A PRIMAL-DUAL INTERIOR-POINT ALGORITHM BASED ON A NEW KERNEL FUNCTION

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Abstract

We propose a new primal-dual interior-point algorithm based on a new kernel function for linear optimization problems. New search directions and proximity functions are proposed based on the kernel function. We show that the new algorithm has $\mathcal{O}(\sqrt{n} \log n \log(n/\epsilon))$ and $\mathcal{O}(\sqrt{n} \log(n/\epsilon))$ iteration bounds for large-update and small-update methods, respectively, which are currently the best known bounds for such methods.

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1. Introduction

In this paper we propose a new primal-dual interior-point method (IPM) for the following standard linear optimization (LO) problem:

$$\min\{c^T x : Ax = b, \, x \geq 0\}, \tag{1.1}$$

where $A \in \mathbf{R}^{m \times n}$, c, $x \in \mathbf{R}^n$, and $b \in \mathbf{R}^m$. Without loss of generality, we assume that $m \leq n$ and $\text{rank}(A) = m$. The dual problem to (1.1) is

$$\max\{b^T y : A^T y + s = c, \, s \geq 0\}, \tag{1.2}$$

where $y \in \mathbf{R}^m$ and $s \in \mathbf{R}^n$.

Since Karmarkar's paper [13] in 1984, IPMs have shown their efficiency in solving large-scale linear programming problems with a wide variety of successful applications. In this paper we propose a new primal-dual interior-point algorithm

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which is the most efficient from a computational point of view [3]. It is generally agreed that the iteration complexity of the algorithm is an appropriate measure for its efficiency [10].

Most polynomial-time interior-point algorithms for LO are based on the logarithmic kernel function [3, 9, 12, 19]. Peng *et al.* [15, 16] proposed a new variant of IPMs based on self-regular kernel functions and achieved so far the best known complexity, $\mathcal{O}(\sqrt{n} \log n \log(n/\epsilon))$ for large-update methods with a specific self-regular function. Roos *et al.* [4–8, 10, 11] proposed new primal-dual IPMs for LO problems based on eligible kernel functions. They also proposed a scheme for analysing the algorithms based on four conditions on the kernel function [5] and obtained an $\mathcal{O}(\sqrt{n} \log(n/\epsilon))$ iteration bound for small-update IPMs in all cases and an $\mathcal{O}(\sqrt{n} \log n \log(n/\epsilon))$ iteration bound for large-update methods with a specific kernel function [4]. At present, the open question in this area is the existence of a kernel function whose complexity bound for large-update methods is the same as or better than the best known complexity result. Recently, Amini and Haseli [1] proposed a generalized version of the kernel function in [5], achieved the best known iteration bound for large-update methods for LO and extended to linear complementarity problems [2]. Motivated by their work, we define a new kernel function and propose a new primal-dual interior-point algorithm for LO based on this kernel function. We analyse the complexity for large-update and small-update methods based on the kernel function and obtain $\mathcal{O}(\sqrt{n} \log n \log(n/\epsilon))$ and $\mathcal{O}(\sqrt{n} \log(n/\epsilon))$ iteration bounds, respectively, which are the best known results so far.

The paper is organized as follows. In Section 2 we recall the generic IPM and the motivation for the new algorithm. In Section 3 we define a new kernel function and give its properties which are essential for complexity analysis. In Section 4 we derive the complexity result for the new algorithm.

We use the following notation throughout the paper. We let $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of n-dimensional nonnegative vectors and positive vectors, respectively. For $x, s \in \mathbf{R}^n$, $x_{\min}$ and xs denote the smallest component of the vector x and the componentwise product of the vectors x and s, respectively. We let e denote the n-dimensional vector $(1, 1, \ldots, 1)$. For $x, s \in \mathbf{R}^n_{++}$ and $\mu > 0$, we define $v := \sqrt{xs/\mu}$, $v^{-1} := \sqrt{\mu e/(xs)}$ whose ith components are $\sqrt{x_i s_i/\mu}$ and $\sqrt{\mu/(x_i s_i)}$, respectively. We denote by X the diagonal matrix obtained from a vector x, that is, $X = \operatorname{diag}(x)$. For $f(x), g(x) : \mathbf{R}_{++} \to \mathbf{R}_{++}$, we write $f(x) = \mathcal{O}(g(x))$ if there exists a positive constant c_1 such that $f(x) \le c_1 g(x)$ and $f(x) = \Theta(g(x))$ if there exist positive constants c_2 and c_3 such that $c_2 g(x) \le f(x) \le c_3 g(x)$, for all $x > 0$.

2. Preliminaries

In this section we recall some basic concepts and the generic IPM. Without loss of generality, we assume that both (1.1) and (1.2) satisfy the interior-point condition (IPC) [17], that is, there exists (x^0, y^0, s^0) such that

$$Ax^0 = b, x^0 > 0, \quad A^T y^0 + s^0 = c, s^0 > 0.$$

By the duality theorem [17, Theorem II.2], finding an optimal solution of (1.1) and (1.2) is equivalent to solving the following system:

$$Ax = b, x \geq 0, \quad A^T y + s = c, s \geq 0, \quad xs = 0. \tag{2.1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (2.1) by the parameterized equation $xs = \mu e$ with $\mu > 0$. We now consider the following system:

$$Ax = b, x > 0, \quad A^T y + s = c, s > 0, \quad xs = \mu e. \tag{2.2}$$

If the IPC holds, then (2.2) has a unique solution for each $\mu > 0$ [14]. We denote this solution by $(x(\mu), y(\mu), s(\mu))$ and call $x(\mu)$ the μ-centre of (1.1) and $(y(\mu), s(\mu))$ the μ-centre of (1.2). The set of μ-centres, $\mu > 0$, is called the central path of (1.1) and (1.2) [18]. The limit of the central path as μ goes to zero exists and since the limit point satisfies (2.1), it naturally yields the optimal solution for (1.1) and (1.2) [17]. Primal-dual IPMs follow the central path approximately and approach the solution of (1.1) and (1.2) as μ goes to zero.

For given $(x, y, s) := (x^0, y^0, s^0)$, by applying Newton's method to system (2.2) we have the following Newton system:

$$A\Delta x = 0, \quad A^T \Delta y + \Delta s = 0, \quad s\Delta x + x\Delta s = \mu e - xs. \tag{2.3}$$

Since A has full row rank, (2.3) has a unique solution $(\Delta x, \Delta y, \Delta s)$ which is called the search direction [17]. By taking a step along the search direction $(\Delta x, \Delta y, \Delta s)$, one constructs a new iterate (x_+, y_+, s_+),

$$x_+ = x + \alpha \Delta x, \quad y_+ = y + \alpha \Delta y, \quad s_+ = s + \alpha \Delta s, \tag{2.4}$$

for some $\alpha > 0$.

To motivate our new algorithm we define the following scaled vectors: for $(x, s) > 0$ and $\mu > 0$,

$$d_x := v\Delta x/x, \quad d_s := v\Delta s/s. \tag{2.5}$$

Using (2.5), we can rewrite system (2.3) as follows:

$$\bar{A}d_x = 0, \quad \bar{A}^T \Delta y + d_s = 0, \quad d_x + d_s = v^{-1} - v, \tag{2.6}$$

where $\bar{A} := (1/\mu)AV^{-1}X$, $V := \text{diag}(v)$, and $X := \text{diag}(x)$. Note that the right-hand side of the third equation in (2.6) equals the negative gradient of the logarithmic barrier function $\Psi_l(v)$, that is,

$$d_x + d_s = -\nabla \Psi_l(v), \tag{2.7}$$

where

$$\Psi_l(v) := \sum_{i=1}^{n} \psi_l(v_i), \quad \psi_l(t) = \sum_{i=1}^{n} ((t^2 - 1)/2 - \log t).$$

We call ψ_l the classical logarithmic kernel function of $\Psi_l(v)$. In this paper we replace $\psi_l(t)$ with a new kernel function $\psi(t)$ which will be defined in (3.2). In what follows, and throughout the paper we assume that $\tau \geq 1$.

The generic interior-point algorithm works as follows. Assume that we are given a strictly feasible point (x, y, s) which is in a τ-neighbourhood of the given μ-centre. Then we decrease μ to $\mu_+ = (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$, and then solve the Newton system (2.3) to obtain the unique search direction. The positivity of a new iterate is ensured by an appropriate choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, y_+, s_+) that is in a τ-neighbourhood of the μ_+-centre, and then we let $\mu := \mu_+$ and $(x, y, s) := (x_+, y_+, s_+)$. Then μ is again reduced by the factor $1 - \theta$ and we solve the Newton system targeting at the new μ_+-centre, and so on. This process is repeated until μ is small enough, say until $n\mu < \varepsilon$.

Generic primal-dual algorithm for LO

Input:
> a threshold parameter $\tau > 0$;
> an accuracy parameter $\varepsilon > 0$;
> a fixed barrier-update parameter θ, $0 < \theta < 1$;
> (x^0, s^0) and $\mu^0 := 1$ such that $\Psi_l(\sqrt{x^0 s^0/\mu^0}) \leq \tau$;

begin
> $x := x^0$; $s := s^0$; $\mu := \mu^0$;
> while $n\mu \geq \varepsilon$ do
> begin
>> $\mu := (1 - \theta)\mu$;
>> while $\Psi_l(v) > \tau$ do
>> begin
>>> solve system (2.3) for Δx, Δy, Δs;
>>> determine a step size α;
>>> $x := x + \alpha \Delta x$;
>>> $s := s + \alpha \Delta s$;
>>> $y := y + \alpha \Delta y$;
>>> $v := \sqrt{xs/\mu}$;
>> end
> end
end

REMARK 2.1. If θ is a constant independent of the dimension of the problem, n, for example $\theta = 1/2$, then we call the algorithm a large-update method. If θ depends on n, for example $\theta = 1/\sqrt{n}$, then the algorithm is called a small-update method.

3. The new kernel function

In this section we define a new kernel function which is neither self-regular nor logarithmic and gives the best known complexity result. We call $\psi : \mathbf{R}_{++} \to \mathbf{R}_+$ a

kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\psi'(1) = \psi(1) = 0, \quad \psi''(t) > 0, t > 0, \quad \lim_{t \to 0^+} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty. \quad (3.1)$$

Now we define a new function $\psi(t)$ as follows:

$$\psi(t) := \frac{t^2 - 1}{2} + \frac{e^{p(g(t)-1)} - 1}{pq}, \quad p \geq 1, q \geq 1, t > 0, \quad (3.2)$$

where $g(t) := e^{q(t^{-1}-1)}$. Then we have the following:

$$\begin{aligned}
\psi'(t) &= t - e^{p(g(t)-1)} g(t) t^{-2}, \\
\psi''(t) &= 1 + e^{p(g(t)-1)} g(t) t^{-4} (pqg(t) + q + 2t), \\
\psi'''(t) &= -e^{p(g(t)-1)} g(t) t^{-6} \big(p^2 q^2 g(t)^2 + 3pq^2 g(t) \\
&\quad + 6pqg(t)t + q^2 + 6qt + 6t^2\big).
\end{aligned} \quad (3.3)$$

From (3.3), $\psi(t)$ is clearly a kernel function and

$$\psi''(t) > 1, \quad t > 0. \quad (3.4)$$

In this paper, we replace the function $\Psi_l(v)$ in (2.7) with the function $\Psi(v)$ defined as follows:

$$d_x + d_s = -\nabla\Psi(v), \quad (3.5)$$

where $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$ with $\psi(t)$ defined by (3.2). Hence the new search direction $(\Delta x, \Delta y, \Delta s)$ is obtained by solving the following modified Newton system:

$$A\Delta x = 0, \quad A^T \Delta y + \Delta s = 0, \quad s\Delta x + x\Delta s = -\mu v \nabla\Psi(v). \quad (3.6)$$

Note that d_x and d_s are orthogonal because the vector d_x belongs to the null space and d_s to the row space of the matrix $\bar{A}$. Since d_x and d_s are orthogonal, we have

$$\Psi(v) = 0 \Leftrightarrow v = \mathbf{e} \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow d_x = d_s = 0 \Leftrightarrow x = x(\mu) \text{ and } s = s(\mu).$$

We use $\Psi(v)$ as the proximity function to measure the distance between the current iterate and the μ-centre for given $\mu > 0$. We also define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \tfrac{1}{2}\|\nabla\Psi(v)\| = \tfrac{1}{2}\|d_x + d_s\|. \quad (3.7)$$

LEMMA 3.1. *For $\psi(t)$, the following results hold:*

(i) $\psi(t)$ *is exponentially convex, that is,*

$$\psi(\sqrt{t_1 t_2}) \leq \tfrac{1}{2}(\psi(t_1) + \psi(t_2)), \quad t_1, t_2 > 0;$$

(ii) $\psi''(t)$ *is monotonically decreasing, $t > 0$;*
(iii) $t\psi''(t) - \psi'(t) > 0, t > 0$;
(iv) $2(\psi''(t))^2 - \psi'(t)\psi'''(t) > 0, t > 0.$

PROOF. For (i), by [16, Lemma 2.1.2], it suffices to show that $\psi(t)$ satisfies $t\psi''(t) + \psi'(t) > 0$, $t > 0$. For $t > 0$, using (3.3),

$$t\psi''(t) + \psi'(t) = 2t + e^{p(g(t)-1)}g(t)t^{-3}(pqg(t) + q + t) > 0,$$

with $g(t)$ defined as before. For (ii), from (3.3), $\psi'''(t) < 0$, $t > 0$. For (iii), using (3.3), we have for $t > 0$,

$$t\psi''(t) - \psi'(t) = e^{p(g(t)-1)}g(t)t^{-3}(pqg(t) + q + 3t) > 0.$$

For (iv), using (3.3), we have for $t > 0$,

$$\begin{aligned}
2(\psi''(t))^2 - \psi'(t)\psi'''(t) = {}& 2 + e^{p(g(t)-1)}g(t)t^{-8}\big(10pqg(t)t^4 + 10qt^4 \\
& + 14t^5 + p^2q^2 e^{p(g(t)-1)}g(t)^3 + q^2 e^{p(g(t)-1)}g(t) \\
& + 2e^{p(g(t)-1)}g(t)t^2 + pq^2 e^{p(g(t)-1)}g(t)^2 \\
& + 2pq e^{p(g(t)-1)}g(t)^2 t + 2q e^{p(g(t)-1)}g(t)t \\
& + p^2 q^2 g(t)^2 t^3 + 3pq^2 e^{p(g(t)-1)}g(t)t^3\big) > 0.
\end{aligned}$$

This completes the proof. □

LEMMA 3.2. *For $\psi(t)$, the following results hold:*
  (i) $(t-1)^2/2 \leq \psi(t) \leq (\psi'(t))^2/2$, $t > 0$;
  (ii) $\psi(t) \leq (pq + q + 3)(t-1)^2/2$, $p \geq 1$, $q \geq 1$, $t \geq 1$.

PROOF. For (i), using the first condition of (3.1) and (3.4),

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\, d\zeta\, d\xi \geq \int_1^t \int_1^\xi d\zeta\, d\xi = \frac{1}{2}(t-1)^2,$$

which proves the first inequality. The second inequality is obtained as follows:

$$\begin{aligned}
\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\, d\zeta\, d\xi &\leq \int_1^t \int_1^\xi \psi''(\xi)\psi''(\zeta)\, d\zeta\, d\xi \\
&= \int_1^t \psi''(\xi)\psi'(\xi)\, d\xi = \int_1^t \psi'(\xi)\, d\psi'(\xi) = \frac{1}{2}(\psi'(t))^2.
\end{aligned}$$

For (ii), using Taylor's theorem and noting that $\psi(1) = \psi'(1) = 0$, $\psi'''(t) < 0$, for all $t > 0$, and $\psi''(1) = pq + q + 3$,

$$\begin{aligned}
\psi(t) &= \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{3!}\psi'''(\xi)(\xi-1)^3 \\
&= \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{3!}\psi'''(\xi)(\xi-1)^3 < \frac{pq+q+3}{2}(t-1)^2,
\end{aligned}$$

for some ξ, $1 \leq \xi \leq t$. This completes the proof. □

LEMMA 3.3 ([5, Lemma 2.4]). *If $\psi(t)$ satisfies Lemma 3.1(ii) and (iii), then $\psi(t)$ satisfies*

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0, \quad t > 1, \beta > 1.$$

REMARK 3.4. Let $\varphi(t) = (e^{p(g(t)-1)} - 1)/(pq)$. Then $\psi(t) = (t^2 - 1)/2 + \varphi(t)$. Since $\varphi'(t) = -e^{p(g(t)-1)}g(t)t^{-2} < 0$, $\varphi(t)$ is monotonically decreasing with respect to $t > 0$.

Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$ and $\rho : [0, \infty) \to (0, 1]$ the inverse function of $-\psi'(t)/2$ for $t \in (0, 1]$. Then we have the following lemma.

LEMMA 3.5. *We have:*

(i) $\sqrt{2u + 1} \leq \varrho(u) \leq 1 + \sqrt{2u}$, $u \geq 0$;
(ii) $\rho(z) \geq 1/(1 + q^{-1}\log(1 + p^{-1}\log(1 + 2z)))$, $z \geq 0$.

PROOF. For (i), let $u = \psi(t)$ for $t \geq 1$. Then $\varrho(u) = t$, $t \geq 1$. Using Lemma 3.2(i), we have $u = \psi(t) \geq (t - 1)^2/2$. Then

$$t = \varrho(u) \leq 1 + \sqrt{2u}.$$

By the definition of $\varphi(t)$, we have $u = \psi(t) = (t^2 - 1)/2 + \varphi(t)$. Using Remark 3.4 and $\varphi(1) = 0$, we have $\varphi(t) \leq 0$, $t \geq 1$. Hence,

$$t^2/2 = u + 1/2 - \varphi(t) \geq u + 1/2.$$

Thus

$$t = \varrho(u) \geq \sqrt{2u + 1}.$$

For (ii), let $z = -\psi'(t)/2$ for $0 < t \leq 1$. Due to the definition of ρ, $\rho(z) = t$ for $z \geq 0$. From (3.3), we have $-t + e^{p(g(t)-1)}g(t)t^{-2} = 2z$ and

$$e^{p(g(t)-1)}g(t)t^{-2} = t + 2z \leq 1 + 2z, \quad 0 < t \leq 1. \tag{3.8}$$

By taking the logarithm of both sides of (3.8), we obtain

$$p(g(t) - 1) + q(t^{-1} - 1) - 2\log t \leq \log(1 + 2z).$$

Since $0 < t \leq 1$,

$$g(t) \leq 1 + p^{-1}\log(1 + 2z). \tag{3.9}$$

Taking the logarithm of both sides of (3.9), we obtain

$$q(t^{-1} - 1) \leq \log(1 + p^{-1}\log(1 + 2z))$$

and hence

$$t^{-1} \leq 1 + q^{-1}\log(1 + p^{-1}\log(1 + 2z)).$$

That is,

$$\rho(z) = t \geq \frac{1}{1 + q^{-1}\log(1 + p^{-1}\log(1 + 2z))}.$$

This completes the proof. □

LEMMA 3.6. *Let $\beta \geq 1$. Then $\psi(\beta t) \leq \psi(t) + t^2(\beta^2 - 1)/2$.*

PROOF. Using Remark 3.4, we have $\varphi(\beta t) - \varphi(t) \leq 0$ for $\beta \geq 1$. Hence

$$
\begin{aligned}
\psi(\beta t) &= ((\beta t)^2 - 1)/2 + \varphi(\beta t) \\
&= (t^2 - 1)/2 + \varphi(t) + (\beta^2 t^2 - t^2)/2 + \varphi(\beta t) - \varphi(t) \\
&= \psi(t) + (t^2\beta^2 - t^2)/2 + \varphi(\beta t) - \varphi(t) \\
&\leq \psi(t) + t^2(\beta^2 - 1)/2.
\end{aligned}
$$

This completes the proof. □

Using Lemma 3.3, we have the following. The reader can refer to [5, Theorem 3.2] for the proof.

LEMMA 3.7. *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$, $t \geq 1$. Then*

$$
\Psi(\beta v) \leq n\psi(\beta\varrho(\Psi(v)/n)), \quad v \in \mathbf{R}^{\mathbf{n}}_{++}, \beta \geq 1.
$$

In the following theorem we compute two upper bounds for the effect of a μ-update on the value of $\Psi(v)$.

THEOREM 3.8. *Let $0 \leq \theta < 1$ and $v_+ = v/\sqrt{1 - \theta}$. If $\Psi(v) \leq \tau$, then:*
  (i) $\Psi(v_+) \leq (2\tau + \theta\sqrt{8n\tau} + \theta n)/(2(1 - \theta))$;
  (ii) $\Psi(v_+) \leq (\sqrt{n}\theta + \sqrt{2\tau})^2(pq + q + 3)/(2(1 - \theta))$.

PROOF. For (i), using Lemma 3.6 with $\beta = 1/\sqrt{1 - \theta}$, we obtain

$$
\Psi(v_+) = \Psi(\beta v) \leq \sum_{i=1}^{n} \left(\psi(v_i) + \frac{1}{2}(\beta^2 - 1)v_i^2 \right) = \Psi(v) + \frac{\theta\|v\|^2}{2(1 - \theta)}. \tag{3.10}
$$

By the first inequality in Lemma 3.2(i) and the Cauchy–Schwarz inequality,

$$
2\Psi(v) = 2\sum_{i=1}^{n} \psi(v_i) \geq \sum_{i=1}^{n}(v_i - 1)^2 = \|v\|^2 - 2e^T v + n \geq (\|v\| - \|e\|)^2.
$$

Hence,

$$
\|v\| \leq \|e\| + \sqrt{2\Psi(v)} = \sqrt{n} + \sqrt{2\Psi(v)}. \tag{3.11}
$$

From (3.10), (3.11), and $\Psi(v) \leq \tau$ we have

$$
\begin{aligned}
\Psi(v_+) &\leq \Psi(v) + \frac{\theta}{2(1 - \theta)}(2\Psi(v) + \sqrt{8n\Psi(v)} + n) \\
&\leq \frac{1}{2(1 - \theta)}(2\tau + \theta\sqrt{8n\tau} + \theta n).
\end{aligned}
$$

For (ii), since $1/\sqrt{1 - \theta} \geq 1$ and $\varrho(\Psi(v)/n) \geq 1$, we have $\varrho(\Psi(v)/n)/\sqrt{1 - \theta} \geq 1$. Using Lemma 3.7 with $\beta = 1/\sqrt{1 - \theta}$, Lemma 3.2(ii), the second inequality in

Lemma 3.5(i), and $\Psi(v) \leq \tau$,

$$
\begin{aligned}
\Psi(v_+) &\leq n\psi\left(\frac{\varrho(\Psi(v)/n)}{\sqrt{1-\theta}}\right) \leq \frac{(pq+q+3)n}{2}\left(\frac{\varrho(\Psi(v)/n)}{\sqrt{1-\theta}} - 1\right)^2 \\
&= \frac{(pq+q+3)n}{2}\left(\frac{\varrho(\Psi(v)/n) - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \\
&\leq \frac{(pq+q+3)n}{2}\left(\frac{1 + \sqrt{2\tau/n} - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \\
&\leq \frac{(pq+q+3)n}{2}\left(\frac{\theta + \sqrt{2\tau/n}}{\sqrt{1-\theta}}\right)^2 \\
&= \frac{pq+q+3}{2(1-\theta)}(\sqrt{n}\theta + \sqrt{2\tau})^2,
\end{aligned}
\tag{3.12}
$$

where the last inequality holds because

$$
1 - \sqrt{1-\theta} = \theta/(1 + \sqrt{1-\theta}) \leq \theta, \quad 0 < \theta < 1.
$$

This completes the proof. □

Define

$$
\bar{\Psi}_0 := \frac{2\tau + \theta\sqrt{8n\tau} + \theta n}{2(1-\theta)}, \quad \tilde{\Psi}_0 := \frac{pq+q+3}{2(1-\theta)}(\sqrt{n}\theta + \sqrt{2\tau})^2.
\tag{3.13}
$$

We will use $\bar{\Psi}_0$ and $\tilde{\Psi}_0$ for the upper bounds of $\Psi(v)$ for large-update and small-update methods, respectively.

REMARK 3.9. By Theorem 3.8 and the assumption that $\Psi(v) \leq \tau$ just before the update of μ, $\Psi(v_+) \leq \min\{\bar{\Psi}_0, \tilde{\Psi}_0\}$. For large-update methods with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, we have $\bar{\Psi}_0 = \mathcal{O}(n)$, and for small-update methods with $\tau = \mathcal{O}(1)$ and $\theta = \Theta(1/\sqrt{n})$, we have $\tilde{\Psi}_0 = \mathcal{O}(pq + q)$.

4. Complexity analysis

In this section we compute a proper step size and the decrease of the proximity function during an inner iteration and give complexity results for the algorithm. For fixed μ, if we take a step size α, then we have a new iterate $x_+ = x + \alpha\Delta x$, $s_+ = s + \alpha\Delta s$. Using (2.5),

$$
x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x)
$$

and
$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s).$$

Thus we have
$$v_+ := \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.$$

Define, for $\alpha > 0$,
$$f(\alpha) = \Psi(v_+) - \Psi(v).$$

Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By Lemma 3.1(i),
$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \leq \tfrac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

Hence we have $f(\alpha) \leq f_1(\alpha)$, where
$$f_1(\alpha) := \tfrac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v). \tag{4.1}$$

We have
$$f(0) = f_1(0) = 0.$$

Taking the derivative of $f_1(\alpha)$ with respect to α, we obtain
$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi'(v_i + \alpha[d_x]_i)[d_x]_i + \psi'(v_i + \alpha[d_s]_i)[d_s]_i),$$

where $[d_x]_i$ and $[d_s]_i$ denote the ith components of the vectors d_x and d_s, respectively. Using (3.5) and (3.7),
$$f_1'(0) = \tfrac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\tfrac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta^2(v). \tag{4.2}$$

Differentiating $f_1'(\alpha)$ with respect to α, we obtain
$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi''(v_i + \alpha[d_s]_i)[d_s]_i^2). \tag{4.3}$$

Since $f_1''(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in α unless $d_x = d_s = 0$.

LEMMA 4.1. *Let $\delta(v)$ be defined as in* (3.7). *Then*
$$\delta(v) \geq \sqrt{\Psi(v)/2}.$$

PROOF. Using Lemma 3.2(i),
$$\delta(v)^2 = \frac{1}{4}\|\nabla\Psi(v)\|^2 = \frac{1}{4}\sum_{i=1}^{n}(\psi'(v_i))^2 \geq \frac{\Psi(v)}{2}.$$

Hence $\delta(v) \geq \sqrt{\Psi(v)/2}$.   □

For notational convenience we denote $\delta := \delta(v)$ and $\Psi := \Psi(v)$.

LEMMA 4.2 (Modification of [5, Lemma 4.1]). *Let $f_1(\alpha)$ be defined as in* (4.1) *and δ as in* (3.7). *Then*

$$f_1''(\alpha) \leq 2\delta^2 \psi''(v_{\min} - 2\alpha\delta). \tag{4.4}$$

LEMMA 4.3 (Modification of [5, Lemma 4.2]). *If the step size α satisfies the inequality*

$$-\psi'(v_{\min} - 2\alpha\delta) + \psi'(v_{\min}) \leq 2\delta, \tag{4.5}$$

then

$$f_1'(\alpha) \leq 0.$$

LEMMA 4.4 (Modification of [5, Lemma 4.3]). *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of $-\psi'(t)/2$ for all $t \in (0, 1]$. Then, in the worst case, the largest step size $\bar{\alpha}$ satisfying* (4.5) *is given by*

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)).$$

LEMMA 4.5 (Modification of [5, Lemma 4.4]). *Let ρ and $\bar{\alpha}$ be defined as in Lemma* 4.4. *Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

For later reference, we define

$$\tilde{\alpha} := \frac{1}{\psi''(\rho(2\delta))}. \tag{4.6}$$

Then $\tilde{\alpha} \leq \bar{\alpha}$.

LEMMA 4.6 ([16, Lemma 1.3.3]). *Suppose that $h(t)$ is a twice-differentiable convex function with*

$$h(0) = 0, \quad h'(0) < 0,$$

and such that $h(t)$ attains its (global) minimum at $t^* > 0$, and $h''(t)$ is increasing with respect to t. Then for any $t \in [0, t^*]$,

$$h(t) \leq \frac{th'(0)}{2}.$$

LEMMA 4.7 (Modification of [5, Lemma 4.5]). *If the step size α is such that $\alpha \leq \tilde{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

PROOF. Let the univariate function h be such that

$$h(0) = f_1(0) = 0, \quad h'(0) = f_1'(0) = -2\delta^2, \quad h''(\alpha) = 2\delta^2 \psi''(v_{\min} - 2\alpha\delta).$$

Then $h(t)$ is twice differentiable, $h(0) = 0$, and $h'(0) < 0$. Since $h''(\alpha) > 0$, $h(t)$ is strictly convex and hence has a global minimum at some $\alpha^* > 0$. From (4.4), we have $f_1''(\alpha) \leq h''(\alpha)$. As a result, we have $f_1'(\alpha) \leq h'(\alpha)$ and $f_1(\alpha) \leq h(\alpha)$. Taking $\alpha \leq \tilde{\alpha}$

and using (4.5), we obtain

$$
\begin{aligned}
h'(\alpha) &= h'(0) + \int_0^\alpha h''(\zeta)\, d\zeta \\
&= -2\delta^2 + 2\delta^2 \int_0^\alpha \psi''(v_{\min} - 2\xi\delta)\, d\xi \\
&= -2\delta^2 - \frac{2\delta^2}{2\delta} \int_0^\alpha \psi''(v_{\min} - 2\xi\delta)\, d(v_{\min} - 2\xi\delta) \\
&= -2\delta^2 - \delta(\psi'(v_{\min} - 2\alpha\delta) - \psi'(v_{\min})) \\
&\le -2\delta^2 + 2\delta^2 = 0.
\end{aligned}
$$

Hence we have $\tilde\alpha \le \alpha^*$. Since

$$
h'''(\alpha) = -4\delta^3 \psi'''(v_{\min} - 2\alpha\delta) > 0,
$$

$h''(\alpha)$ is monotonically increasing in α. Thus, using Lemma 4.6,

$$
f_1(\alpha) \le h(\alpha) \le \frac{\alpha h'(0)}{2} = -\alpha\delta^2.
$$

Since $f(\alpha) \le f_1(\alpha)$, the lemma is proved. $\qquad \square$

Using Lemma 4.7 and (4.6), we have the following lemma.

LEMMA 4.8 (Modification of [5, Lemma 4.6]). *Let $\tilde\alpha$ be defined as in* (4.6). *Then*

$$
f(\tilde\alpha) \le -\frac{\delta^2}{\psi''(\rho(2\delta))}. \tag{4.7}
$$

Using Lemma 3.1(iv), we obtain the following.

LEMMA 4.9 (Modification of [5, Lemma 4.10]). *The right-hand side in* (4.7) *is monotonically decreasing with respect to δ.*

Denote the value of $\Psi(v)$ after the μ-update as Ψ_0. Then

$$
\Psi_0 \le \bar\Psi_0 \quad \text{and} \quad \Psi_0 \le \tilde\Psi_0, \tag{4.8}
$$

where $\bar\Psi_0$ and $\tilde\Psi_0$ are defined in (3.13). Define

$$
L(\Psi, p) := 1 + p^{-1} \log(1 + 2\sqrt{2\Psi}). \tag{4.9}
$$

Then $L(\Psi, p)$ is monotonically increasing with respect to Ψ and $L(\Psi, p) \ge 1$.

THEOREM 4.10. *Let $\tilde\alpha$ be defined as in* (4.6), *$\mathfrak{X} := 1 + q^{-1} \log(L(\Psi, p))$, and $\mathfrak{X}_0 := 1 + q^{-1} \log(L(\Psi_0, p))$. Assume that $\Psi > \tau \ge 1$. Then for $p \ge 1$ and $q \ge 1$,*

$$
f(\tilde\alpha) \le -\frac{\sqrt{\Psi}}{10 L(\Psi_0, p) \mathfrak{X}_0^4 (pq L(\Psi_0, p) + q + 2)},
$$

where $L(\Psi_0, p) := 1 + p^{-1} \log(1 + 2\sqrt{2\Psi_0})$.

PROOF. Using Lemma 3.5(ii),

$$\rho\left(2\sqrt{\frac{\Psi}{2}}\right) \geq \frac{1}{1 + q^{-1}\log(1 + p^{-1}\log(1 + 2\sqrt{2\Psi}))}. \tag{4.10}$$

Using (4.10), $\psi'''(t) < 0$, for all $t > 0$, (3.3), and the assumption,

$$\begin{aligned}
\psi''\left(\rho\left(2\sqrt{\frac{\Psi}{2}}\right)\right) &\leq \psi''\left(\frac{1}{\mathfrak{X}}\right) \\
&= 1 + \left(pqL(\Psi, p) + q + \frac{2}{\mathfrak{X}}\right)(1 + 2\sqrt{2\Psi})L(\Psi, p)\mathfrak{X}^4 \\
&\leq 2(1 + \sqrt{2\Psi})L(\Psi, p)\mathfrak{X}^4(pqL(\Psi, p) + q + 2) \\
&\leq 5\sqrt{\Psi}L(\Psi, p)\mathfrak{X}^4(pqL(\Psi, p) + q + 2), \tag{4.11}
\end{aligned}$$

where the last inequality follows from the assumption $\Psi \geq 1$ and a simple calculation. Using Lemmas 4.8, 4.9, 4.1, and (4.11),

$$\begin{aligned}
f(\tilde{\alpha}) &\leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \\
&\leq -\frac{\Psi/2}{\psi''(\rho(2\sqrt{\Psi/2}))} \\
&\leq -\frac{\Psi/2}{5\sqrt{\Psi}L(\Psi, p)\mathfrak{X}^4(pqL(\Psi, p) + q + 2)} \\
&= -\frac{\sqrt{\Psi}}{10L(\Psi, p)\mathfrak{X}^4(pqL(\Psi, p) + q + 2)} \\
&\leq -\frac{\sqrt{\Psi}}{10L(\Psi_0, p)\mathfrak{X}_0^4(pqL(\Psi_0, p) + q + 2)},
\end{aligned}$$

where the last inequality follows from the assumption $\Psi_0 \geq \Psi$. This completes the proof. $\square$

LEMMA 4.11 ([15, Lemma 14]). *Let* $t_0, t_1, \ldots, t_K$ *be a sequence of finitely many positive numbers such that*

$$t_{k+1} \leq t_k - \gamma t_k^{1-\tilde{\beta}}, \quad k = 0, 1, \ldots, K - 1,$$

where $\gamma > 0$ *and* $0 < \tilde{\beta} \leq 1$. *Then* $K \leq \lfloor t_0^{\tilde{\beta}}/(\gamma\tilde{\beta})\rfloor$.

After the μ-update we denote the subsequent values in the same outer iteration as $\Psi_k, k = 1, 2, \ldots, K$. Then

$$\Psi_{K-1} > \tau, \quad 0 \leq \Psi_K \leq \tau.$$

Denote

$$N(\Psi, p, q) := L(\Psi, p)(1 + q^{-1} \log(L(\Psi, p)))^4 (pqL(\Psi, p) + q + 2), \quad (4.12)$$

where $L(\Psi, p)$ is defined in (4.9). Then $N(\Psi, p, q)$ is monotonically increasing with respect to Ψ.

LEMMA 4.12. *Let* $\bar{\Psi}_0$ *and* $\tilde{\Psi}_0$ *be defined as in* (3.13) *and* K_1 *and* K_2 *be the total number of inner iterations in the outer iteration for large-update and small-update methods, respectively. Then for* $p \geq 1$ *and* $q \geq 1$,

$$K_1 \leq 20N(\bar{\Psi}_0, p, q)\bar{\Psi}_0^{1/2} \quad and \quad K_2 \leq 20N(\tilde{\Psi}_0, p, q)\tilde{\Psi}_0^{1/2}.$$

PROOF. By Theorem 4.10, Lemma 4.11 with $\tilde{\beta} := 1/2$ and $\gamma := 1/(10N(\Psi_0, p, q))$, and (4.8),

$$K_1 \leq 20N(\Psi_0, p, q)\Psi_0^{1/2} \leq 20N(\bar{\Psi}_0, p, q)\bar{\Psi}_0^{1/2}$$

and

$$K_2 \leq 20N(\Psi_0, p, q)\Psi_0^{1/2} \leq 20N(\tilde{\Psi}_0, p, q)\tilde{\Psi}_0^{1/2}.$$

This completes the proof. $\square$

THEOREM 4.13. *Let an LO be given and* $\bar{\Psi}_0$, $\tilde{\Psi}_0$ *be defined as in* (3.13). *Then for* $p \geq 1$, $q \geq 1$, $0 < \theta < 1$, *and* $\tau \geq 1$, *the total number of iterations required to obtain an approximate solution with* $n\mu < \epsilon$ *is bounded by:*

  (i) $\lceil (20/\theta)N(\bar{\Psi}_0, p, q)\bar{\Psi}_0^{1/2} \log(n/\epsilon) \rceil$, *for large-update methods;*
  (ii) $\lceil (20/\theta)N(\tilde{\Psi}_0, p, q)\tilde{\Psi}_0^{1/2} \log(n/\epsilon) \rceil$, *for small-update methods.*

PROOF. If the parameter μ has the initial value $\mu^0 = 1$ and is updated by multiplying by $1 - \theta$ with $0 < \theta < 1$, then after at most

$$\lceil (1/\theta) \log(n/\epsilon) \rceil$$

iterations we have $n\mu < \epsilon$ [17]. For the total number of iterations, we multiply the number of inner iterations by the number of outer iterations. Hence the total numbers of iterations for large-update and small-update methods are bounded by

$$\lceil (20/\theta)N(\bar{\Psi}_0, p, q)\bar{\Psi}_0^{1/2} \log(n/\epsilon) \rceil \quad \text{and} \quad \lceil (20/\theta)N(\tilde{\Psi}_0, p, q)\tilde{\Psi}_0^{1/2} \log(n/\epsilon) \rceil,$$

respectively. This completes the proof. $\square$

REMARK 4.14. By Remark 3.9 and Theorem 4.13, for large-update methods, taking $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, the algorithm has

$$\mathcal{O}(\mathfrak{Y}(1 + q^{-1} \log(\mathfrak{Y}))^4 (pq\mathfrak{Y} + q + 2)\sqrt{n} \log(n/\epsilon))$$

iteration complexity where $\mathfrak{Y} := 1 + p^{-1} \log(1 + 2\sqrt{2n})$. In particular, choosing $p = \mathcal{O}(\log(1 + 2\sqrt{2n}))$ and $q = 1$, the algorithm has $\mathcal{O}(\sqrt{n} \log n \log(n/\epsilon))$ iteration complexity which is currently the best known result for large-update methods.

For small-update methods, taking $\tau = \mathcal{O}(1)$ and $\theta = \Theta(1/\sqrt{n})$, the algorithm has

$$\mathcal{O}(\mathfrak{Z}(1 + q^{-1} \log(\mathfrak{Z}))^4 (pq\mathfrak{Z} + q + 2)\sqrt{n} \log(n/\epsilon))$$

iteration complexity where $\mathfrak{Z} := 1 + p^{-1} \log(1 + 2\sqrt{2(pq + q)})$. Choosing $p = 1$ and $q = 1$, we have $\mathcal{O}(\sqrt{n} \log(n/\epsilon))$ iteration complexity which is the best known complexity result for such methods.

5. Concluding remarks

Motivated by recent work of Amini and Haseli [1], we propose a new primal-dual interior-point algorithm for LO problems based on a new kernel function and analyse the iteration complexity of the algorithm. We obtain $\mathcal{O}(\sqrt{n} \log n \log(n/\epsilon))$ and $\mathcal{O}(\sqrt{n} \log(n/\epsilon))$ iteration bounds for large-update and small-update methods, respectively. These are the best known results for such methods.

Future research might focus on the extension to semidefinite optimization, second-order cone optimization, and complementarity problems. Numerical tests will be another topic for future research.

Acknowledgements

References

[1] K. Amini and A. Haseli, "A new proximity function generating the best known iteration bounds for both large-update and small-update interior point methods", *ANZIAM J.* **49** (2007) 259–270.

[2] K. Amini and M. R. Peyghami, "Exploring complexity of large update interior-point methods for $P_*(\kappa)$ linear complementarity problem based on kernel function", *Appl. Math. Comput.* **207** (2009) 501–513.

[3] E. D. Andersen, J. Gondzio, Cs. Meszaros and X. Xu, "Implementation of interior point methods for large scale linear programming", in: *Interior point methods of mathematical programming*, (ed. T. Terlaky), (Kluwer Academic Publisher, Dordrecht, 1996) 189–252.

[4] Y. Q. Bai, M. El Ghami and C. Roos, "A new efficient large-update primal-dual interior-point method based on a finite barrier", *SIAM J. Optim.* **13** (2003) 766–782.

[5] Y. Q. Bai, M. El Ghami and C. Roos, "A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization", *SIAM J. Optim.* **15** (2004) 101–128.

[6] Y. Q. Bai, G. Lesaja, C. Roos, G. Q. Wang and M. El Ghami, "A class of large-update and small-update primal-dual interior-point algorithms for linear optimization", *J. Optim. Theory and Appl.* **138** (2008) 341–359.

[7]   Y. Q. Bai and C. Roos, "A primal-dual interior point method based on a new kernel function with linear growth rate", *Proceedings of the 9th Australian Optimization Day*, Perth, Australia, 2002.

[8]   Y. Q. Bai and C. Roos, "A polynomial-time algorithm for linear optimization based on a new simple kernel function", *Optim. Methods Software* **18** (2003) 631–646.

[9]   D. den Hertog, *Interior point approach to linear, quadratic and convex programming*, Volume 277 of *Mathematics and its Applications* (Kluwer Academic Publishers, Dordrecht, 1994).

[10] M. El Ghami, I. Ivanov, J. B. M. Melissen, C. Roos and T. Steihaug, "A polynomial-time algorithm for linear optimization based on a new class of kernel functions", *J. Comput. Appl. Math.* **224** (2009) 500–513.

[11] M. El Ghami and C. Roos, "Generic primal-dual interior point methods based on a new kernel function", *RAIRO Oper. Res.* **42** (2008) 199–213.

[12] C. C. Gonzaga, "Path following methods for linear programming", *SIAM Rev.* **34** (1992) 167–227.

[13] N. K. Karmarkar, "A new polynomial-time algorithm for linear programming", *Combinatorica* **4** (1984) 373–395.

[14] M. Kojima, S. Mizuno and A. Yoshise, "A primal-dual interior-point algorithm for linear programming", in: *Progress in mathematical programming: interior point and related methods*, (ed. N. Megiddo), (Springer, New York, 1989) 29–47.

[15] J. Peng, C. Roos and T. Terlaky, "Self-regular functions and new search directions for linear and semidefinite optimization", *Math. Program.* **93** (2002) 129–171.

[16] J. Peng, C. Roos and T. Terlaky, *Self-regularity: a new paradigm for primal-dual interior-point algorithms* (Princeton University Press, Princeton, NJ, 2002).

[17] C. Roos, T. Terlaky and J. Ph. Vial, *Theory and algorithms for linear optimization: an interior approach* (Wiley, Chichester, UK, 1997).

[18] G. Sonnevend, "An analytic center for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming", *System modeling and optimization: proceeding of the 12th IFIP-conference*, Budapest, Hungary, September 1985, Volume 84 of *Lecture Notes in Control and Information Sciences* (eds A. Prekopa, J. Szleezsan and B. Strazicky), (Springer, Berlin, 1986) 866–876.

[19] N. J. Todd, "Recent developments and new directions in linear programming", in: *Mathematical programming: recent developments and applications*, (eds M. Iri and K. Tanabe), (Kluwer Academic Publishers, Dordrecht, 1989) 109–157.