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CONGRUENCE PRIMES FOR SIEGEL MODULAR FORMS OF PARAMODULAR LEVEL AND APPLICATIONS TO THE BLOCH–KATO CONJECTURE

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Abstract. It has been well established that congruences between automorphic forms have far-reaching applications in arithmetic. In this paper, we construct congruences for Siegel–Hilbert modular forms defined over a totally real field of class number 1. As an application of this general congruence, we produce congruences between paramodular Saito–Kurokawa lifts and non-lifted Siegel modular forms. These congruences are used to produce evidence for the Bloch–Kato conjecture for elliptic newforms of square-free level and odd functional equation.

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1. Introduction. While congruences between automorphic forms are inherently interesting in their own right to those studying arithmetic properties of automorphic forms, they are of wider interest due to their far-ranging applications in arithmetic. Beginning with Ribet's remarkable proof of the converse of Herbrand's theorem using congruences between Eisenstein series and elliptic cusp forms [**21**], congruences between automorphic forms have been a powerful tool in relating special values of L-functions and interesting arithmetic groups, such as class groups, or more generally, Selmer groups. Congruences between elliptic modular forms were used in Wiles' proof of the Iwasawa main conjecture for totally real fields [**33**] as well as in the proof of Gross' conjecture for totally real fields F and narrow ring class characters by Darmon–Dasgupta–Pollack [**12**].

In addition to the profound applications found in congruences between elliptic modular forms, more recently great progress has been made in studying congruences between automorphic forms on other reductive algebraic groups. The most striking example of this is Skinner and Urban's proof of the Iwasawa main conjecture for GL(2) [**29**]. In this work, they study congruences between automorphic forms on U(2, 2). For other work on congruences between automorphic forms on unitary groups, one can see [**9**, **19**, **20**].

In this paper, we will be focused on congruence results for automorphic forms on symplectic groups. There is a good deal of literature already dealing with congruences for Siegel modular forms: [**2**, **3**, **6**, **7**, **8**, **18**]. These previous results have been limited to Siegel modular forms defined over $\mathbf{Q}$; in this paper, our general congruence result is valid for Siegel modular forms defined over totally real fields of class number 1, thus providing the first results over a field other than $\mathbf{Q}$. Moreover, previous results have been restricted to forms with full level or congruence level where our result is valid for general levels. For instance, in the case of $n=2$, our result applies to cusp forms of paramodular level. This is particularly significant in that paramodular level is in some sense the "correct" level to work with for Siegel modular forms. For instance, one has a nice local newform theory for Siegel modular forms of paramodular level [22] and the paramodular conjecture relates abelian surfaces with Siegel modular forms of paramodular level [11]. Thus, our extension to more general levels should be of considerable interest.

The first main result of the paper concerns congruences for Siegel modular forms of general genus. Let $F/\mathbf{Q}$ be a totally real field of class number 1, $\mathcal{O}$ the ring of integers of F, and $\mathfrak{M} \subset \mathcal{O}$ an ideal. Let ℓ be a rational prime and $\lambda \mid \ell$ a prime of $\mathcal{O}$. Let $\Gamma_n(\mathfrak{M}) \subset \mathrm{Sp}_{2n}(\mathcal{O})$ be the principal congruence subgroup and $\Gamma^{(n)} \supset \Gamma_n(\mathfrak{M})$ any congruence subgroup. Fix a tuple $k=(k, \ldots, k) \in \mathbf{Z}^{[F:\mathbf{Q}]}$ and an eigenform $f \in S_k(\Gamma^{(n)})$. The main congruence result states that (up to some minor technical conditions) if there exists a proper ideal $\mathfrak{N} \subset \mathcal{O}$ and a Hecke character χ of conductor $\mathfrak{N}$ so that $\mathrm{val}_\lambda(L^{\mathfrak{N}}_{\mathrm{alg}}(2n+1-k, f, \chi; \mathrm{st})) < 0$, then there exists a $g \in S_k(\Gamma_n(\mathfrak{N}))$ orthogonal to f under the Petersson inner product with $f \equiv g \pmod{\lambda}$. If one further assumes $\ell \nmid [\Gamma^{(n)} : \Gamma_n(\mathfrak{N})]$, then one can choose g to be orthogonal to f. One can see Theorem 5.1 for the precise statement of the general congruence result. This result is analogous to the main congruence result found in [9] which deals with Hilbert Hermitian automorphic forms. Our result stands apart from other congruence results for Siegel modular forms in that this result is able to deal with Hilbert–Siegel modular forms where previous results were restricted to Siegel modular forms defined over $\mathbf{Q}$, and as noted above we are able to handle more general levels. The restriction that F has class number 1 can likely be removed and may be explored in future work.

The second part of the paper deals with the special case that $F=\mathbf{Q}$ and $n=2$. In this case, we specialize the congruence subgroup to the paramodular group and the automorphic form f to a paramodular Saito–Kurokawa lift of an elliptic newform $\phi \in S^{\mathrm{new}}_{2k-2}(\Gamma_0(M))$ with odd functional equation. The general congruence result specializes as follows. Let $\phi \in S^{\mathrm{new},-}_{2k-2}(\Gamma_0(M))$ be a newform, $\ell \nmid M$, and $\overline{\rho}_{\phi,\lambda}$ irreducible. Assume $\mathrm{ord}_\lambda(L_{\mathrm{alg}}(k, \phi)) > 0$ and one can choose a fundamental discriminant D and a Hecke character χ of conductor N so that λ does not divide a certain product of special values of L-functions. We prove there exists a Siegel eigenform g with irreducible Galois representation so that the paramodular Saito–Kurokawa lift $f_{\phi,D}$ of ϕ has eigenvalues congruent to those of g modulo λ. One can see Corollary 6.7 for the precise statement given in terms of the CAP ideal of $f_{\phi,D}$. The analogous result for congruence level Saito–Kurokawa lifts can be found in [2]. This congruence result, and the bound on the CAP ideal, are used in the final section to give evidence for the Bloch–Kato conjecture for ϕ. The main point to note here is that previous results (see [2, 8] for example) required ϕ to be of weight $2k-2$ with k even in order to have a congruence level Saito–Kurokawa lift to construct the congruence. Our current result removes this restriction on k and replaces it with ϕ having odd functional equation. While this is the same condition if $M=1$, it is not necessarily the same condition for $M>1$. Thus, we obtain evidence for the Bloch–Kato conjecture for a significant new class of newforms ϕ.

A natural future direction for this work is to reproduce these congruences in ℓ-adic families. This is current ongoing work.

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2. Notation. Fix a totally real field $F/\mathbf{Q}$ with class number 1. Let $d=[F:\mathbf{Q}]$. We let $\mathcal{O}_F$ be the ring of integers of F, D_F the discriminant of F, and $\mathfrak{d}(F/\mathbf{Q})$ the different. Let $\mathbf{f}$ denotes the set of finite places of F and $\mathbf{a}$ the infinite places. For a place v of F, we write F_v for the completion of F at v and $\mathcal{O}_v$ for the valuation ring of F_v. We write $\mathbf{A}_F$ for the adeles of F.

For $z\in\mathbf{C}$, we set $e(z)=e^{2\pi iz}$. We define the characters $e_{\mathbf{A}_F}:\mathbf{A}_F\to S^1$ and $e_v:F_v\to S^1$ as usual so that $e_{\mathbf{A}_F}(x)=\prod_v e_v(x_v)$, $e_v(x_v)=e(x_v)$ for $v\in\mathbf{a}$, and $e_{\mathbf{A}_F}(F)=1$. See [28, 18.2] for the details. We set $e_{\mathbf{a}}(x)=e\left(\sum_{v\in\mathbf{a}}x_v\right)$ for $x\in\mathbf{A}_F$ or $x\in\mathbf{C}^{\mathbf{a}}$.

2.1. Relevant groups and actions. Set $\omega_n=\begin{bmatrix}0_n & -1_n\\ 1_n & 0_n\end{bmatrix}$ and define the general symplectic group associated with ω_n by

$$\mathrm{GSp}(2n)=\{g\in\mathrm{GL}(2n):{}^t g\omega_n g=\mu_n(g)\omega_n,\ \mu_n(g)\in\mathrm{GL}(1)\}.$$

For $R\subset\mathbf{R}$ a ring, we set $\mathrm{GSp}_{2n}^{+}(R)$ to be the subgroup of $\mathrm{GSp}_{2n}(R)$ consisting of elements g with $\mu_n(g)>0$. We set $G_n=\ker(\mu_n)$. Let P_n denotes the Siegel parabolic subgroup of G_n, that is, $P_n=M_nN_n$ where

$$M_n=\left\{\begin{bmatrix}A & 0_n\\ 0_n & \hat{A}\end{bmatrix}:A\in\mathrm{GL}(n)\right\},$$
$$N_n=\left\{\begin{bmatrix}1_n & x\\ 0_n & 1_n\end{bmatrix}:x\in S_n\right\}$$

with $\hat{x}={}^t x^{-1}$ for $x\in\mathrm{Mat}(n)$ and

$$S_n=\{g\in\mathrm{Mat}(n):{}^t g=g\}.$$

Set

$$\mathfrak{h}_n=\{z\in\mathrm{Mat}_n(\mathbf{C}):{}^t z=z,\ \Im(z)>0\}.$$

For $v\in\mathbf{a}$, $g=\begin{bmatrix}a_g & b_g\\ c_g & d_g\end{bmatrix}\in\mathrm{GSp}_{2n}^{+}(F_v)$, and $z\in\mathfrak{h}_n$ we define

$$gz=(a_gz+b_g)(c_gz+d_g)^{-1}\in\mathfrak{h}_n$$

and

$$j(g,z)=\det(c_gz+d_g).$$

For $g=(g_v)\in G_n(\mathbf{A}_F)$ with $g_v\in\mathrm{GSp}_{2n}^{+}(F_v)$ for all $v\in\mathbf{a}$ and $z=(z_v)_{v\in\mathbf{a}}\in\mathfrak{h}_n^{\mathbf{a}}$, we set

$$gz=(g_vz_v)_{v\in\mathbf{a}}$$

and

$$j(g,z)=(j(g_v,z_v))_{v\in\mathbf{a}}.$$

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Given $k=(k_v)_{v\in\mathbf{a}}\in\mathbf{Z}^{\mathbf{a}}$ and $x=(x_v)_{v\in\mathbf{a}}\in\mathbf{C}^{\mathbf{a}}$, we set

$$x^k=\prod_{v\in\mathbf{a}} x_v^{k_v}.$$

Fix an integral ideal $\mathfrak{N}\subset\mathcal{O}_F$. For each $v\in\mathbf{f}$, we set

$$\mathcal{K}_{n,v}(\mathfrak{N}_v)=\{g\in G_n(\mathcal{O}_v): g\equiv 1_{2n}\pmod{\mathfrak{N}_v}\}.$$

Note that $\mathcal{K}_{n,v}(\mathfrak{N}_v)=G_n(\mathcal{O}_v)$ for all but finitely many places, namely, $\mathcal{K}_{n,v}(\mathfrak{N}_v)=G_n(\mathcal{O}_v)$ if $v\nmid\mathfrak{N}$. Put $\mathcal{K}_{n,\mathbf{f}}(\mathfrak{N})=\prod_{v\in\mathbf{f}}\mathcal{K}_{n,v}(\mathfrak{N})$. Set $\mathcal{K}_{n,\mathbf{a}}=\{\kappa\in G_n(\mathbf{R}^{\mathbf{a}}):\kappa(\mathbf{i}_{n,\mathbf{a}})=\mathbf{i}_{n,\mathbf{a}}\}$ where we write $\mathbf{i}_{n,\mathbf{a}}$ for the element of $\mathfrak{h}_n^{\mathbf{a}}$ with $\mathbf{i}_n$ in each component. Set $\mathcal{K}_n(\mathfrak{N})=\mathcal{K}_{n,\mathbf{a}}\mathcal{K}_{n,\mathbf{f}}(\mathfrak{N})$ and $\Gamma_n(\mathfrak{N})=G_n(F)\cap\mathcal{K}_n(\mathfrak{N})$. Note that $\Gamma_n(\mathfrak{N})$ is the usual principal congruence subgroup in $G_n(\mathcal{O}_F)$.

In the case that $n=2$, $F=\mathbf{Q}$, and $\mathfrak{N}=N$, we will also make use of the paramodular group. We define the paramodular group of level N to be the subgroup of $G_2(\mathbf{Q})$ defined by

$$\Gamma^{\text{para}}(N)=\left\{\begin{bmatrix} * & N* & * & * \\ * & * & * & N^{-1}* \\ * & N* & * & * \\ N* & N* & N* & * \end{bmatrix}\in \mathrm{Sp}_4(\mathbf{Q})\right\}.$$

Observe that $\Gamma_2(N)\subset\Gamma^{\text{para}}(N)$.

Set $S_{\mathbf{a}}=\prod_{v\in\mathbf{a}}S_n(F_v)$. We have a pairing $S_{\mathbf{a}}\times S_{\mathbf{a}}\to S^1$ given by $(x,y)\mapsto e_{\mathbf{a}}(\mathrm{tr}\,(xy))$. We define a lattice L_n in $S_n(F)$ (respectively, $L_{n,v}$ in $S_n(F_v)$) by $L_n=S_n(F)\cap\mathrm{Mat}_n(\mathfrak{N}\mathcal{O}_F)$ (respectively, $L_{n,v}=S_n(F_v)\cap\mathrm{Mat}_n(\mathfrak{N}\mathcal{O}_{F,v})$). Set

$$L'=\{x\in S_n(F):\mathrm{tr}\,(xL_n)\subset\mathcal{O}_F\}.$$

Note that since L' is an $\mathcal{O}_F$-lattice in $S_n(F)$, we have L'_v makes sense for $v\in\mathbf{f}$. Set $\mathcal{L}_n=\mathfrak{N}^{-1}\mathfrak{d}(F/\mathbf{Q})^{-1}L'$.

2.2. Modular forms. Fix $k=(k_v)_{v\in\mathbf{a}}\in\mathbf{Z}^{\mathbf{a}}$. Let $f:\mathfrak{h}_n^{\mathbf{a}}\to\mathbf{C}$ be a function. Given $g\in\mathrm{GSp}_{2n}(\mathbf{A}_F)$ with $g_v\in\mathrm{GSp}_{2n}^+(F_v)$ for all $v\in\mathbf{a}$, we define a function $f|_k g$ on $\mathfrak{h}_n^{\mathbf{a}}$ via

$$(f|_k g)(z)=\mu_n(g)^{nk/2}j(g,z)^{-k}f(gz).$$

In general, we drop k from the notation $f|_k$ as it will be clear from context. Let $\Gamma^{(n)}$ be a congruence subgroup of $G_n(F)$ of level $\mathfrak{N}$, that is, $\Gamma^{(n)}(\mathfrak{N})\subset\Gamma^{(n)}$. We let $M_k(\Gamma^{(n)})$ denote the finite dimensional $\mathbf{C}$-vector space of complex-valued holomorphic functions f on $\mathfrak{h}_n^{\mathbf{a}}$ satisfying $f|_k g=f$ for every $g\in\Gamma^{(n)}$. (We also require holomorphic at the cusps in the case $n=1$ and $F=\mathbf{Q}$.) We let $S_k(\Gamma^{(n)})$ denote the space of cusp forms in $M_k(\Gamma^{(n)})$.

Given $f_1,f_2\in M_k(\Gamma^{(n)}(\mathfrak{N}))$ with at least one in $S_k(\Gamma^{(n)}(\mathfrak{N}))$, we set

$$\langle f_1,f_2\rangle=\frac{1}{[G_n(\mathcal{O}_F):\Gamma^{(n)}(\mathfrak{N})]}\int_{\Gamma^{(n)}(\mathfrak{N})\backslash\mathfrak{h}_n^{\mathbf{a}}} f_1(z)\overline{f_2(z)}(\det\Im(z))^k d\mu z,$$

where

$$d\mu z=\Im(z)^{-n-1}\bigwedge_{v\in\mathbf{a}}\bigwedge_{\alpha\leq\beta}(dx_{\alpha,\beta}^v\wedge dy_{\alpha,\beta}^v)$$

with $z_v=(x_{\alpha,\beta}^v+iy_{\alpha,\beta}^v)$.

Let $f \in M_k(\Gamma^{(n)})$. The function f has a Fourier expansion of the form:

$$f(z) = \sum_{T \in \mathcal{L}_n} a(T; f) e_{\mathbf{a}}(\mathrm{tr}\,(Tz)),$$

where $a(T; f) \in \mathbf{C}$. Given a ring $\mathcal{O}$, we write $M_k(\Gamma^{(n)}; \mathcal{O})$ to denote the collection of $f \in M_k(\Gamma^{(n)})$ so that $a(T; f) \in \mathcal{O}$ for all $T \in \mathcal{L}_n$.

2.3. Congruences. Given an ideal $\mathcal{I} \subset \mathcal{O}$ and $f_1, f_2 \in M_k(\Gamma^{(n)}; \mathcal{O})$, we say f_1 and f_2 are congruent modulo $\mathcal{I}$ and write

$$f_1 \equiv f_2 \pmod{\mathcal{I}},$$

if

$$a(T; f_1) - a(T; f_2) \in \mathcal{I}$$

for all $T \in \mathcal{L}_n$.

2.4. *L*-functions. Given L-function with Euler product of the form:

$$L(s) = \prod_{v \in \mathbf{f}} L_v(s),$$

we write

$$L^{\mathfrak{N}}(s) = \prod_{\substack{v \in \mathbf{f} \\ v \nmid \mathfrak{N}}} L_v(s)$$

and

$$L_{\mathfrak{N}}(s) = \prod_{\substack{v \in \mathbf{f} \\ v \mid \mathfrak{N}}} L_v(s).$$

Let χ be a Hecke character of F of conductor $\mathfrak{N}$. We define the Dirichlet L-function attached to χ as:

$$L^{\mathfrak{N}}(s, \chi) = \prod_{\substack{v \in \mathbf{f} \\ v \nmid \mathfrak{N}}} (1 - \chi(\varpi_v)|\varpi_v|^s)^{-1}.$$

Let $f \in S_k(\Gamma^{(n)})$ be an eigenform. Following [4], we can associate an automorphic form $f_{\mathbf{A}_F}$ and a cuspidal automorphic representation π_f of $\mathrm{PGSp}_{2n}(\mathbf{A}_F)$ to f. Moreover, π_f can be decomposed into local components as $\pi_f = \bigotimes' \pi_{f,v}$ with $\pi_{f,v}$ an Iwahori spherical representation of $\mathrm{PGSp}_{2n}(F_v)$ for $v \nmid \mathfrak{N}$. For $v \nmid \mathfrak{N}$, the representation $\pi_{f,v}$ is given by $\pi(\chi_0, \chi_1, \ldots, \chi_n)$ for χ_i unramified characters of $F_v^\times$. Let $\alpha_0(v; f) = \chi_0(\varpi_v), \ldots, \alpha_n(v; f) = \chi_n(\varpi_v)$ be the v-Satake parameters of f normalized so that

$$\alpha_0(v; f)^2 \alpha_1(v; f) \cdots \alpha_n(v; f) = 1.$$

We drop f and/or v in the notation for the Satake parameters when they are clear from context. Set $\widetilde{\alpha}_0 = |\varpi_v|^{\frac{2nk - n(n+1)}{4}} \alpha_0$ and

$$L_v(X, f; \mathrm{spin}) = (1 - \widetilde{\alpha}_0 X) \prod_{j=1}^{n} \prod_{1 \leq i_1 \leq \cdots \leq i_j \leq n} (1 - \widetilde{\alpha}_0 \alpha_{i_1} \cdots \alpha_{i_j} X).$$

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The Spinor L-function associated with f is given by

$$L^{\mathfrak{N}}(s,f;\mathrm{spin}) = \prod_{\substack{v\in\mathbf{f}\\ v\nmid\mathfrak{N}}} L_v(|\varpi_v|^{-s},f;\mathrm{spin})^{-1}.$$

One should note that in the case f is an elliptic modular form, we denote the Spinor L-function as $L(s,f)$. This is the usual L-function attached to an elliptic modular form.

In the case f is an elliptic modular form, we will make use of the following integrality result.

THEOREM 2.1. [17, 31] *Let $k\geq 2$, $M\geq 1$, and ℓ a prime with $\ell > k$ and $\ell\nmid M$. Let $f\in S_k(\Gamma_1(M);\mathcal{O})$ be a newform where $\mathcal{O}$ is a finite extension of $\mathbf{Z}_\ell$. For each integer j with $0<j<k$ and every Dirichlet character χ, one has*

$$L_{\mathrm{alg}}(j,f,\chi) := \frac{L(j,f,\chi)}{\tau(\chi)(2\pi i)^j\Omega_f^{\pm}} \in \mathcal{O}_\chi,$$

where $\mathcal{O}_\chi$ is the finite extension of $\mathcal{O}$ generated by the values of χ, $\tau(\chi)$ is the Gauss sum of χ, and we normalize with Ω_f^+ if $\chi(-1)=(-1)^{j-1}$ and Ω_f^- if $\chi(-1)=(-1)^j$.

For $v\in\mathbf{f}$ and $f\in S_k(\Gamma^{(n)})$ an eigenform, set

$$L_v(X,f;\mathrm{st}) = (1-X)\prod_{i=1}^{n}(1-\alpha_i(v;f)X)(1-\alpha_i(v;f)^{-1}X).$$

We define the standard L-function associated with f by

$$L^{\mathfrak{N}}(s,f;\mathrm{st}) = \prod_{\substack{v\in\mathbf{f}\\ v\nmid\mathfrak{N}}} L_v(|\varpi_v|^{s},f;\mathrm{st})^{-1}.$$

Given a Hecke character χ, we define the twisted standard L-function associated with f by

$$L^{\mathfrak{N}}(s,f,\chi;\mathrm{st}) = \prod_{\substack{v\in\mathbf{f}\\ v\nmid\mathfrak{N}}} L_v(\chi(\varpi_v)|\varpi_v|^{s},f;\mathrm{st})^{-1}.$$

2.5. Galois representations. We will make use of the following two results giving the existence of Galois representations associated with cuspidal elliptic newforms and cuspidal Siegel eigenforms of genus 2. Note that we take geometric conventions so the Frob_p below refer to geometric and not arithmetic Frobenius. The first result is well known due to Deligne et al.

THEOREM 2.2. *Let $k\geq 2$ and let $\phi\in S_k^{\mathrm{new}}(\Gamma)$ be a newform with $\Gamma\subset\mathrm{SL}_2(\mathbf{Z})$ a congruence subgroup of level M. Let $\mathbf{Q}(\phi)$ be the number field generated by the eigenvalues of ϕ, λ a prime of $\mathbf{Q}(\phi)$ over ℓ, and E the completion of $\mathbf{Q}(\phi)$ at λ. Then there exists a continuous, irreducible representation $(\rho_{\phi,\lambda}, V_{\phi,\lambda})$ of $G_{\mathbf{Q}}$ where $V_{\phi,\lambda}$ is a two-dimensional E-vector space such that $(\rho_{\phi,\lambda}, V_{\phi,\lambda})$ is unramified at all primes $p\nmid\ell M$ and*

$$\det(1_2-\rho_{\phi,\lambda}(\mathrm{Frob}_p)p^{-s}) = L_p(s,\phi)$$

for all $p\nmid\ell M$.

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THEOREM 2.3. [32, Theorem I] *Let $f \in S_k(\Gamma^{(n)})$ be an eigenform with $\Gamma \subset G_2(F)$ a congruence subgroup of level M, $\mathbf{Q}(f)$ the number field generated by the Hecke eigenvalues of f, and λ a prime of $\mathbf{Q}(f)$ over ℓ. There exists a finite extension E of the completion of $\mathbf{Q}(f)_\lambda$ of $\mathbf{Q}(f)$ at λ and a continuous semi-simple Galois representation:*

$$\rho_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_4(E)$$

unramified away from ℓM so that for all $p \nmid \ell M$ we have

$$\det(1_4 - \rho_{f,\lambda}(\mathrm{Frob}_p)p^{-s}) = L_p(s, f, \mathrm{spin}).$$

3. Siegel Eisenstein series.

3.1. Definition. We fix an integer $k > n$. Let χ be a Hecke character of F satisfying

$$\chi_{\mathbf{a}}(x) = (x/|x|)^k \tag{3.1}$$

for $k = (k, \ldots, k) \in \mathbf{Z}^{\mathbf{a}}$ and

$$\chi_v(x) = 1 \ \text{ if } v \notin \mathbf{a}, x \in \mathcal{O}_v^\times, x - 1 \in \mathfrak{N}. \tag{3.2}$$

We set $\chi_{\mathfrak{N}} = \prod_{v|\mathfrak{N}} \chi_v$.

Let $\mathfrak{b}$ and $\mathfrak{c}$ be integral ideals in $\mathcal{O}_F$. For $v \in \mathbf{f}$, define

$$\mathcal{K}_{n,v}(\mathfrak{b}, \mathfrak{c}) = \left\{ g \in G_n(F_v) : \det(g) \in \mathcal{O}_v^\times, a_g, d_g \in \mathrm{Mat}_n(\mathcal{O}_v), b_g \in \mathfrak{b}, c_g \in \mathfrak{c} \right\}.$$

Put $\mathcal{K}_{n,\mathbf{f}}(\mathfrak{b}, \mathfrak{c}) = \prod_{v \in \mathbf{f}} \mathcal{K}_{n,v}(\mathfrak{b}, \mathfrak{c})$ and $\mathcal{K}_n(\mathfrak{b}, \mathfrak{c}) = \mathcal{K}_{n,\mathbf{a}} \mathcal{K}_{n,\mathbf{f}}(\mathfrak{b}, \mathfrak{c})$. Set $\Gamma_n(1, \mathfrak{N}) = G_n(F) \cap \mathcal{K}_n(1, \mathfrak{N})$.

For $p = \begin{bmatrix} a_p & b_p \\ 0 & d_p \end{bmatrix} \in P_n(F_v)$, set

$$\delta_{P_n,v}(p) = |\det(d_p)|_{F_v}^2$$

and $\delta_{P_n} = \prod_v \delta_{P_n,v}$. Define the function μ_v by setting $\mu_v(x_v) = 0$ if $x_v \notin P_n(F_v)\mathcal{K}(1, \mathfrak{N})$ and for $p_v \kappa_v \in P_n(F_v)\mathcal{K}_{n,v}(1, \mathfrak{N})$, we set

$$\mu_{P_n,v}(p_v \kappa_v) = \begin{cases} \chi_v(\det d_{p_v})^{-1} & v \nmid \mathfrak{N}, v \notin \mathbf{a} \\ \chi_v(\det d_{p_v})^{-1} \chi_v(\det d_{\kappa_v})^{-1} & v \mid \mathfrak{N}, v \notin \mathbf{a} \\ \chi_v(\det d_{p_v})^{-1} j(\kappa_v, \mathbf{i}_{n,v})^{-k} & v \in \mathbf{a} \end{cases}$$

and $\mu_{P_n} = \prod_v \mu_{P_n,v}$.

We define the Siegel Eisenstein series of weight $k \in \mathbf{Z}^{\mathbf{a}}$, level $\mathcal{K}(1, \mathfrak{N})$, and character χ by

$$E(g, s, k, \chi, \mathcal{K}) = \sum_{\gamma \in P_n(F) \backslash G_n(F)} \mu_{P_n}(\gamma g) \delta_{P_n}(\gamma g)^{-s}.$$

This gives a holomorphic function on $\{s \in \mathbf{C} : \Re(s) > (n+1)/2\}$ and extends to a meromorphic function on $\mathbf{C}$ with at most finitely many poles. To ease notation, we write this Eisenstein series as $E(g, s; \chi)$.

We associate a classical Siegel Eisenstein series $E(z, s; \chi)$ to $E(g, s; \chi)$ in the usual way. In particular, for $r \in G_n(\mathbf{A}_{F,\mathbf{f}})$, we define a function $E_r(z, s; \chi)$ on $\mathfrak{h}_n^{\mathbf{a}} \times \mathbf{C}$ by setting

$$E_r(z, s; \chi) = j(g_\infty, \mathbf{i}_{n,\mathbf{a}})^k E(rg_\infty, s; \chi),$$

where $z = g_\infty \mathbf{i}_{n,\mathbf{a}}$. Note that for $s = (n+1-k)/2$, we have $E_r(z, s; \chi)$ is a holomorphic modular form of weight k for $k > n+1$ [26, Proposition 4.1].

3.2. Fourier coefficients. In this section, we study the Fourier coefficients of Siegel Eisenstein series defined in the previous section.

Define

$$\Lambda^{\mathfrak{N}}(s) = \Lambda^{\mathfrak{N}}(s, \chi) = L^{\mathfrak{N}}(2s, \chi) \prod_{j=0}^{[n/2]} L^{\mathfrak{N}}(4s - 2j, \chi^2)$$

and

$$D(g, s; \chi) = \pi^{-d(n(n+2))/4} \Lambda^{\mathfrak{N}}(s) E(g, s; \chi).$$

Set $D(g; \chi) := D(g, (n+1-k)/2; \chi)$. We have $D(z; \chi) \in M_k(\Gamma_n(1, \mathfrak{N}), \overline{\mathbf{Q}})$ via [26, Proposition 4.1] where we write $D(z; \chi)$ for the classical Eisenstein series associated with $D(g; \chi)$. As we are interested in congruences, we need more precise results than this about the Fourier coefficients. However, this normalized Eisenstein series will be used below and in the next section so we make note of it here.

In order to study the Fourier coefficients of the Eisenstein series above, we follow Shimura and shift to the Eisenstein series:

$$E^*(g, s; \chi) = E(g\omega_{n,\mathbf{f}}^{-1}, s; \chi).$$

We write $D^*(z, \chi)$ for the normalized classic Eisenstein series associated with $E^*(g, (n+1-k)/2; \chi)$.

One has a Fourier expansion for E^* of the form:

$$E^*\left(\begin{bmatrix} q & \sigma\hat{q} \\ 0 & \hat{q} \end{bmatrix}, s; \chi\right) = \sum_{h \in S_n(F)} c(h, q, s) e_{\mathbf{A}_F}(\mathrm{tr}\,(h\sigma))$$

for $q \in \mathrm{GL}_n(\mathbf{A}_F)$ and $\sigma \in S_n(\mathbf{A}_F)$.

LEMMA 3.1. [28, Lemma 18.7(2)] *For $t = \mathrm{diag}[q_1, \hat{q}_1]$ with $q_1 \in \mathrm{GL}_n(\mathbf{A}_{F,\mathbf{f}})$ and $z = x + iy \in \mathfrak{h}^{\mathbf{a}}$, put $c_t(h, y, s) = \det(y)^{-k/2} c(h, q, s)$ with $q \in \mathrm{GL}_n(\mathbf{A}_F)$ so that $q_{\mathbf{f}} = q_1$ and $q_{\mathbf{a}} = y^{1/2}$. Then,*

$$E_t^*(z, s) = \sum_{h \in S_n(F)} c_t(h, y, s) e_{\mathbf{a}}(\mathrm{tr}\,(hx)).$$

We have the following proposition that calculates the Fourier coefficients $c(h, q, s)$. We will define the terms appearing in the formula after the proposition.

PROPOSITION 3.2. [28, Proposition 18.14] *Suppose that $\mathfrak{N} \neq \mathcal{O}_F$. Then $c(h, q, s) \neq 0$ only if $(q^*hq)_v \in (\mathfrak{N}\mathfrak{d}(F/\mathbf{Q}))_v^{-1} L_v'$ for every $v \in \mathbf{f}$ in which case we have*

$$c(h, q, s) = c(S)\chi(\det(q))^{-1}\, \mathrm{N}(\mathfrak{N})^{-n\lambda} |\det(qq^*)_{\mathbf{f}}|_F^{\lambda - s} |\det(qq^*)_{\mathbf{a}}|^s$$
$$\cdot\, \alpha^{\mathfrak{N}}(\omega q^* hq, 2s, \chi)\, \Xi(qq^*, h, s + k/2, s - k/2),$$

where $\omega \in \mathbf{A}_{F,\mathbf{f}}^\times$ such that $\omega\mathcal{O}_F = \mathfrak{d}(F/\mathbf{Q})$, and $\lambda = (n+1)/2$.

The value $c(S)$ arises from comparing the local and global measures and is given by

$$c(S) = |D_F|^{-n(n+1)/4}.$$

The function Ξ is defined by

$$\Xi(y, w; t, t') = \prod_{v \in \mathbf{a}} \xi(y_v, w_v; t_v, t'_v)$$

for $t, t' \in \mathbf{C}^{\mathbf{a}}$, $y \in S_n(\mathbf{A}_F)$, $y_v > 0$ for $v \in \mathbf{a}$, $w \in S_n(\mathbf{A}_F)$ and ξ is defined by

$$\xi(y, h; s, s') = \int_{S_n(F_v)} e_v(\mathrm{tr}\,(-hx)) \det(x+iy)^{-s} \det(x-iy)^{-s'} dx$$

for $s, s' \in \mathbf{C}$, $0 < y \in S_n(F_v)$, $h \in S_n(F_v)$, $v \in \mathbf{a}$.

For each $v \in \mathbf{f}$, take an element $\delta_v \in F_v$ so that $\delta_v \mathcal{O}_v = \mathfrak{d}(F/\mathbf{Q})_v$. The function $\alpha_{\mathfrak{N}}$ is defined by

$$\alpha_{\mathfrak{N}}(\zeta, s, \chi) = \prod_{v \nmid \mathfrak{N}} \alpha(\zeta_v, s, \chi),$$

where

$$\alpha(\zeta_v, s, \chi) = \sum_{\sigma \in S_n(F_v)/\Lambda_v} e_v(\delta_v^{-1} \mathrm{tr}\,(\zeta\sigma)) \chi^*(\nu_1(\sigma)) \nu[\sigma]^{-s}.$$

Note we are not defining all the terms used to define α as we have the following result.

PROPOSITION 3.3. [28, Proposition 19.2] *Let $r = rank(h)$ and $g^* h g = \mathrm{diag}[h', 0]$ with $g \in \mathrm{GL}_n(F)$ and $h' \in S_r(F)$. Set $c = (-1)^{r/2}\det(h')$ if $r > 0$ and $c = 1$ if $r = 0$. Let ρ_h be the quadratic Hecke character corresponding to $F(c^{1/2})/F$ if $r > 0$ and $\rho_h = 1$ if $r = 0$. We have*

$$\alpha^{\mathfrak{N}}(\omega q^* h q, 2s, \chi) = \frac{\Lambda_h^{\mathfrak{N}}(s)}{\Lambda^{\mathfrak{N}}(s)} \prod_{v \in \mathbf{c}} f_{h,q,v}(\chi(\varpi_v)|\varpi_v|^{2s})$$

with $\mathbf{c} \subset \mathbf{f}$ a finite set, $f_{h,q,v} \in \mathbf{Z}[x]$ with constant term 1 independent of χ, and

$$\Lambda_h^{\mathfrak{N}}(s) = \begin{cases} L^{\mathfrak{N}}(2s-n+r/2, \chi\rho_h) \prod_{j=1}^{[(n-1)/2]} L^{\mathfrak{N}}(4s-2n+r+2j-1, \chi^2) & r \in 2\mathbf{Z}, \\ \prod_{j=1}^{[(n-r+1)/2]} L^{\mathfrak{N}}(4s-2n+r+2j-2, \chi^2) & r \notin 2\mathbf{Z}. \end{cases}$$

We will now specialize to the case $s = (n+1-k)/2$ and q with $q_{\mathbf{a}} = y^{1/2}$. Note that since y is symmetric, $q_{\mathbf{a}} q_{\mathbf{a}}^* = y$. Using this, we have

$$c\left(h, q, \frac{n+1-k}{2}\right) = |D_F|^{-\frac{n(n+1)}{4}} \chi(\det(q))^{-1} \mathrm{N}(\mathfrak{N})^{-\frac{n(n+1)}{2}} |\det(qq^*)_{\mathbf{f}}|_F^{\frac{k}{2}} |\det(y)|^{\frac{n+1-k}{2}}$$
$$\cdot \alpha_{\mathfrak{N}}(\omega q^* h q, n+1-k, \chi) \Xi\left(y, h, \frac{n+1}{2}, \frac{n+1-2k}{2}\right).$$

The next step is to evaluate Ξ.

To evaluate Ξ, we turn to [25]. Note that we are interested in Case K with $\mathbf{K} = \mathbf{R}$ so that $V_m = S_n(\mathbf{R})$. In this paper, Shimura studies

$$\xi(g, h; \alpha, \beta) = \int_{S_n(\mathbf{R})} e(\mathrm{tr}\,(-hx)) \det(x+ig)^{-\alpha} \det(x-ig)^{-\beta} dx,$$

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which corresponds to ξ exactly upon renaming variables. Thus, we want to evaluate $\xi\left(y, h, \frac{n+1}{2}, \frac{n+1-2k}{2}\right)$.

Define

$$\Gamma_t(a) = \pi^{t(t-1)/4} \prod_{j=0}^{t-1} \Gamma(a - j/2).$$

Let $V(a, b, c)$ be the set of elements $h \in S_n(\mathbf{R})$ with a positive eigenvalues, b negative eigenvalues, and c copies of 0 as an eigenvalue. We have via [25, (4.34K)] that for $h \in V(a, b, c)$:

$$\begin{aligned}\xi(g, h; \alpha, \beta) = |\sigma|^{-1} i^{n(\beta-\alpha)} 2^{\varphi} \pi^{\psi} \Gamma_c(\alpha + \beta - \kappa) \Gamma_{n-b}(\alpha)^{-1} \Gamma_{n-a}(\beta)^{-1} \\ \cdot \det(g)^{\kappa-\alpha-\beta} d_+(hg)^{\alpha-\kappa+\iota b/4} \\ \cdot d_-(hg)^{\beta-\kappa+\iota a/4} \omega(2\pi g, h; \alpha, \beta),\end{aligned}$$

where

$$\begin{aligned}\varphi &= (2a - n)\alpha + (2b - n)\beta + (n + c)\kappa + \iota ab/2 \\ \psi &= a\alpha + b\beta + c + (\iota/2)(c(c-1) - ab),\end{aligned}$$

$\kappa = (n+1)/2$, $\sigma = 2^{n(n-1)/2}$, $\iota = [\mathbf{K} : \mathbf{R}] = 1$, and we use $d_+(g)$ to denote the product of the positive eigenvalues of g (similarly for d_-). Observe that [1, Lemma VIII.14] gives the Fourier coefficients vanish if h is not positive semi-definite as long as $k > n$. Note that this result is for the Hermitian Siegel Eisenstein series, but the argument follows verbatim for our case. Thus, we can assume $b = 0$. The above equation reduces to

$$\begin{aligned}\xi(g, h; \alpha, \beta) = |\sigma|^{-1} i^{n(\beta-\alpha)} 2^{\varphi} \pi^{\psi} \Gamma_c(\alpha + \beta - \kappa) \Gamma_n(\alpha)^{-1} \Gamma_{n-a}(\beta)^{-1} \\ \cdot \det(g)^{\kappa-\alpha-\beta} d_+(hg)^{\alpha-\kappa} \omega(2\pi g, h; \alpha, \beta).\end{aligned}$$

We can apply [25, (4.35K)] to conclude that

$$\omega(2\pi g, h; \alpha, c/2) = \omega(2\pi g, h; (n+1)/2, \beta) = 2^{-a(n+1)/2} \pi^{ac/2} e(i\mathrm{tr}\,(gh)).$$

We now specialize all of this to the case $\alpha = \frac{n+1}{2}$ and $\beta = \frac{n+1-2k}{2}$. Since $c = n - a$, this gives

$$\begin{aligned}\varphi &= \frac{(n+1)(2a + c - n) + 2nk}{2} = \frac{na + a + 2nk}{2} \\ \psi &= \frac{a(n+1) + c(c+1)}{2} = \frac{n^2 + n + a^2 - na}{2}.\end{aligned}$$

We can simplify the equation for ξ:

$$\xi(y, h; \alpha, \beta) = |\sigma|^{-1} i^{nk} 2^{\varphi} \pi^{\psi} \Gamma_n(\alpha)^{-1} \det(y)^{-\beta} \omega(2\pi y, h; \alpha, \beta).$$

Observe that $\psi + \frac{ac}{2} = n(n+1)/2$ and $\varphi - n(n-1)/2 - a(n+1)/2 = nk - n(n+1)/2$. Thus,

$$\xi(y, h; \alpha, \beta) = 2^{\frac{2nk - n(n+1)}{2}} i^{-nk} \pi^{\frac{n(n+1)}{2}} \Gamma_n(\alpha)^{-1} \det(y)^{-\beta} e(i\mathrm{tr}\,(yh)).$$

Combining all of this, we have

$$c(h,q,\,(n+1-k)/2) = 2^{\frac{(2nk-n(n+1))d}{2}} i^{-nkd} |D_F|^{-\frac{n(n+1)}{4}} \chi(\det(q))^{-1}\, \mathrm{N}(\mathfrak{N})^{-\frac{n(n+1)}{2}}$$
$$\cdot\, |\det(qq^*)_{\mathbf{f}}|_F^{\frac{k}{2}} \det(y)^{k/2} \alpha^{\mathfrak{N}}(\omega q^* hq,\, n+1-k,\, \chi) \pi^{\frac{n(n+2)d}{4}} \mathcal{P}_n^{-d} e_{\mathbf{a}}(i\mathrm{tr}\,(yh)),$$

where

$$\mathcal{P}_n = \prod_{j=0}^{[(n+1)/2]} j! \prod_{j=0}^{[(n+1)/2]-1} \frac{(2j+1)!!}{2^{j+1}},$$

where

$$n!! = \begin{cases} n(n-2)\cdots 5\cdot 3\cdot 1 & \text{if } n>0 \text{ and odd} \\ n(n-2)\cdots 6\cdot 4\cdot 2 & \text{if } n>0 \text{ and even} \\ 1 & \text{if } n=0. \end{cases}$$

Write the Fourier expansion of D as:

$$D^*\left(\begin{bmatrix} q & \sigma\hat{q} \\ 0 & \hat{q}\end{bmatrix};\chi\right) = \sum_{h\in S_n(F)} \tilde{c}(h,\,q) e_{\mathbf{A}_F}(\mathrm{tr}\,(h\sigma))$$

for $q \in \mathrm{GL}_n(\mathbf{A}_F)$ and $\sigma \in S_n(\mathbf{A}_F)$. Then for q chosen as above, we have

$$\tilde{c}(h,\,q) = 2^{\frac{(2nk-n(n+1))d}{2}} i^{-nkd} |D_F|^{-\frac{n(n+1)}{4}} \chi(\det(q))^{-1}\, \mathrm{N}(\mathfrak{N})^{-\frac{n(n+1)}{2}}$$
$$\cdot\, |\det(qq^*)_{\mathbf{f}}|_F^{\frac{k}{2}} \det(y)^{k/2} \tilde{\alpha}^{\mathfrak{N}}(\omega q^* hq,\, n+1-k,\, \chi) \mathcal{P}_n^{-d} e_{\mathbf{a}}(i\mathrm{tr}\,(yh)),$$

where $\tilde{\alpha}^{\mathfrak{N}}(\omega q^* hq,\, n+1-k,\, \chi) = \Lambda^{\mathfrak{N}}\left(\frac{n+1-k}{2}\right) \alpha^{\mathfrak{N}}(\omega q^* hq,\, n+1-k,\, \chi)$. Using the notation as in Lemma 3.1, we have

$$D_t^*(z) = \sum_{\substack{h\in S_n(F)\\ h\geq 0}} \tilde{c}_t(h) e_{\mathbf{a}}(\mathrm{tr}\,(hz)),$$

where

$$\tilde{c}_t(h) = \det(y)^{-k/2}\tilde{c}(h,\,q)$$
$$= 2^{\frac{(2nk-n(n+1))d}{2}} i^{-nkd} |D_F|^{-\frac{n(n+1)}{4}} \chi(\det(q))^{-1}\, \mathrm{N}(\mathfrak{N})^{-\frac{n(n+1)}{2}}$$
$$\cdot\, |\det(qq^*)_{\mathbf{f}}|_F^{\frac{k}{2}} \tilde{\alpha}^{\mathfrak{N}}(\omega q^* hq,\, n+1-k,\, \chi) \mathcal{P}_n^{-d}.$$

As we are assuming $h_F = 1$, we can take $q \in G_n(\mathbf{A}_F)$ such that $q_{\mathbf{f}} = 1_n$ and $q_{\mathbf{a}} = y^{1/2}$, and so

$$\tilde{c}_{1_{2n}}(h) = 2^{\frac{(2nk-n(n+1))d}{2}} i^{-nkd} |D_F|^{-\frac{n(n+1)}{4}}\, \mathrm{N}(\mathfrak{N})^{-\frac{n(n+1)}{2}} \tilde{\alpha}^{\mathfrak{N}}(\omega q^* hq,\, n+1-k,\, \chi) \mathcal{P}_n^{-d},$$

where we have used that $\chi(\det(y)^{1/2}) = \chi_{\mathbf{a}}(\det(y)^{1/2}) = 1$ since $y > 0$.

PROPOSITION 3.4. *Let F be a totally real field such that $h_F = 1$. Let $k > n+1$ and let ℓ be an odd prime with $\ell \nmid D_F\, \mathrm{N}(\mathfrak{N}) n!$. There exists a finite extension $\mathcal{O}$ of $\mathbf{Z}_\ell$ so that*

$$\tilde{c}_{1_{2n}}(h) \in \mathcal{O}$$

for all $h \in S_n(F)$.

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Proof. This is mostly clear from the calculations above. It only remains to show that $\tilde{\alpha}^{\mathfrak{N}}(\omega q^* hq, n+1-k, \chi) \in \mathcal{O}_E$ for E some finite extension of $\mathbf{Q}_\ell$. However, this follows immediately from results on special values of Dirichlet L-functions and the fact that $k > n+1$; see for example [9, Proposition 3.1]. □

4. Pullbacks and inner product formula. Let $E(z;\chi) := E_{1_{4n}}(z, (2n+1-k)/2; \chi)$ be the classic Siegel Eisenstein series of weight k and level $\Gamma_{2n}(1, \mathfrak{N})$ defined on $\mathfrak{h}_{2n}^{\mathbf{a}}$ as given in the previous section and $D(z, \chi)$ the normalized Eisenstein series associated with $E(z, \chi)$. In this section, we consider the pullback of this Eisenstein series to $\mathfrak{h}_n^{\mathbf{a}} \times \mathfrak{h}_n^{\mathbf{a}}$ and specialize an inner product relation of Shimura to our setting.

We have an embedding $\iota : \mathfrak{h}_n^{\mathbf{a}} \times \mathfrak{h}_n^{\mathbf{a}} \hookrightarrow \mathfrak{h}_{2n}^{\mathbf{a}}$ via $(z, w) \mapsto \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix}$. We have $\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$ embeds in $\mathrm{Sp}(4n)$ via

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \times \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}.$$

Under these maps, given a modular form $f \in M_k(\Gamma^{(2n)})$ for a congruence subgroup $\Gamma^{(2n)}$, we can restrict f to a function on $\mathfrak{h}_n^{\mathbf{a}} \times \mathfrak{h}_n^{\mathbf{a}}$ via $f(\iota(z, w))$. In fact, one has $f(\iota(z, w))$ is a weight k modular form in of z and w independently. In particular, we have that $E(\iota(z, w); \chi)$ is a weight k and level $\Gamma_n(1, \mathfrak{N})$ modular form in z and w independently. Moreover, one knows that $E(\iota(z, w); \chi)$ is a cusp form in each variable, see [14].

Observe that the Fourier expansion of f can be written as:

$$f(\iota(z, w)) = \sum_{T_1, T_2 \in \mathcal{L}_n} \left(\sum_{T \in \mathcal{L}_{2n}(T_1, T_2)} a(T; f) \right) e_{\mathbf{a}}(\mathrm{Tr}(T_1 z)) e_{\mathbf{a}}(\mathrm{Tr}(T_2 w)),$$

where $a(T; f)$ is the T-th Fourier coefficient of f and for $T_1, T_2 \in \mathcal{L}_n$ we define

$$\mathcal{L}_{2n}(T_1, T_2) = \left\{ T \in \mathcal{L}_{2n} : T = \begin{bmatrix} T_1 & b \\ b & T_2 \end{bmatrix} \right\}.$$

From this, we immediately have that if f has Fourier coefficients in some ring $\mathcal{O}$, so does $f(\iota(z, w))$.

We define an element $\sigma = (\sigma_v)_{v \in \mathbf{f}} \in G_{2n}(\mathbf{A}_{F,\mathbf{f}})$ by setting

$$\sigma_v = \begin{cases} \tau_n & v \mid \mathfrak{N} \\ 1_{4n} & v \nmid \mathfrak{N} \end{cases},$$

where τ_n is defined by

$$\tau_n = \begin{bmatrix} 1_{2n} & 0_{2n} \\ \rho_n & 1_{2n} \end{bmatrix}$$

with

$$\rho_n = \begin{bmatrix} 0_n & 1_n \\ 1_n & 0_n \end{bmatrix}.$$

We use strong approximation to obtain an element ρ of $G_{2n}(F) \cap \mathcal{K}_{2n}(1, \mathfrak{N})\sigma$ so that $a(\sigma\rho^{-1})_v - 1_{2n} \in \mathrm{Mat}_{2n}(\mathfrak{N}\mathcal{O}_{F,v})$ for every $v \mid \mathfrak{N}$.

REMARK 4.1. Observe that the map $h \mapsto h|_{\omega_n}$ gives an isomorphism from $S_k(\Gamma_n(\mathfrak{N}))$ to $S_k(\Gamma_n(\mathfrak{N}))$, so if we view h as having level $\Gamma_n(\mathfrak{N})$, then $(h|_{\omega_n})^c$ also has level $\Gamma_n(\mathfrak{N})$.

Let $f \in S_k(\Gamma)$ be an eigenform with Γ a congruence subgroup of level $\mathfrak{M}$ with $\mathfrak{M} \mid \mathfrak{N}$. Observe that $\Gamma_n(\mathfrak{N}) \subset \Gamma$ and $\Gamma_n(\mathfrak{N}) \subset \Gamma_n(1, \mathfrak{N})$. This means we can view f and $(E|_\rho)(\iota(z, w); \chi)$ as cusp forms of level $\Gamma_n(\mathfrak{N})$ where we are viewing $(E|_\rho)(\iota(z, w); \chi)$ as a cusp form in the w variable. The inner product formula we are interested in is given in [27, (6.17)]. Specialized to our case, it gives

$$\begin{aligned}[]
[\mathrm{Sp}_{2n}(\mathcal{O}_F) : \Gamma_n(\mathfrak{N})]\langle \Lambda^{\mathfrak{N}}(2n+1-k)(E|_\rho)(\iota(z, w); \chi), (f|_{\omega_n})^c(w)\rangle \\
= 2c_{n,k}(n-k+1/2)L^{\mathfrak{N}}(2n+1-k, f, \chi; \mathrm{st}) f(z).
\end{aligned}$$

For $\ell > 2n-1$, we have $c_{n,k}(n-k+1/2) = \pi^{dn(n+1)/2}u$ for u an ℓ-adic unit. Thus, we have

$$\langle (D|_\rho)(\iota(z, w); \chi), (f|_{\omega_n})^c(w)\rangle \tag{4.1}$$

$$= \frac{2u}{[\mathrm{Sp}_{2n}(\mathcal{O}_F) : \Gamma_n(\mathfrak{N})]} \cdot \frac{L^{\mathfrak{N}}(2n+1-k, f, \chi; \mathrm{st})}{\pi^{dn(n+1)/2}} f(z). \tag{4.2}$$

In the next section, we will make use of this inner product formula to produce a congruence between a eigenform $f \in S_k(\Gamma)$ and a cuspidal eigenform that is orthogonal to f.

5. Congruence. In this section, we combine the results of the previous section to produce the desired congruence. In the following section, we will specialize the congruence to a particular case of interest.

Given $\Gamma_1 \subset \Gamma_2$, we define the trace operator:

$$\begin{aligned}
\mathrm{tr}_{\Gamma_1}^{\Gamma_2} : M_k(\Gamma_1) &\to M_k(\Gamma_2) \\
f &\mapsto \sum_{\gamma \in \Gamma_1 \backslash \Gamma_2} f|_\gamma.
\end{aligned}$$

Note that if f has $\mathcal{O}$-integral Fourier coefficients for $\mathcal{O}$ some finite extension of $\mathbf{Z}_\ell$, the q-expansion principle implies that $\mathrm{tr} f$ also has $\mathcal{O}$-integral Fourier coefficients.

Let $\Gamma^{(n)}$ be a congruence subgroup of level $\mathfrak{M}$ with $\mathfrak{M} \mid \mathfrak{N}$. Recall that $(D|_\rho)(\iota(z, w); \chi)$ is a cusp form of weight k and level $\Gamma_n(1, \mathfrak{N})$ in z and w independently as noted above. In particular, we have $\Gamma_n(\mathfrak{N}) \subset \Gamma_n(1, \mathfrak{N})$, so we can view $(D|_\rho)(\iota(z, w); \chi)$ as a cusp form in z and w of weight k and level $\Gamma_n(\mathfrak{N})$.

Fix $f \in S_k(\Gamma^{(n)}; \mathcal{O})$ be an eigenform for $\mathcal{O}$ a sufficiently large finite extension of $\mathbf{Z}_\ell$ with maximal ideal λ and uniformizer ϖ. Moreover, assume the Fourier coefficients of f

are real. We will enlarge $\mathcal{O}$ as necessary in the proof but continue to denote the larger ring as $\mathcal{O}$. Let $f = f_1, f_2, \ldots, f_r$ be an orthogonal basis of eigenforms of $S_k(\Gamma_n(\mathfrak{N}))$ with each $f_i \in S_k(\Gamma_n(\mathfrak{N}); \mathcal{O})$. Write

$$(D|_\rho)(\iota(z,w);\chi) = \sum_{i,j=1}^{m} c_{i,j} f_i(z)(f_j|_{\omega_n})^c(w),$$

which is possible via Remark 4.1. We have via [8, Proposition 5.1] that $c_{i,j} = 0$ unless $i = j$. Note that the reference given is for $F = \mathbf{Q}$ and $\Gamma = \mathrm{Sp}_{2n}(\mathbf{Z})$, but the exact same proof carries over to our situation. Thus, we can write

$$(D|_\rho)(\iota(z,w);\chi) = c_1 f(z)(f|_{\omega_n})^c(w) + \sum_{i=2}^{r} c_i f_i(z)(f_i|_{\omega_n})^c(w),$$

where $c_i := c_{i,i}$. We can now use the fact the f_i form an orthogonal basis combined with equation (4.1) to obtain

$$c_1 = \frac{2u}{[G_n(\mathcal{O}_F):\Gamma_n(\mathfrak{M})]} \cdot \frac{L^{\mathfrak{N}}(2n+1-k, f, \chi; \mathrm{st})}{\pi^{dn(n+1)/2}\langle (f|_{\omega_n})^c, (f|_{\omega_n})^c\rangle}$$

for u an ℓ-adic unit. Observe that we have $(f|_{\omega_n})^c = f^c|_{\omega_n^{-1}}$ via [27, p. 562]. Moreover, since we are assuming f has real Fourier coefficients, we have $(f|_{\omega_n})^c = f|_{\omega_n^{-1}}$. Thus, we have

$$c_1 = \frac{2u}{[G_n(\mathcal{O}_F):\Gamma_n(\mathfrak{M})]} \cdot \frac{L^{\mathfrak{N}}(2n+1-k, f, \chi; \mathrm{st})}{\pi^{dn(n+1)/2}\langle f, f\rangle}.$$

Write

$$L^{\mathfrak{N}}_{\mathrm{alg}}(2n+1-k, f, \chi; \mathrm{st}) = \frac{L^{\mathfrak{N}}(2n+1-k, f, \chi; \mathrm{st})}{\pi^{dn(n+1)/2}\langle f, f\rangle} \in \overline{\mathbf{Q}}.$$

THEOREM 5.1. *Let $F/\mathbf{Q}$ be a totally real number field of class number 1. Let $\mathfrak{M} \subset \mathcal{O}_F$ be an integral ideal and $\Gamma^{(n)} \supset \Gamma_n(\mathfrak{M})$ a congruence subgroup. Let $k = (k, \ldots, k) \in \mathbf{Z}^{\mathbf{a}}$ with $k > n+1$. Let ℓ be a rational prime with $\ell \nmid D_F(2n-1)!$. Let $f \in S_k(\Gamma^{(n)}; \mathcal{O} \cap \mathbf{R})$ be an eigenform where $\mathcal{O}$ is a suitably large finite extension of $\mathbf{Z}_\ell$ with maximal ideal λ and uniformizer ϖ with f normalized so that there exists T_0 so that $\mathrm{val}_\lambda(a(T_0; f|_{\omega_n^{-1}})) = 0$. If there exists a proper ideal $\mathfrak{N} \subset \mathcal{O}_F$, $\mathfrak{M} \mid \mathfrak{N}$, $\ell \nmid \mathrm{N}(\mathfrak{N})$, a Hecke character χ defined as in (3.1) and (3.2) so that*

$$\mathrm{val}_\lambda(L^{\mathfrak{N}}_{\mathrm{alg}}(2n+1-k, f, \chi; \mathrm{st})) = -b < 0,$$

then there exists a $g \in S_k(\Gamma_n(\mathfrak{N})) \cap (\mathbf{C}f)^\perp$ so that $f \equiv g \pmod{\lambda^b}$. Moreover, if $\ell \nmid [\Gamma^{(n)} : \Gamma_n(\mathfrak{N})]$ then there exists $g \in S_k(\Gamma^{(n)}) \cap (\mathbf{C}f)^\perp$ so that $f \equiv g \pmod{\lambda^b}$.

Proof. As above, write

$$(D|_\rho)(\iota(z,w);\chi) = c_1 f(z)(f|_{\omega_n})^c(w) + \sum_{i=2}^{r} c_i f_i(z)(f_i|_{\omega_n})^c(w). \tag{5.1}$$

Our hypothesis allows us to write $c_1 = \varpi^{-b}\alpha$ for $\mathrm{val}_\varpi(\alpha) = 0$. Expanding each side of (5.1) in its Fourier expansion in terms of the w variable, multiplying both sides by ϖ^b, and equating the T_0th Fourier coefficients, we have

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$$\sum_{T_1\in\mathcal{L}_n}\left(\sum_{T\in\mathcal{L}_{2n}(T_1,T_0)}\varpi^b a(T;\,(D|_\rho)\circ\iota)\right)e(\mathrm{Tr}(T_1z))$$
$$=\alpha a(T_0;\,(f|_{\omega_n})^c)f(z)+\sum_{i=2}^{r}\varpi^b c_i a(T;\,(f_i|_{\omega_n})^c)f_i(z).$$

Observe that since $a(T;\,(D|_\rho)\circ\iota)\in\mathcal{O}$ for all T, we have $\varpi^b a(T;\,(D|_\rho)\circ\iota)\equiv 0 \pmod{\varpi^b}$ for all T. Thus,

$$f(z)\equiv(\alpha a(T;\,(f_{\omega_n})^c))^{-1}\sum_{i=2}^{r}\varpi^b c_i a(T;\,(f_i|_{\omega_n})^c)f_i(z)\pmod{\lambda^b}.$$

Setting

$$h(z)=(\alpha a(T;\,(f_{\omega_n})^c))^{-1}\sum_{i=2}^{r}\varpi^b c_i a(T;\,(f_i|_{\omega_n})^c)f_i(z),$$

we have $h\in S_k(\Gamma_n(\mathfrak{N});\,\mathcal{O})$ and $f\equiv h \pmod{\lambda^b}$. To obtain a congruence to a form in $S_k(\Gamma^{(n)})$, we use that if h has Fourier coefficients in $\mathcal{O}$, so does $h|_\gamma$ for each $\gamma\in\Gamma^{(n)}$ by the q-expansion principle. Since $(f-h)/\varpi^b$ is a modular form with coefficients in $\mathcal{O}$, so is $(f|_\gamma-h|_\gamma)/\varpi^b$. This gives $f|_\gamma\equiv h|_\gamma \pmod{\lambda^b}$ for each $\gamma\in\Gamma^{(n)}$. Thus, taking traces and using that f has level $\Gamma^{(n)}$, we obtain

$$f\equiv g\pmod{\lambda^b},$$

where

$$g(z)=\frac{1}{[\Gamma^{(n)}:\Gamma_n(\mathfrak{N})]}\mathrm{tr}^{\Gamma^{(n)}}_{\Gamma_n(\mathfrak{N})}h.$$ □

REMARK 5.2. The authors are unaware of any conjectures that relate the divisibility of the denominator of an L-value to congruences between automorphic forms. As such, this result does not fit into any larger framework that the authors are aware of.

6. Congruences of Saito–Kurokawa lifts of paramodular level.

In this section, we apply the results of the previous section to the case that f is a Saito–Kurokawa lift of paramodular level, also known as a Gritsenko lift. We specialize to the case that $F=\mathbf{Q}$, $n=2$, and $\mathfrak{M}=M$.

6.1. Existence and basic facts.

Let $k\geq 2$ and $M\geq 1$. We let $S^{\mathrm{new},-}_{2k-2}(\Gamma_0(M))$ denote the subspace of $S^{\mathrm{new}}_{2k-2}(\Gamma_0(M))$ such that the sign of the functional equation of the associated L-function is -1.

THEOREM 6.1. [15, 16, 30] *Let $k\geq 2$ and $M\geq 1$. Let $\phi\in S^{\mathrm{new},-}_{2\kappa-2}(\Gamma_0(M))$ be a newform. Given a negative fundamental discriminant D, there exists a nonzero cuspidal Siegel eigenform $f_{\phi,D}\in S_\kappa(\Gamma^{\mathrm{para}}(M))$ satisfying*

$$L^M(s,f_{\phi,D};\,\mathrm{spin})=\zeta^M(s-\kappa+1)\zeta^M(s-\kappa+2)L^M(s,\phi).$$

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If $\mathcal{O}$ is a ring that can be embedded into $\mathbf{C}$ *and* ϕ *has Fourier coefficients in* $\mathcal{O}$*, the lift* $f_{\phi,D}$ *can be normalized to have Fourier coefficients in* $\mathcal{O}$*. If* $\mathcal{O}$ *is a DVR,* $f_{\phi,D}$ *can be normalized to have Fourier coefficients in* $\mathcal{O}$ *with at least one Fourier coefficient in* $\mathcal{O}^\times$*.*

Note that we have

$$(f_{\phi,D})^\sigma = f_{\phi^\sigma,D}$$

for all $\sigma \in \mathrm{Aut}(\mathbf{C})$. In particular, if we take a newform ϕ we know that ϕ has real Fourier coefficients, so $f_{\phi,D}$ also has real Fourier coefficients. Moreover, we have the following result.

Let Σ be a set of finite primes containing the primes dividing M. Write $\mathrm{SK}(\phi)$ for the subspace of $S_k(\Gamma^{\mathrm{para}}(M))$ spanned by common eigenforms f away from Σ such that

$$L^\Sigma(s, f, \mathrm{spin}) = \zeta^\Sigma(s-k+1)\zeta^\Sigma(s-k+2)L^\Sigma(s,\phi).$$

We have via [24, Theorem 5.2] that $\dim_{\mathbf{C}} \mathrm{SK}(\phi) = 1$ if M is odd and square-free. We let $S_k^{\mathrm{SK}}(\Gamma^{\mathrm{para}}(M))$ be the space generated by all the $\mathrm{SK}(\phi)$ and $S_k^{\text{N-SK}}(\Gamma^{\mathrm{para}}(M))$ the orthogonal complement of $S_k^{\mathrm{SK}}(\Gamma^{\mathrm{para}}(M))$ in $S_k(\Gamma^{\mathrm{para}}(M))$.

One has for a Hecke character χ, the standard L-function of $f_{\phi,D}$ factors as:

$$L^M(5-k, f_{\phi,D}, \chi; \mathrm{st}) = L^M(3-k,\chi)L^M(1,f,\chi)L^M(2,f,\chi),$$

where we have specialized to the special value that arises in Theorem 5.1.

The following result on the structure of the Galois representation of a paramodular Saito–Kurokawa lift follows immediately from the factorization of the associated Spinor L-function.

Corollary 6.2. *Let* ε *be the* ℓ*-adic cyclotomic character and let* $f_{\phi,D} \in \mathrm{SK}(\phi)$ *be an eigenform. Then,*

$$\rho_{f_{\phi,D},\lambda} \simeq \varepsilon^{2-k} \oplus \rho_{\phi,\lambda} \oplus \varepsilon^{1-k}.$$

Corollary 6.3. *Let* $\mathbf{T}^S_{\mathbf{Z}}$ *and* $\mathbf{T}_{\mathbf{Z}}$ *be the standard Hecke algebras acting on the space of cusp form* $S_k(\Gamma^{\mathrm{para}}(M))$ *and* $S^-_{2k-2}(\Gamma_0(M))$*, respectively. There exists a surjection* $\Phi: \mathbf{T}^S_{\mathbf{Z}} \to \mathbf{T}_{\mathbf{Z}}$ *satisfying*

$$T(f_{\phi,D}) = f_{\Phi(T)\phi,D}$$

for all $T \in \mathbf{T}^S_{\mathbf{Z}}$*.*

We will make use of the relation between $\langle f_{\phi,D}, f_{\phi,D}\rangle$ and $\langle \phi,\phi\rangle$.

Corollary 6.4. [10, Corollary 4.6] *Let* $\phi \in S^{\mathrm{new}}_{2k-2}(\Gamma_0(M))$ *be a newform,* D *be a negative fundamental discriminant so that* $M \nmid D$ *and* D *is a square modulo* $4M$*, and* $f_{\phi,D} \in S_k(\Gamma^{\mathrm{para}}(M))$ *as above. Then,*

$$\frac{\langle f_{\phi,D}, f_{\phi,D}\rangle}{\langle \phi,\phi\rangle} = \mathcal{C}_{k,M}\frac{|a(D;\varphi_D^{\mathrm{alg}})|^2 L(4,\chi_M)L(k,\phi)}{\pi^5|D|^{k-3/2}L(k-1,\phi,\chi_D)},$$

where

$$\mathcal{C}_{k,M} = \frac{3\cdot 5\cdot (k-1)M^2(M+1)[G_1(\mathbf{Z}):\Gamma_0(M)]}{2^3(M-1)[G_2(\mathbf{Z}):\Gamma^{\mathrm{para}}(M)]}.$$

6.2. Constructing the congruence. The goal of this section is to produce a congruence between a paramodular Saito–Kurokawa lift $f_{\phi,D} \in S_k^{\text{SK}}(\Gamma^{\text{para}}(M))$ and a Siegel modular form $g \in S_k^{\text{N-SK}}(\Gamma^{\text{para}}(M))$. As the case of k even is covered by the main theorem of [2], we restrict ourselves to the case k is odd here.

Let $f_1 = f_{\phi,D}, \ldots, f_{r+m}$ be an orthogonal basis of $S_k(\Gamma^{\text{para}}(M))$ consisting of eigenforms away from M with $f_1, \ldots, f_r \in S_k^{\text{SK}}(\Gamma^{\text{para}}(M))$. We enlarge our ℓ-adic ring $\mathcal{O}$ so that this basis is defined over $\mathcal{O}$. As in Section 5, we choose a character χ and $N > 1$ so that $M \mid N$ and write

$$(D|_\rho)(\iota(z, w); \chi) = c_1 f_{\phi,D}(z)(f_{\phi,D}|_{\omega_n})^c(w) + \sum_{i=2}^{r+m} c_i f_i(z)(f_i|_{\omega_n})^c(w).$$

As we desire a congruence between $f_{\phi,D}$ and a form in $S_k^{\text{N-SK}}(\Gamma^{\text{para}}(M))$, we apply a certain Hecke operator to $(D|_\rho)(\iota(z, w); \chi)$ in the z-variable. The Hecke operator is given by the following theorem; see also [2, Theorem 6.4] for the congruence level case.

THEOREM 6.5. *Let M be square-free, $\ell > k$ a prime, and $\ell \nmid M$. Let $\phi \in S_{2k-2}^{\text{new},-}(\Gamma_0(M); \mathcal{O})$ be a newform with $\overline{\rho}_{\phi,\lambda}$ irreducible. If $M = 1, 3$, we further assume that ϕ is ordinary at ℓ. Then there exists T_ϕ^S defined over $\mathcal{O}$ so that $T_\phi^S f_{\phi,D} = \alpha_\phi f_{\phi,D}$ with $\alpha_\phi = u_\phi \frac{\langle \phi,\phi \rangle}{\Omega_\phi^+ \Omega_\phi^-}$ for $u_\phi \in \mathcal{O}^\times$ and $T_\phi^S f_i = 0$ for $i = 2, \ldots, r$.*

Proof. The exact same argument used to prove [2, Theorem 6.4] applies in this case. □

This allows us to write

$$T_\phi^S(D|_\rho)(\iota(z, w); \chi) = c_1 \alpha_\phi f_{\phi,D}(f_{\phi,D}|_{\omega_n})^c(w) + \sum_{i=r+1}^{m} c_i T_\phi^S f_i(z)(f_i|_{\omega_n})^c(w).$$

We now study $c_1\alpha_\phi$ to produce our congruence as above. We know since $D < 0$, $\chi_D(-1) = -1$ so the fact that k is odd gives that Ω_f^+ is the correct period for $L(k, \phi)$ and $L(k-1, \phi, \chi_D)$ so,

$$\frac{L(k-1, \phi, \chi_D)}{L(k, \phi)} = \frac{\tau(\chi_D) L_{\text{alg}}(k-1, \phi, \chi_D)}{(2\pi i) L_{\text{alg}}(k, \phi)}.$$

We also have

$$\frac{L^N(1, \phi, \chi) L^N(2, \phi, \chi)}{\Omega_\phi^+ \Omega_\phi^-} = \frac{\tau(\chi)^2 (2\pi i)^3 L_{\text{alg}}(1, \phi, \chi) L_{\text{alg}}(2, \phi, \chi)}{L_N(1, \phi, \chi) L_N(2, \phi, \chi)}.$$

We apply the calculation of c_1 in Section 5 with Corollary 6.4 and the factorization of the standard L-function of a paramodular Saito–Kurokawa lift to obtain

$$\begin{aligned}
\alpha_\phi c_1 &= \mathcal{C}_{k,M,N,D} \frac{\pi^2 L^N(3-k, \chi) L^N(1, \phi, \chi) L^N(2, \phi, \chi) L(k-1, \phi, \chi_D)}{L(4, \chi_M) L(k, \phi) \Omega_\phi^+ \Omega_\phi^-} \\
&= \mathcal{C}_{k,M,N,D,\chi} \frac{L^N(3-k, \chi) L_{\text{alg}}^N(1, \phi, \chi) L_{\text{alg}}^N(2, \phi, \chi) L_{\text{alg}}(k-1, \phi, \chi_D)}{L(4, \chi_M) L_{\text{alg}}(k, \phi)},
\end{aligned}$$

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where

$$\mathcal{C}_{k,M,N,D} = \frac{uu_\phi 2^4(M-1)|D|^{k-3/2}}{3\cdot 5\cdot (k-1)M^2(M+1)[\mathrm{SL}_2(\mathbf{Z}):\Gamma_0(M)][\Gamma^{\mathrm{para}}(M):\Gamma_2(N)]|a(D;\varphi_D^{\mathrm{alg}})|^2}$$

and

$$\mathcal{C}_{k,M,N,D,\chi} = 2^2\tau(\chi_D)\tau(\chi)^2\mathcal{C}_{k,M,N,D}.$$

Observe that if we choose an odd prime $\ell > k$ so that $\ell \nmid DN(M-1)$ then,

$$\mathrm{ord}_\varpi(\mathcal{C}_{k,M,N,D,\chi}) \leq 0.$$

These calculations, along with Theorem 5.1, give the following theorem.

THEOREM 6.6. *Let k and M be positive integers with $k \geq 6$ and M odd and squarefree. Let $\phi \in S_{2k-2}^{\mathrm{new},-}(\Gamma_0(M))$ be a newform. Let $\ell > k$ be a prime with $\ell \nmid M$ and $\overline{\rho}_{\phi,\lambda}$ irreducible. Furthermore, if $M = 1$ or $M = 3$, assume ϕ is ordinary at ℓ. Let $\mathcal{O}$ be a sufficiently large finite extension of $\mathbf{Z}_\ell$ with maximal ideal λ and uniformizer ϖ. Assume that* $\mathrm{ord}_\lambda(L_{\mathrm{alg}}(k,\phi)) > 0$. *If there exists a negative fundamental discriminant D so that $\ell \nmid D$, $M \nmid D$, D is a square modulo $4M$, and an integer $N > 1$ so that $M \mid N$, $\ell \nmid N[\Gamma^{\mathrm{para}}(M):\Gamma_2(N)]$, and a Dirichlet character χ of conductor N so that*

$$\mathrm{ord}_\lambda\left(\frac{L^N(3-k,\chi)L_{\mathrm{alg}}^N(1,\phi,\chi)L_{\mathrm{alg}}^N(2,\phi,\chi)L_{\mathrm{alg}}(k-1,\phi,\chi_D)}{L(4,\chi_M)L_{\mathrm{alg}}(k,\phi)}\right) = -b < 0,$$

then there exists a nonzero $g \in S_k^{\mathrm{N\text{-}SK}}(\Gamma^{\mathrm{para}}(M))$ so that

$$f_{\phi,D} \equiv g \pmod{\lambda^b}.$$

6.3. CAP ideal and a lower bound. Our goal is to apply Theorem 6.6 to give evidence for the Bloch–Kato conjecture in some new cases. In order to do this, we introduce the CAP ideal of $f_{\phi,D}$, an ideal measuring congruences between $f_{\phi,D}$ and eigenforms with irreducible Galois representations. We use Theorem 6.6 to give a lower bound on the index of this ideal; this will allow us to give a bound on an appropriate Selmer group in Section 7.2.

Let $\mathbf{T}_{\mathcal{O}}^{\mathrm{N\text{-}SK},\Sigma}$ denote the image of $\mathbf{T}_{\mathcal{O}}^{S,\Sigma}$ in $\mathrm{End}_{\mathbf{C}}(S_k^{\mathrm{N\text{-}SK}}(\Gamma^{\mathrm{para}}(M)))$. Let $\Psi : \mathbf{T}_{\mathcal{O}}^{S,\Sigma} \to \mathbf{T}_{\mathcal{O}}^{\mathrm{N\text{-}SK},\Sigma}$ denote the canonical $\mathcal{O}$-algebra surjection. Write $\mathrm{Ann}(f_{\phi,D})$ for the annihilator of $f_{\phi,D}$ in $\mathbf{T}_{\mathcal{O}}^{S,\Sigma}$ and observe we have an isomorphism:

$$\mathbf{T}_{\mathcal{O}}^{S,\Sigma}/\,\mathrm{Ann}(f_{\phi,D}) \cong \mathcal{O}.$$

As Ψ is surjective, we have that $\Psi(\mathrm{Ann}(f_{\phi,D}))$ is an ideal in $\mathbf{T}_{\mathcal{O}}^{\mathrm{N\text{-}SK},\Sigma}$. We refer to $\Psi(\mathrm{Ann}(f_{\phi,D}))$ as the CAP ideal associated with $f_{\phi,D}$. From our work above, we have that (under the conditions given above), if f is an eigenform of weight k and level $\Gamma^{\mathrm{para}}(M)$ with reducible Galois representation so that $f \equiv_{\mathrm{ev},\Sigma} f_{\phi,D} \pmod{\lambda}$ for some finite set of places Σ, then $f \in S_k^{\mathrm{SK}}(\Gamma^{\mathrm{para}}(M))$.

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One has that there exists an $r \in \mathbf{Z}_{\geq 0}$ so that the following diagram commutes

$$\begin{array}{ccc}
\mathbf{T}_{\mathcal{O}}^{S,\Sigma} & \xrightarrow{\quad\Psi\quad} & \mathbf{T}_{\mathcal{O}}^{\text{N-SK},\Sigma} \\
\downarrow & & \downarrow \\
\mathbf{T}_{\mathcal{O}}^{S,\Sigma}/\operatorname{Ann}(f_{\phi,D}) & \xrightarrow{\Psi} & \mathbf{T}_{\mathcal{O}}^{\text{N-SK},\Sigma}/\phi(\operatorname{Ann}(f_{\phi,D})) \\
\downarrow & & \downarrow \simeq \\
\mathcal{O} & \longrightarrow & \mathcal{O}/\varpi^r\mathcal{O}.
\end{array}$$

Note that all of the maps in the above diagram are $\mathcal{O}$-algebra surjections.

COROLLARY 6.7. *With r as in the above diagram and b as in Theorem 6.6, if we assume $\ell \nmid (p^2-1)$ for all $p \mid M$ we have $r \geq b$.*

Proof. Assume that $b > r$. Note that the condition $\ell \nmid (p^2-1)$ for all $p \mid M$ ensures that any eigenform congruent to $f_{\phi,D}$ cannot be a weak endoscopic lift, so must have irreducible Galois representation. See [2, Theorem 7.4] for this; the proof carries over verbatim to our situation. The rest of the proof follows exactly as in [2, Corollary 7.6]. □

7. Selmer group and applications to Bloch–Kato.

7.1. Selmer group: definition. In this section, we define the relevant Selmer group that will be used to state the Bloch–Kato conjecture for our situation, and for which we will provide a lower bound. We follow [5] for our Selmer group definitions.

For a number field K and a topological $G_K = \operatorname{Gal}(\overline{K}/K)$-module $\mathscr{M}$ with a continuous action of G_K, we consider the group $H^1_{\text{cont}}(G_K, \mathscr{M})$ of cohomology classes of continuous cocycles $G_K \to \mathscr{M}$. We write $H^1(K, \mathscr{M})$ to denote $H^1_{\text{cont}}(G_K, \mathscr{M})$ to ease the notation.

Let Σ be a finite set of rational primes containing ℓ and the primes dividing M. Let G_Σ denote the Galois group of the maximal Galois extension $\mathbf{Q}_\Sigma$ of $\mathbf{Q}$ unramified outside of Σ. Let E be a finite extension of $\mathbf{Q}_\ell$ and $\mathcal{O}_E$ be its ring of integers. Let V be a finite dimensional E-vector space with a continuous G_Σ-action. In the case that $\dim_E(V) = n$, we will write this action as $\rho : G_\Sigma \to \operatorname{GL}_n(E)$. Let $T \subset V$ be a G_Σ-stable $\mathcal{O}_E$-lattice. Set $W := V/T \cong T \otimes_{\mathcal{O}_E} E/\mathcal{O}_E$.

We write B_{crys} for the ring of ℓ-adic periods ([13]). For every $p \in \Sigma$ and a G_Σ-module $\mathscr{M}$ define

$$H^1_{\text{un}}(\mathbf{Q}_p, \mathscr{M}) := \ker\{H^1(\mathbf{Q}_p, \mathscr{M}) \xrightarrow{\text{res}} H^1(I_p, \mathscr{M})\},$$

where I_p is the inertia group at p. We define the local p-Selmer group for V as:

$$H^1_f(\mathbf{Q}_p, V) := \begin{cases} H^1_{\text{un}}(\mathbf{Q}_p, V) & p \in \Sigma \smallsetminus \ell \\ \ker\{H^1(\mathbf{Q}_\ell, V) \to H^1(\mathbf{Q}_\ell, V \otimes B_{\text{crys}})\} & p = \ell. \end{cases}$$

For every p, define $H^1_f(\mathbf{Q}_p, W)$ to be the image of $H^1_f(\mathbf{Q}_p, V)$ under the natural map $H^1(\mathbf{Q}_p, V) \to H^1(\mathbf{Q}_p, W)$. Using the fact, that $\operatorname{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p) \cong \hat{\mathbf{Z}}$ has cohomological dimension one, one has that if W is unramified at p and $p \neq \ell$, then $H^1_f(\mathbf{Q}_p, W) = H^1_{\text{un}}(\mathbf{Q}_p, W)$.

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We are now in a position to define the Selmer group of interest to us. For any set $\Sigma' \subset \Sigma \smallsetminus \ell$, let

$$\mathrm{Sel}_\Sigma(\Sigma', W) := \ker\left\{ H^1(G_\Sigma, W) \xrightarrow{\mathrm{res}} \bigoplus_{p \in \Sigma' \cup \{\ell\}} \frac{H^1(\mathbf{Q}_p, W)}{H^1_f(\mathbf{Q}_p, W)} \right\}.$$

In the case that $\Sigma' = \emptyset$, we write $\mathrm{Sel}_\Sigma(W)$ for $\mathrm{Sel}_\Sigma(\emptyset, W)$.

For a $\mathbf{Z}_p$-module $\mathscr{M}$, let $\mathscr{M}^\vee$ denote the Pontryagin dual of $\mathscr{M}$ defined as:

$$\mathscr{M}^\vee = \mathrm{Hom}_{\mathrm{cont}}(\mathscr{M}, \mathbf{Q}_p/\mathbf{Z}_p).$$

We denote the Pontryagin dual of $\mathrm{Sel}_\Sigma(\Sigma', W)$ by $X_\Sigma(\Sigma', W)$, that is,

$$X_\Sigma(\Sigma', W) = (\mathrm{Sel}_\Sigma(\Sigma', W))^\vee.$$

We need the following lemma for what follows.

LEMMA 7.1. [23], [29, Section 3] *The module $X_\Sigma(\Sigma', W)$ is a finitely generated $\mathcal{O}$-module and if the mod λ reduction $\bar{\rho}$ of ρ is absolutely irreducible, then the length of $X_\Sigma(\Sigma', W)$ as an $\mathcal{O}$-module is independent of the choice of the lattice T.*

7.2. Bloch–Kato. We begin by stating the specialization of the Bloch–Kato conjecture to the case of interest here. Let $\phi \in S^{\mathrm{new}}_{2k-2}(\Gamma_0(M))$ be a newform and $(\rho_{\phi,\lambda}, V_{\phi,\lambda})$ be the λ-adic Galois representation associated with it. Let $V_{\phi,\lambda}(k-2)$ denote the representation space of $\rho = \rho_{\phi,\lambda} \otimes \varepsilon_\ell^{k-2}$ of $G_{\mathbf{Q}}$ where we recall ε_ℓ is the ℓ-adic cyclotomic character. Let $T_{\phi,\lambda}(k-2) \subset V_{\phi,\lambda}(k-2)$ be some choice of a $G_{\mathbf{Q}}$-stable lattice. Set $W_{\phi,\lambda}(k-2) = V_{\phi,\lambda}(k-2)/T_{\phi,\lambda}(k-2)$. The action of $G_{\mathbf{Q}}$ on $V_{\phi,\lambda}(k-2)$ factors through G_Σ.

The λ-part of the Bloch–Kato conjecture can be phrased as follows. For a detailed discussion of how this follows from the more general conjecture, the reader is advised to consult [2, Section 3.8].

CONJECTURE 7.2. *(λ-part of Bloch–Kato) Let $\phi \in S^{\mathrm{new}}_{2k-2}(\Gamma_0(M))$ be a newform with M odd and square-free. Let K be a number field containing $\mathbf{Q}(\phi)$ with ring of integers $\mathcal{O}_K$. Let ℓ be an odd prime, λ a prime of $\mathcal{O}_K$ dividing ℓ, and $\mathcal{O}$ the valuation ring of K_λ. Let $(\rho_{\phi,\lambda}, V_{\phi,\lambda})$ be the ℓ-adic Galois representation attached to ϕ. Then,*

$$\#X_\Sigma(\Sigma', W_{\phi,\lambda}(k-2))\mathcal{O} = L^\Sigma_{\mathrm{alg}}(k, \phi)\mathcal{O},$$

where Σ' consists of the primes dividing M and $\Sigma = \Sigma' \cup \{\ell\}$.

Recall that for an $\mathcal{O}$-module $\mathscr{M}$, we set

$$\mathrm{ord}_\ell(\#\mathscr{M}) = [\mathcal{O}/\lambda : \mathbf{F}_\ell]\,\mathrm{length}_{\mathcal{O}}(\mathscr{M}).$$

If we assume the mod λ reduction of ρ is absolutely irreducible, then $\mathrm{ord}_\ell(\#X_\Sigma(\Sigma', W_{\phi,\lambda}(k-2)))$ is independent of the choice of $T_{\phi,\lambda}(k-2)$.

THEOREM 7.3. [2, Theorem 8.8] *With the setup as above with $\overline{\rho}$ assumed to be irreducible, we have*

$$\mathrm{ord}_\ell(\#X_\Sigma(\Sigma', W_{\phi,\lambda}(k-2))) \geq \mathrm{ord}_\ell(\#\mathbf{T}^{\mathrm{N\text{-}SK}}_{\mathfrak{m}_{f_\phi}}/\Psi(\mathrm{Ann}(f_\phi))).$$

Combining this theorem with Theorem 6.6 and Corollary 6.7, we obtain the following theorem.

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THEOREM 7.4. *Let k and M be positive integers with $k \geq 6$ and M odd and square-free. Let $\Sigma' = \{p \mid M\}$. Let $\phi \in S_{2k-2}^{\text{new},-}(\Gamma_0(M))$ be a newform. Let $\ell > 2k-2$ be a prime with $\ell \nmid M \prod_{p \in \Sigma}(p^2-1)$ and $\overline{\rho}_{\phi,\lambda}$ irreducible. Set $\Sigma' = \Sigma \cup \{\ell\}$. Furthermore, if $M = 1$ or $M = 3$, assume ϕ is ordinary at ℓ. Let $\mathcal{O}$ be a sufficiently large finite extension of $\mathbf{Z}_\ell$ with maximal ideal λ and uniformizer ϖ. Assume that $\mathrm{ord}_\lambda(L_{\mathrm{alg}}(k,\phi)) > 0$. If there exists a negative fundamental discriminant D, so that $\ell \nmid D$, $M \nmid D$, D is a square modulo $4M$, and an integer $N > 1$ so that $M \mid N$, $\ell \nmid N[\Gamma^{\mathrm{para}}(M) : \Gamma_2(N)]$, and a Dirichlet character χ of conductor N so that*

$$\mathrm{ord}_\lambda\left(\frac{L^N(3-k,\chi)L_{\mathrm{alg}}^N(1,\phi,\chi)L_{\mathrm{alg}}^N(2,\phi,\chi)L_{\mathrm{alg}}(k-1,\phi,\chi_D)}{L(4,\chi_M)L_{\mathrm{alg}}(k,\phi)}\right) = -b < 0,$$

then,

$$\mathrm{ord}_\ell(\# X_\Sigma(\Sigma', W_{\phi,\lambda}(k-2))) \geq b.$$

If one can choose D, N, and χ, so that

$$\mathrm{ord}_\lambda(L^N(3-k,\chi)L_{\mathrm{alg}}^N(1,\phi,\chi)L_{\mathrm{alg}}^N(2,\phi,\chi)L_{\mathrm{alg}}(k-1,\phi,\chi_D)) = 0,$$

then we have

$$\mathrm{ord}_\ell(\# X_\Sigma(\Sigma', W_{\phi,\lambda}(k-2))) \geq \mathrm{ord}_\ell(\#\mathcal{O}/L_{\mathrm{alg}}^\Sigma(k,\phi)),$$

that is,

$$L_{\mathrm{alg}}^\Sigma(k,\phi)\mathcal{O} \subset \# X_\Sigma(\Sigma', W_{\phi,\lambda}(k-2))\mathcal{O}.$$

Proof. The only thing to note here is that since $\ell \nmid \prod_{p\in\Sigma}(p^2-1)$, we have $L_\Sigma(k,\phi)$ is a λ-adic unit. Thus, $L_{\mathrm{alg}}^\Sigma(k,\phi)\mathcal{O} = L_{\mathrm{alg}}(k,\phi)\mathcal{O}$. □

Note that the above theorem gives one direction of the λ-part of the Bloch–Kato conjecture for $W_{\phi,\lambda}(k-2)$ under the hypotheses of the theorem. While this theorem looks completely analogous to [2, Corollary 8.9], there is an important difference in the case that $M > 1$. In the case that $M = 1$, there is no difference between the congruence level and paramodular level Saito–Kurokawa lift as the requirement that k be even is equivalent to the requirement the sign of the functional equation be negative. However, in the case that $M > 1$, these are no longer equivalent. Thus, using the paramodular Saito–Kurokawa lift, we are able to consider forms $\phi \in S_{2k-2}^{\text{new},-}(\Gamma_0(M))$ for which k is not required to be even and apply Theorem 7.4 to give evidence for the Bloch–Kato conjecture in this case, a significant strengthening of previous results.

REFERENCES

1. M. Agarwal, p-adic L-functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, PhD Thesis (University of Michigan, Ann Arbor, MI, 2007).

2. M. Agarwal and J. Brown, On the Bloch-Kato conjecture for elliptic modular forms of square-free level, *Math. Z.* **276**(3–4) (2014), 889–924.

3. M. Agarwal and K. Klosin, Yoshida lifts and the Bloch-Kato conjecture for the convolution L-function, *J. Number Theory* **133**(8) (2013), 2496–2537.

4. M. Asgari and R. Schmidt, Siegel modular forms and representations, *Manuscripta Math.* **104**(2) (2001), 173–200.

5. S. Bloch and K. Kato, L-functions and Tamagawa numbers of motives, in *The Grothendieck Festschrift, Vol. I*, Progress in Mathematics, vol. 86 (Birkhäuser Boston, Boston, MA, 1990), 333–400.

https://doi.org/10.1017/S0017089520000439 Published online by Cambridge University Press

6. S. Böcherer, N. Dummigan and R. Schulze-Pillot, Yoshida lifts and Selmer groups, *J. Math. Soc. Japan* **64**(4) (2012), 1353–1405.

7. J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, *Compos. Math.* **143**(2) (2007), 290–322.

8. J. Brown, On the cuspidality of pullbacks of Siegel Eisenstein series and applications to the Bloch-Kato conjecture, *Int. Math. Res. Not.* **2011**(7) (2011), 1706–1756.

9. J. Brown and K. Klosin, Congruence primes for automorphic forms on unitary groups and applications to the arithmetic of Ikeda lifts, *Kyoto J. Math.* **60**(1) (2020), 179–217.

10. J. Brown and A. Pitale, Special values of L-functions for Saito-Kurokawa lifts, *Math. Proc. Cambridge Philos. Soc.* **155**(2) (2013), 237–255.

11. A. Brumer and K. Kramer, Paramodular abelian varieties of odd conductor, *Trans. Amer. Math. Soc.* **366**(5) (2014), 2463–2516.

12. S. Dasgupta, H. Darmon and R. Pollack, Hilbert modular forms and the Gross-Stark conjecture, *Ann. Math. (2)* **174**(1) (2011), 439–484.

13. J.-M. Fontaine, Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate, *Ann. Math. (2)* **115**(3) (1982), 529–577.

14. P. Garrett, On the arithmetic of Siegel-Hilbert cuspforms: Petersson inner products and Fourier coefficients, *Invent. Math.* **107**(3) (1992), 453–481.

15. V. Gritsenko, Irrationality of the moduli spaces of polarized abelian surfaces, in *Abelian varieties (Egloffstein, 1993)* (de Gruyter, Berlin, 1995), 63–84. With an appendix by the author and K. Hulek.

16. B. Gross, W. Kohnen and D. Zagier, Heegner points and derivatives of L-series. II, *Math. Ann.* **278**(1–4) (1987), 497–562.

17. H. Hida, Galois representations and the theory of p-adic Hecke algebras, *Sūgaku* **39**(2) (1987), 124–139. Sugaku Expositions **2**(1) (1989), 75–102.

18. H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, *Math. Z.* **259**(1) (2008), 97–111.

19. K. Klosin, Congruences among modular forms on U(2, 2) and the Bloch-Kato conjecture, *Ann. Inst. Fourier (Grenoble)* **59**(1) (2009), 81–166.

20. K. Klosin, The Maass space for $U(2, 2)$ and the Bloch-Kato conjecture for the symmetric square motive of a modular form, *J. Math. Soc. Japan* **67**(2) (2015), 797–860.

21. K. Ribet, A modular construction of unramified p-extensions of$Q(\mu_p)$, *Invent. Math.* **34**(3) (1976), 151–162.

22. B. Roberts and R. Schmidt, *Local newforms for GSp(4)*, Lecture Notes in Mathematics, vol. 1918 (Springer, Berlin, 2007).

23. K. Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147 (Princeton University Press, Princeton, NJ, 2000). Hermann Weyl Lectures. The Institute for Advanced Study.

24. R. Schmidt, On classical Saito-Kurokawa liftings, *J. Reine Angew. Math.* **604** (2007), 211–236.

25. G. Shimura, Confluent hypergeometric functions on tube domains, *Math. Ann.* **260**(3) (1982), 269–302.

26. G. Shimura, Nearly holomorphic functions on Hermitian symmetric spaces, *Math. Ann.* **278**(1–4) (1987), 1–28.

27. G. Shimura, Eisenstein series and zeta functions on symplectic groups, *Invent. Math.* **119**(3) (1995), 539–584.

28. G. Shimura, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics, vol. 93 (Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997).

29. C. Skinner and E. Urban, The Iwasawa main conjectures for GL_2, *Invent. Math.* **195**(1) (2014), 1–277.

30. N. Skoruppa and D. Zagier, Jacobi forms and a certain space of modular forms, *Invent. Math.* **94**(1) (1988), 113–146.

31. V. Vatsal, Canonical periods and congruence formulae, *Duke Math. J.* **98**(2) (1999), 397–419.

32. R. Weissauer, Four dimensional Galois representations, *Astérisque* (302) (2005), 67–150. Formes automorphes. II. Le cas du groupe GSp(4).

33. A. Wiles, The Iwasawa conjecture for totally real fields, *Ann. Math. (2)* **131**(3) (1990), 493–540.