS" package, we can fit a robust linear regression model using M and MM estimators based on the chosen method argument (M by default). Also, by entering only the intercept into the model formula, this function returns the M estimator of location (Huber by default). Moreover, the robust (psi) functions (12), (13) and (15) can be specified by an argument psi with possible values psi.huber, psi.hampel and psi.bisquare, respectively. For more details on robust statistical methods with R, the reader may be referred to Jurečková *et al.* (2019).

3.4 Algorithms for robust estimation of the random coefficients regression model

We present the steps that are required to obtain the robust M and MM estimation of the purely RCR model as it was presented in section 2.1. The algorithm is slightly different than the algorithm presented in Pitselis (2014). Also, an additional step is added (Step 5) in the case we want to obtain the robust estimation of the actual response RCR model in (10).

3.4.1 Algorithm for M estimation

Step 1: Following Huber & Dutter (1974), we can obtain M estimators of $\boldsymbol{\beta}$ in (3), as solutions of the following minimisation form:

$$Q(\boldsymbol{\beta}, S) = \frac{1}{n}\sum_{i=1}^{n}\left[\rho\left(\frac{Y_i - f_i(\boldsymbol{\beta})}{S}\right) + A\right] S \to \min, \tag{16}$$

where $f_i(\boldsymbol{\beta}) = \sum_{j=0}^{p} x_{ij}\beta_j$, with $\rho(0) = 0$ and $A > 0$. Differentiating (16) with respect to $\boldsymbol{\beta}$ and S we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\psi\left(\frac{r_i}{S}\right)\frac{\partial f_i(\hat{\boldsymbol{\beta}})}{\partial \beta_j} = 0 \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}\chi\left(\frac{r_i}{S}\right) = A\,,$$

where $r_i = X_i - f_i(\boldsymbol{\beta})$, with $\psi(t) = \rho'(t)$ and $\chi(t) = t\psi(t) - \rho(t)$. Note that $\chi(t)$ has an absolute minimum at $t = 0$, namely, $\chi(0) = 0$. We also assume that ψ and χ are continuous. In order to obtain consistency of the scale estimator in the normal model and to recapture the classical estimators for the classical choice $\rho(t) = \frac{1}{2}t^2$, we take (see Huber 1981):

$$A = \frac{(n-p-1)}{n}E_\Phi(\chi),$$

where E_Φ denotes the expectation taken with respect to a normal distribution.

Step 2: Based on robust estimate of $\boldsymbol{\beta}$ and scale S obtained in step 1, winsorize the residuals:

$$r_i^R = \begin{cases} \psi\left(\frac{r_i}{S}\right) S, & \text{for } r_i < -cS, \\ r_i, & \text{for } |r_i| \le cS, \\ cS, & \text{for } r_i > cS, \end{cases}$$

and obtain $\dot{\boldsymbol{r}}^R = \boldsymbol{r}^R * \boldsymbol{r}^R$ with $\boldsymbol{r}^R = (r_1^R, \ldots, r_n^R)'$ as in (5).

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Step 3: Estimate the coefficients $\dot{\boldsymbol{\sigma}}$ in the regression setting (6), i.e.,

$$\dot{\boldsymbol{r}}^R = \dot{\boldsymbol{M}}\dot{\boldsymbol{X}}\dot{\boldsymbol{\sigma}} + \boldsymbol{w} = \boldsymbol{G}\,\dot{\boldsymbol{\sigma}} + \boldsymbol{w},$$

and obtain

$$\hat{\dot{\boldsymbol{\sigma}}} = (\boldsymbol{G}^T(\dot{\boldsymbol{\Omega}}^R)^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}^T(\dot{\boldsymbol{\Omega}}^R)^{-1}\dot{\boldsymbol{r}}^R,$$

where

$$\widehat{\dot{\boldsymbol{\Omega}}}^R = \widehat{\boldsymbol{\Omega}}^R * \widehat{\boldsymbol{\Omega}}^R, \quad \text{with} \quad \widehat{\boldsymbol{\Omega}}^R = \boldsymbol{M}\widehat{\boldsymbol{\Lambda}}^R\boldsymbol{M},$$

and $\widehat{\boldsymbol{\Lambda}}^R = \text{diag}(\hat{\lambda}_{11}, ..., \hat{\lambda}_{nn})$, with $\hat{\lambda}_{ii}^R = \hat{\sigma}_1^2 + \sum_k^p \hat{\sigma}_k^2 X_{ki}^2$.

Step 4: Then the robust estimator of $\widehat{\boldsymbol{\beta}}$ in (4) and scale S_2 can be obtained by the following minimisation problem:

$$Q(\beta, S_2) = (1/n)\sum_{i=1}^{n}\left\{\rho\left[\widehat{\lambda}_{ii}^{1/2}[Y_i - f_i(\boldsymbol{\beta})]/S_2\right] + a_2\right\}S_2,$$

where $a_2 = [(n-p)/n]E_\Phi(\chi)$, with E_Φ and $\rho(.)$ as defined before.

Step 5: To obtain a robust estimator $\boldsymbol{b}^R$ of the actual response $\boldsymbol{b}$ in (9), we substitute the non-robust estimators by the robust ones, i.e.,

$$\hat{\boldsymbol{b}}^R = \boldsymbol{L}\hat{\boldsymbol{\beta}}^R + \boldsymbol{\Lambda}^R\boldsymbol{Z}'(\boldsymbol{\Lambda}^R)^{-1}\hat{\boldsymbol{u}}^R,$$

where the elements of $\boldsymbol{u}^R = (\hat{u}_i^R, \hat{u}_2^R, ..., \hat{u}_n^R)'$ are the winsorized residuals as in Step 2,

$$\hat{u}_i^R = S\psi\left(\frac{u_i}{S}\right), \quad i = 1, 2, ..., n.$$

3.4.2 Algorithm for MM estimation

The algorithm for obtaining the MM estimation of the RCR model is similar to the algorithm for the M estimation, except for Steps 1 and 4, which are now replaced by the three-stage MM estimation, defined in section 3.3, as a modified version of the iterated weighted least squares algorithm used for the M estimation (see Huber (1981) and Yohai (1987)). The MM estimation is based on bisquare ρ function given in (14).

Besides the M estimators and the MM estimators, there are a lot of other robust estimators, which can be applied to the RCR model. Each time, the appropriateness of a robust estimator depends on how much of our data are contaminated, the efficiency of the regression estimators that we would like to obtain and how the appearance of outlier events depends on the design matrix of the regression model. For more details on these estimators, the reader may be referred among others to Hampel *et al.* (1986), Rousseeuw & Leroy (1987), Yohai & Zamar (1997), Gervini & Yohai (2002) and Maronna *et al.* (2006).

Remark 2. *In general, when the design matrix consists of dummy variables, M estimation shows a consistency on the expected ultimate losses and reserves in the presence of artificial extreme events and performs slightly better than the MM estimation. This may be due to the fact that estimators, which are robust to outliers of the design matrix (e.g. the MM estimators), in addition to dependent variables, they can also robustify independent variables (in our case these are fixed dummy variables).*

Remark 3. *The excess of reserves (bias) that appeared due to the robustification of our RCR model can be distributed equivalently to all accident years ultimate reserves, or according to expert recommendations. Parameter estimators from robust approaches are asymptotically biased, when error distributions are not symmetric. Treatment of bias appearing into the robustified model is necessary. In this spirit, Wang et al. (2005) proposed a distribution-free method for correcting this bias.*

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4 Numerical Illustrations

In this section, we apply the robust non-random and random linear regression models, first, to the widely used dataset of Taylor & Ashe (1983), and second, to a recent dataset from the motor business line of a non-life insurance company of the Greek insurance industry. The datasets are presented in Appendices A and B.

For the applications that follow with both datasets, we use the model of Christofides (1997), expressed under the RCR formulation (11). The same model can also be embedded within the quantile regression framework, as presented in Badounas and Pitselis (2020) for longitudinal data. After dealing with different design matrices, we selected a model, which incorporates 12 independent variables (without intercept), assuming a linear relationship for the parameters of development periods with the same slope, defined by

$$Y_{ij} = a_i + d_j + \epsilon_{ij}, \quad \text{for } i, j \text{ from 1 to 10},$$

where $d_0 = d$, $d_j = s \times j$, for $j > 1$ and ϵ_{ij} is the error term.

Next, we provide estimations with fixed and RCR methods for the ultimate reserves of each accident year, as well as for the total ultimate reserves in the absence and presence of outliers. The robust M and MM estimation with fixed and random coefficients was implemented into the R statistical software (2020), as a combination of own RCR procedures with the rlm function of "MASS" package.

4.1 Claims data from Taylor and Ashe (1983)

First, we have to investigate if data are contaminated by outlying observations, which may have a large impact on the least squares estimates and consequently in the loss reserving estimation. This can be achieved by using diagnostic plots, such as the Q–Q plot and the plot of standardised residuals versus fitted values. Second, in order to study the impact of outlying values, we create one and two artificial outliers, just by multiplying one or two observations of Taylor and Ashe data by **10**, a technique that has been used in similar studies (see Verdonck *et al.* (2009), Verdonck and Debruyne (2011), Pitselis *et al.*, (2015)).

4.1.1 Original data

Applying the least squares estimation with the original data, the Q–Q plot of standardised residuals versus the normal quantiles and the standardised residuals versus fitted values in Figure 2 (left panels) show that the observation Y_{44} (which corresponds to the accident year 4 and development year 4) is an outlier event. The same is evident from the standardised residuals versus fitted values for the robust M and MM estimation (middle and right panels), which also identify observation Y_{44} as an outlier.

The left panel of Table 1 shows the values of overall reserves based on least squares (LS) estimation, M estimation (Huber, Hampel, Bisquare functions) and MM estimation using the original data. In the last row, we observe that the values of total reserves based on robust estimations are slightly different (smaller) than the value of total reserves based on least squares estimation with fixed coefficient and this is due to the robustification of the outlier event Y_{44}.

In the sequel, applying regression and robust regression models with random coefficients (right panel, last row of Table 1), we observe an increase of the total reserves, in comparison with total reserves obtained with fixed coefficients, which is due to the involvement of the weights $\widehat{\lambda}_{ii}$ in (8). Also, Figure 3 indicates that the values of ultimate reserves show a similar behaviour across each accident year, except for some small fluctuations in fixed coefficients estimation for year 4 and between years 7 and 8.

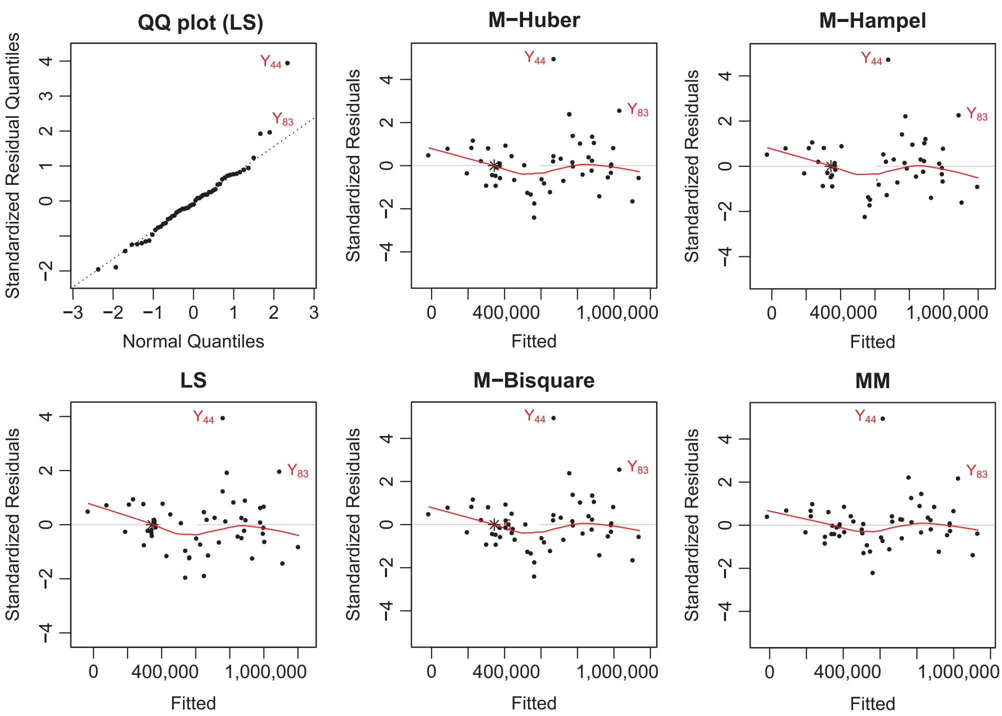

Figure 2. Taylor and Ashe original data: Diagnostic plots for non-robust LS estimation (left panels) and robust M and MM estimation (middle and right panels).

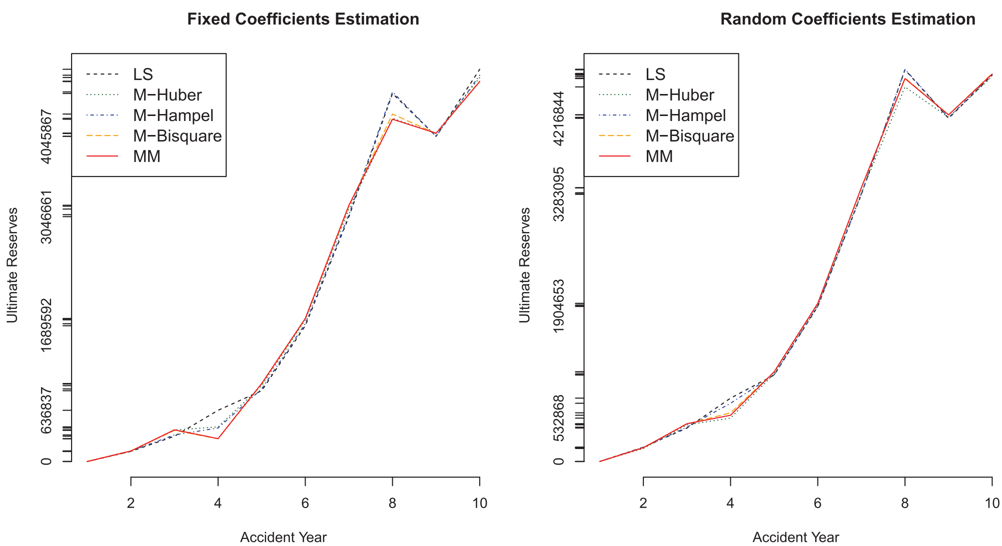

Figure 3. Estimated ultimate reserves against year with fixed coefficients (left panel) and random coefficients (right panel) robust regression models – Taylor and Ashe original data.

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Table 1. Ultimate reserves per accident year – Taylor and Ashe original data.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
Year	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
1	0	0	0	0	0	0	0	0	0	0
2	126, 286	124, 078	130, 686	131, 302	130, 445	175, 710	159, 788	175, 259	169, 633	169, 071
3	314, 225	391, 496	330, 120	387, 279	394, 237	411, 070	450, 802	421, 434	461, 358	464, 198
4	636, 837	431, 236	416, 852	282, 566	284, 627	778, 239	532, 868	715, 516	598, 145	568, 307
5	880, 729	943, 666	902, 954	964, 314	970, 329	1, 062, 396	1, 077, 508	1, 065, 885	1, 100, 532	1, 102, 213
6	1, 689, 592	1, 762, 837	1, 714, 614	1, 774, 110	1, 779, 877	1, 904, 653	1, 922, 421	1, 909, 081	1, 941, 639	1, 942, 960
7	3, 046, 661	3, 141, 402	3, 071, 721	3, 177, 001	3, 188, 477	3, 283, 095	3, 302, 396	3, 288, 528	3, 361, 586	3, 364, 850
8	4, 579, 337	4, 269, 931	4, 598, 915	4, 323, 342	4, 258, 811	4, 813, 098	4, 602, 725	4, 819, 669	4, 711, 126	4, 705, 457
9	4, 045, 867	4, 084, 922	4, 046, 166	4, 080, 907	4, 084, 846	4, 216, 844	4, 229, 114	4, 224, 884	4, 260, 512	4, 259, 409
10	4, 880, 559	4, 777, 990	4, 806, 389	4, 732, 644	4, 727, 172	4, 748, 307	4, 731, 813	4, 759, 142	4, 768, 417	4, 761, 075
Total	20, 200, 094	19, 927, 558	20, 018, 418	19, 853, 465	19, 818, 822	21, 393, 411	21, 009, 436	21, 379, 399	21, 372, 946	21, 337, 541

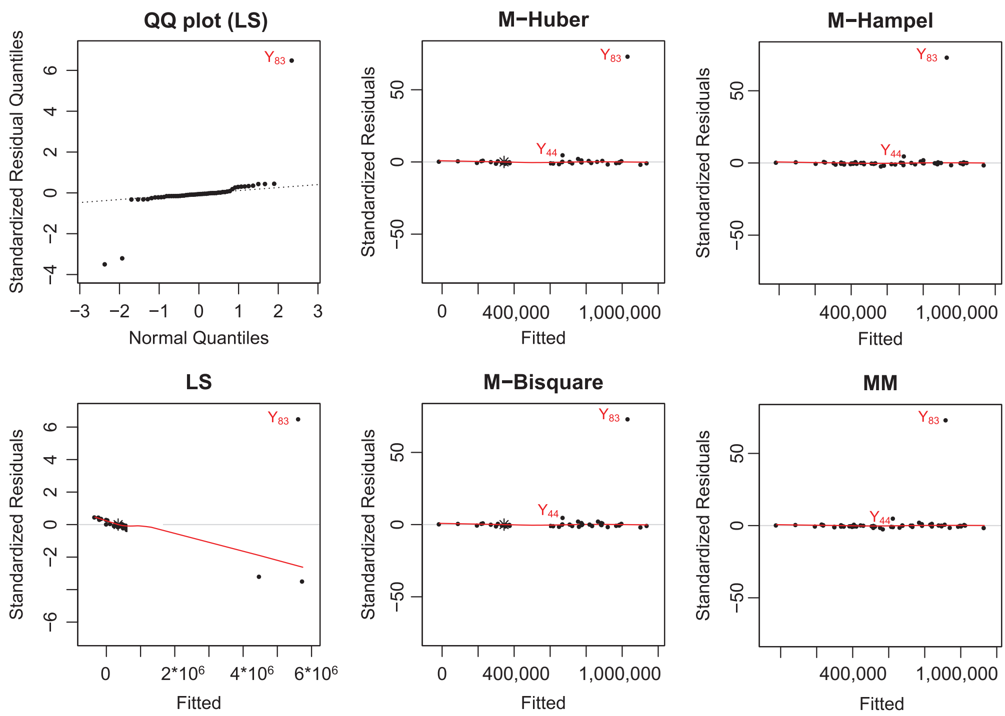

Figure 4. Taylor and Ashe data with one artificial outlier: Diagnostic plots for non-robust LS estimation (left panels) and robust M and MM estimation (middle and right panels).

4.1.2 One artificial outlier

In order to create an artificial outlier, we multiply the observation Y_{83} by **10**, i.e., $Y_{83} = 14{,}433{,}70$**0** (which corresponds to accident year 8 and development year 3). Figure 4 displays residual diagnostics in the presence of one artificial outlier for the non-robust LS and robust M and MM estimation. We can easily observe that claims Y_{83} is identified as an outlier.

In the presence of one artificial outlier (Y_{83}), the LS model leads to a significant overestimation of the total reserves, which is more evident in LS estimation with fixed coefficients (Table 1: 20,200,094, Table 2: 61,590,318). Moreover, Figure 5 displays an abrupt jump in LS estimates of year 8 due to the origin of the selected outlier. On the other hand, robust estimates of the total ultimate reserves in Table 2 lie on the same levels with the corresponding values of Table 1. Especially, M-Huber estimates of total reserves are less affected by the outlier, giving similar values for fixed (Table 1: 19,927,558, Table 2: 19,926,951) and random coefficient models (Table 1: 21,009,436, Table 2: 21,011,074).

When presenting reserve estimates, it is equally important to obtain their standard errors. Table 3 presents the reserve estimates and their standard errors for different estimation methods applied to Taylor and Ashe data with one artificial outlier (Y_{83}). Although the incremental claims of the initial data are not negative, there are cases where robust estimation under the fixed coefficient regression produces few negative predicted outstanding claims. We can see that for all reserve estimates, the RCR models give smaller standard error values than those produced by fixed coefficients regression, due to the involvement of the weights $\hat{\lambda}_{ii}$ in (8) for the estimation of the generalised least squares estimator of $\widehat{\sigma}$ in (7). We also observe that either for fixed coefficient (left panel) or RCR models (right panel), the robust methods give smaller standard error values than the LS ones.

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Table 2. Ultimate reserves per accident year – Taylor and Ashe data with one outlier (Y_{83}).

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
Year	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
1	0	0	0	0	0	0	0	0	0	0
2	173, 511	124, 134	127, 831	128, 614	128, 178	129, 433	159, 811	129, 221	166, 024	165, 948
3	426, 082	391, 266	334, 537	386, 470	389, 955	319, 657	450, 645	345, 947	458, 253	458, 872
4	838, 191	431, 164	389, 300	274, 290	275, 454	643, 318	532, 969	399, 475	554, 261	543, 682
5	1, 208, 880	943, 576	897, 034	956, 762	959, 952	886, 411	1, 077, 368	907, 126	1, 088, 803	1, 089, 338
6	2, 204, 219	1, 762, 727	1, 702, 383	1, 761, 427	1, 764, 722	1, 691, 514	1, 922, 249	1, 712, 554	1, 924, 879	1, 925, 325
7	3, 852, 202	3, 139, 522	3, 048, 348	3, 152, 244	3, 158, 626	3, 039, 642	3, 302, 193	3, 056, 154	3, 333, 041	3, 334, 158
8	36, 195, 439	4, 271, 882	3, 287, 922	3, 349, 442	3, 353, 652	34, 863, 786	4, 605, 369	3, 274, 845	3, 538, 405	3, 538, 697
9	6, 373, 179	4, 084, 782	3, 957, 033	4, 015, 499	4, 019, 712	3, 974, 341	4, 228, 871	3, 938, 116	4, 200, 110	4, 200, 337
10	10, 318, 615	4, 777, 900	4, 575, 685	4, 573, 994	4, 576, 749	4, 660, 216	4, 731, 600	4, 488, 421	4, 636, 722	4, 636, 037
Total	61, 590, 318	19, 926, 951	18, 320, 074	18, 598, 743	18, 627, 000	50, 208, 318	21, 011, 074	18, 251, 859	19, 900, 499	19, 892, 395

Table 3. Values of estimated reserves and standard errors (s.e.) – Taylor and Ashe data with one outlier (Y_{83}).

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
Year	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
$Y_{2,10}$	173, 511	124, 134	127, 831	128, 614	128, 178	129, 433	159, 811	129, 221	166, 024	165, 948
s.e.	1, 864, 433	203, 242	202, 167	195, 096	191, 014	126, 725	79, 840	85, 986	84, 500	84, 556
$Y_{3,9}$	270, 291	248, 529	221, 112	245, 598	247, 265	214, 178	274, 102	226, 320	277, 571	277, 867
s.e.	1, 854, 817	202, 243	201, 123	194, 204	190, 103	120, 768	85, 796	82, 767	79, 792	79, 873
$Y_{3,10}$	155, 791	142, 737	113, 425	140, 873	142, 689	105, 479	176, 543	119, 627	180, 682	181, 005
s.e.	1, 894, 085	206, 321	205, 385	197, 849	193, 826	134, 990	92, 576	92, 316	87, 948	87, 997
$Y_{4,8}$	393, 897	249, 514	237, 453	196, 155	196, 394	323, 139	275, 216	239, 851	281, 642	278, 090
s.e.	1, 850, 508	201, 796	200, 655	193, 804	189, 694	128, 691	87, 689	84, 714	118, 556	117, 248
$Y_{4,9}$	279, 397	143, 721	129, 767	91, 430	91, 818	214, 439	177, 656	133, 158	184, 754	181, 227
s.e.	1, 885, 148	205, 393	204, 415	197, 019	192, 978	140, 536	93, 568	92, 962	123, 528	122, 257
$Y_{4,10}$	164, 897	37, 929	22, 080	−13, 295	−12, 758	105, 740	80, 097	26, 465	87, 865	84, 365
s.e.	1, 927, 477	209, 792	209, 009	200, 956	196, 997	154, 067	100, 405	102, 343	129, 458	128, 226
$Y_{5,7}$	473, 969	394, 583	385, 788	396, 279	396, 852	384, 652	415, 681	386, 821	417, 534	417, 628
s.e.	1, 853, 187	202, 074	200, 946	194, 052	189, 948	122, 523	73, 120	82, 721	68, 090	68, 066
$Y_{5,8}$	359, 470	288, 790	278, 102	291, 553	292, 276	275, 952	318, 122	280, 128	320, 645	320, 766
s.e.	1, 882, 893	205, 159	204, 170	196, 809	192, 764	133, 177	79, 146	89, 967	75, 297	75, 254
$Y_{5,9}$	244, 970	182, 998	170, 415	186, 828	187, 700	167, 253	220, 562	173, 435	223, 756	223, 903
s.e.	1, 920, 484	209, 065	208, 250	200, 305	196, 333	145, 799	86, 268	98, 550	83, 664	83, 599
$Y_{5,10}$	130, 471	77, 205	62, 729	82, 103	83, 124	58, 554	123, 003	66, 742	126, 868	127, 041
s.e.	1, 965, 506	213, 749	213, 137	204, 501	200, 613	159, 924	94, 238	108, 152	92, 877	92, 790
$Y_{6,6}$	669, 843	564, 130	555, 849	561, 736	562, 096	555, 701	579, 568	555, 897	578, 753	578, 789
s.e.	1, 865, 810	203, 384	202, 316	195, 223	191, 145	126, 413	43, 618	88, 542	53, 133	52, 818
$Y_{6,7}$	555, 343	458, 338	448, 163	457, 011	457, 520	447, 002	482, 009	449, 204	481, 865	481, 927
s.e.	1, 890, 197	205, 917	204, 963	197, 488	193, 457	134, 955	51, 622	94, 151	60, 644	60, 347
$Y_{6,8}$	440, 844	352, 545	340, 477	352, 285	352, 944	338, 303	384, 450	342, 511	384, 976	385, 065
s.e.	1, 922, 610	209, 286	208, 481	200, 503	196, 535	145, 749	60, 726	101, 271	69, 490	69, 204
$Y_{6,9}$	326, 344	246, 753	232, 790	247, 560	248, 368	229, 603	286, 891	235, 818	288, 087	288, 203
s.e.	1, 962, 651	213, 452	212, 827	204, 234	200, 341	158, 334	70, 505	109, 609	79, 224	78, 944
$Y_{6,10}$	211, 845	140, 960	125, 104	142, 835	143, 793	120, 904	189, 331	129, 125	191, 198	191, 341
s.e.	2, 009, 865	218, 368	217, 951	208, 645	204, 837	172, 319	80, 715	118, 908	89, 559	89, 279
$Y_{7,5}$	928, 283	787, 735	777, 274	787, 187	787, 877	778, 355	794, 263	776, 091	797, 729	797, 849
s.e.	1, 894, 197	206, 333	205, 397	197, 860	193, 837	147, 753	100, 196	104, 487	86, 339	86, 453
$Y_{7,6}$	813, 783	681, 942	669, 587	682, 462	683, 301	669, 656	696, 704	669, 398	700, 840	700, 986
s.e.	1, 912, 720	208, 258	207, 408	199, 582	195, 595	153, 381	103, 120	108, 140	90, 083	90, 181
$Y_{7,7}$	699, 284	576, 150	561, 901	577, 737	578, 726	560, 957	599, 145	562, 705	603, 951	604, 124
s.e.	1, 939, 330	211, 025	210, 296	202, 060	198, 124	161, 300	107, 186	113, 303	95, 246	95, 324
$Y_{7,8}$	584, 784	470, 358	454, 215	473, 011	474, 150	452, 257	501, 586	456, 013	507, 063	507, 262
s.e.	1, 973, 699	214, 602	214, 026	205, 265	201, 393	171, 192	112, 270	119, 782	101, 614	101, 668
$Y_{7,9}$	470, 284	364, 565	346, 529	368, 286	369, 574	343, 558	404, 027	349, 320	410, 174	410, 400
s.e.	2, 015, 431	218, 948	218, 555	209, 165	205, 367	182, 738	118, 240	127, 376	108, 975	109, 002

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Table 3. Continued.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
Year	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
$Y_{7,10}$	355, 785	258, 773	238, 843	263, 561	264, 998	234, 859	306, 468	242, 627	313, 285	313, 538
s.e.	2, 064, 080	224, 019	223, 835	213, 722	210, 007	195, 644	124, 970	135, 898	117, 142	117, 142
$Y_{8,4}$	5, 514, 276	927, 646	792, 762	792, 668	792, 821	5, 306, 639	950, 587	787, 914	796, 153	796, 115
s.e.	1, 951, 772	212, 320	211, 646	203, 220	199, 307	1, 526, 963	120, 105	856, 754	230, 518	226, 654
$Y_{8,5}$	5, 399, 776	821, 854	685, 075	687, 942	688, 245	5, 197, 940	853, 028	681, 221	699, 264	699, 252
s.e.	1, 963, 531	213, 543	212, 922	204, 316	200, 425	1, 527, 311	121, 766	857, 037	231, 461	227, 611
$Y_{8,6}$	5, 285, 277	716, 061	577, 389	583, 217	583, 669	5, 089, 240	755, 469	574, 528	602, 375	602, 390
s.e.	1, 983, 300	215, 601	215, 068	206, 162	202, 307	1, 527, 920	124, 457	857, 534	233, 038	229, 209
$Y_{8,7}$	5, 170, 777	610, 269	469, 703	478, 492	479, 093	4, 980, 541	657, 910	467, 835	505, 486	505, 528
s.e.	2, 010, 844	218, 470	218, 057	208, 736	204, 930	1, 528, 790	128, 111	858, 244	235, 235	231, 436
$Y_{8,8}$	5, 056, 277	504, 476	362, 017	373, 766	374, 517	4, 871, 842	560, 351	361, 142	408, 598	408, 666
s.e.	2, 045, 848	222, 118	221, 856	212, 013	208, 267	1, 529, 920	132, 650	859, 167	238, 035	234, 274
$Y_{8,9}$	4, 941, 778	398, 684	254, 331	269, 041	269, 941	4, 763, 142	462, 792	254, 449	311, 709	311, 804
s.e.	2, 087, 937	226, 507	226, 424	215, 960	212, 285	1, 531, 309	137, 986	860, 303	241, 418	237, 699
$Y_{8,10}$	4, 827, 278	292, 892	146, 644	164, 316	165, 365	4, 654, 443	365, 232	147, 756	214, 820	214, 942
s.e.	2, 136, 692	231, 596	231, 715	220, 541	216, 945	1, 532, 957	144, 031	861, 650	245, 359	241, 689
$Y_{9,3}$	1, 197, 396	880, 871	871, 531	868, 476	868, 480	877, 240	870, 066	865, 690	864, 124	864, 060
s.e.	2, 078, 360	225, 508	225, 385	215, 061	211, 370	286, 403	75, 174	210, 395	101, 358	100, 470
$Y_{9,4}$	1, 082, 896	775, 079	763, 845	763, 751	763, 904	768, 541	772, 507	758, 997	767, 236	767, 198
s.e.	2, 081, 505	225, 836	225, 726	215, 356	211, 671	286, 754	76, 144	210, 600	102, 019	101, 139
$Y_{9,5}$	968, 397	669, 286	656, 158	659, 025	659, 328	659, 842	674, 948	652, 304	670, 347	670, 335
s.e.	2, 092, 303	226, 963	226, 898	216, 370	212, 702	288, 490	78, 774	211, 672	104, 110	103, 245
$Y_{9,6}$	853, 897	563, 494	548, 472	554, 300	554, 752	551, 142	577, 388	545, 611	573, 458	573, 473
s.e.	2, 110, 637	228, 876	228, 888	218, 092	214, 454	291, 585	829, 05	213, 600	107, 548	106, 703
$Y_{9,7}$	739, 398	457, 701	440, 786	449, 575	450, 176	442, 443	479, 829	438, 918	476, 569	476, 611
s.e.	2, 136, 312	231, 557	231, 674	220, 506	216, 909	295, 997	88, 327	216, 359	112, 209	111, 387
$Y_{9,8}$	624, 898	351, 909	333, 100	344, 849	345, 600	333, 744	382, 270	332, 225	379, 681	379, 749
s.e.	2, 169, 069	234, 978	235, 229	223, 589	220, 043	301, 668	94, 820	219, 919	117, 948	117, 151
$Y_{9,9}$	510, 398	246, 117	225, 414	240, 124	241, 024	225, 044	284, 711	225, 532	282, 792	282, 887
s.e.	2, 208, 592	239, 109	239, 518	227, 314	223, 828	308, 530	102, 179	224, 241	124, 615	123, 842
$Y_{9,10}$	395, 899	140, 324	117, 727	135, 399	136, 448	116, 345	187, 152	118, 839	185, 903	186, 025
s.e.	2, 254, 525	243, 912	244, 503	231, 651	228, 232	316, 504	110, 230	229, 284	132, 072	131, 320
$Y_{10,2}$	1, 604, 511	954, 048	939, 154	927, 123	926, 831	952, 599	915, 970	925, 485	902, 746	902, 564
s.e.	2, 469, 950	266, 483	267, 880	252, 078	248, 945	560, 111	152, 491	435, 210	205, 228	203, 377
$Y_{10,3}$	1, 490, 012	848, 255	831, 468	822, 398	822, 255	843, 900	818, 411	818, 792	805, 858	805, 702
s.e.	2, 459, 035	265, 337	266, 696	251, 040	247, 894	558, 689	151, 313	434, 353	204, 074	202, 221
$Y_{10,4}$	1, 375, 512	742, 463	723, 782	717, 672	717, 679	735, 200	720, 852	712, 099	708, 969	708, 840
s.e.	2, 454, 595	264, 872	266, 214	250, 618	247, 466	557, 978	150, 991	433, 918	203, 643	201, 792
$Y_{10,5}$	1, 261, 012	636, 670	616, 096	612, 947	613, 104	626, 501	623, 292	605, 406	612, 080	611, 977
s.e.	2, 456, 666	265, 089	266, 439	250, 815	247, 665	557, 982	151, 531	433, 905	203, 939	202, 094

Table 3. Continued.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
Year	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
$Y_{10,6}$	1, 146, 513	530, 878	508, 409	508, 222	508, 528	517, 802	525, 733	498, 713	515, 191	515, 115
s.e.	2, 465, 232	265, 988	267, 368	251, 629	248, 491	558, 699	152, 923	434, 315	204, 960	203, 125
$Y_{10,7}$	1, 032, 013	425, 085	400, 723	403, 496	403, 952	409, 102	428, 174	392, 021	418, 303	418, 253
s.e.	2, 480, 224	267, 560	268, 995	253, 056	249, 935	560, 127	155, 145	435, 145	206, 694	204, 872
$Y_{10,8}$	917, 514	319, 293	293, 037	298, 771	299, 376	300, 403	330, 615	285, 328	321, 414	321, 391
s.e.	2, 501, 528	269, 796	271, 307	255, 083	251, 989	562, 261	158, 162	436, 395	209, 124	207, 320
$Y_{10,9}$	803, 014	213, 500	185, 351	194, 045	194, 800	191, 704	233, 056	178, 635	224, 525	224, 529
s.e.	2, 528, 985	272, 678	274, 286	257, 698	254, 636	565, 093	161, 928	438, 060	212, 226	210, 442
$Y_{10,10}$	688, 514	107, 708	77, 665	89, 320	90, 224	83, 005	135, 497	71, 942	127, 636	127, 667
s.e.	2, 562, 395	276, 186	277, 912	260, 883	257, 859	568, 612	166, 394	440, 135	215, 971	214, 209

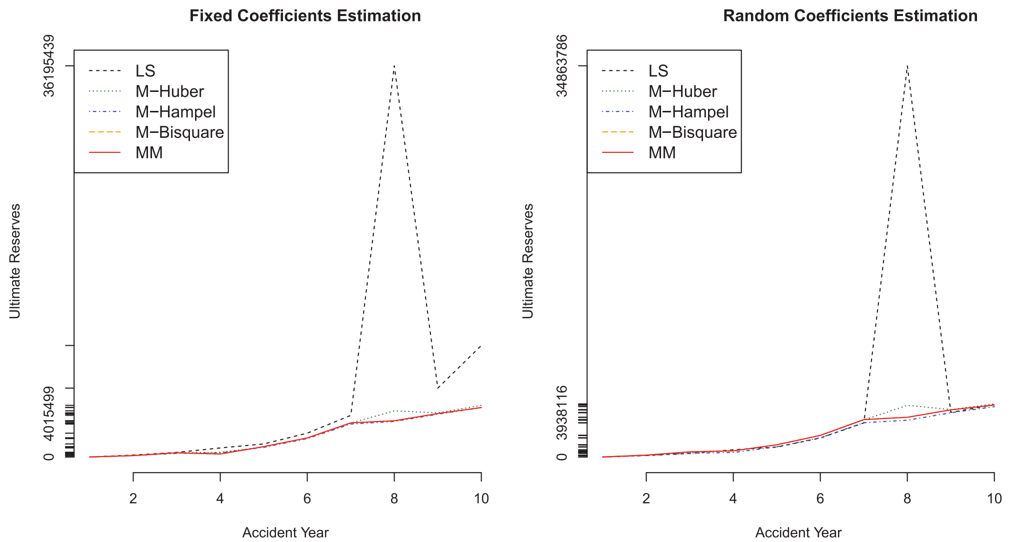

Figure 5. Estimated ultimate reserves against year with fixed coefficients (left panel) and random coefficients (right panel) robust regression models – Taylor and Ashe data with one outlier.

4.1.3 Two artificial outliers

As a second artificial outlier, we multiply the observation Y_{44} by **10**, i.e., $Y_{44} = 15{,}624{,}000$. Figure 6 displays the outlier diagnostics for LS, M and MM estimation in the presence of two artificial outliers. We can easily observe that claims Y_{83} and Y_{44} of the original data are identified as outliers. The existence of a second outlier (Y_{44}) affects the values of total reserves, which is again more evident in LS estimation with fixed coefficients (Table 1: 20,200,094, Table 2: 61,590,318, Table 3: 67,151,105). A moderate jump is also obvious in LS estimates of Figure 7 by the addition of a second outlier in year 4.

From Table 4, we can observe that robust estimates do not significantly change, with M-Huber being again the most appropriate method, either for fixed (Table 1: 19,927,558, Table 2: 19,926,951, Table 3: 19,926,809) or random coefficients estimation (Table 1: 21,009,436, Table 2: 21,011,074, Table 3: 21,011,325). In addition, Table 5 presents the reserve estimates and their standard errors for different estimation methods applied to Taylor and Ashe data with two artificial outliers (Y_{83}

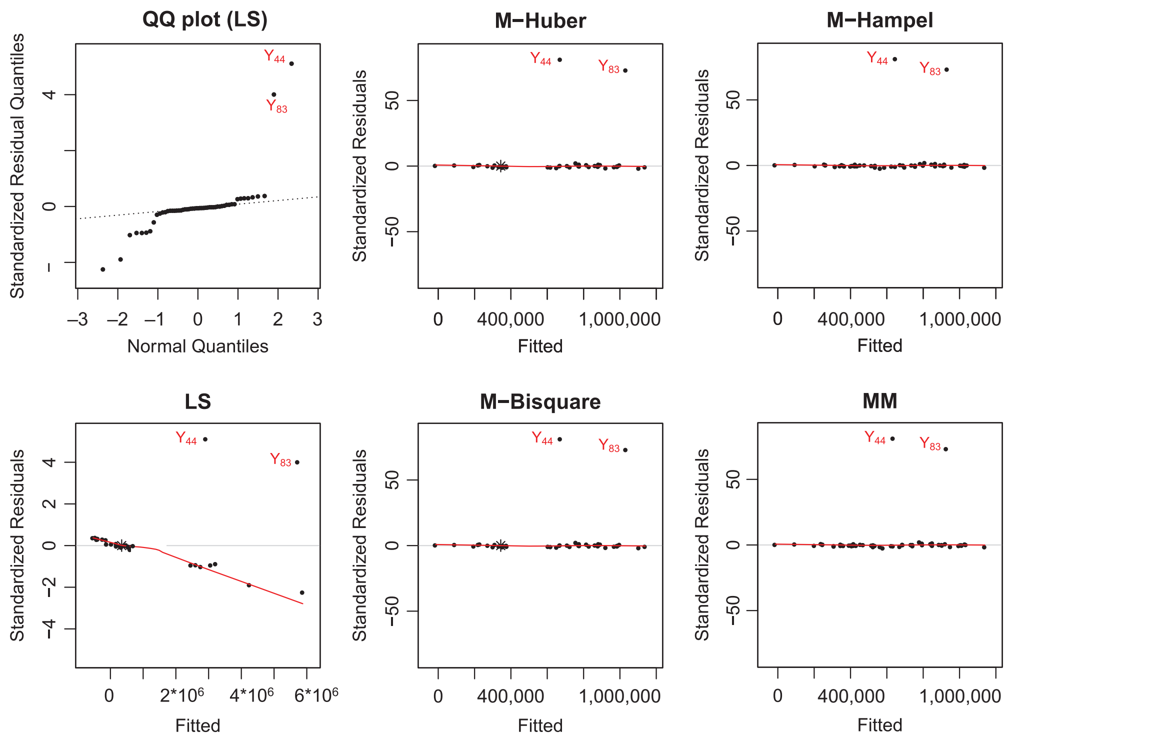

Figure 6. Taylor and Ashe data with two artificial outliers: Diagnostic plots for non-robust LS estimation (left panels) and robust M and MM estimation (middle and right panels).

and Y_{44}). Under both regression methods, we can see that the robust methods (fixed and random) provide smaller standard error values than the LS ones for all reserve estimates, while if we change the position of outlying observations, the order of standard error values follows the same pattern.

In general, LS models are very sensitive to outlier events (very large values) and in most times, they result in an overestimation of total reserves. However, there are cases where they underestimate the total reserves, depending on the position and the size of the outliers. For example, multiplying the observation Y_{51} by **10**, i.e., $Y_{51} = 4{,}431{,}60\mathbf{0}$ (which corresponds to accident year 5 and development year 1), the least squares with fixed coefficients underestimates the total reserves (14,354,701), while the random coefficient model provides a moderate overestimation of total reserves (24,171,414). The robust total reserves remain close to their initial estimates obtained without artificial outliers (see Table 6).

4.1.4 The behaviour of the run-off triangle in the presence of an outlier in each location

Since the influence of an outlier depends on its location on the triangle, it is appropriate to see the behaviour of the run-off triangle in the presence of an outlier in each location. Therefore, following the idea of Verdonck *et al.* (2009), we contaminate the data by multiplying each observation (cell) by 10 and observe the total ultimate reserve for each outlying observation, separately.

In the following, we compare the values of reserves under LS estimation with the values of reserves under robust estimation (M-Huber, M-Hampel, M-Bisquare, MM) for fixed and RCR models, respectively. The first line of Table 7 illustrates the values of the total reserves for the LS and robust estimation based on the original data (no artificial outliers). From the second line and

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Table 4. Ultimate reserves per accident year – Taylor and Ashe data with two outliers (Y_{83} and Y_{44}).

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
Year	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
1	0	0	0	0	0	0	0	0	0	0
2	36, 235	124, 145	133, 498	128, 729	128, 178	156, 730	159, 815	163, 720	167, 206	167, 138
3	162, 633	391, 218	327, 174	385, 698	389, 955	375, 396	450, 614	439, 306	461, 169	461, 374
4	6, 490, 831	431, 147	249, 606	274, 065	275, 454	6, 755, 540	532, 962	371, 421	384, 728	384, 489
5	748, 602	943, 555	897, 771	956, 113	959, 952	1, 004, 773	1, 077, 340	1, 060, 189	1, 093, 070	1, 093, 223
6	1, 695, 490	1, 762, 702	1, 702, 038	1, 760, 728	1, 764, 722	1, 846, 349	1, 922, 214	1, 900, 278	1, 930, 263	1, 930, 294
7	3, 361, 643	3, 139, 156	3, 045, 656	3, 150, 938	3, 158, 626	3, 237, 834	3, 302, 153	3, 274, 933	3, 337, 430	3, 337, 947
8	35, 856, 286	4, 272, 257	3, 272, 507	3, 348, 562	3, 353, 652	35, 119, 101	4, 605, 855	3, 521, 538	3, 539, 696	3, 539, 654
9	6, 518, 530	4, 084, 750	3, 938, 257	4, 014, 622	4, 019, 712	4, 321, 193	4, 228, 820	4, 186, 553	4, 200, 142	4, 200, 080
10	12, 280, 853	4, 777, 880	4, 520, 390	4, 573, 463	4, 576, 749	5, 236, 266	4, 731, 550	4, 670, 559	4, 620, 550	4, 620, 248
Total	67, 151, 105	19, 926, 809	18, 086, 898	18, 592, 917	18, 627, 000	58, 053, 183	21, 011, 325	19, 588, 498	19, 734, 253	19, 734, 447

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Table 5. Values of estimated reserves and standard errors (s.e.) – Taylor and Ashe data with two outliers (Y_{83} and Y_{44}).

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
$Y_{2,10}$	36, 235	124, 145	133, 498	128, 729	128, 178	156, 730	159, 815	163, 720	167, 206	167, 138
s.e.	2, 982, 818	203, 326	222, 149	195, 860	191, 014	377, 636	79, 849	78, 222	80, 189	80, 341
$Y_{3,9}$	156, 735	248, 506	217, 285	245, 228	247, 265	239, 777	274, 087	268, 813	278, 848	278, 949
s.e.	2, 967, 433	202, 328	221, 336	194, 971	190, 103	352, 882	85, 797	73, 359	75, 715	75, 891
$Y_{3,10}$	5, 898	142, 712	109, 889	140, 470	142, 689	135, 619	176, 527	170, 494	182, 320	182, 425
s.e.	3, 030, 256	206, 406	224, 665	198, 603	193, 826	397, 062	92, 578	82, 476	83, 462	83, 610
$Y_{4,8}$	2, 314, 448	249, 509	190, 598	196, 112	196, 394	2, 356, 005	275, 215	222, 125	224, 770	224, 687
s.e.	2, 960, 540	201, 880	220, 971	194, 573	189, 694	1, 012, 807	87, 697	188, 967	112, 750	111, 403
$Y_{4,9}$	2, 163, 610	143, 716	83, 202	91, 355	91, 818	2, 251, 847	177, 654	123, 807	128, 243	128, 163
s.e.	3, 015, 958	205, 477	223, 906	197, 775	192, 978	1, 027, 075	93, 577	192, 241	117, 462	116, 162
$Y_{4,10}$	2, 012, 773	37, 922	24, 194	−13,403	−12,758	2, 147, 689	80, 094	25, 488	31, 715	31, 639
s.e.	3, 083, 679	209, 877	227, 512	201, 698	196, 997	1, 044, 606	100, 414	196, 249	123, 083	121, 834
$Y_{5,7}$	413, 406	394, 579	385, 537	396, 164	396, 852	407, 430	415, 676	412, 525	418, 059	418, 092
s.e.	2, 964, 826	202, 159	221, 198	194, 821	189, 948	324, 430	73, 121	67, 186	64, 628	64, 673
$Y_{5,8}$	262, 569	288, 786	278, 141	291, 407	292, 276	303, 272	318, 115	314, 207	321, 531	321, 568
s.e.	3, 012, 351	205, 243	223, 714	197, 567	192, 764	360, 818	79, 148	74, 717	71, 472	71, 502
$Y_{5,9}$	111, 732	182, 992	170, 745	186, 649	187, 700	199, 114	220, 555	215, 888	225, 004	225, 044
s.e.	3, 072, 490	209, 150	226, 915	201, 049	196, 333	402, 915	86, 271	83, 432	79, 417	79, 431
$Y_{5,10}$	−39, 106	77, 198	63, 348	81, 892	83, 124	94, 957	122, 994	117, 570	128, 476	128, 520
s.e.	3, 144, 520	213, 833	230, 771	205, 230	200, 613	449, 118	94, 242	92, 999	88, 165	88, 164
$Y_{6,6}$	640, 773	564, 128	555, 200	561, 660	562, 096	577, 585	579, 564	576, 693	579, 108	579, 107
s.e.	2, 985, 021	203, 469	222, 266	195, 987	191, 145	325, 127	43, 624	67, 966	50, 485	50, 185
$Y_{6,7}$	489, 935	458, 334	447, 804	456, 903	457, 520	473, 428	482, 003	478, 374	482, 580	482, 583
s.e.	3, 024, 036	206, 002	224, 335	198, 243	193, 457	355, 316	51, 628	74, 158	57, 610	57, 338
$Y_{6,8}$	339, 098	352, 540	340, 408	352, 146	352, 944	369, 270	384, 443	380, 056	386, 053	386, 059
s.e.	3, 075, 892	209, 371	227, 096	201, 246	196, 535	392, 437	60, 732	81, 786	66, 003	65, 754
$Y_{6,9}$	188, 261	246, 747	233, 011	247, 388	248, 368	265, 112	286, 882	281, 737	289, 525	289, 535
s.e.	3, 139, 953	213, 536	230, 526	204, 965	200, 341	434, 718	70, 513	90, 488	75, 241	75, 009
$Y_{6,10}$	37, 423	140, 953	125, 615	142, 631	143, 793	160, 954	189, 322	183, 418	192, 997	193, 011
s.e.	3, 215, 488	218, 452	234, 594	209, 360	204, 837	480, 799	80, 724	99, 985	85, 050	84, 828
$Y_{7,5}$	937, 367	787, 677	776, 100	787, 050	787, 877	800, 034	794, 260	791, 619	797, 558	797, 635
s.e.	3, 030, 436	206, 417	224, 675	198, 613	193, 837	356, 763	100, 194	75, 108	81, 925	82, 143
$Y_{7,6}$	786, 530	681, 883	668, 704	682, 292	683, 301	695, 876	696, 700	693, 300	701, 030	701, 111
s.e.	3, 060, 070	208, 343	226, 252	200, 329	195, 595	378, 200	103, 119	79, 470	85, 481	85, 685
$Y_{7,7}$	635, 692	576, 089	561, 308	577, 535	578, 726	591, 718	599, 139	594, 981	604, 502	604, 587
s.e.	3, 102, 641	211, 110	228, 526	202, 798	198, 124	407, 439	107, 186	85, 437	90, 385	90, 572
$Y_{7,8}$	484, 855	470, 296	453, 911	472, 778	474, 150	487, 560	501, 579	496, 663	507, 974	508, 062
s.e.	3, 157, 627	214, 686	231, 475	205, 992	201, 393	442, 937	112, 270	92, 698	96, 433	96, 599
$Y_{7,9}$	334, 018	364, 502	346, 515	368, 020	369, 574	383, 402	404, 018	398, 344	411, 447	411, 538

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Table 5. Continued.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
s.e.	3, 224, 393	219, 032	235, 075	209, 879	205, 367	483, 317	118, 242	100, 976	103, 423	103, 568
$Y_{7,10}$	183, 180	258, 709	239, 119	263, 263	264, 998	279, 245	306, 457	300, 026	314, 919	315, 014
s.e.	3, 302, 223	224, 103	239, 296	214, 420	210, 007	527, 460	124, 973	110, 040	111, 179	111, 302
$Y_{8,4}$	5, 574, 839	927, 703	789, 690	792, 638	792, 821	5, 329, 488	950, 661	798, 033	795, 254	795, 237
s.e.	3, 122, 547	212, 404	229, 592	203, 954	199, 307	1, 770, 152	120, 147	381, 459	219, 458	215, 355
$Y_{8,5}$	5, 424, 001	821, 910	682, 293	687, 881	688, 245	5, 225, 330	853, 100	699, 714	698, 726	698, 713
s.e.	3, 141, 359	213, 628	230, 601	205, 046	200, 425	1, 773, 026	121, 808	382, 029	220, 351	216, 264
$Y_{8,6}$	5, 273, 164	716, 116	574, 897	583, 123	583, 669	5, 121, 172	755, 540	601, 395	602, 199	602, 189
s.e.	3, 172, 987	215, 686	232, 302	206, 886	202, 307	1, 777, 926	124, 498	383, 002	221, 843	217, 783
$Y_{8,7}$	5, 122, 327	610, 322	467, 501	478, 366	479, 093	5, 017, 014	657, 979	503, 077	505, 671	505, 665
s.e.	3, 217, 053	218, 554	234, 678	209, 451	204, 930	1, 784, 835	128, 152	384, 375	223, 924	219, 899
$Y_{8,8}$	4, 971, 489	504, 529	360, 105	373, 609	374, 517	4, 912, 857	560, 419	404, 758	409, 143	409, 141
s.e.	3, 273, 054	222, 202	237, 711	212, 717	208, 267	1, 793, 729	132, 690	386, 145	226, 575	222, 595
$Y_{8,9}$	4, 820, 652	398, 735	252, 708	268, 851	269, 941	4, 808, 699	462, 858	306, 440	312, 615	312, 617
s.e.	3, 340, 390	226, 592	241, 374	216, 651	212, 285	1, 804, 581	138, 026	388, 305	229, 779	225, 850
$Y_{8,10}$	4, 669, 815	292, 942	145, 312	164, 094	165, 365	4, 704, 541	365, 298	208, 121	216, 087	216, 092
s.e.	3, 418, 392	231, 681	245, 641	221, 218	216, 945	1, 817, 353	144, 070	390, 850	233, 511	229, 640
$Y_{9,3}$	1, 342, 747	880, 871	868, 169	868, 479	868, 480	904, 701	870, 064	867, 434	862, 865	862, 845
s.e.	3, 325, 069	225, 593	240, 539	215, 755	211, 370	672, 715	75, 184	144, 025	96, 354	95, 462
$Y_{9,4}$	1, 191, 910	775, 078	760, 773	763, 721	763, 904	800, 544	772, 504	769, 116	766, 337	766, 320
s.e.	3, 330, 100	225, 921	240, 813	216, 049	211, 671	674, 680	76, 154	144, 408	96, 981	96, 097
$Y_{9,5}$	1, 041, 072	669, 284	653, 376	658, 964	659, 328	696, 386	674, 943	670, 797	669, 809	669, 796
s.e.	3, 347, 376	227, 047	241, 755	217, 060	212, 702	681, 951	78, 784	145, 855	98, 963	98, 099
$Y_{9,6}$	890, 235	563, 491	545, 980	554, 206	554, 752	592, 228	577, 383	572, 478	573, 282	573, 272
s.e.	3, 376, 707	228, 961	243, 358	218, 776	214, 454	694, 360	82, 915	148, 335	102, 222	101, 384
$Y_{9,7}$	739, 398	457, 697	438, 584	449, 449	450, 176	488, 070	479, 822	474, 160	476, 754	476, 748
s.e.	3, 417, 784	231, 641	245, 608	221, 183	216, 909	711, 640	88, 338	151, 798	106, 642	105, 835
$Y_{9,8}$	588, 560	351, 903	331, 188	344, 692	345, 600	383, 912	382, 262	375, 841	380, 226	380, 224
s.e.	3, 470, 190	235, 063	248, 487	224, 257	220, 043	733, 445	94, 831	156, 177	112, 084	111, 310
$Y_{9,9}$	437, 723	246, 110	223, 791	239, 934	241, 024	279, 755	284, 701	277, 523	283, 698	283, 700
s.e.	3, 533, 421	239, 194	251, 975	227, 971	223, 828	759, 387	102, 190	161, 399	118, 408	117, 668
$Y_{9,10}$	286, 886	140, 316	116, 395	135, 177	136, 448	175, 597	187, 140	179, 204	187, 170	187, 175
s.e.	3, 606, 906	243, 998	256, 045	232, 296	228, 232	789, 056	110, 242	167, 384	125, 481	124, 774
$Y_{10,2}$	1, 967, 889	954, 050	931, 851	927, 192	926, 831	998, 439	915, 970	912, 225	899, 506	899, 458
s.e.	3, 951, 555	266, 569	275, 366	252, 672	248, 945	1, 316, 775	152, 511	292, 751	195, 107	193, 239
$Y_{10,3}$	181, 7051	848, 256	824, 454	822, 435	822, 255	894, 281	818, 409	813, 907	802, 978	802, 933
s.e.	3, 934, 092	265, 424	274, 379	251, 636	247, 894	1, 311, 927	151, 333	291, 804	194, 012	192, 140
$Y_{10,4}$	1, 666, 214	742, 463	717, 058	717, 677	717, 679	790, 123	720, 849	715, 588	706, 450	706, 409
s.e.	3, 926, 989	264, 959	273, 977	251, 215	247, 466	1, 309, 819	151, 011	291, 387	193, 603	191, 732
$Y_{10,5}$	1, 515, 377	636, 669	609, 662	612, 920	613, 104	685, 965	623, 288	617, 270	609, 922	609, 885

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Table 5. Continued.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
s.e.	3, 930, 303	265, 176	274, 165	251, 412	247, 665	1, 310, 463	151, 551	291, 500	193, 883	192, 019
$Y_{10,6}$	1, 364, 539	530, 876	502, 266	508, 163	508, 528	581, 807	525, 728	518, 951	513, 394	513, 361
s.e.	3, 944, 007	266, 074	274, 939	252, 224	248, 491	1, 313, 856	152, 943	292, 144	194, 850	192, 998
$Y_{10,7}$	1, 213, 702	425, 082	394, 869	403, 405	403, 952	477, 650	428, 167	420, 632	416, 867	416, 837
s.e.	3, 967, 993	267, 647	276, 296	253, 647	249, 935	1, 319, 976	155, 165	293, 314	196, 493	194, 659
$Y_{10,8}$	1, 062, 865	319, 288	287, 473	298, 648	299, 376	373, 492	330, 607	322, 314	320, 339	320, 313
s.e.	4, 002, 076	269, 883	278, 227	255, 671	251, 989	1, 328, 786	158, 182	295, 005	198, 797	196, 984
$Y_{10,9}$	912, 027	213, 495	180, 077	193, 890	194, 800	269, 334	233, 046	223, 995	223, 811	223, 788
s.e.	4, 046, 002	272, 765	280, 720	258, 280	254, 636	1, 340, 232	161, 948	297, 208	201, 737	199, 951
$Y_{10,10}$	761, 190	107, 701	72, 680	89, 133	90, 224	165, 176	135, 486	125, 677	127, 283	127, 264
s.e.	4, 099, 454	276, 273	283, 760	261, 458	257, 859	1, 354, 249	166, 414	299, 910	205, 288	203, 530

Table 6. Ultimate reserves per accident year – Taylor and Ashe original data.

Total Ultimate Reserves – Fixed Regression Coefficients					Total Ultimate Reserves – Random Regression Coefficients				
Non-Robust	Robust				Non-Robust	Robust			
LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
14,354,701	19,627,842	20,271,926	20,173,379	20,151,695	24,171,414	21,176,467	21,300,606	23,433,894	22,636,531

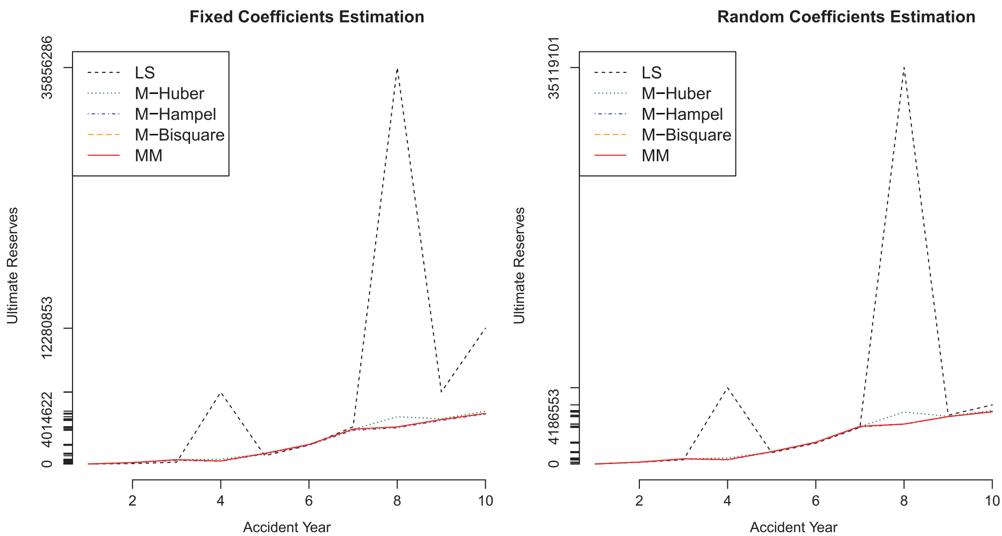

Figure 7. Estimated ultimate reserves against year with fixed coefficients (left panel) and random coefficients (right panel) robust regression models – Taylor and Ashe data with two outliers.

on, the first column indicates which observation is contaminated (multiplied by 10) and can be considered as an outlier.

From the results presented in Table 7, we observe that most of times, the LS estimation cannot handle a large claim (outlier) and produces an overestimation of outstanding reserves, but

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Table 7. Total ultimate reserves – Taylor and Ashe data with each observation multiplied by 10.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
–	20, 200, 094	**19,927,558**	**20,018,418**	**19,853,465**	**19,818,822**	21, 393, 411	**21,009,436**	**21,379,399**	**21,372,946**	**21,337,541**
Y_{11}	7, 391, 245	**19,242,815**	**20,723,710**	**20,607,904**	**20,585,061**	29, 668, 148	**20,497,122**	2, 096, 788	**22,577,651**	**21,597,176**
Y_{12}	−2,431,328	**18,583,682**	**19,690,482**	**19,593,757**	**19,570,382**	11, 429, 108	**19,196,465**	**19,527,691**	**20,544,610**	**20,473,615**
Y_{13}	7, 294, 907	**18,884,588**	**19,654,870**	**19,535,621**	**19,502,917**	15, 344, 617	**19,509,198**	**19,535,611**	**20,470,289**	**20,390,864**
Y_{14}	14, 034, 978	**19,291,783**	**19,764,447**	**19,638,735**	**19,605,123**	17, 929, 268	**19,967,138**	**19,633,745**	**20,550,435**	**20,477,673**
Y_{15}	17, 882, 845	**19,801,498**	**20,004,959**	**19,881,724**	**19,861,619**	19, 181, 448	**20,539,092**	**19,580,818**	**20,776,016**	**20,728,640**
Y_{16}	22, 484, 545	**19,974,424**	**19,915,858**	**19,777,125**	**19,730,410**	19, 762, 157	**20,906,371**	**19,382,681**	**20,834,988**	**20,794,161**
Y_{17}	22, 007, 211	**20,585,877**	**20,603,934**	**20,115,081**	**20,085,269**	21, 453, 795	**21,553,614**	**21,316,891**	**21,205,759**	**21,186,460**
Y_{18}	23, 099, 864	**20,795,063**	**20,713,920**	**20,042,358**	**20,026,398**	22, 696, 492	**21,892,087**	23, 083, 846	**21,351,729**	**21,227,282**
Y_{19}	26, 810, 528	**20,407,957**	**19,442,317**	**19,314,559**	**19,263,548**	28, 511, 410	**21,543,785**	26, 976, 897	**21,154,801**	**21,139,540**
$Y_{1,10}$	22, 745, 628	**20,879,209**	**21,628,695**	**20,151,689**	**19,684,187**	22, 835, 029	**22,017,498**	**22,801,316**	**22,348,705**	**22,296,053**
Y_{21}	9, 414, 173	**18,533,182**	**19,895,805**	**19,775,818**	**19,761,087**	29, 761, 229	**20,288,159**	470, 398	**22,890,325**	**22,196,888**
Y_{22}	−1,322,434	**18,664,483**	**19,617,713**	**19,545,895**	**19,527,540**	13, 780, 874	**19,720,607**	**19,584,291**	**20,753,956**	**20,697,718**
Y_{23}	5, 281, 407	**19,483,917**	**20,091,422**	**20,022,164**	**19,990,917**	16, 421, 945	**20,390,079**	**19,676,003**	**20,994,335**	**20,934,060**
Y_{24}	11, 203, 253	**19,928,090**	**20,397,819**	**20,044,924**	**19,994,647**	18, 695, 019	**21,011,530**	**19,553,284**	**21,168,364**	**21,136,853**
Y_{25}	20, 542, 521	**19,979,248**	**20,017,917**	**19,898,945**	23, 219, 956	20, 795, 031	**20,857,867**	**20,195,346**	**21,108,250**	**20,661,811**
Y_{26}	23, 133, 893	**20,460,982**	**20,281,284**	**20,125,905**	**20,091,453**	21, 723, 317	**21,307,008**	**21,444,592**	**21,393,497**	**21,344,209**
Y_{27}	29, 442, 542	**20,329,276**	**19,845,632**	**19,658,236**	**19,595,131**	25, 237, 224	**21,394,905**	**22,220,020**	**21,265,210**	**21,326,259**
Y_{28}	27, 089, 317	**21,106,402**	**20,278,694**	**20,173,255**	**20,148,517**	25, 651, 683	**21,961,740**	25, 065, 240	**19,474,473**	**21,902,656**
Y_{29}	34, 759, 646	**20,209,365**	**19,140,092**	**18,955,963**	**18,908,130**	29, 047, 046	**21,535,100**	24, 397, 708	**18,838,571**	**22,127,266**
Y_{31}	12, 872, 114	**18,626,751**	**19,778,161**	**19,523,806**	**19,460,386**	25, 677, 686	**20,879,155**	24, 629, 503	23, 365, 815	**22,680,890**
Y_{32}	1, 226, 607	**19,361,435**	**20,062,717**	**19,931,910**	**19,909,763**	17, 936, 918	**20,614,014**	**20,530,932**	**21,416,245**	**21,331,194**
Y_{33}	10, 411, 855	**19,691,026**	**20,079,445**	**19,939,581**	**19,916,092**	20, 475, 856	**20,857,618**	**20,534,719**	**21,427,408**	**21,340,942**
Y_{34}	17, 967, 022	**19,928,428**	**20,070,301**	**19,861,744**	**19,809,948**	22, 410, 402	**21,024,823**	**20,431,214**	**21,396,795**	**21,289,911**
Y_{35}	24, 836, 361	**20,130,302**	**19,921,173**	**19,772,570**	**19,743,395**	22, 933, 653	**21,277,822**	**20,747,585**	**21,524,900**	**21,446,744**
Y_{36}	22, 337, 299	**20,871,085**	**21,411,073**	**20,350,037**	**20,296,380**	22, 866, 256	**21,873,470**	**22,683,021**	**22,272,692**	**22,138,794**
Y_{37}	31, 567, 186	**20,693,570**	**19,813,927**	**19,713,881**	**19,697,043**	25, 728, 603	**21,718,345**	**21,919,264**	**21,066,356**	**21,203,059**

Table 7. Continued.

Y_{38}	28,973,786	**21,497,030**	**20,197,075**	**20,231,393**	**20,231,310**	27,932,362	**22,133,078**	26,333,518	**19,699,848**	**21,821,812**
Y_{41}	14,152,813	**19,271,979**	**20,087,031**	**20,166,324**	**20,147,008**	26,757,353	**21,595,586**	23,472,309	**23,059,879**	**22,576,250**
Y_{42}	5,589,156	**19,889,406**	**20,406,041**	**20,267,485**	**20,233,961**	23,176,020	**21,018,704**	**21,807,168**	**21,426,472**	**21,398,817**
Y_{43}	16,464,816	**19,882,889**	**19,989,311**	**19,910,831**	**19,885,947**	23,954,479	**21,356,660**	**21,848,409**	**21,488,085**	**21,463,026**
Y_{44}	25,760,880	**19,928,195**	**19,830,578**	**19,856,564**	**19,816,826**	27,657,882	**21,011,772**	**21,373,197**	**21,176,013**	**21,168,252**
Y_{45}	23,450,966	**20,833,726**	**20,543,024**	**20,231,948**	28,049,208	22,186,486	**21,946,817**	**22,558,796**	**21,855,383**	**23,092,647**
Y_{46}	27,347,493	**21,203,704**	**20,342,644**	**20,059,410**	**20,041,938**	25,782,582	**22,208,981**	24,429,412	**21,859,425**	**21,866,480**
Y_{47}	26,115,039	**21,878,274**	**20,654,141**	**20,317,290**	**20,300,373**	23,409,699	**22,576,970**	24,038,874	**21,356,301**	**21,694,930**
Y_{51}	14,354,701	**19,627,842**	**20,271,926**	**20,173,379**	**20,151,695**	24,171,414	**21,176,467**	**21,300,606**	23,433,894	**22,636,531**
Y_{52}	15,413,705	**19,490,190**	**19,780,186**	**19,635,387**	**19,599,050**	23,585,217	**20,419,268**	**20,479,934**	**21,480,893**	**21,423,534**
Y_{53}	21,654,925	**19,947,153**	**19,971,665**	**19,852,445**	**19,820,470**	25,862,889	**21,028,013**	**20,064,974**	**21,448,394**	**21,405,966**
Y_{54}	27,770,356	**20,206,188**	**19,924,999**	**19,797,082**	48,098,657	25,286,897	**21,301,825**	**20,325,749**	**21,570,099**	**22,457,850**
Y_{55}	29,393,185	**20,812,197**	**20,243,072**	**20,136,109**	**20,101,710**	24,429,523	**21,813,791**	**22,744,596**	**21,867,894**	**21,819,580**
Y_{56}	32,710,137	**21,014,402**	**20,092,578**	**20,003,600**	**19,980,923**	25,272,967	**22,045,416**	23,061,937	**21,801,005**	**21,871,876**
Y_{61}	17,793,965	**19,621,852**	**20,026,785**	**19,893,098**	**19,873,643**	28,048,016	**20,300,764**	24,997,699	**22,773,335**	**22,605,871**
Y_{62}	20,398,120	**19,958,187**	**20,001,866**	**19,870,211**	**19,848,061**	27,740,464	**20,490,273**	**19,746,228**	**20,935,348**	**20,883,708**
Y_{63}	27,473,986	**20,314,216**	**20,045,282**	**19,914,712**	57,740,064	27,590,289	**20,902,921**	**19,775,909**	**20,967,737**	**20,799,425**
Y_{64}	33,848,889	**20,512,662**	**19,931,861**	**19,796,771**	**19,774,440**	27,828,777	**21,144,595**	23,098,052	**20,937,165**	**20,890,997**
Y_{65}	38,079,030	**20,774,603**	**19,862,324**	**19,732,349**	**19,708,589**	28,479,109	**21,465,689**	26,108,605	**21,010,791**	**20,965,512**
Y_{71}	21,305,422	**20,115,864**	**20,015,204**	**19,910,244**	**19,863,572**	29,607,759	**21,752,429**	25,240,285	**21,776,929**	25,443,532
Y_{72}	27,653,100	**20,630,250**	**20,368,104**	**20,174,125**	**20,139,175**	30,873,437	**21,764,895**	**19,905,450**	**21,200,300**	**21,182,475**
Y_{73}	39,619,647	**20,366,280**	**19,658,238**	**19,441,652**	**19,419,295**	34,991,051	**21,337,424**	**19,121,265**	**20,485,321**	**20,474,116**
Y_{74}	47,351,383	**20,342,076**	**19,336,529**	**19,136,904**	77,764,110	34,218,981	**21,411,178**	24,356,886	**20,463,564**	37,795,675
Y_{81}	25,239,737	**22,331,352**	**20,561,704**	**20,838,956**	**20,904,328**	29,974,676	23,623,604	30,501,686	**22,936,891**	**22,837,596**
Y_{82}	41,756,474	**22,574,803**	**20,489,821**	**20,629,178**	85,870,975	41,970,956	24,259,503	**21,138,662**	**21,677,331**	86,884,760
Y_{83}	61,590,318	**19,926,951**	**18,320,074**	**18,598,743**	**18,627,000**	50,208,318	**21,011,074**	**18,251,859**	**19,900,499**	**19,892,395**
Y_{91}	32,576,983	33,222,366	**19,939,001**	NA	46,935,038	34,928,986	34,706,882	34,898,680	NA	34,909,531
Y_{92}	58,818,585	55,770,055	**20,073,262**	**19,841,644**	**19,812,847**	57,679,270	56,483,232	56,863,864	**21,220,875**	**21,186,549**
$Y_{10,1}$	48,065,228	47,792,692	47,883,552	47,718,599	47,683,956	49,258,545	48,874,570	49,244,533	49,238,080	49,202,675

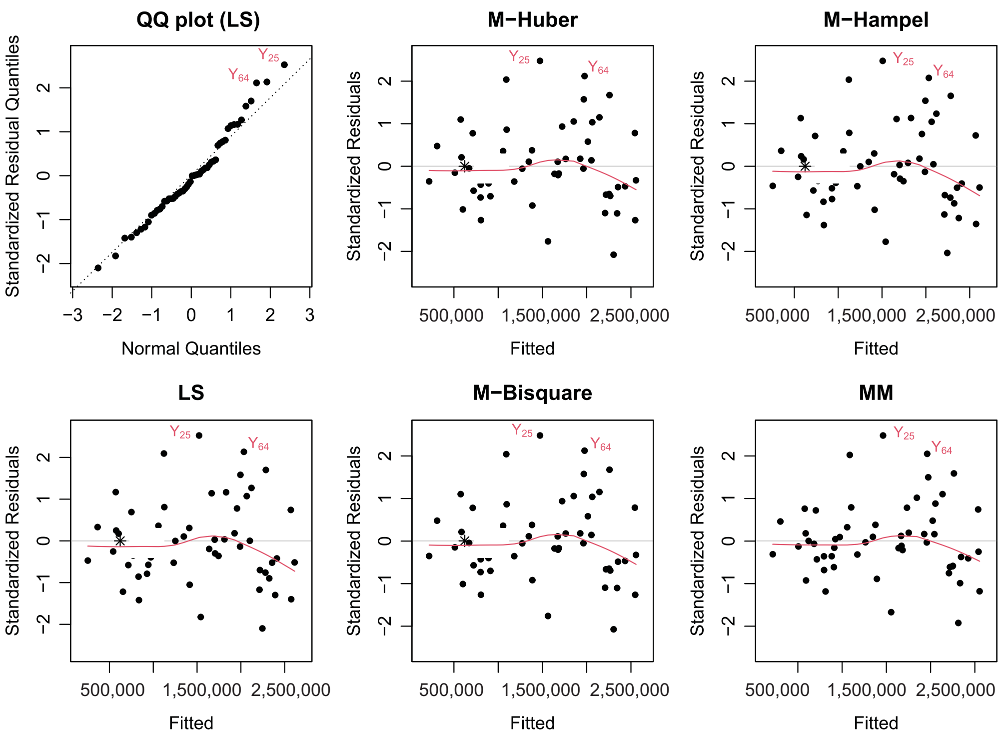

Figure 8. Claims data from motor business line: Diagnostic plots for non-robust LS estimation (left panels) and robust M and MM estimation (middle and right panels).

there are also cases where we have an underestimation of the reserves, depending on the location of that outlier. In particular, the presence of outlying observations in positions Y_{12} and Y_{22} cannot be captured by the fixed coefficients LS model and leads to an extreme underestimation of total reserves. On the contrary, robust estimation seems to robustify very well most of the outliers, except those that appear in the lower left corner (Y_{91}, Y_{92}, $Y_{10,1}$). In the same lower corner some convergence issues appeared with the M-Bisquare estimation algorithm (NA values). Also, M-Hampel estimation under the fixed coefficient regression and MM estimation under both regression models failed to robustify some outliers. The boldface values correspond to the total ultimate reserves obtained with the best performing robust methods for each regression cases. In both cases, it can be concluded the superiority of M-Huber estimates, which are less affected by outlying observations.

Remark 4. *Although we use different estimation methods, Verdonck et al. (2009) faced similar situations and explained why robust methods can fail in some borderline and they suggested some adjustments for the outlying values of the triangle, before applying the robust chain-ladder method. Also, based on the influence function and robust estimators, Verdonck et al. (2011) presented a diagnostic tool that detects automatically the most influential claims in a given run-off triangle and they showed that a robust version of the generalised linear model can provide very satisfactory results on reserves estimation.*

4.2 Claims data from motor business line

In this section, we will apply our methods on a run-off triangle from the motor business line of a non-life insurance company operating in Greece. The claims are from 2010 to 2019 in Euro

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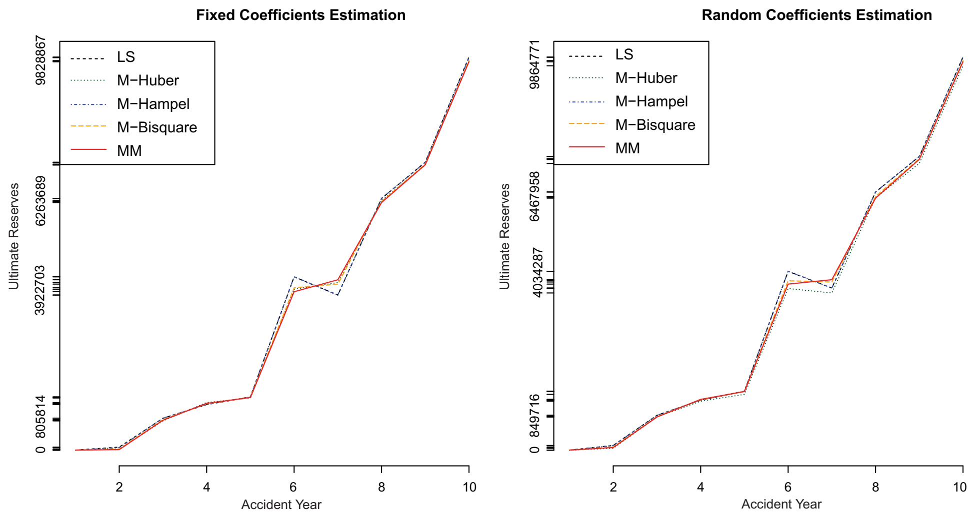

Figure 9. Estimated ultimate reserves against year with fixed coefficients (left panel) and random coefficients (right panel) robust regression models – Claims data from motor business line.

currency, presented in Appendix B. The diagnostic plots of Figure 8 indicate that values (Y_{25} and Y_{64}) lie in higher levels than the others and can be considered as (real) outliers. However these outliers do not exceed much the borderline of the tuning constant $k = 2$ (Huber function), compared with the artificial outliers in Taylor and Ashe data (see Figures 4 and 6). This is also evident in Figure 9, which displays the values of ultimate reserves per accident year for non-robust and robust estimation. We observe that both regression models show a similar behaviour, with some small fluctuations in accident year 6.

Finally, Table 9 presents the reserve estimates and their standard errors for non-robust and robust estimation. Although the initial data do not contain negative incremental claims, there are cases where the regression model produces few negative predicted outstanding claims. However, the ultimate reserves per accident year (Table 8) are all positive. We also observe that there is not much difference between the standard errors resulting from the LS model in comparison with the standard errors resulting from robust methods due to the fact that the LS model is not affected much by the outlying observations Y_{25} and Y_{64}. In any case, we can see that the RCR models produce lower standard error values in comparison with the fixed ones. We also note that the regression nature of the proposed methods may lead to negative incremental reserve estimates.

5 Conclusions

In this paper, we investigated how random coefficients regression models can be incorporated in loss reserving techniques. These models provide a fair value for the estimation of outstanding reserves in cases where we have indications that the run-off patterns are changing. We relaxed the assumption that the development factors are constant by proposing a random coefficients regression model. As already mentioned, loss reserving under least square estimators with fixed or random coefficients is sensitive to outlier events and might lead to the overestimation or sometimes to the underestimation of the total reserves, depending on the position and the size of the outlying observations.

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Table 8. Ultimate reserves per accident year – Claims data from motor business line.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
Year	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
1	0	0	0	0	0	0	0	0	0	0
2	75, 493	19, 967	61, 662	27, 952	12, 610	119, 288	47, 100	115, 581	80, 161	67, 059
3	805, 814	739, 592	807, 761	771, 711	756, 411	892, 532	852, 146	893, 267	862, 150	849, 716
4	1, 152, 689	1, 203, 718	1, 155, 451	1, 176, 947	1, 183, 705	1, 281, 082	1, 255, 762	1, 282, 123	1, 297, 651	1, 303, 127
5	1, 344, 074	1, 318, 996	1, 347, 475	1, 336, 353	1, 332, 103	1, 512, 272	1, 430, 717	1, 513, 549	1, 506, 217	1, 503, 379
6	4, 384, 060	4, 071, 829	4, 387, 820	4, 108, 058	4, 008, 436	4, 589, 070	4, 147, 458	4, 590, 472	4, 347, 502	4, 261, 365
7	3, 922, 703	4, 240, 764	3, 926, 328	4, 200, 730	4, 310, 468	4, 159, 287	4, 034, 287	4, 160, 621	4, 315, 341	4, 374, 630
8	6, 366, 965	6, 313, 957	6, 369, 471	6, 291, 052	6, 263, 689	6, 624, 648	6, 475, 701	6, 625, 527	6, 510, 782	6, 467, 958
9	7, 283, 661	7, 212, 657	7, 282, 587	7, 236, 827	7, 220, 443	7, 536, 255	7, 358, 914	7, 535, 710	7, 480, 316	7, 461, 134
10	9, 945, 544	9, 830, 481	9, 931, 043	9, 855, 819	9, 828, 867	10, 088, 295	9, 864, 771	10, 082, 453	9, 996, 624	9, 967, 031
Total	35, 281, 003	34, 951, 961	35, 269, 598	35, 005, 449	34, 916, 731	36, 802, 728	35, 466, 858	36, 799, 302	36, 396, 743	36, 255, 399

Table 9. Values of estimated reserves and standard errors (s.e.) – Claims data from motor business line.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
$Y_{2,10}$	75, 493	19, 967	61, 662	27, 952	12, 610	119, 288	47, 100	115, 581	80, 161	67, 059
s.e.	409, 370	396, 034	419, 177	402, 797	364, 565	178, 314	154, 931	185, 627	181, 734	184, 211
$Y_{3,9}$	547, 677	515, 063	548, 517	530, 800	523, 260	586, 308	568, 065	586, 624	571, 497	565, 430
s.e.	407, 258	394, 002	417, 101	400, 656	362, 170	180, 339	176, 760	187, 658	182, 942	184, 957
$Y_{3,10}$	258, 136	224, 529	259, 244	240, 911	233, 151	306, 224	284, 082	306, 643	290, 653	284, 286
s.e.	415, 880	402, 303	425, 581	409, 396	371, 931	196, 222	190, 676	204, 187	199, 241	201, 514
$Y_{4,8}$	673, 771	691, 774	674, 423	682, 205	684, 677	707, 111	702, 571	707, 355	713, 394	715, 521
s.e.	406, 312	393, 091	416, 171	399, 696	361, 095	162, 327	169, 322	168, 914	164, 702	166, 495
$Y_{4,9}$	384, 230	401, 239	385, 150	392, 316	394, 568	427, 027	418, 587	427, 374	432, 550	434, 376
s.e.	413, 918	400, 413	423, 650	407, 407	369, 714	177, 797	182, 111	185, 015	180, 576	182, 621
$Y_{4,10}$	94, 689	110, 705	95, 878	102, 427	104, 459	146, 944	134, 604	147, 393	151, 707	153, 231
s.e.	423, 212	409, 362	432, 797	416, 825	380, 189	195, 370	196, 901	203, 303	198, 589	200, 913
$Y_{5,7}$	770, 330	765, 550	770, 778	768, 921	768, 189	798, 193	783, 655	798, 359	797, 819	797, 562
s.e.	406, 900	393, 657	416, 750	400, 293	361, 764	109, 336	105, 553	113, 673	111, 779	113, 345
$Y_{5,8}$	480, 789	475, 016	481, 505	479, 033	478, 080	518, 110	499, 671	518, 378	516, 976	516, 417
s.e.	413, 423	399, 937	423, 163	406, 906	369, 154	128, 349	122, 459	133, 476	131, 157	132, 976
$Y_{5,9}$	191, 248	184, 482	192, 233	189, 144	187, 971	238, 026	215, 688	238, 397	236, 132	235, 272
s.e.	421, 676	407, 883	431, 285	415, 269	378, 463	149, 293	141, 287	155, 284	152, 509	154, 610
$Y_{5,10}$	−98,293	−106,052	−97,040	−100,745	−102,138	−42,057	−68,296	−41,584	−44,711	−45,872
s.e.	431, 562	417, 403	441, 020	425, 280	389, 551	171, 462	161, 366	178, 364	175, 115	177, 515
$Y_{6,6}$	1, 455, 894	1, 395, 434	1, 456, 109	1, 401, 389	1, 381, 905	1, 477, 981	1, 397, 458	1, 478, 056	1, 431, 187	1, 414, 563
s.e.	409, 672	396, 325	419, 475	403, 103	364, 908	179, 983	148, 672	187, 150	183, 611	186, 549
$Y_{6,7}$	1, 166, 353	1, 104, 900	1, 166, 836	1, 111, 500	1, 091, 796	1, 197, 898	1, 113, 475	1, 198, 075	1, 150, 344	1, 133, 418
s.e.	415, 027	401, 481	424, 741	408, 531	370, 967	190, 084	159, 011	197, 671	193, 927	196, 980
$Y_{6,8}$	876, 812	814, 366	877, 564	821, 612	801, 687	917, 814	829, 492	918, 094	869, 500	852, 273
s.e.	422, 143	408, 333	431, 745	415, 742	378, 988	202, 894	171, 979	211, 011	207, 007	210, 209
$Y_{6,9}$	587, 271	523, 831	588, 291	531, 723	511, 578	637, 730	545, 508	638, 113	588, 657	571, 128
s.e.	430, 935	416, 799	440, 403	424, 645	388, 850	217, 936	187, 029	226, 674	222, 363	225, 745
$Y_{6,10}$	297, 730	233, 297	299, 019	241, 834	221, 469	357, 647	261, 525	358, 132	307, 813	289, 984
s.e.	441, 302	426, 782	450, 620	435, 137	400, 417	234, 780	203, 702	244, 212	239, 558	243, 145
$Y_{7,5}$	1, 377, 636	1, 433, 130	1, 377, 569	1, 424, 844	1, 443, 684	1, 393, 423	1, 382, 340	1, 393, 389	1, 421, 332	1, 431, 967
s.e.	415, 905	402, 326	425, 605	409, 421	371, 959	213, 295	234, 563	221, 606	216, 051	219, 020
$Y_{7,6}$	1, 088, 095	1, 142, 595	1, 088, 297	1, 134, 955	1, 153, 575	1, 113, 340	1, 098, 356	1, 113, 408	1, 140, 489	1, 150, 822
s.e.	419, 972	406, 242	429, 607	413, 542	376, 544	219, 968	239, 726	228, 562	222, 915	225, 975
$Y_{7,7}$	798, 554	852, 061	799, 024	845, 066	863, 466	833, 256	814, 373	833, 427	859, 645	869, 677
s.e.	425, 814	411, 868	435, 359	419, 460	383, 112	229, 287	247, 038	238, 276	232, 494	235, 682
$Y_{7,8}$	509, 013	561, 527	509, 752	555, 177	573, 357	553, 173	530, 389	553, 446	578, 802	588, 533
s.e.	433, 361	419, 135	442, 793	427, 101	391, 562	240, 947	256, 318	250, 426	244, 469	247, 817
$Y_{7,9}$	219, 472	270, 993	220, 479	265, 288	283, 248	273, 089	246, 406	273, 465	297, 958	307, 388
s.e.	442, 524	427, 959	451, 825	436, 374	401, 776	254, 625	267, 359	264, 678	258, 507	262, 043

Table 9. Continued.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
$\gamma_{7,10}$	−70,069	−19,542	−68,793	−24,600	−6,861	−6,995	−37,577	−6,516	17, 115	26, 243
s.e.	453, 206	438, 247	462, 362	447, 177	413, 624	270, 015	279, 953	280, 711	274, 293	278, 039
$\gamma_{8,4}$	1, 778, 189	1, 773, 597	1, 777, 742	1, 768, 388	1, 765, 140	1, 786, 629	1, 777, 050	1, 786, 447	1, 772, 642	1, 767, 428
s.e.	428, 546	414, 499	438, 049	422, 227	386, 175	168, 807	177, 045	175, 495	171, 626	173, 561
$\gamma_{8,5}$	1, 488, 648	1, 483, 062	1, 488, 469	1, 478, 499	1, 475, 031	1, 506, 545	1, 493, 067	1, 506, 466	1, 491, 799	1, 486, 283
s.e.	431, 128	416, 985	440, 593	424, 841	389, 066	174, 375	181, 501	181, 298	177, 337	179, 364
$\gamma_{8,6}$	1, 199, 107	1, 192, 528	1, 199, 197	1, 188, 611	1, 184, 922	1, 226, 462	1, 209, 084	1, 226, 485	1, 210, 955	1, 205, 139
s.e.	435, 469	421, 165	444, 870	429, 235	393, 917	183, 340	188, 817	190, 637	186, 522	188, 693
$\gamma_{8,7}$	909, 566	901, 994	909, 924	898, 722	894, 813	946, 378	925, 100	946, 504	930, 112	923, 994
s.e.	441, 517	426, 989	450, 831	435, 355	400, 656	195, 233	198, 676	203, 026	198, 702	201, 060
$\gamma_{8,8}$	620, 025	611, 460	620, 652	608, 833	604, 704	666, 295	641, 117	666, 523	649, 268	642, 849
s.e.	449, 202	434, 391	458, 412	443, 129	409, 191	209, 558	210, 721	217, 944	213, 363	215, 942
$\gamma_{8,9}$	330, 484	320, 925	331, 380	318, 944	314, 595	386, 211	357, 133	386, 542	368, 425	361, 704
s.e.	458, 444	443, 293	467, 532	452, 472	419, 411	225, 851	224, 602	234, 910	230, 031	232, 858
$\gamma_{8,10}$	40, 943	30, 391	42, 107	29, 055	24, 486	106, 127	73, 150	106, 561	87, 581	80, 560
s.e.	469, 149	453, 604	478, 104	463, 290	431, 198	243, 719	239, 999	253, 514	248, 303	251, 398
$\gamma_{9,3}$	1, 923, 851	1, 918, 452	1, 922, 777	1, 919, 214	1, 917, 937	1, 922, 324	1, 913, 806	1, 921, 897	1, 917, 991	1, 916, 648
s.e.	456, 341	441, 267	465, 456	450, 347	417, 090	259, 756	268, 714	268, 963	260, 084	261, 929
$\gamma_{9,4}$	1, 634, 310	1, 627, 918	1, 633, 505	1, 629, 326	1, 627, 828	1, 642, 241	1, 629, 823	1, 641, 916	1, 637, 148	1, 635, 504
s.e.	457, 031	441, 932	466, 138	451, 045	417, 852	260, 905	269, 555	270, 169	261, 289	263, 160
$\gamma_{9,5}$	1, 344, 769	1, 337, 384	1, 344, 232	1, 339, 437	1, 337, 719	1, 362, 157	1, 345, 839	1, 361, 935	1, 356, 305	1, 354, 359
s.e.	459, 402	444, 216	468, 479	453, 441	420, 469	264, 511	272, 439	273, 942	265, 045	266, 993
$\gamma_{9,6}$	1, 055, 228	1046, 849	1, 054, 960	1, 049, 548	1, 047, 610	1082, 074	1, 061, 856	1, 081, 954	1, 075, 461	1, 073, 214
s.e.	463, 428	448, 094	472, 454	457, 510	424, 906	270, 474	277, 305	280, 180	271, 249	273, 320
$\gamma_{9,7}$	765, 687	756, 315	765, 687	759, 659	757, 501	801, 990	777, 873	801, 973	794, 618	792, 069
s.e.	469, 066	453, 524	478, 022	463, 206	431, 106	278, 644	284, 049	288, 722	279, 736	281, 972
$\gamma_{9,8}$	476, 146	465, 781	476, 415	469, 770	467, 392	521, 906	493, 889	521, 992	513, 774	510, 925
s.e.	476, 258	460, 453	485, 129	47, 071	438, 996	288, 833	292, 542	299, 371	290, 306	292, 743
$\gamma_{9,9}$	186, 605	175, 246	187, 142	179, 881	177, 283	241, 823	209, 906	242, 011	232, 931	229, 780
s.e.	484, 936	468, 813	493, 709	479, 233	448, 485	300, 836	302, 637	311, 912	302, 742	305, 409
$\gamma_{9,10}$	−102,936	−115,288	−102,130	−110,008	−112,826	−38,261	−74,078	−37,970	−47,913	−51,365
s.e.	495, 021	478, 530	503, 685	489, 412	459, 475	314, 445	314, 179	326, 127	316, 824	319, 745
$\gamma_{10,2}$	2, 263, 225	2, 254, 413	2, 260, 539	2, 254, 646	2, 252, 533	2, 241, 256	2, 232, 019	2, 240, 197	2, 234, 110	2, 232, 027
s.e.	542, 322	524, 109	550, 544	537, 098	510, 547	349, 777	325, 828	362, 465	355, 889	360, 256
$\gamma_{10,3}$	1, 973, 684	1, 963, 879	1, 971, 267	1, 964, 758	1, 962, 423	1, 961, 173	1, 948, 036	1, 960, 216	1, 953, 266	1, 950, 882
s.e.	539, 925	521, 800	548, 167	534, 684	507, 975	346, 845	322, 941	359, 406	352, 901	357, 228
$\gamma_{10,4}$	1, 684, 143	1, 673, 344	1, 681, 994	1, 674, 869	1, 672, 314	1, 681, 089	1, 664, 053	1, 680, 235	1, 672, 423	1, 669, 737
s.e.	538, 950	520, 860	547, 201	533, 702	506, 929	345, 767	321, 759	358, 285	351, 810	356, 125
$\gamma_{10,5}$	1, 394, 602	1, 382, 810	1, 392, 722	1, 384, 980	1, 382, 205	1, 401, 005	1, 380, 069	1, 400, 254	1, 391, 579	1, 388, 593
s.e.	539, 405	521, 298	547, 652	534, 160	507, 417	346, 561	322, 301	359, 119	352, 635	356, 966

Table 9. Continued.

	Fixed Regression Coefficients					Random Regression Coefficients				
	Non-Robust	Robust				Non-Robust	Robust			
	LS	M-Huber	M-Hampel	M-Bisquare	MM	LS	M-Huber	M-Hampel	M-Bisquare	MM
$Y_{10,6}$	1, 105, 060	1, 092, 276	1, 103, 449	1, 095, 091	1, 092, 096	1120, 922	1, 096, 086	1, 120, 273	1, 110, 736	1, 107, 448
s.e.	541, 286	523, 111	549, 517	536, 054	509, 435	349, 213	324, 559	361, 894	355, 362	359, 736
$Y_{10,7}$	815, 519	801, 741	814, 177	805, 202	801, 987	840, 838	812, 102	840, 292	829, 893	826, 303
s.e.	544, 577	526, 284	552, 782	539, 370	512, 965	353, 681	328, 497	366, 568	359, 947	364, 392
$Y_{10,8}$	525, 978	511, 207	524, 904	515, 313	511, 878	560, 754	528, 119	560, 311	549, 049	545, 158
s.e.	549, 255	530, 792	557, 422	544, 082	517, 976	359, 899	334, 055	373, 069	366, 321	370, 863
$Y_{10,9}$	236, 437	220, 673	235, 632	225, 424	221, 769	280, 671	244, 135	280, 329	268, 206	264, 014
s.e.	555, 284	536, 602	563, 403	550, 152	524, 426	367, 777	341, 155	381, 302	374, 393	379, 055
$Y_{10,10}$	−53,104	−69,862	−53,641	−64,464	−68,340	587	−39,848	348	−12,638	−17,131
s.e.	562, 620	543, 673	570, 683	557, 538	532, 261	377, 211	349, 702	391, 160	384, 055	388, 860

In order to remediate the effect of outliers to the estimation of total reserves, robust versions of fixed and random coefficients regression models were applied. The superiority of robust estimators in comparison with the non-robust estimators was evaluated by data implementations, where in most cases, the M-Huber robust estimator had the best performance. In the presence of artificial outliers, robust methods led to satisfactory estimation results as well as to lower standard errors of reserve estimates. Of course, we have to mention that in the final step of reserves estimation, the bias term due to robustification of outliers (if there are real large values and not artificial outliers) must be added to the value of the total reserve. The main advantages of applying M and MM estimators in claims reserving are their simplicity, the availability of robust procedures in R packages and the interactivity of these packages with other software (e.g. Excel) for an easier implementation of datasets in actuarial practice.

However, there are cases where our models are not appropriate and alternative robust estimators should be used, such as, when the distribution of the error term in a regression model is asymmetric or the claims are heavy-tailed distributed. For asymmetric errors, Bianco *et al.* (2005) proposed a generalised version of MM estimates in case where the distribution of errors is in a class of exponential families, including the log-gamma distribution. For cases where the claims are heavy-tailed distributed, Dornheim and Brazauskas (2011) proposed the so-called corrected adaptively truncated likelihood procedure with symmetric or asymmetric log-location-scale errors that provides high robustness against outliers while achieving high efficiency for the assumed long-tailed model when none of the claims are truncated. Further investigation on how our models can be implemented in these cases could be the topic for future research.

Acknowledgements. The authors thank the anonymous referees for their valuable comments and acknowledge the partial support from the University of Piraeus Research Center. This work is partially based on the PhD thesis of the first author.

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Appendix A Claims Data from Taylor and Ashe (1983)

	1	2	3	4	5	6	7	8	9	10
1	357,848	766,940	610,542	482,940	527,326	574,398	146,342	139,950	227,229	67,948
2	352,118	884,021	933,894	1,183,289	445,745	320,996	527,804	266,172	425,046	
3	290,507	1,001,799	926,219	1,016,654	750,816	146,923	495,992	280,405		
4	310,608	1,108,250	776,189	1,562,400	272,482	352,053	206,286			
5	443,160	693,190	991,983	769,488	504,851	470,639				
6	396,132	937,085	847,498	805,037	705,960					
7	440,832	847,631	1,131,398	1,063,269						
8	359,480	1,061,648	1,443,370							
9	376,686	986,608								
10	344,014									

Appendix B Recent Claims Data from Non-Life LoB

	1	2	3	4	5	6	7	8	9	10
1	674,526	2,816,693	2,021,829	1,943,845	1,711,712	1,519,730	1,850,053	537,995	457,335	98,294
2	974,520	1,958,442	2,103,564	1,825,928	2,399,323	1,052,405	747,174	241,985	475,218	
3	940,002	2,107,094	2,863,319	2,538,894	1,602,877	1,056,410	1,401,421	364,656		
4	658,858	2,271,063	2,550,733	2,228,533	922,265	1,252,816	1,009,052			
5	656,028	1,987,034	1,989,773	1,574,610	1,384,682	1,179,991				
6	851,695	2,446,490	2,030,418	2,735,539	1,628,220					
7	658,129	1,585,558	2,203,499	2,028,546						
8	551,818	2,203,066	2,386,707							
9	864,217	1,921,885								
10	622,542									

Cite this article: Badounas I, Bozikas A and Pitselis G (2022). A robust random coefficient regression representation of the chain-ladder method, *Annals of Actuarial Science*, **16**, 151–182. https://doi.org/10.1017/S1748499521000154